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Invariant differential operators for non-compact Lie algebras parabolically related to conformal Lie algebras

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Abstract: In the present paper we continue the project of systematic construction of invariant differential operators for non-compact semisimple Lie groups. Our starting points is the class of algebras, which we call 'conformal Lie algebras' (CLA), which have very similar properties to the conformal algebras of Minkowski space-time, though our aim is to go beyond this class in a natural way. For this we introduce the new notion of *parabolic relation* between two non-compact semisimple Lie algebras $\mathcal{G}$ and $\mathcal{G}'$ that have the same complexification and possess maximal parabolic subalgebras with the same complexification. Thus, we consider the exceptional algebra $E_{7(7)}$ which is parabolically related to the CLA $E_{7(-25)}$, the parabolic subalgebras including $E_{6(6)}$ and $E_{6(-26)}$. Other interesting examples are the orthogonal algebras $so(p,q)$ all of which are parabolically related to the conformal algebra $so(n,2)$ with $p+q=n+2$, the parabolic subalgebras including the Lorentz subalgebra $so(n-1,1)$ and its analogs $so(p-1,q-1)$. We consider also $E_{6(6)}$ and $E_{6(2)}$ which are parabolically related to the hermitian symmetric case $E_{6(-14)}$, the parabolic subalgebras including real forms of $sl(6)$.

We also give a formula for the number of representations in the main multiplets valid for CLAs and all algebras that are parabolically related to them. In all considered cases we give the main multiplets of indecomposable elementary representations including the necessary data for all relevant invariant differential operators. In the case of $so(p,q)$ we give also the reduced multiplets. We should stress that the multiplets are given in the most economic way in pairs of *shadow fields*. Furthermore we should stress that the classification of all invariant differential operators includes as special cases all possible *conservation laws* and *conserved currents*, unitary or not.

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1 Introduction

Invariant differential operators play very important role in the description of physical symmetries — starting from the early occurrences in the Maxwell, d'Allembert, Dirac, equations, (for more examples cf., e.g., [1]), to the latest applications of (super-)differential operators in conformal field theory, supergravity and string theory (for reviews, cf. e.g., [2, 3].

For example, applications of invariant differential operators in supersymmetry involved the study of multiplets, superfields and supercurrents [4–8], of harmonic superspaces [9–19], of auxiliary fields of supergravity [20, 21], on the coupling of supersymmetric Yang-Mills theories to supergravity [22–24], twistor formulation of superstrings [25–27], Landau-Ginzburg description of $N = 2$ minimal models [28, 29], in various other applications to superstrings and supergravity [30–34].

Invariant differential operators played important role in the group-theoretical approach to conformal field theory [35–38], e.g., in the derivation of operator product expansion of two scalar fields.

Invariant super-differential operators were crucial in the derivation of the classification of positive energy unitary irreducible representations of extended conformal supersymmetry in 4D [39–41], later in 3D & 5D [42], in 6D [42, 43], (see also [44, 45]), then for the derivation of the character formulae in 2D [46]. Later applications include [47–71].

Special mentioning requires the applications of exceptional groups, cf. [72–94], since they play important role in the present paper. Exceptional groups recently appeared also as symmetries of Freudenthal dual Lagrangians, as investigated, e.g., in [95].

Finally, among our motivations are the mathematical developments — see the relevant mathematical references: [96–120], and others throughout the text.

Thus, it is important for the applications in physics to study systematically such operators. In a recent paper [121] we started the systematic explicit construction of invariant differential operators. We gave an explicit description of the building blocks, namely, the *parabolic subgroups and subalgebras* from which the necessary representations are induced. Thus we have set the stage for study of different non-compact groups.

Since the study and description of detailed classification should be done group by group we had to decide which groups to study first. A natural choice would be non-compact groups that have *discrete series* of representations. By the Harish-Chandra criterion [122, 123] these are groups where holds:

$$\mathrm{rank}\, G = \mathrm{rank}\, K,$$

where K is the *maximal compact subgroup* of the non-compact group G. Another formulation is to say that the Lie algebra $\mathcal{G}$ of G has a compact Cartan subalgebra.

Example: the groups $\mathrm{SO}(p,q)$ have discrete series, *except* when both p,q are *odd* numbers.

This class is still rather big, thus, we decided to consider a subclass, namely, the class of *Hermitian symmetric spaces*. The practical criterion is that in these cases, the *maximal compact subalgebra* $\mathcal{K}$ is of the form:

$$\mathcal{K} = so(2) \oplus \mathcal{K}' \ . \tag{1.1}$$

The Lie algebras from this class are:

$$so(n,2),\ \ sp(n,R),\ \ su(m,n),\ \ so^*(2n),\ \ E_{6(-14)}\,,\ \ E_{7(-25)} \tag{1.2}$$

These groups/algebras have *highest/lowest weight representations*, and relatedly holomorphic *discrete series representations*.

The most widely used of these algebras are the *conformal algebras* $so(n,2)$ in n-dimensional Minkowski space-time. In that case, there is a maximal *Bruhat decomposition* [124]:

$$\begin{aligned} so(n,2) &= \mathcal{P} \oplus \tilde{\mathcal{N}} = \mathcal{M} \oplus \mathcal{A} \oplus \mathcal{N} \oplus \tilde{\mathcal{N}}\,, \\ \mathcal{M} &= so(n-1,1)\,, \quad \dim \mathcal{A} = 1, \quad \dim \mathcal{N} = \dim \tilde{\mathcal{N}} = n \end{aligned} \tag{1.3}$$

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that has direct physical meaning, namely, $so(n-1,1)$ is the *Lorentz algebra* of n-dimensional Minkowski space-time, the subalgebra $\mathcal{A} = so(1,1)$ represents the *dilatations*, the conjugated subalgebras $\mathcal{N}$, $\tilde{\mathcal{N}}$ are the algebras of *translations*, and *special conformal transformations*, both being isomorphic to n-dimensional Minkowski space-time. The subalgebra $\mathcal{P} = \mathcal{M} \oplus \mathcal{A} \oplus \mathcal{N}$ $(\cong \mathcal{M} \oplus \mathcal{A} \oplus \tilde{\mathcal{N}})$ is a maximal parabolic subalgebra.[1]

There are other special features which are important. In particular, the complexification of the maximal compact subgroup is isomorphic to the complexification of the first two factors of the Bruhat decomposition:

$$\mathcal{K}^{\mathbb{C}} = so(n,\mathbb{C}) \oplus so(2,\mathbb{C}) \ \cong so(n-1,1)^{\mathbb{C}} \oplus so(1,1)^{\mathbb{C}} = \mathcal{M}^{\mathbb{C}} \oplus \mathcal{A}^{\mathbb{C}} \ . \tag{1.4}$$

In particular, the coincidence of the complexification of the semi-simple subalgebras:

$$\mathcal{K}'^{\mathbb{C}} = \mathcal{M}^{\mathbb{C}} \tag{1.5}$$

means that the sets of finite-dimensional (nonunitary) representations of $\mathcal{M}$ are in 1-to-1 correspondence with the finite-dimensional (unitary) representations of $so(n)$. The latter leads to the fact that the corresponding induced representations are representations of finite $\mathcal{K}$-type [122, 123].

It turns out that some of the hermitian-symmetric algebras share the above-mentioned special properties of $so(n,2)$. That is why, in view of applications to physics, these algebras should be called '*conformal Lie algebras*' (CLA), (or groups).

This subclass consists of:

$$so(n,2), \quad sp(n,\mathbb{R}), \quad su(n,n), \quad so^*(4n), \quad E_{7(-25)} \tag{1.6}$$

the corresponding analogs of Minkowski space-time V being:

$$\mathbb{R}^{n-1,1}, \quad \mathrm{Sym}(n,\mathbb{R}), \quad \mathrm{Herm}(n,\mathbb{C}), \quad \mathrm{Herm}(n,\mathbb{Q}), \quad \mathrm{Herm}(3,\mathbb{O}) \ . \tag{1.7}$$

The corresponding groups are also called '*Hermitian symmetric spaces of tube type*' [125]. The same class was identified from different considerations in [126] called there '*conformal groups of simple Jordan algebras*'. In fact, the relation between Jordan algebras and division algebras was known long time ago. Our class was identified from still different considerations also in [127] where they were called '*simple space-time symmetries generalizing conformal symmetry*'. For more references on Jordan algebras relevant in our approach cf., e.g., [128–144, 148].

We have started the study of the above class in the framework of the present approach in the cases: $so(n,2)$, $su(n,n)$, $sp(n,\mathbb{R})$, $E_{7(-25)}$, in [149–152], resp., and we have considered also the algebra $E_{6(-14)}$, [153, 154].

In the present paper we are mainly interested in non-compact Lie algebras (and groups) that are 'parabolically' related to the conformally Lie algebras.

• *Definition:* Let $\mathcal{G}, \mathcal{G}'$ be two non-compact semisimple Lie algebras with the same complexification $\mathcal{G}^{\mathbb{C}} \cong \mathcal{G}'^{\mathbb{C}}$. We call them *parabolically related* if they have parabolic subalgebras $\mathcal{P} = \mathcal{M} \oplus \mathcal{A} \oplus \mathcal{N}$, $\mathcal{P}' = \mathcal{M}' \oplus \mathcal{A}' \oplus \mathcal{N}'$, such that: $\mathcal{M}^{\mathbb{C}} \cong \mathcal{M}'^{\mathbb{C}}$ $(\Rightarrow \mathcal{P}^{\mathbb{C}} \cong \mathcal{P}'^{\mathbb{C}})$.$\diamondsuit$

[1]The precise general definition is given in section 2.

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Certainly, there are many such parabolic relationships for any given algebra $\mathcal{G}$. Furthermore, two algebras $\mathcal{G}, \mathcal{G}'$ may be parabolically related with different parabolic subalgebras. For example, the exceptional Lie algebras $E_{6(6)}$ and $E_{6(2)}$ are considered in section 7 (as related also to $E_{6(-14)}$) with maximal parabolics such that $\mathcal{M}^{\mathbb{C}} \cong \mathcal{M}'^{\mathbb{C}} \cong sl(6,\mathbb{C})$. But these two algebras are related also by another pair of maximal parabolics $\tilde{\mathcal{P}}^{\mathbb{C}}$, $\tilde{\mathcal{P}}'^{\mathbb{C}}$ such that $\tilde{\mathcal{M}}^{\mathbb{C}} \cong \tilde{\mathcal{M}}'^{\mathbb{C}} \cong sl(3,\mathbb{C}) \oplus sl(3,\mathbb{C}) \oplus sl(2,\mathbb{C})$, cf. [121], (11.4), (11.7).

Another interesting example are the algebras $so^*(2m)$ and $so(p,q)$ which have a series of maximal parabolics with $\mathcal{M}$-factors [121],:

$$\begin{aligned} \mathcal{M}_j &= su^*(2j) \oplus so^*(2m-4j)\,, \qquad j \leq \left[\tfrac{m}{2}\right]\,, \\ \mathcal{M}'_j &= sl(2j,\mathbb{R}) \oplus so(p-2j,q-2j)\,, \quad j \leq \left[\tfrac{q}{2}\right] \leq \left[\tfrac{p}{2}\right]\,, \end{aligned} \tag{1.8}$$

whose complexifications coincide for $p+q=2m$

$$(\mathcal{M}_j)^{\mathbb{C}} = (\mathcal{M}'_j)^{\mathbb{C}} = sl(2j,\mathbb{C}) \oplus so(2m-4j,\mathbb{C})\,, \quad j \leq \left[\tfrac{q}{2}\right] \leq \left[\tfrac{m}{2}\right] = \left[\tfrac{p+q}{4}\right]\,. \tag{1.9}$$

As we know only for $m=2n$, i.e., for $so^*(4n)$ is fulfilled relation (1.5), with $\mathcal{M} = \mathcal{M}_n = \,= su^*(2n)$ from (1.8), (recalling that $\mathcal{K}' \cong su(2n)$). Obviously, $so(p,q)$ is parabolically related to $so^*(4n)$ with this $\mathcal{M}$-factor only when $p=q=2n$, i.e., $\mathcal{G}' = so(2n,2n)$ with $\mathcal{M}'_n = sl(2n,\mathbb{R})$ (which is outside the range of (1.9)).

We leave the classification of the parabolic relations between the non-compact semisimple Lie algebras for a subsequent publication. In the present paper we consider mainly algebras parabolically related to conformal Lie algebras with maximal parabolics fulfilling (1.5). We summarize the relevant cases in table 1, where we have included also the algebra $E_{6(-14)}$; we display only the semisimple part $\mathcal{K}'$ of $\mathcal{K}$; $sl(n,\mathbb{C})_{\mathbb{R}}$ denotes $sl(n,\mathbb{C})$ as a real Lie algebra, (thus, $(sl(n,\mathbb{C})_{\mathbb{R}})^{\mathbb{C}} = sl(n,\mathbb{C}) \oplus sl(n,\mathbb{C})$); e_6 denotes the compact real form of E_6 ; and we have imposed restrictions to avoid coincidences or degeneracies due to well known isomorphisms: $so(1,2) \cong sp(1,\mathbb{R}) \cong su(1,1)$, $so(2,2) \cong so(1,2) \oplus so(1,2)$, $su(2,2) \cong so(4,2)$, $sp(2,\mathbb{R}) \cong so(3,2)$, $so^*(4) \cong so(3) \oplus so(2,1)$, $so^*(8) \cong so(6,2)$.

After this extended introduction we give the outline of the paper. In section 2 we give the preliminaries, actually recalling and adapting facts from [121]. We add a remark on *conservation laws* and *conserved currents* which are an integral part of our approach. In section 3 we consider the case of the pseudo-orthogonal algebras $so(p,q)$ which are parabolically related to the conformal algebra $so(n,2)$ for $p+q=n+2$. We add historical remarks and a remark on shadow representations. In section 4 we consider the algebras $su^*(4k)$ and $sl(4k,\mathbb{R})$ as parabolically related to the CLA $su(2k,2k)$. In section 5 we consider the algebra $sp(r,r)$ as parabolically related to the CLA $sp(2r)$ (of rank $2r$). In section 6 we consider the algebra $E_{7(7)}$ as parabolically related to the CLA $E_{7(-25)}$. In section 7 we consider the algebras $E_{6(6)}$ and $E_{6(2)}$ as parabolically related to the hermitian symmetric case $E_{6(-14)}$. In section 8 we give Summary and Outlook.

2 Preliminaries

Let G be a semisimple non-compact Lie group, and K a maximal compact subgroup of G. Then we have an *Iwasawa decomposition* $G = KA_0N_0$, where A_0 is Abelian simply con-

$\mathcal{G}$	$\mathcal{K}'$	$\mathcal{M}$ $\dim V$	$\mathcal{G}'$	$\mathcal{M}'$
$so(n,2)$ $n \geq 3$	$so(n)$	$so(n-1,1)$ n	$so(p,q)$, $p+q=$ $=n+2$	$so(p-1,q-1)$
$su(2k,2k)$ $k \geq 2$	$su(2k) \oplus su(2k)$	$sl(2k,\mathbb{C})_{\mathbb{R}}$ $(2k)^2$	$su^*(4k)$ $sl(4k,\mathbb{R})$	$su^*(2k) \oplus su^*(2k)$ $sl(2k,\mathbb{R}) \oplus sl(2k,\mathbb{R})$
$sp(2r,\mathbb{R})$ rank $=2r \geq 4$	$su(2r)$	$sl(2r,\mathbb{R})$ $r(2r+1)$	$sp(r,r)$	$su^*(2r)$
$so^*(4n)$ $n \geq 3$	$su(2n)$	$su^*(2n)$ $n(2n-1)$	$so(2n,2n)$	$sl(2n,\mathbb{R})$
$E_{7(-25)}$	e_6	$E_{6(-26)}$ 27	$E_{7(7)}$	$E_{6(6)}$
below not CLA!				
$E_{6(-14)}$	$so(10)$	$su(5,1)$ 21	$E_{6(6)}$ $E_{6(2)}$	$sl(6,\mathbb{R})$ $su(3,3)$

Table 1. Table of conformal Lie algebras (CLA) $\mathcal{G}$ with $\mathcal{M}$-factor fulfilling (1.5) and the corresponding parabolically related algebras $\mathcal{G}'$.

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nected vector subgroup of G, N_0 is a nilpotent simply connected subgroup of G preserved by the action of A_0. Further, let M_0 be the centralizer of A_0 in K. Then the subgroup $P_0 = M_0 A_0 N_0$ is a *minimal parabolic subgroup* of G. A *parabolic subgroup* $P = M'A'N'$ is any subgroup of G which contains a minimal parabolic subgroup.

Further, let $\mathcal{G}, \mathcal{K}, \mathcal{P}, \mathcal{M}, \mathcal{A}, \mathcal{N}$ denote the Lie algebras of G, K, P, M, A, N, resp.

For our purposes we need to restrict to *maximal parabolic subgroups* $P = MAN$, i.e. rank $A = 1$, resp. to *maximal parabolic subalgebras* $\mathcal{P} = \mathcal{M} \oplus \mathcal{A} \oplus \mathcal{N}$ with $\dim \mathcal{A} = 1$.

Let ν be a (non-unitary) character of A, $\nu \in \mathcal{A}^*$, parameterized by a real number d, called the *conformal weight* or energy.

Further, let μ fix a discrete series representation D^μ of M on the Hilbert space V_μ, or the finite-dimensional (non-unitary) representation of M with the same Casimirs.

We call the induced representation $\chi = \mathrm{Ind}_P^G(\mu \otimes \nu \otimes 1)$ an *elementary representation* of G [37]. (These are called *generalized principal series representations* (or *limits thereof*) in [155].) Their spaces of functions are:

$$\mathcal{C}_\chi = \{\mathcal{F} \in C^\infty(G, V_\mu) \mid \mathcal{F}(gman) = \\ = e^{-\nu(H)} \cdot D^\mu(m^{-1})\,\mathcal{F}(g)\} \tag{2.1}$$

where $a = \exp(H) \in A'$, $H \in \mathcal{A}'$, $m \in M'$, $n \in N'$. The representation action is the *left regular action*:

$$(\mathcal{T}^\chi(g)\mathcal{F})(g') = \mathcal{F}(g^{-1}g')\,, \quad g, g' \in G\,. \tag{2.2}$$

• An important ingredient in our considerations are the *highest/lowest weight representations* of $\mathcal{G}^{\mathbb{C}}$. These can be realized as (factor-modules of) Verma modules V^Λ over $\mathcal{G}^{\mathbb{C}}$, where $\Lambda \in (\mathcal{H}^{\mathbb{C}})^*$, $\mathcal{H}^{\mathbb{C}}$ is a Cartan subalgebra of $\mathcal{G}^{\mathbb{C}}$, weight $\Lambda = \Lambda(\chi)$ is determined uniquely from χ [156].

Actually, since our ERs may be induced from finite-dimensional representations of $\mathcal{M}$ the Verma modules are always reducible. Thus, it is more convenient to use *generalized Verma modules* $\tilde{V}^\Lambda$ such that the role of the highest/lowest weight vector v_0 is taken by the (finite-dimensional) space $V_\mu\, v_0$. For the generalized Verma modules (GVMs) the reducibility is controlled only by the value of the conformal weight d. Relatedly, for the intertwining differential operators only the reducibility w.r.t. non-compact roots is essential.

• One main ingredient of our approach is as follows. We group the (reducible) ERs with the same Casimirs in sets called *multiplets*. The multiplet corresponding to fixed values of the Casimirs may be depicted as a connected graph, the *vertices* of which correspond to the reducible ERs and the *lines (arrows)* between the vertices correspond to intertwining operators. The explicit parametrization of the multiplets and of their ERs is important for understanding of the situation. The notion of multiplets was introduced in [157, 158] and applied to representations of $\mathrm{SO}_o(p,q)$ and $\mathrm{SU}(2,2)$, resp., induced from their minimal parabolic subalgebras. Then it was applied to the conformal superalgebra [159], to infinite-dimensional (super-)algebras [160–163], to quantum groups [164].[2]

Remark: Note that the multiplets of Verma modules include in general more members, since there enter Verma modules which are induced from infinite-dimensional representations of $\mathcal{M}$ but nevertheless have the same Casimirs. The main multiplets in this case contain as many members as the Weyl group $W(\mathcal{G}^{\mathbb{C}})$ of $\mathcal{G}^{\mathbb{C}}$. For example, for $su(2,2)$ the maximal multiplets contain 24 members ($|W(sl(\ell,\mathbb{C}))| = \ell!$), which were considered in [158] and the $su(2,2)$ sextets of ERs induced from the maximal parabolic with $\mathcal{M} = sl(2,\mathbb{C})$ are submerged in the 24-member multiplets.$\diamondsuit$

In fact, the multiplets contain explicitly all the data necessary to construct the intertwining differential operators. Actually, the data for each intertwining differential operator consists of the pair (β, m), where β is a (non-compact) positive root of $\mathcal{G}^{\mathbb{C}}$, $m \in \mathbb{N}$, such that the *BGG Verma module reducibility condition* (for highest weight modules) is fulfilled:

$$(\Lambda + \rho, \beta^\vee) = m\,, \quad \beta^\vee \equiv 2\beta/(\beta,\beta) \tag{2.3}$$

where ρ is half the sum of the positive roots of $\mathcal{G}^{\mathbb{C}}$. When the above holds then the Verma module with shifted weight $V^{\Lambda - m\beta}$ (or $\tilde{V}^{\Lambda - m\beta}$ for GVM and β non-compact) is embedded

[2]For other applications we refer to [165–168].

in the Verma module V^Λ (or $\tilde{V}^\Lambda$). This embedding is realized by a singular vector v_s expressed by a polynomial $\mathcal{P}_{m,\beta}(\mathcal{G}^-)$ in the universal enveloping algebra $(U(\mathcal{G}_-))\, v_0$, $\mathcal{G}^-$ is the subalgebra of $\mathcal{G}^\mathbb{C}$ generated by the negative root generators [169]. More explicitly, [156], $v^s_{m,\beta} = \mathcal{P}_{m,\beta}\, v_0$ (or $v^s_{m,\beta} = \mathcal{P}_{m,\beta}\, V_\mu\, v_0$ for GVMs).[3]

Then there exists [156] an *intertwining differential operator* of order $m = m_\beta$:

$$\mathcal{D}_{m,\beta} : \mathcal{C}_{\chi(\Lambda)} \longrightarrow \mathcal{C}_{\chi(\Lambda - m\beta)} \tag{2.4}$$

given explicitly by:

$$\mathcal{D}_{m,\beta} = \mathcal{P}_{m,\beta}(\widehat{\mathcal{G}^-}) \tag{2.5}$$

where $\widehat{\mathcal{G}^-}$ denotes the *right action* on the functions $\mathcal{F}$.

Thus, in each such situation we have an *invariant differential equation* of order $m = m_\beta$:

$$\mathcal{D}_{m,\beta}\, f = f' , \qquad f \in \mathcal{C}_{\chi(\Lambda)} , \quad f' \in \mathcal{C}_{\chi(\Lambda - m\beta)} . \tag{2.6}$$

In most of these situations the invariant operator $\mathcal{D}_{m,\beta}$ has a non-trivial invariant kernel in which a subrepresentation of $\mathcal{G}$ is realized. Thus, studying the equations with trivial r.h.s.:

$$\mathcal{D}_{m,\beta}\, f = 0 , \qquad f \in \mathcal{C}_{\chi(\Lambda)} , \tag{2.7}$$

is also very important. For example, in many physical applications in the case of first order differential operators, i.e., for $m = m_\beta = 1$, equations (2.7) are called *conservation laws*, and the elements $f \in \ker \mathcal{D}_{m,\beta}$ are called *conserved currents*.

The above construction works also for the *subsingular vectors* $v_{\rm ssv}$ of Verma modules. Such a vector is also expressed by a polynomial $\mathcal{P}_{\rm ssv}(\mathcal{G}^-)$ in the universal enveloping algebra: $v^s_{\rm ssv} = \mathcal{P}_{\rm ssv}(\mathcal{G}^-)\, v_0$, cf. [172, 173]. Thus, there exists a *conditionally invariant differential operator* given explicitly by: $\mathcal{D}_{\rm ssv} = \mathcal{P}_{\rm ssv}(\widehat{\mathcal{G}^-})$, and a *conditionally invariant differential equation*, for many more details, see [172, 173]. (Note that these operators/equations are not of first order.)

Below in our exposition we shall use the so-called Dynkin labels:

$$m_i \equiv (\Lambda + \rho, \alpha_i^\vee) , \quad i = 1, \dots, n, \tag{2.8}$$

where $\Lambda = \Lambda(\chi)$, ρ is half the sum of the positive roots of $\mathcal{G}^\mathbb{C}$.

We shall use also the so-called Harish-Chandra parameters:

$$m_\beta \equiv (\Lambda + \rho, \beta) , \tag{2.9}$$

where β is any positive root of $\mathcal{G}^\mathbb{C}$. These parameters are redundant, since they are expressed in terms of the Dynkin labels, however, some statements are best formulated in their terms.[4]

[3]For explicit expressions for singular vectors we refer to [170, 171].

[4]Clearly, both the Dynkin labels and Harish-Chandra parameters have their origin in the BGG reducibility condition (2.3).

3 The pseudo-orthogonal algebras $so(p, q)$

3.1 Choice of parabolic subalgebra

Let $\mathcal{G} = so(p, q)$, $p \geq q$, $p + q > 4$.[5] Most of the results here are known for $q = 1, 2$, cf. [149, 174–176], and the purpose of the consideration is to extend those for arbitrary q.

For fixed p, q this algebra has at least q maximal parabolic subalgebras [121]. For example, when $p > q$ there are the following possibilities for $\mathcal{M}$-factor (cf. (7.11) of [121]):

$$\mathcal{M}_j^{\max} = sl(j, \mathbb{R}) \oplus so(p - j, q - j)\,, \quad j = 1, 2, \ldots, q\ . \tag{3.1}$$

(There are more choices when $p = q$.)

We would like to consider a case, which would relate parabolically all $\mathcal{G} = so(p, q)$ for $p + q$-fixed. Thus, in order in order to include the case $q = 1$ (where there is only one parabolic which is both minimal and maximal), we choose the case $j = 1$:

$$\mathcal{M} = \mathcal{M}_1^{\max} = so(p - 1, q - 1)\ . \tag{3.2}$$

Then we have:

$$\dim \mathcal{N} = \dim \tilde{\mathcal{N}} = p + q - 2\ . \tag{3.3}$$

With this choice we get for the conformal algebra exactly the Bruhat decomposition in (1.3).

We label the signature of the ERs of $\mathcal{G}$ as follows:

$$\begin{aligned} \chi &= \{\, n_1\,, \ldots, \, n_h\,;\, c\,\}\,, \\ n_j &\in \mathbb{Z}/2\,, \quad c = d - \tfrac{p+q-2}{2}\,, \quad h \equiv \left[\tfrac{p+q-2}{2}\right], \\ |n_1| &< n_2 < \cdots < n_h\,, \qquad\qquad p + q \ \ \text{even}\,, \\ 0 &< n_1 < n_2 < \cdots < n_h\,, \qquad p + q \ \ \text{odd}\,, \end{aligned} \tag{3.4}$$

where the last entry of χ labels the characters of $\mathcal{A}$, and the first h entries are labels of the finite-dimensional nonunitary irreps of $\mathcal{M} \cong so(p - 1, q - 1)$.

The reason to use the parameter c instead of d will become clear below.

3.2 Main multiplets

Following results of [149, 174–176] we present the main multiplets (which contain the maximal number of ERs with this parabolic) with the explicit parametrization of the ERs

[5] We shall explain the last restriction at the end of this section.

in the multiplets in a simple way (helped by the use of the signature entry c):

$$\begin{aligned}
\chi_1^\pm &= \{\epsilon\, n_1\,,\ldots,\, n_h\,;\, \pm n_{h+1}\}\,, \\
&\quad n_h < n_{h+1}\,, \\
\chi_2^\pm &= \{\epsilon\, n_1\,,\ldots,\, n_{h-1}\,,\, n_{h+1}\,;\, \pm n_h\} \\
\chi_3^\pm &= \{\epsilon\, n_1,\ldots, n_{h-2}, n_h, n_{h+1}\,;\, \pm n_{h-1}\} \\
&\ldots \\
\chi_{h-1}^\pm &= \{\epsilon\, n_1\,, n_2\,, n_4\,,\ldots,\, n_h\,,\, n_{h+1}\,;\, \pm n_3\} \\
\chi_h^\pm &= \{\epsilon\, n_1\,, n_3\,,\ldots,\, n_h\,,\, n_{h+1}\,;\, \pm n_2\} \\
\chi_{h+1}^\pm &= \{\epsilon\, n_2\,, n_3\,,\ldots,\, n_h\,,\, n_{h+1}\,;\, \pm n_1\} \\
\epsilon &= \begin{cases} \pm\,, & p+q \;\; even \\ 1, & p+q \;\; odd \end{cases}
\end{aligned} \tag{3.5}$$

($\epsilon = \pm$ is correlated with $\chi^\pm$). Clearly, the multiplets correspond 1-to-1 to the finite-dimensional irreps of $so(p+q,\mathbb{C})$ with signature $\{n_1,\ldots,n_h,n_{h+1}\}$ and we are able to use previous results due to the parabolic relation between the $so(p,q)$ algebras for $p+q$-fixed.

Note that the two representations in each pair $\chi^\pm$ were called *shadow fields* in the 1970s, see more on this towards the end of this section.

Further, the number of ERs in the corresponding multiplets is equal to $2\left[\frac{p+q}{2}\right] = 2(h+1)$. This maximal number is equal to the following ratio of numbers of elements of Weyl groups:

$$|W(\mathcal{G}^{\mathbb{C}},\mathcal{H}^{\mathbb{C}})|\,/\,|W(\mathcal{M}^{\mathbb{C}},\mathcal{H}_m^{\mathbb{C}})|\,, \tag{3.6}$$

where $\mathcal{H}^{\mathbb{C}}$, $\mathcal{H}_m^{\mathbb{C}}$ are Cartan subalgebras of $\mathcal{G}^{\mathbb{C}}$, $\mathcal{M}^{\mathbb{C}}$, resp.

The above formula actually holds for all conformal Lie algebras and those parabolically related to them. More precisely, we have:

- *The number of elements of the main multiplets of a conformal Lie algebra $\mathcal{G}$ with $\mathcal{M}$-factor fulfilling* (1.5) *is given by* (3.6). *The same number holds for any algebra $\mathcal{G}'$ parabolically related to $\mathcal{G}$ w.r.t. $\mathcal{M}$.*$\diamondsuit$

Further, we denote by $\mathcal{C}_i^\pm$ the representation space with signature $\chi_i^\pm$.

We first give the multiplets pictorially in figures 1 and 2 for $p+q$ even and odd, resp., and then explain notations and results:

The ERs in the multiplet are related by *intertwining integral and differential operators.*

The *integral operators* were introduced by Knapp and Stein [177, 178]. They correspond to elements of the restricted Weyl group of $\mathcal{G}$. In fact, these operators are defined for any ER, not only for the reducible ones, the general action being in the context of (3.4), (3.5):

$$\begin{aligned}
G \;:\; & \mathcal{C}_\chi \longrightarrow \mathcal{C}_{\chi'}\,, \\
\chi &= \{\, n_1\,,\ldots,\, n_h\,;\, c\,\}\,, \\
\chi' &= \{\,(-1)^{p+q+1} n_1\,,\ldots,\, n_h\,;\, -c\,\}\,.
\end{aligned} \tag{3.7}$$

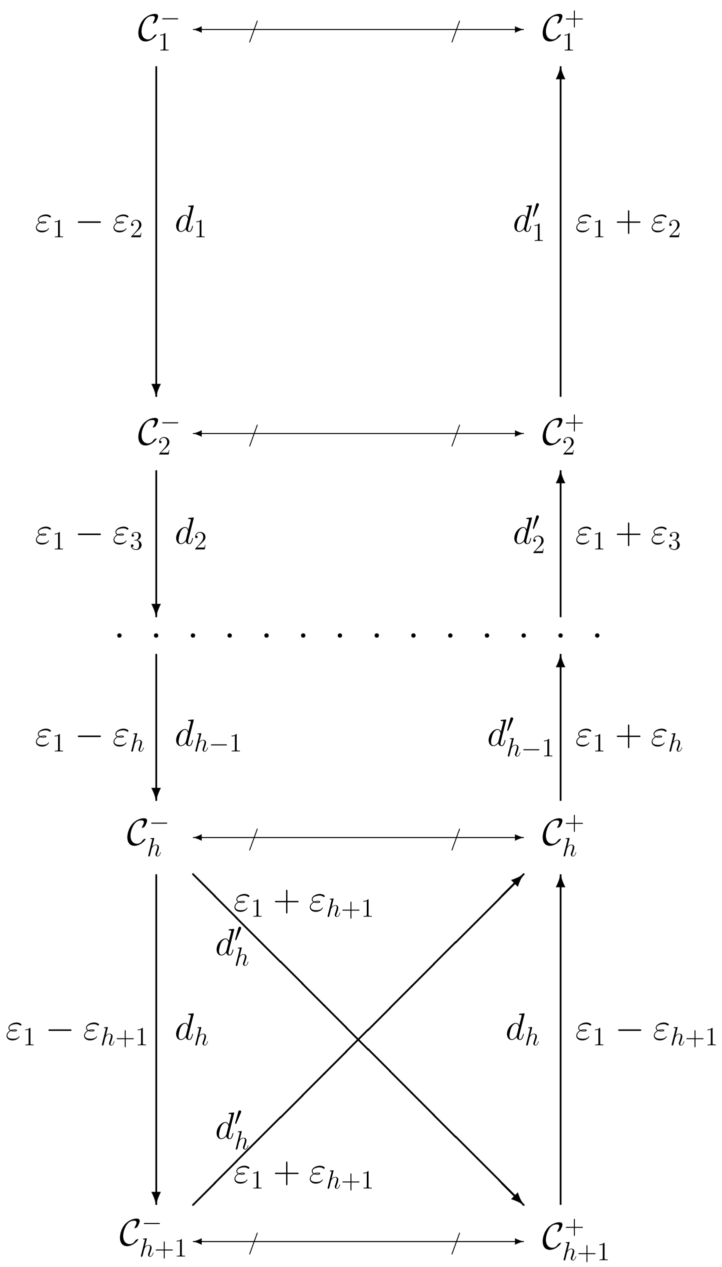

Figure 1. Main multiplet for $\mathrm{SO}(p,q)$ for $p+q = 2h+2 \geq 6$, with maximal parabolic subalgebra $\mathcal{P} = \mathcal{M} \oplus \mathcal{A} \oplus \mathcal{N}$ where $\mathcal{M}^{\mathbb{C}} = so(2h, \mathbb{C})$ (arrows are differential operators d_i, d'_i, dashed arrows are integral operators) $\varepsilon_1 \pm \varepsilon_k$, are the non-compact roots.

These operators intertwine the pairs $\mathcal{C}_i^{\pm}$ (cf. (3.5)):

$$G_i^{\pm} : \ \mathcal{C}_i^{\mp} \longrightarrow \mathcal{C}_i^{\pm} \, , \quad i = \ 1, \ldots, 1+h \ . \tag{3.8}$$

In the conformal setting (both Euclidean $q = 1$ and Minkowskian $q = 2$) the integral kernel of the Knapp-Stein operator is given by the conformal two-point function [37].

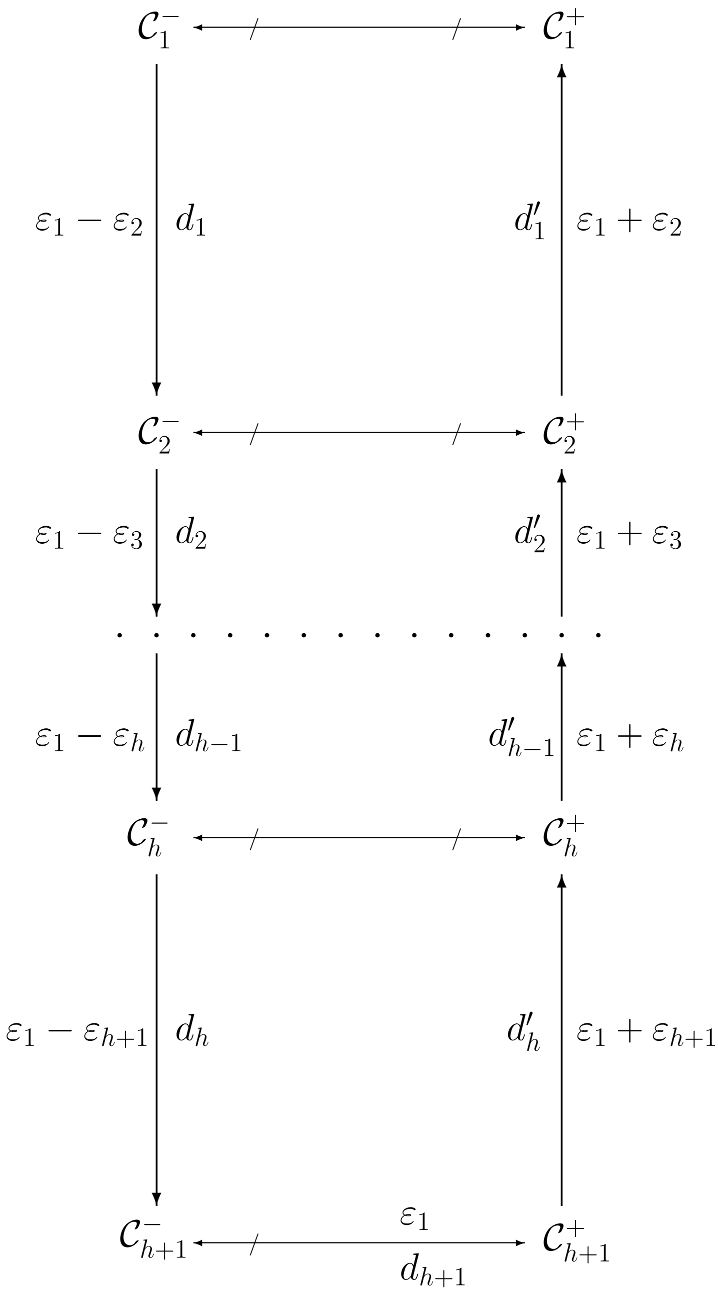

Figure 2. Main multiplet for $\mathrm{SO}(p,q)$ for $p+q = 2h+3 \geq 5$, with maximal parabolic subalgebra $\mathcal{P} = \mathcal{M} \oplus \mathcal{A} \oplus \mathcal{N}$ where $\mathcal{M}^{\mathbb{C}} = so(2h+1, \mathbb{C})$ (arrows are differential operators d_i, d'_i, dashed arrows are integral operators) $\varepsilon_1 \pm \varepsilon_k$, ε_1 are the non-compact roots.

The *intertwining differential operators* correspond to non-compact positive roots of the root system of $so(p+q, \mathbb{C})$, cf. [156]. In the current context, compact roots of $so(p+q, \mathbb{C})$ are those that are roots also of the subalgebra $so(p+q-2, \mathbb{C})$, the rest of the roots are non-compact. In more detail, we briefly recall the root systems:

For $p+q=2h+2$ even, the positive root system of $so(2h+2,\mathbb{C})$ may be given by vectors $\varepsilon_i \pm \varepsilon_j$, $1 \leq i < j \leq h+1$, where ε_i form an orthonormal basis in $\mathbb{R}^{h+1}$, i.e., $(\varepsilon_i,\varepsilon_j)=\delta_{ij}$. The non-compact roots may be taken as $\varepsilon_1 \pm \varepsilon_i$, $2 \leq i \leq h+1$. The root $\varepsilon_1-\varepsilon_i$ corresponds to the operator d_{i-1}, the root $\varepsilon_1+\varepsilon_i$ corresponds to the operator d'_{i-1}.

For $p+q=2h+3$ odd, the positive root system of $so(2h+3,\mathbb{C})$ may be given by vectors $\varepsilon_i \pm \varepsilon_j$, $1 \leq i < j \leq h+1$, ε_k, $1 \leq k \leq h+1$. The non-compact roots may be taken as $\varepsilon_1 \pm \varepsilon_i$, ε_1. The root $\varepsilon_1 - \varepsilon_i$ corresponds to the operator d_{i-1}, the root $\varepsilon_1+\varepsilon_i$ corresponds to the operator d'_{i-1}. The root ε_1 has a special position since it intertwines the same ERs that are intertwined by the Knapp-Stein integral operator G^+_{h+1}. The latter means that G^+_{h+1} degenerates to the differential operator d_{h+1}, and this degenerations determines that $d_{h+1} \sim \Box^{n_1}$, (for $n_1 \in \mathbb{N}$), where $\Box$ is the d'Alembert operator, as explained explicitly for the case $so(3,2)$ in [179]. (The latter phenomenon happens for the Knapp-Stein integral operators at critical points, but usually there is no non-compact root involved, cf., e.g., [37].)

The degrees of these intertwining differential operators are given just by the differences of the c entries [176]:

$$\begin{aligned}
\deg d_i &= \deg d'_i = n_{h+2-i} - n_{h+1-i}\,, \qquad i=1,\ldots,h\,, \\
\deg d'_h &= n_2 + n_1\,, \quad p+q \ \text{ even}\,, \\
\deg d_{h+1} &= 2n_1\,, \qquad p+q \ \text{ odd}\,.
\end{aligned} \tag{3.9}$$

where d'_h is omitted from the first line for $(p+q)$ even. By our construction they are equal to the Harish-Chandra parameters for the non-compact roots:

$$\deg d_i = m_{\varepsilon_1-\varepsilon_{i+1}}\,, \tag{3.10}$$

$$\deg d'_i = m_{\varepsilon_1+\varepsilon_{i+1}}\,, \qquad i=1,\ldots,h\,,$$

$$\deg d_{h+1} = m_{\varepsilon_1}\,. \tag{3.11}$$

Matters are arranged so that in every multiplet only the ER with signature χ_1^- contains a *finite-dimensional nonunitary subrepresentation* in a subspace $\mathcal{E}$. The latter corresponds to the finite-dimensional unitary irrep of $so(p+q)$ with signature $\{n_1,\ldots,n_h,n_{h+1}\}$. The subspace $\mathcal{E}$ is annihilated by the operator G^+_1, and is the image of the operator G^-_1.

Although the diagrams are valid for arbitrary $so(p,q)$ $(p+q\geq 5)$ the contents is very different. We comment only on the ER with signature χ_1^+. In all cases it contains an UIR of $so(p,q)$ realized on an invariant subspace $\mathcal{D}$ of the ER χ_1^+. That subspace is annihilated by the operator G^-_1, and is the image of the operator G^+_1. (Other ERs contain more UIRs.)

If $pq \in 2\mathbb{N}$ the mentioned UIR is a discrete series representation. Other ERs contain more discrete series UIRs. The number of discrete series is given by the formula [155]:

$$|W(\mathcal{G}^{\mathbb{C}},\mathcal{H}^{\mathbb{C}})|\,/\,|W(\mathcal{K}^{\mathbb{C}},\mathcal{H}^{\mathbb{C}})|\,, \tag{3.12}$$

where $\mathcal{H}^{\mathbb{C}}$ is a Cartan subalgebra of both $\mathcal{G}^{\mathbb{C}}$ and $\mathcal{K}^{\mathbb{C}}$.

And if $q=2$ the invariant subspace $\mathcal{D}$ is the direct sum of two subspaces $\mathcal{D}=\mathcal{D}^+\oplus\mathcal{D}^-$, in which are realized a *holomorphic discrete series representation* and its conjugate *anti-holomorphic discrete series representation*, resp. These are contained only in χ_1^+ and count for two series in the formula (3.12). Furthermore, any holomorphic discrete series representation is infinitesimally equivalent to a *lowest weight GVM* of the conformal algebra $so(p,2)$, while an anti-holomorphic discrete series representation is infinitesimally equivalent to a *highest weight GVM*.

Highest/lowest weight GVMs are related to other pairs besides χ_1^+.

A detailed analysis of these occurrences is done for the conformal algebra $so(3,2)$ in [149] and for $so(4,2)$ in [149, 175].

3.3 Reduced multiplets

Besides the main multiplets which are 1-to-1 with the finite-dimensional irreps of $so(p+q,\mathbb{C})$, there are other multiplets which we describe here.

• We start with the case $p+q$ even. In this case there are $h+1(=(p+q)/2)$ multiplets — doublets — each consisting of a pair with signatures $\tilde\chi^\pm$ given explicitly as follows:

$$\begin{aligned}
\tilde\chi_1^\pm &= \{\pm n_1\,,\ldots,\,n_h\,;\,\pm n_h\} \\
\tilde\chi_2^\pm &= \{\pm n_1\,,\ldots,\,n_{h-1}\,,\,n_{h+1}\,;\,\pm n_{h-1}\} \\
\tilde\chi_3^\pm &= \{\pm n_1,\ldots,n_{h-2},n_h,n_{h+1}\,;\,\pm n_{h-2}\} \\
&\ldots \\
\chi_{h-1}^\pm &= \{\pm n_1\,,n_2\,,n_4\,,\ldots,\,n_h\,,\,n_{h+1}\,;\,\pm n_2\} \\
\tilde\chi_h^\pm &= \{\pm n_1\,,n_3\,,\ldots,\,n_h\,,\,n_{h+1}\,;\,\pm n_1\}\,, \qquad n_1\neq 0 \\
\tilde\chi_{h+1}^\pm &= \{\mp n_1\,,n_3\,,\ldots,\,n_h\,,\,n_{h+1}\,;\,\pm n_1\}\,, \qquad n_1\neq 0
\end{aligned} \tag{3.13}$$

Clearly, the signature $\tilde\chi_i^\pm$ may be obtained from $\chi_i^\pm$ by setting the corresponding Harish-Chandra parameter equal to zero:

$$m_{\varepsilon_1\pm\varepsilon_{i+1}} = \deg d_i = \deg d'_i = n_{h+2-i} - n_{h+1-i} = 0\,, \qquad i=1,\ldots,h-1\,,$$

$$m_{\varepsilon_1-\varepsilon_{h+1}} = \deg d_h = n_2 - n_1 = 0\,, \quad \text{for } \tilde\chi_h^\pm\,, \tag{3.14}$$

$$m_{\varepsilon_1+\varepsilon_{h+1}} = \deg d'_h = n_2 + n_1 = 0\,, \quad \text{for } \tilde\chi_{h+1}^\pm\,. \tag{3.15}$$

Although written compactly as (3.5) no pair is related to any other pair. This may be seen easily as follows. Consider (3.5) and set formally $n_{h+1} = n_h$. The signatures $\chi_1^\pm$ and $\chi_2^\pm$ coincide are become equal to $\tilde\chi_1^\pm$, but the rest of the signatures $\chi_i^\pm$, $i\geq 3$ are not allowed in our class, e.g.,

$$\chi_3^\pm \;\longrightarrow \{\epsilon\, n_1,\ldots,n_{h-2},n_h,n_h\,;\,\pm n_{h-1}\}$$

is not allowed since it violates (3.4) due to equality of two $\mathcal{M}$-signature entries (n_h). Thus, from the whole multiplet only the pair $\tilde\chi_1^\pm$ remains in our class. Similarly for the rest of the pairs.

Inside a fixed pair $\tilde\chi_i^\pm$, $i=1,\ldots,h+1$, act two operators: a Knapp-Stein integral operator from $\tilde\chi_i^+$ to $\tilde\chi_i^-$, and a differential operator from $\tilde\chi_i^-$ to $\tilde\chi_i^+$. In more detail:

• Let first $i=1,\ldots,h-1$. Inside a fixed pair $\tilde\chi_i^\pm$, acts the Knapp-Stein integral operator G_i^- (3.8) (coinciding with G_{i+1}^- for this signature), and a differential operator $\tilde d_i$ of degree

$2n_{h+1-i}$ which is a degeneration of the Knapp-Stein integral operator G^+_i (coinciding with G^+_{i+1} for this signature). For this differential operator for $n_1 = 0$ we have: $\tilde{d}_i \sim \Box^{n_{h+1-i}}$, ($n_{h+1-i} \in \mathbb{N}$).[6]

• Inside the fixed pair $\tilde{\chi}^\pm_h$ acts the Knapp-Stein integral operator G^-_h (3.8) (coinciding with G^-_{h+1} for this signature), and the differential operator d'_h of degree $2n_1$ (cf. the previous subsection) which in addition is a degeneration of the Knapp-Stein integral operator G^+_h (coinciding with G^+_{h+1} for this signature).

• Inside the fixed pair $\tilde{\chi}^\pm_{h+1}$ acts the Knapp-Stein integral operator G^-_{h+1} (3.8) (coinciding with G^+_h for this signature), and the differential operator d_h of degree $2n_2$ which in addition is a degeneration of the Knapp-Stein integral operator G^+_{h+1} (coinciding with G^-_h for this signature).

• We continue with the case $p+q$ odd. In this case there are h doublets[7] with signatures $\hat{\chi}^\pm$ given similarly to the even case as follows:

$$\begin{aligned}
\hat{\chi}^\pm_1 &= \{n_1\,, \ldots,\, n_h\,;\, \pm n_h\} \\
\hat{\chi}^\pm_2 &= \{n_1\,, \ldots,\, n_{h-1}\,,\, n_{h+1}\,;\, \pm n_{h-1}\} \\
\hat{\chi}^\pm_3 &= \{n_1, \ldots, n_{h-2}, n_h, n_{h+1}\,;\, \pm n_{h-2}\} \\
&\ldots \\
\hat{\chi}^\pm_h &= \{n_1\,, n_3\,, \ldots,\, n_h\,,\, n_{h+1}\,;\, \pm n_1\}
\end{aligned} \tag{3.16}$$

The signature $\hat{\chi}^\pm_i$ may be obtained from $\chi^\pm_i$ by setting the corresponding Harish-Chandra parameter equal to zero:

$$m_{\varepsilon_1 \pm \varepsilon_{i+1}} = \deg d_i = \deg d'_i = n_{h+2-i} - n_{h+1-i} = 0\,, \qquad i = 1, \ldots, h\,. \tag{3.17}$$

Inside a fixed pair $\hat{\chi}^\pm_i$, $i = 1, \ldots, h$, acts the Knapp-Stein integral operator G^-_i (3.8) (coinciding with G^-_{i+1} for this signature), and a differential operator $\hat{d}_i$ of degree $2n_{h+1-i}$ which is a degeneration of the Knapp-Stein integral operator G^+_i (coinciding with G^+_{i+1} for this signature). For the differential operators we have $\hat{d}_i \sim \Box^{n_{h+1-i}}$, (when $n_{h+1-i} \in \mathbb{N}$). The difference with the even situation is only for $i = h$, where the degeneration of G^+_{h+1} was present already in the main multiplet.

If $pq \in 2\mathbb{N}$ the representations $\tilde{\chi}^+_1$, $\hat{\chi}^+_1$, contain an UIR called limits of the discrete series representations. And if $q = 2$ that UIR is the direct sum of two subspaces in which are realized *limits of holomorphic discrete series representation* and its conjugate *limits of anti-holomorphic discrete series representation*, resp. The latter do not happen in any other doublet, while limits of discrete series representations happen in other doublets. (For more on this see [149] for $so(3,2)$ and [149, 175] for $so(4,2)$.)

[6]For $so(4,2)$, ($h = 2, i = 1$), when $n_1 = 0, n_2 = 1$ the latter d'Alembert operator arises also as a conditionally invariant differential operator due to the presence of a subsingular vector in the corresponding Verma module [172, 173].

[7]In the case $so(3,2)$ there are two additional doublets [149] involving the two singleton representations, which are special for $so(3,2)$.

3.4 Remarks on shadow fields and history

• We labelled the signature of the ERs in (3.4) as

$$\chi = \{\, n_1\,, \dots,\, n_h\,;\, c\,\}$$

using the parameter c instead of the conformal weight $d = c + \frac{p+q-2}{2}$. This was used already in [37] since the multiplets were given more economically in terms of pairs of ERs in which the parameter c just changes sign. (Also mathematicians use the parameter c due to the fact that in its terms the representation parameter space looks simple: the principal unitary series representation induced from a maximal parabolic is given by $c = i\rho$, $\rho \in \mathbb{R}$; the supplementary series of unitary representations is given by $-s < c < s$, $s \in \mathbb{R}$, etc.)

Otherwise in the current context we should use for each Knapp-Stein operators conjugated doublet of shadow fields:

$$\begin{aligned} \chi^+ &= [\, n_1\,, \dots,\, n_h\,;\, d\,]\,, \qquad n_j \in \mathbb{Z}/2\,, \\ \chi^- &= [\,(-1)^{p+q+1} n_1\,, \dots,\, n_h\,;\, d_{\text{shadow}} = p+q-2-d\,]\;. \end{aligned} \tag{3.18}$$

The reason the representations $\chi^\pm$ in the 1970s were called "shadow fields" in the context of the conformal algebra $so(n,2)$ is that the sum of their conformal weights equals the dimension n of Minkowski space-time - isomorphic to $\mathcal{N}$ or $\tilde{\mathcal{N}}$, cf. (3.3). This continues to be true for general $so(p,q)$:

$$d + d_{\text{shadow}} = p + q - 2 = n\,, \tag{3.19}$$

and also for all conformal Lie algebras considered in the next sections.

Shadow fields appear all the time in conformal field theory. For example, in [180] we showed that in the generic case each field on the AdS bulk has *two* boundary fields which are shadow fields being related by a integral Knapp-Stein operator. Later Klebanov-Witten [181] showed that these two boundary fields are related by a Legendre transform.

For a current discussion on shadow fields we refer to [182].

• The diagram for $p+q$ even appeared first for the Euclidean conformal group in four-dimensional space-time $\mathrm{SU}^*(4) \cong \mathrm{Spin}(5,1)$ in [174]. Later it was generalised to the Minkowskian conformal group in four-dimensional space-time $\mathrm{SO}(4,2)$ in [175]. In both cases, the three $(= (p+q)/2)$ doublets (from the previous subsection) were also given together the corresponding degeneration of the Knapp-Stein integral operators.

The exposition above including figures 1 & 2 follows the exposition for Euclidean case $so(n+1,1)$ in [176]. Later the results were generalised to the Minkowskian case $so(n,2)$ [149].

• Actually, the case of Euclidean conformal group in arbitrary dimensions $\mathrm{SO}(p,1)$ was studied in [37] for representations of $\mathcal{M} = so(p-1)$ which are symmetric traceless tensors. This means in (3.4) we should set $n_1 = n_2 = \cdots = n_{h-1} = 0$, and then only the first two pairs $\chi_1^\pm, \chi_2^\pm$ in (3.5) are possible. Thus from the two figures only the upper quadrants are relevant, and were given in [37], cf. figure 1 there.

• Above we restricted to $p+q \geq 5$. The excluded cases are: $so(3,1)$, $so(2,2) \cong so(2,1) \oplus so(2,1)$, $so(2,1)$, ($so(1,1)$ is abelian).

In the case $so(3,1) \cong sl(2,\mathbb{C})$ the multiplet in general contains only four ERs, and is in fact representable by the diagram in the case of symmetric traceless tensors of $so(p,1)$, $p > 3$, cf. [37], appendix B.

The case $so(2,1) \cong sl(2,\mathbb{R})$ is special and must be treated separately. But in fact, it is contained in what we presented already. In that case the multiplets contain only two ERs which may be depicted by the top pair $\chi_1^\pm$ in both figures. (Formally, set $h = 0$ in both figures.) They have the properties that we described, including the (anti)holomorphic discrete series which are present in this case. That case was the first given already in 1946-7 independently by Gel'fand et al [183] and Bargmann [184].

4 The Lie algebras $su^*(2n)$ and $sl(n,\mathbb{R})$

4.1 Case $su^*(2n)$

Let $\mathcal{G} = su^*(2n)$. It has maximal compact subalgebra $\mathcal{K} = sp(n)$, and thus $\mathcal{G}$ does not have discrete series representations (as $\mathrm{rank}\,\mathcal{K} = n < \mathrm{rank}\,su^*(2n) = 2n-1$).

The algebra $\mathcal{G} = su^*(2n)$ has $n-1$ maximal parabolic subalgebras with $\mathcal{M}$-factors (cf. (5.8) from [121]):

$$\mathcal{M}_k^{\max} = su^*(2k) \oplus su^*(2(n-k)) , \qquad 1 \le k \le n-1 , \tag{4.1}$$

with complexification:

$$(\mathcal{M}_k^{\max})^{\mathbb{C}} = sl(2k,\mathbb{C}) \oplus sl(2(n-k),\mathbb{C}) . \tag{4.2}$$

We would like to relate parabolically this algebra with the appropriate conformal Lie algebra, namely, with $su(n,n)$. It was considered in [150] with $\mathcal{M}$-factor: $\mathcal{M}' = sl(n,\mathbb{C})_{\mathbb{R}}$ which has complexification:

$$\mathcal{M}'^{\mathbb{C}} = sl(n,\mathbb{C}) \oplus sl(n,\mathbb{C}) . \tag{4.3}$$

Clearly, the latter expression can match (4.2) only if $n = 2k$, i.e., n must be *even*.

Thus, we set $n = 2k$ and consider:

$$\begin{aligned} \mathcal{G} &= su^*(4k) , \\ \mathcal{M} &= su^*(2k) \oplus su^*(2k) , \\ \mathcal{M}^{\mathbb{C}} &= sl(2k,\mathbb{C}) \oplus sl(2k,\mathbb{C}) . \end{aligned} \tag{4.4}$$

4.2 Case $sl(n,\mathbb{R})$

Let $sl(n,\mathbb{R})$. Its maximal compact subalgebra is $\mathcal{K} = so(n)$, and thus it does not have discrete series representations. It has $[\frac{n}{2}]$ maximal parabolic subalgebras obtained by deleting a node from its standard Dynkin diagram and taking into account the symmetry (cf. [121]):

$$\mathcal{M}_j = sl(j,\mathbb{R}) \oplus sl(n-j,\mathbb{R}) , \quad 1 \le j \le \left[\tfrac{n}{2}\right] . \tag{4.5}$$

We would like to match this with (4.3). Obviously this can happen only for $n = 4k$ and $j = n/2 = 2k$, so we consider:

$$\begin{aligned} \mathcal{G} &= sl(4k,\mathbb{R})\,, \\ \mathcal{M} &= sl(2k,\mathbb{R}) \oplus sl(2k,\mathbb{R})\,, \\ \mathcal{M}^{\mathbb{C}} &= sl(2k,\mathbb{C}) \oplus sl(2k,\mathbb{C})\ . \end{aligned} \tag{4.6}$$

4.3 Representations and multiplets

Above we have chosen the $\mathcal{M}$-factors of the Lie algebras $su^*(4k)$ and $sl(4k,\mathbb{R})$ so that they are parabolically related to the conformal Lie algebra $su(2k,2k)$ with $\mathcal{M}$-factor $\mathcal{M}^{\mathbb{C}} = sl(2k,\mathbb{C}) \oplus sl(2k,\mathbb{C})$, cf. (4.4), (4.6), thus, we shall discuss them together.

The signature of the ERs of both $\mathcal{G}$ may be denoted as:

$$\begin{aligned} \chi &= \{\, n_1\,, \ldots,\, n_{2k-1}\,,\, n_{2k+1}\, \ldots,\, n_{4k-1}\,;\, c\,\}\,, \\ & n_j \in \mathbb{N}\,, \quad c = d - 2k\,, \end{aligned} \tag{4.7}$$

same as for $su(2k,2k)$.

The Knapp-Stein restricted Weyl reflection mapping χ to its shadow is given by:

$$\begin{aligned} G \ &: \ \mathcal{C}_\chi \ \longrightarrow \ \mathcal{C}_{\chi'}\ , \\ \chi' &= \{(n_1,\ldots,n_{2k-1},n_{2k+1},\ldots,n_{4k-1})^*; -c\}\,, \\ & (n_1,\ldots,n_{2k-1},n_{2k+1},\ldots,n_{4k-1})^* \ \doteq \\ & (n_{2k+1},\ldots,n_{4k-1},n_1,\ldots,n_{2k-1}) \end{aligned} \tag{4.8}$$

Further, we use the root system of the complex algebra $sl(4k,\mathbb{C})$. The positive roots α_{ij} in terms of the simple roots α_i are:

$$\begin{aligned} \alpha_{ij} &= \alpha_i + \alpha_{i+1} + \cdots + \alpha_j\,, \qquad 1 \le i < j \le 4k-1\,, \\ \alpha_{ii} &\equiv \alpha_i\,, \qquad\qquad\qquad\qquad 1 \le i \le 4k-1 \end{aligned} \tag{4.9}$$

from which the non-compact are:

$$\alpha_{ij}\,, \qquad 1 \le i \le 2k\,, \qquad 2k \le j \le 4k-1$$

The correspondence between the signatures χ and the highest weight Λ is through the Dynkin labels:

$$\begin{aligned} n_i &= m_i \equiv (\Lambda+\rho,\alpha_i^\vee) = (\Lambda+\rho,\alpha_i)\,, \qquad i = 1,\ldots,4k-1, \\ c &= -\tfrac{1}{2}(m_{\tilde\alpha} + m_{2k}) = -\tfrac{1}{2}(m_1 + \cdots + m_{2k-1} + 2m_{2k} + m_{2k+1} + \cdots + m_{4k-1}) \end{aligned} \tag{4.10}$$

$\Lambda = \Lambda(\chi)$, $\tilde\alpha = \alpha_1 + \cdots + \alpha_{4k-1}$ is the highest root.

In our diagrams we need also the Harish-Chandra parameters for the non-compact roots using the following notation:

$$m_{ij} \equiv m_{\alpha_{ij}} = m_i + \cdots + m_j\,, \quad i < j$$

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The number of ERs in the corresponding multiplets is according to (3.6):

$$\frac{|W(\mathcal{G}^{\mathbb{C}},\mathcal{H}^{\mathbb{C}})|}{|W(\mathcal{M}^{\mathbb{C}},\mathcal{H}_m^{\mathbb{C}})|} = \frac{|W(sl(4k,\mathbb{C}))|}{|W(sl(2k,\mathbb{C}))|^2} = \frac{(4k)!}{((2k)!)^2} = \binom{4k}{2k} \tag{4.11}$$

(which was given for $su(n,n)$ in [150]).

Below we give the diagrams for the cases $k = 1,2$. Of course, the case $k = 1$ is known long time ago, first as $su^*(4) \cong so(5,1)$, cf. [174], then as $su(2,2) \cong so(4,2)$, cf. [175], and also as $sl(4,\mathbb{R}) \cong so(3,3)$, as we recalled already in the previous section on $so(p,q)$ algebras. We present it here using a new diagram look which can handle the more complicated cases that follow further. In this new look only the invariant differential operators are presented explicitly. The integral Knapp-Stein operators, more precisely the restricted Weyl reflection action is understood by a symmetry of the picture, either w.r.t. a central point, or w.r.t. middle line.

Thus, in figure 3 we give the case $k = 1$, where the Knapp-Stein symmetry is w.r.t. to the bullet in the middle of the figure. Then in figure 4 we give the diagram figure 1 for the special case $h = 2$, as given originally for $so(5,1)$ in [174], and $so(4,2)$ in [175], stressing that both figures 3 and 4 have the same content.

Next we give the case $k = 2$, in figure 5, which applies to $su^*(8)$, $sl(8,\mathbb{R})$ and $su(4,4)$. (For reduced multiplets we refer to [150].) The diagram is very complicated and just to be able to depict all the relevant information we must use the following condensing conventions. Each intertwining differential operator is represented by an arrow accompanied by a symbol $i_{j\ldots\ell}$ encoding the root $\beta_{j\ldots\ell}$ and the number $m_{\beta_{j\ldots\ell}}$ which is involved in the BGG criterion. This notation is used to save space, but it can be used due to the fact that only intertwining differential operators which are non-composite are displayed, and that the data β, m_β, which is involved in the embedding $V^\Lambda \longrightarrow V^{\Lambda - m_\beta,\beta}$ turns out to involve only the m_i corresponding to simple roots, i.e., for each β, m_β there exists $i = i(\beta, m_\beta, \Lambda) \in \{1,\ldots,r\}$, ($r = \mathrm{rank}\,\mathcal{G}$), such that $m_\beta = m_i$. Hence the data $\beta_{j\ldots\ell}$,$m_{\beta_{j\ldots\ell}}$ is represented by $i_{j\ldots\ell}$ on the arrows.

5 The Lie algebras $sp(p,r)$

Let $\mathcal{G} = sp(p,r)$, $p \geq r$. It has maximal compact subalgebra $\mathcal{K} = sp(p) \oplus sp(r)$ and has discrete series representations (as $\mathrm{rank}\,\mathcal{K} = p + r = \mathrm{rank}\,\mathcal{G}$). It has r maximal parabolic subalgebras with $\mathcal{M}$-factors (cf. (9.8) from [121]):

$$\mathcal{M}_j^{\max} = su^*(2j) \oplus sp(p-j, r-j) \ , \quad 1 \leq j \leq r \tag{5.1}$$

with complexification:

$$(\mathcal{M}_j^{\max})^{\mathbb{C}} = sl(2j,\mathbb{C}) \oplus sp(p+r-2j,\mathbb{C}) \ . \tag{5.2}$$

We would like to match this algebra with the appropriate conformal Lie algebra, namely, with $sp(n,\mathbb{R})$. It was considered in [151] with $\mathcal{M}$-factor: $\mathcal{M}' = sl(n,\mathbb{R})$ with

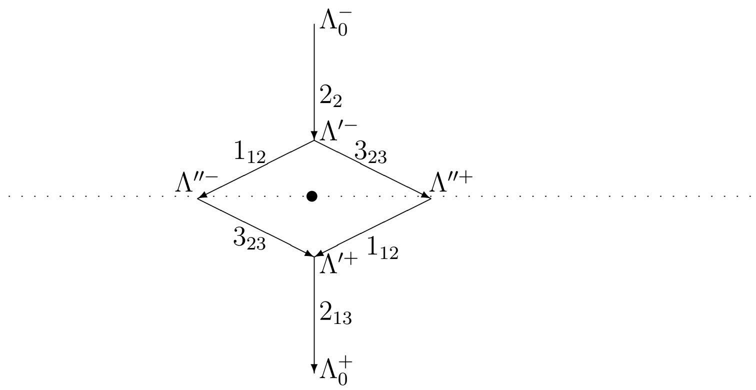


Figure 3. Main multiplets for $su^*(4) \cong so(5,1)$ and $su(2,2) \cong so(4,2)$ with parabolic factor $\mathcal{M}^{\mathbb{C}} = sl(2,\mathbb{C}) \oplus sl(2,\mathbb{C})$. The pairs of shadow fields are symmetric w.r.t. the bullet.

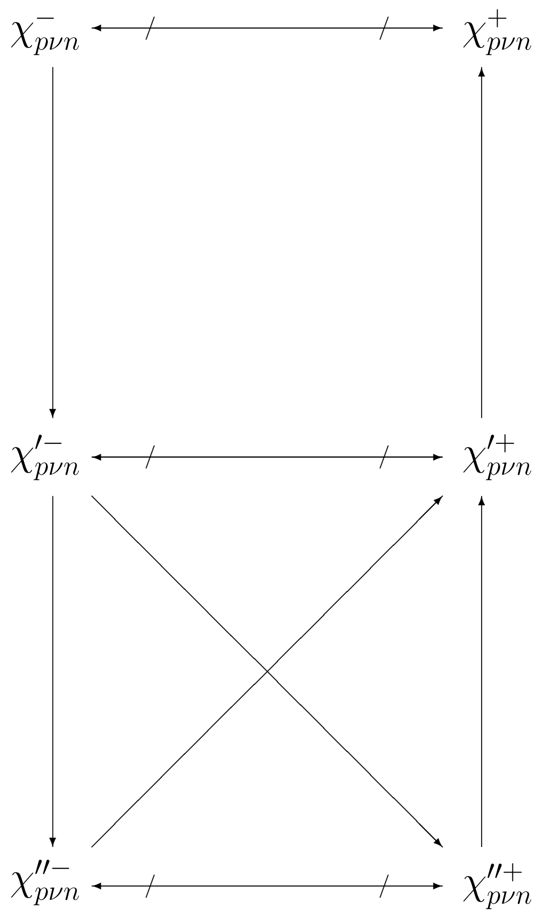


Figure 4. Sextet of partially equivalent ERs and intertwining operators for $so(5,1) \cong su^*(4)$ and $so(4,2) \cong su(2,2)$ cf. [140, 141], resp. (arrows are differential operators, dashed arrows are integral operators).

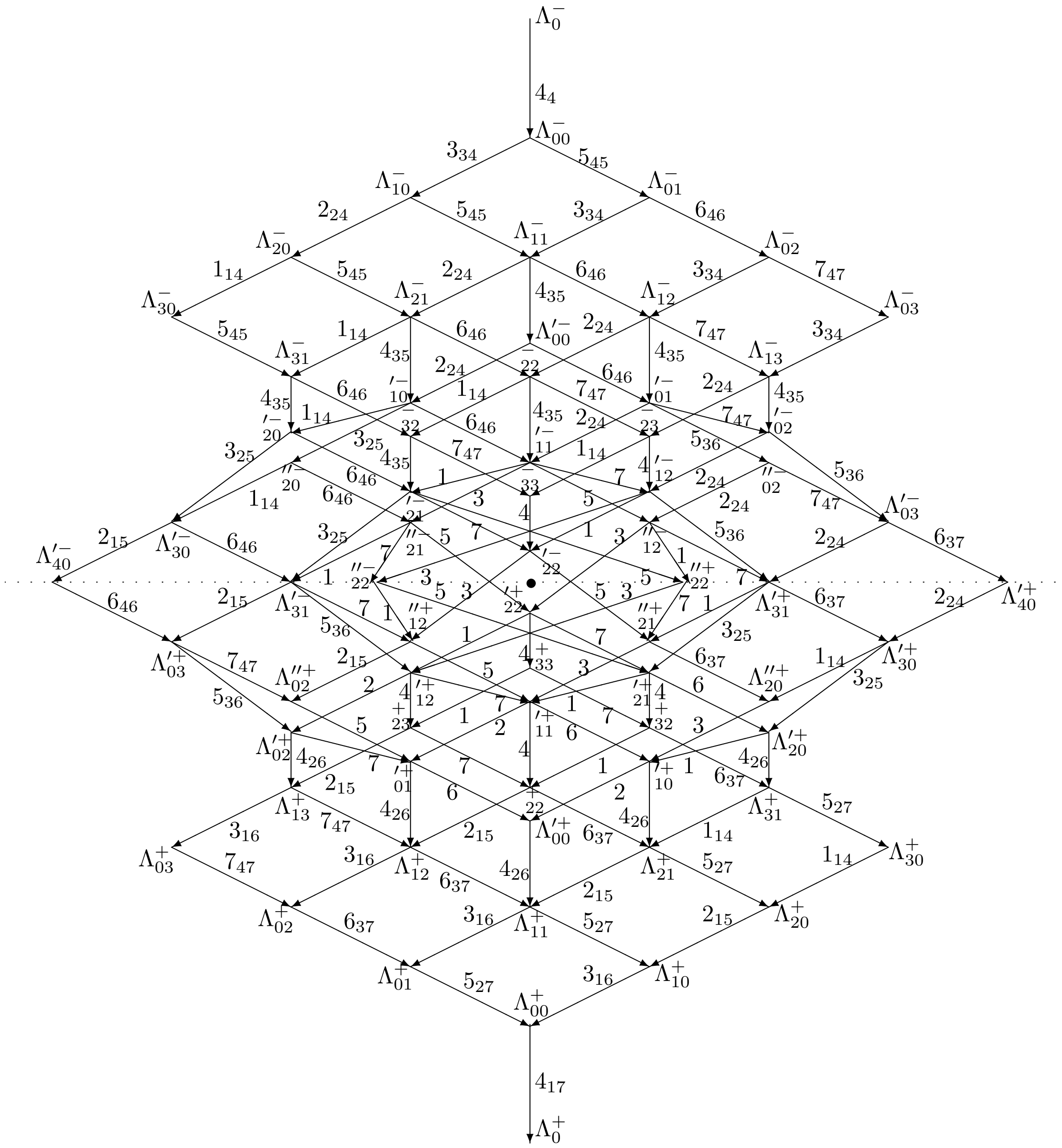


Figure 5. Main multiplets for $su(4,4)$ and $su^*(8)$ with parabolic factor $\mathcal{M}^{\mathbb{C}} = sl(4,\mathbb{C}) \oplus sl(4,\mathbb{C})$.

complexification $\mathcal{M}'^{\mathbb{C}} = sl(n,\mathbb{C})$. Obviously, the latter can match (5.2) only if n is even and $p = r = j = n/2$. Thus, we shall consider

$$\begin{aligned} \mathcal{G} &= sp(r,r)\,, \\ \mathcal{M} &= su^*(2r)\,, \\ \mathcal{M}^{\mathbb{C}} &= sl(2r,\mathbb{C})\ . \end{aligned} \tag{5.3}$$

The signature of the ERs of $\mathcal{G}$ is:

$$\chi = \{\, n_1\,, \ldots,\, n_{2r-1}\,;\, c\,\}\,, \qquad n_j \in \mathbb{N}\,, \qquad c = d - r - \tfrac{1}{2}\ . \tag{5.4}$$

The Knapp-Stein restricted Weyl reflection acts as follows:

$$G: \mathcal{C}_\chi \longrightarrow \mathcal{C}_{\chi'}, \qquad (5.5)$$
$$\chi' = \{ (n_1, \dots, n_{2r-1})^* ; -c \}, \qquad (n_1, \dots, n_{2r-1})^* \doteq (n_{2r-1}, \dots, n_1)$$

In terms of an orthonormal basis ε_i, $i = 1, \dots, n$, the positive roots of $sp(2r, \mathbb{C})$ are:

$$\Delta^+ = \{\varepsilon_i \pm \varepsilon_j, \ \ 1 \le i < j \le 2r; \ \ \ 2\varepsilon_i, \ \ 1 \le i \le 2r\}, \qquad (5.6)$$

the simple roots are:

$$\pi = \{\alpha_i = \varepsilon_i - \varepsilon_{i+1}, \ 1 \le i \le 2r-1; \ \ \ \alpha_{2r} = 2\varepsilon_{2r}\}, \qquad (5.7)$$

the positive non-compact roots are:

$$\beta_{ij} \equiv \varepsilon_i + \varepsilon_j, , \ \ 1 \le i \le j \le 2r, \qquad (5.8)$$

the Harish-Chandra parameters: $m_\beta \equiv (\Lambda + \rho, \beta)$ for the noncompact roots are:

$$m_{\beta_{ij}} = \Big(\sum_{s=i}^{2r} + \sum_{s=j}^{2r} \Big) m_s, \qquad i < j, \qquad (5.9)$$
$$m_{\beta_{ii}} = \sum_{s=i}^{2r} m_s$$

The correspondence between the signatures χ and the highest weight Λ is:

$$n_i = m_i, \quad c = -\tfrac{1}{2}(m_{\tilde{\alpha}} + m_{2r}) = -\tfrac{1}{2}(m_1 + \dots + m_{2r-1} + 2m_{2r}) \qquad (5.10)$$

where $\tilde{\alpha} = \beta_{11}$ is the highest root.

The number of ERs in the corresponding multiplets is according to (3.6):

$$\frac{|W(\mathcal{G}^{\mathbb{C}}, \mathcal{H}^{\mathbb{C}})|}{|W(\mathcal{M}^{\mathbb{C}}, \mathcal{H}^{\mathbb{C}}_m)|} = \frac{|W(sp(2r, \mathbb{C}))|}{|W(sl(2r, \mathbb{C}))|} = \frac{2^{2r}(2r)!}{((2r)!)} = 2^{2r} \qquad (5.11)$$

(which was given for $sp(n, \mathbb{R})$ in [151]).

Below we give pictorially the multiplets for $sp(r, r)$ for $r = 1, 2$, valid also for $sp(2r, \mathbb{R})$. (The case $r = 3$, together with the reduced multiplets and $sp(5, \mathbb{R}$ are given in [151].)

In fact, the case $r = 1$ is known long time as $sp(1,1) \cong so(4,1)$, cf. [37], then later as $sp(2, \mathbb{R}) \cong so(3,2)$, cf. [179], as we recalled already in the previous section on $so(p,q)$ algebras. We present it here using the new diagram look which we already used in the previous section. Thus, in figure 6 we give the case $r = 1$, where the Knapp-Stein symmetry is w.r.t. to the bullet in the middle of the figure. Thus, it is seen that the action of the differential operator indexed by 1_{12} is the same as the Knapp-Stein operator from Λ'^- to Λ'^+, so that the latter operator degenerates as discussed in section 1. Then in figure 7 we give the diagram figure 2 for the special case $h = 1$, stressing that both figures 6 and 7 have the same content.

Finally, in figure 8 we give the case $r = 2$.

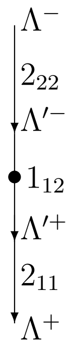


Figure 6. Main multiplets for $sp(1,1) \cong so(4,1)$ and $sp(2,\mathbb{R}) \cong so(3,2)$ with parabolic factor $\mathcal{M}^{\mathbb{C}} = sl(2,\mathbb{C})$.

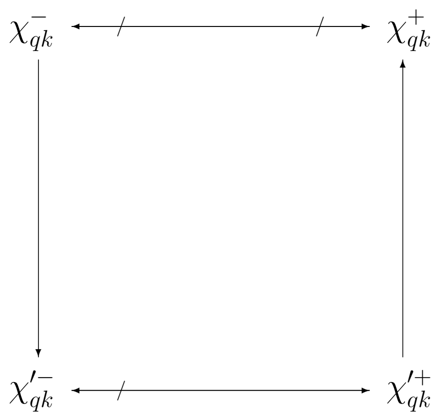
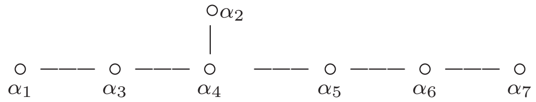


Figure 7. Quartet of partially equivalent ERs and intertwining operators for $so(4,1) = sp(1,1)$ and $so(3,2) \cong sp(2,\mathbb{R})$ cf. [125, 148] resp. (arrows are differential operators, dashed arrows are integral operators).

6 The non-compact Lie algebra $E_{7(7)}$

Let $\mathcal{G} = E_{7(7)}$. This is the split real form of E_7 which is denoted also as E'_7 or EV. The maximal compact subgroup is $\mathcal{K} \cong su(8)$. This algebra has discrete series representations (as $\operatorname{rank}\mathcal{G} = \operatorname{rank}\mathcal{K}$).

It has the following Dynkin-Satake diagram (same as for E_7) [185]:



(6.1)

The real algebra $E_{7(7)}$ has seven maximal parabolics which are obtained by deleting one node as explained in [121]. We choose the one which is most suitable w.r.t. the maximal compact subgroup $\mathcal{K} = su(8)$, as will become clear below. This parabolic is obtained by deleting the root α_7 from the Dynkin-Satake diagram (6.1), i.e., we shall use as $\mathcal{M}$-factor $E_{6(6)}$ (the split real form of E_6).

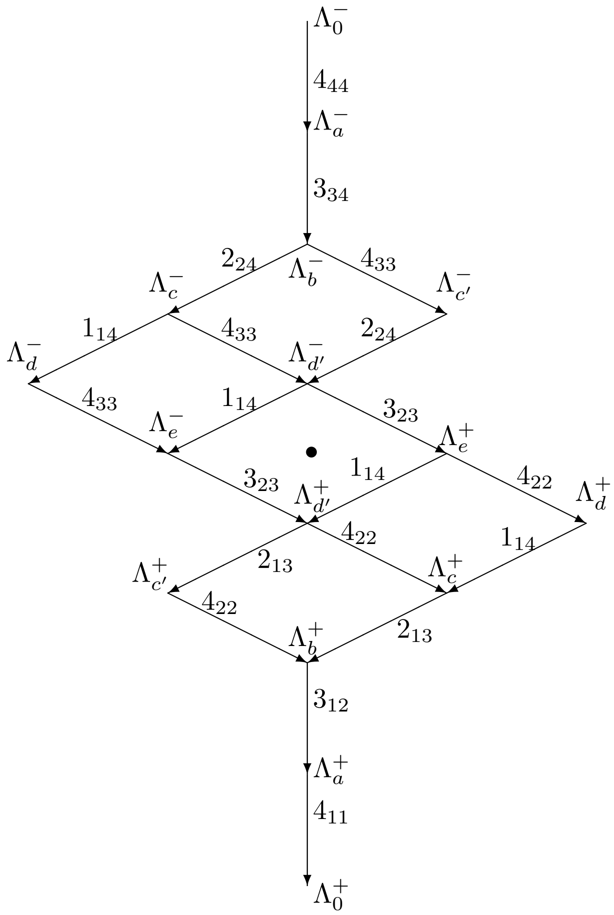


Figure 8. Main multiplets for $sp(2,2)$ and $sp(4,\mathbb{R})$ with parabolic factor $\mathcal{M}^{\mathbb{C}} = sl(4,\mathbb{C})$.

Thus, our *maximal* parabolic is

$$\mathcal{P} = \mathcal{M} \oplus \mathcal{A} \oplus \mathcal{N}\,, \qquad \mathcal{A} \cong so(1,1)\,, \;\; \mathcal{M} \cong E_{6(6)}\,, \;\; \dim_{\mathbb{R}} \mathcal{N} = 27\,, \tag{6.2}$$

cf. (11.17) of [121].

We label the signature of the ERs of $\mathcal{G}$ as follows:

$$\chi = \{\, n_1\,, \ldots\,, n_6\,;\, c\,\}\,, \qquad n_j \in \mathbb{N}\,, \quad c = d - 9 \tag{6.3}$$

where the last entry of χ labels the characters of $\mathcal{A}$, and the first 6 entries are labels of the finite-dimensional nonunitary irreps of $\mathcal{M}$, (or of the finite-dimensional unitary irreps of the compact e_6).

Further, we need the root system of the complex algebra E_7. With Dynkin diagram enumerating the simple roots α_i as in (6.1), the positive roots are:
first there are 21 roots forming the positive root system of $sl(7)$ (with simple roots $\alpha_1, \alpha_3, \alpha_4, \alpha_5,\; \alpha_6, \alpha_7$), then 21 positive roots which are positive roots of the E_6 subalgebra including the non-$sl(7)$ root α_2, and finally the following 21 roots including the

non-E_6 root α_7:

$$\begin{aligned}
&\alpha_2+\alpha_4+\alpha_5+\alpha_6+\alpha_7\,, \quad \alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7\,, \\
&\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7\,, \\
&\alpha_2+\alpha_3+2\alpha_4+\alpha_5+\alpha_6+\alpha_7\,, \quad \alpha_1+\alpha_2+\alpha_3+2\alpha_4+\alpha_5+\alpha_6+\alpha_7\,, \\
&\alpha_2+\alpha_3+2\alpha_4+2\alpha_5+\alpha_6+\alpha_7\,, \quad \alpha_1+\alpha_2+2\alpha_3+2\alpha_4+\alpha_5+\alpha_6+\alpha_7\,, \\
&\alpha_1+\alpha_2+\alpha_3+2\alpha_4+2\alpha_5+\alpha_6+\alpha_7\,, \\
&\alpha_1+\alpha_2+2\alpha_3+2\alpha_4+2\alpha_5+\alpha_6+\alpha_7\,, \\
&\alpha_1+\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+\alpha_6+\alpha_7\,, \\
&\alpha_1+2\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+\alpha_6+\alpha_7\,, \\
&\alpha_2+\alpha_3+2\alpha_4+2\alpha_5+2\alpha_6+\alpha_7\,, \\
&\alpha_1+\alpha_2+\alpha_3+2\alpha_4+2\alpha_5+2\alpha_6+\alpha_7\,, \\
&\alpha_1+\alpha_2+2\alpha_3+2\alpha_4+2\alpha_5+2\alpha_6+\alpha_7\,, \\
&\alpha_1+\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+2\alpha_6+\alpha_7\,, \\
&\alpha_1+2\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+2\alpha_6+\alpha_7\,, \\
&\alpha_1+\alpha_2+2\alpha_3+3\alpha_4+3\alpha_5+2\alpha_6+\alpha_7\,, \\
&\alpha_1+2\alpha_2+2\alpha_3+3\alpha_4+3\alpha_5+2\alpha_6+\alpha_7\,, \\
&\alpha_1+2\alpha_2+2\alpha_3+4\alpha_4+3\alpha_5+2\alpha_6+\alpha_7\,, \\
&\alpha_1+2\alpha_2+3\alpha_3+4\alpha_4+3\alpha_5+2\alpha_6+\alpha_7\,, \\
&2\alpha_1+2\alpha_2+3\alpha_3+4\alpha_4+3\alpha_5+2\alpha_6+\alpha_7=\tilde{\alpha}\,,
\end{aligned} \tag{6.4}$$

where $\tilde{\alpha}$ is the highest root of the E_7 root system.

The differential intertwining operators that give the multiplets correspond to the non-compact roots, and since we shall use the latter extensively, we introduce more compact notation for them. Namely, the non-simple roots will be denoted in a self-explanatory way as follows:

$$\begin{aligned}
\alpha_{ij} &= \alpha_i+\alpha_{i+1}+\cdots+\alpha_j\,, \quad \alpha_{i,j}=\alpha_i+\alpha_j\,, \quad i<j\,, \\
\alpha_{ij,k} &= \alpha_{k,ij}=\alpha_i+\alpha_{i+1}+\cdots+\alpha_j+\alpha_k\,, \quad i<j\,, \\
\alpha_{ij,km} &= \alpha_i+\alpha_{i+1}+\cdots+\alpha_j+\alpha_k+\alpha_{k+1}+\cdots+\alpha_m\,, \\
&\qquad i<j\,, \quad k<m\,, \\
\alpha_{ij,km,4} &= \alpha_i+\alpha_{i+1}+\cdots+\alpha_j+\alpha_k+\alpha_{k+1}+\cdots+\alpha_m+\alpha_4\,, \\
&\qquad i<j\,, \quad k<m\,,
\end{aligned} \tag{6.5}$$

i.e., the non-compact roots will be written as:

$$\alpha_7\,, \quad \alpha_{67}\,, \quad \alpha_{57}\,, \quad \alpha_{47}\,, \quad \alpha_{37}\,, \quad \alpha_{1,37}\,, \tag{6.6a}$$

$$\begin{aligned}
&\alpha_{2,47}\,, \quad \alpha_{27}\,, \quad \alpha_{17}\,, \quad \alpha_{27,4}\,, \quad \alpha_{17,4}\,, \quad \alpha_{27,45}\,, \\
&\alpha_{17,34}\,, \quad \alpha_{17,45}\,, \quad \alpha_{27,46}\,, \quad \alpha_{17,35}\,, \quad \alpha_{17,46}\,, \quad \alpha_{17,36}\,, \\
&\alpha_{17,35,4}\,, \quad \alpha_{17,25,4}\,, \quad \alpha_{17,36,4}\,, \quad \alpha_{17,26,4}\,, \\
&\alpha_{17,36,45}\,, \quad \alpha_{17,26,45}\,, \quad \alpha_{17,26,45,4}\,, \quad \alpha_{17,26,35,4}\,, \quad \alpha_{17,16,35,4}=\tilde{\alpha}\,,
\end{aligned} \tag{6.6b}$$

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where the first six roots in (6.6a) are from the $sl(7)$ subalgebra, and the 21 in (6.6b) are those from (6.4).

Further, we give the correspondence between the signatures χ and the highest weight Λ. The connection is through the Dynkin labels (2.8) m_i, $i = 1, \ldots, 7$, and is given explicitly by:

$$n_i = m_i\,, \quad i = 1, \ldots, 6\,, \qquad (6.7)$$
$$c = -\tfrac{1}{2}(m_{\tilde{\alpha}} + m_7) = -\tfrac{1}{2}(2m_1 + 2m_2 + 3m_3 + 4m_4 + 3m_5 + 2m_6 + 2m_7)$$

Here we note that the simple root system of the $su(8)$ compact subalgebra of $E_{7(7)}$, or equivalently, of the $sl(8)$ subalgebra of E_7, is given by the $sl(7)$ simple roots plus the highest root $\hat{\alpha}$ of the E_6 subalgebra:

$$\alpha_1,\ \alpha_3,\ \alpha_4,\ \alpha_5,\ \alpha_6,\ \alpha_7,\ \hat{\alpha} = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6 \qquad (6.8)$$

Indeed, it is easy to check that:

$$(\alpha_i, \hat{\alpha}) = 0, \quad i = 1, 3, 4, 5, 6, \qquad (\alpha_7, \hat{\alpha}) = -1\ .$$

Now we should connect our considerations with the case of another real form of E_7, namely, the Lie algebra $E_{7(-25)}$, cf. [152]. In that paper we chose as maximal parabolic $\mathcal{P}' = \mathcal{M}' \oplus \mathcal{A}' \oplus \mathcal{N}'$, where $\mathcal{M}' \cong E_{6(-26)}$, $\dim_{\mathbb{R}} \mathcal{N} = 27$, cf. (11.24) of [121].

Since the algebras $E_{7(7)}$ and $E_{7(-25)}$ are parabolically related they have the same signatures, and thus the same main multiplets.

The number of ERs in the corresponding main multiplets is according to (3.6):

$$\frac{|W(\mathcal{G}^{\mathbb{C}}, \mathcal{H}^{\mathbb{C}})|}{|W(\mathcal{M}^{\mathbb{C}}, \mathcal{H}_m^{\mathbb{C}})|} = \frac{|W(E_7)|}{|W(E_6)|} = \frac{2^{10}\, 3^4\, 5.7}{2^7\, 3^4\, 5} = 56 \qquad (6.9)$$

(which was given for $E_{7(-25)}$ in [152]).

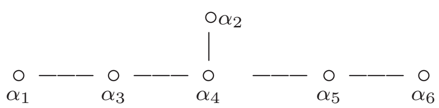
Below we give the main multiplets valid for both algebras in figure 9. For reduced multiplets cf. [152].

7 Two real forms of E_6

7.1 The Lie algebra $E_{6(6)}$

Let $\mathcal{G} = E_{6(6)}$. This is the split real form of E_6 denoted also as E_6' or $E_{\rm I}$. The maximal compact subgroup is $\mathcal{K} \cong sp(4)$. This real form does not have discrete series representations (as $\operatorname{rank} \mathcal{G} \neq \operatorname{rank} \mathcal{K}$).

We use the following Dynkin-Satake diagram (same as for E_6):



(7.1)

The real algebra $E_{6(6)}$ has four maximal parabolics which are obtained by deleting one node as explained in [121]. (Note that deleting node 1 or node 6 produces the same

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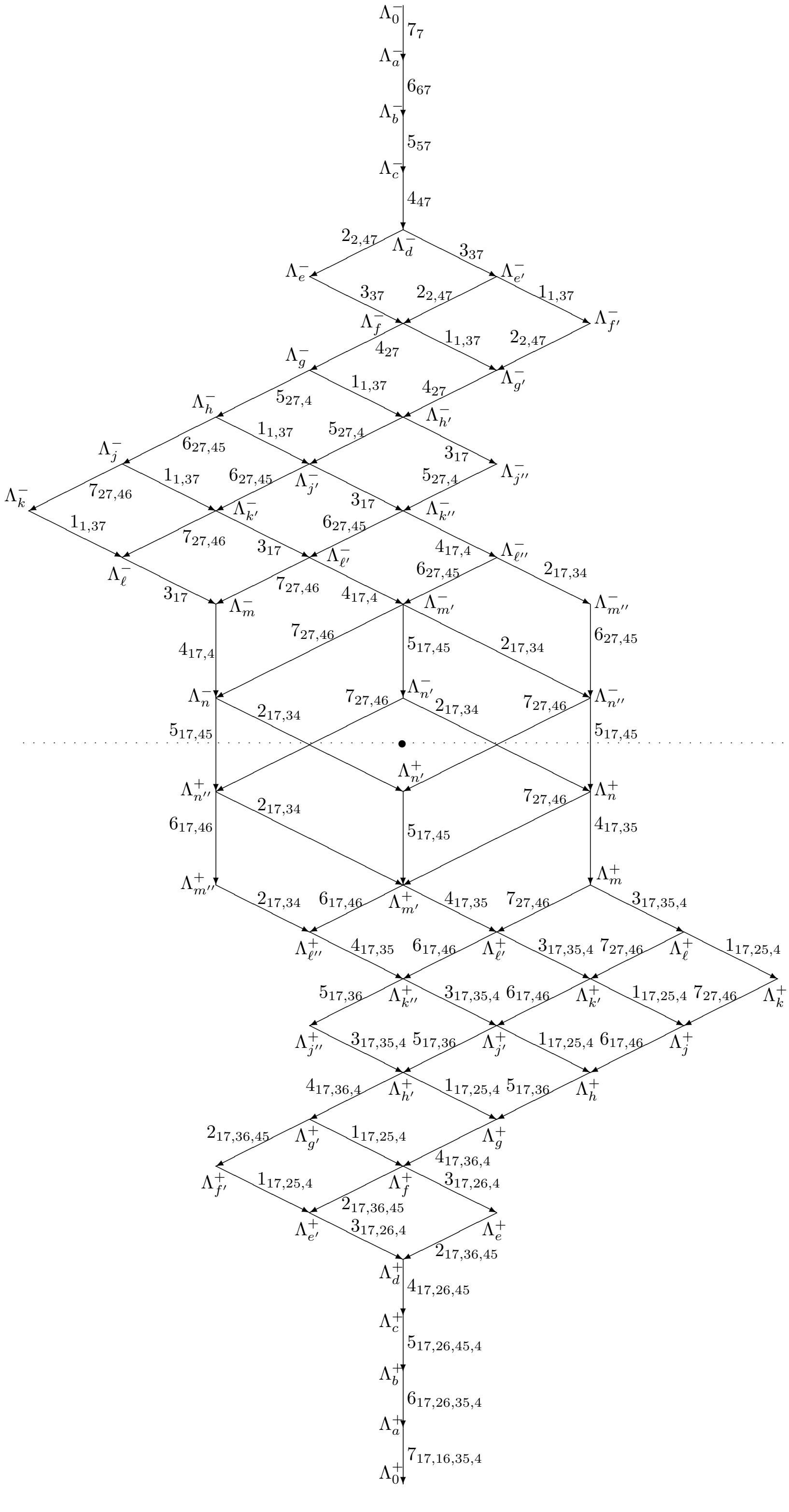

Figure 9. Main type of multiplets for $E_{7(7)}$ and $E_{7(-25)}$ with parabolic factor $\mathcal{M}^{\mathbb{C}} = E_6$.

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parabolic, same for deleting node 3 or node 5.) We choose the parabolic obtained by deleting node 2.

Thus, the *maximal* parabolic is

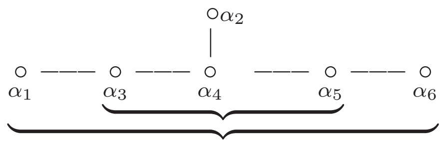
$$\mathcal{P} = \mathcal{M} \oplus \mathcal{A} \oplus \mathcal{N}\,, \qquad \mathcal{A} \cong so(1,1)\,, \quad \mathcal{M} \cong sl(6,\mathbb{R})\,, \quad \dim_{\mathbb{R}} \mathcal{N} = 21\,, \tag{7.2}$$

cf. (11.4) of [121].

7.2 The Lie algebra $E_{6(2)}$

Let $\mathcal{G} = E_{6(2)}$. This is another real form of E_6 sometimes denoted as E_6'', or $E_{\rm II}$. The maximal compact subalgebra is $\mathcal{K} \cong su(6) \oplus su(2)$. This real form has discrete series representations.

The Satake diagram is:



(7.3)

The real algebra $E_{6(2)}$ has four maximal parabolics which are obtained by deleting one node as explained in [121] (taking into account E_6 symmetry as in the previous case). We choose the parabolic obtained by deleting node 2.

Thus, the *maximal* parabolic is

$$\mathcal{P} = \mathcal{M} \oplus \mathcal{A} \oplus \mathcal{N}\,, \qquad \mathcal{A} \cong so(1,1)\,, \quad \mathcal{M} \cong su(3,3)\,, \quad \dim_{\mathbb{R}} \mathcal{N} = 21\,, \tag{7.4}$$

cf. (11.7) of [121].

7.3 Representations and multiplets

We note that the $\mathcal{M}$-factors of the two real forms of E_6 discussed in the previous subsections have the same complexification:

$$sl(6,\mathbb{R})^{\mathbb{C}} = su(3,3)^{\mathbb{C}} = sl(6,\mathbb{C})$$

i.e., they are parabolically related and we can discuss them together.

The signature of the ERs of $\mathcal{G}$ is:

$$\chi = \{\, n_1\,, n_3\,, n_4\,, n_5\,, n_6\,;\, c\}\,, \qquad c = d - \tfrac{11}{2}\,,$$

expressed through the Dynkin labels as:

$$n_i = m_i\,, \quad -c = \tfrac{1}{2} m_{\tilde{\alpha}} =$$

$$\tfrac{1}{2}(m_1 + 2m_2 + 2m_3 + 3m_4 + 2m_5 + m_6)$$

Further, we need the root system of the complex algebra E_6. With Dynkin diagram enumerating the simple roots α_i as in (7.1), the positive roots are:

first there are 15 roots forming the positive root system of $sl(6)$ (with simple roots $\alpha_1, \alpha_3, \alpha_4, \alpha_5, \alpha_6$), then the following 21 roots including the non-$sl(6)$ root α_2:

$$\begin{aligned}
&\alpha_2\,, \quad \alpha_2+\alpha_4\,, \quad \alpha_2+\alpha_3+\alpha_4\,, \quad \alpha_2+\alpha_4+\alpha_5\,, \\
&\alpha_2+\alpha_3+\alpha_4+\alpha_5\,, \quad \alpha_1+\alpha_2+\alpha_3+\alpha_4\,, \quad \alpha_2+\alpha_4+\alpha_5+\alpha_6\,, \\
&\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5\,, \quad \alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6\,, \quad \alpha_2+\alpha_3+2\alpha_4+\alpha_5, \\
&\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6\,, \quad \alpha_1+\alpha_2+\alpha_3+2\alpha_4+\alpha_5\,, \\
&\alpha_2+\alpha_3+2\alpha_4+\alpha_5+\alpha_6\,, \quad \alpha_1+\alpha_2+\alpha_3+2\alpha_4+\alpha_5+\alpha_6\,, \\
&\alpha_1+\alpha_2+2\alpha_3+2\alpha_4+\alpha_5\,, \quad \alpha_2+\alpha_3+2\alpha_4+2\alpha_5+\alpha_6\,, \\
&\alpha_1+\alpha_2+2\alpha_3+2\alpha_4+\alpha_5+\alpha_6\,, \quad \alpha_1+\alpha_2+\alpha_3+2\alpha_4+2\alpha_5+\alpha_6\,, \\
&\alpha_1+\alpha_2+2\alpha_3+2\alpha_4+2\alpha_5+\alpha_6\,, \\
&\alpha_1+\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+\alpha_6\,, \\
&\alpha_1+2\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+\alpha_6 \equiv \tilde{\alpha}\,,
\end{aligned} \tag{7.5}$$

where $\tilde{\alpha}$ is the highest root of the E_6 root system.

Relative to our parabolic subalgebra, the roots in (7.5) are non-compact, while the rest are compact. As before we introduce more condensed notation for the noncompact roots:

$$\begin{aligned}
&\alpha_2\,, \alpha_{14}\,, \alpha_{15}\,, \ \alpha_{16}\,, \alpha_{24}\,, \alpha_{25}\,, \ \alpha_{26} \\
&\alpha_{2,4}\,, \alpha_{2,45}\,, \ \alpha_{2,46}\,, \ \alpha_{25,4}\,, \alpha_{15,4}\,, \ \alpha_{26,4} \\
&\alpha_{16,4}\,, \alpha_{15,34}\,, \alpha_{26,45}\,, \ \alpha_{16,34}\,, \alpha_{16,45} \\
&\alpha_{16,35}\,, \alpha_{16,35,4}\,, \ \alpha_{16,25,4} = \tilde{\alpha}
\end{aligned}$$

Now we should connect our considerations with the case of another real form of E_6, namely, the Lie algebra $E_{6(-14)}$, cf. [153, 154]. In that paper we chose as maximal parabolic $\mathcal{P}' = \mathcal{M}' \oplus \mathcal{A}' \oplus \mathcal{N}'$, where $\mathcal{M}' \cong su(5,1)$, $\dim_{\mathbb{R}} \mathcal{N} = 21$, cf. (11.21) of [121].

Since both the algebras and the maximal parabolics have the same complexification, this means that they are parabolically related, thus, we have the same non-compact roots, the same signatures, and the same multiplets. We show only the main multiplet in figure 10, referring to [153, 154] for the diagrams of reduced multiplets. The main multiplet has 70 members and the figure has the standard E_6 symmetry, namely, conjugation exchanging indices $1 \longleftrightarrow 6$, $3 \longleftrightarrow 5$. The Knapp-Stein operators act pictorially as reflection w.r.t. the dotted line separating the $\mathcal{H}^-\ldots$ members from the $\mathcal{H}^+\ldots$ members. Note that there are five cases when the embeddings correspond to the highest root $\tilde{\alpha}$: $V^{\Lambda^-} \longrightarrow V^{\Lambda^+}$, $\Lambda^+ = \Lambda^- - m_{\tilde{\alpha}}\,\tilde{\alpha}$. In these five cases the weights are denoted as: $\Lambda^\pm_{k''}$, $\Lambda^\pm_{k'}$, $\Lambda^\pm_{\tilde{k}}$, $\Lambda^\pm_{k}$, $\Lambda^\pm_{k^o}$, then: $m_{\tilde{\alpha}} = m_1, m_3, m_4, m_5, m_6$, resp. We recall that Knapp-Stein operators G^+ intertwine the corresponding ERs $\mathcal{T}^-_\chi$ and $\mathcal{T}^+_\chi$. In the above five cases the Knapp-Stein operators G^+ degenerate to differential operators as we discussed earlier.

8 Summary and outlook

In the present paper we continued the project of systematic construction of invariant differential operators for non-compact semisimple Lie groups. Our aim in this paper was

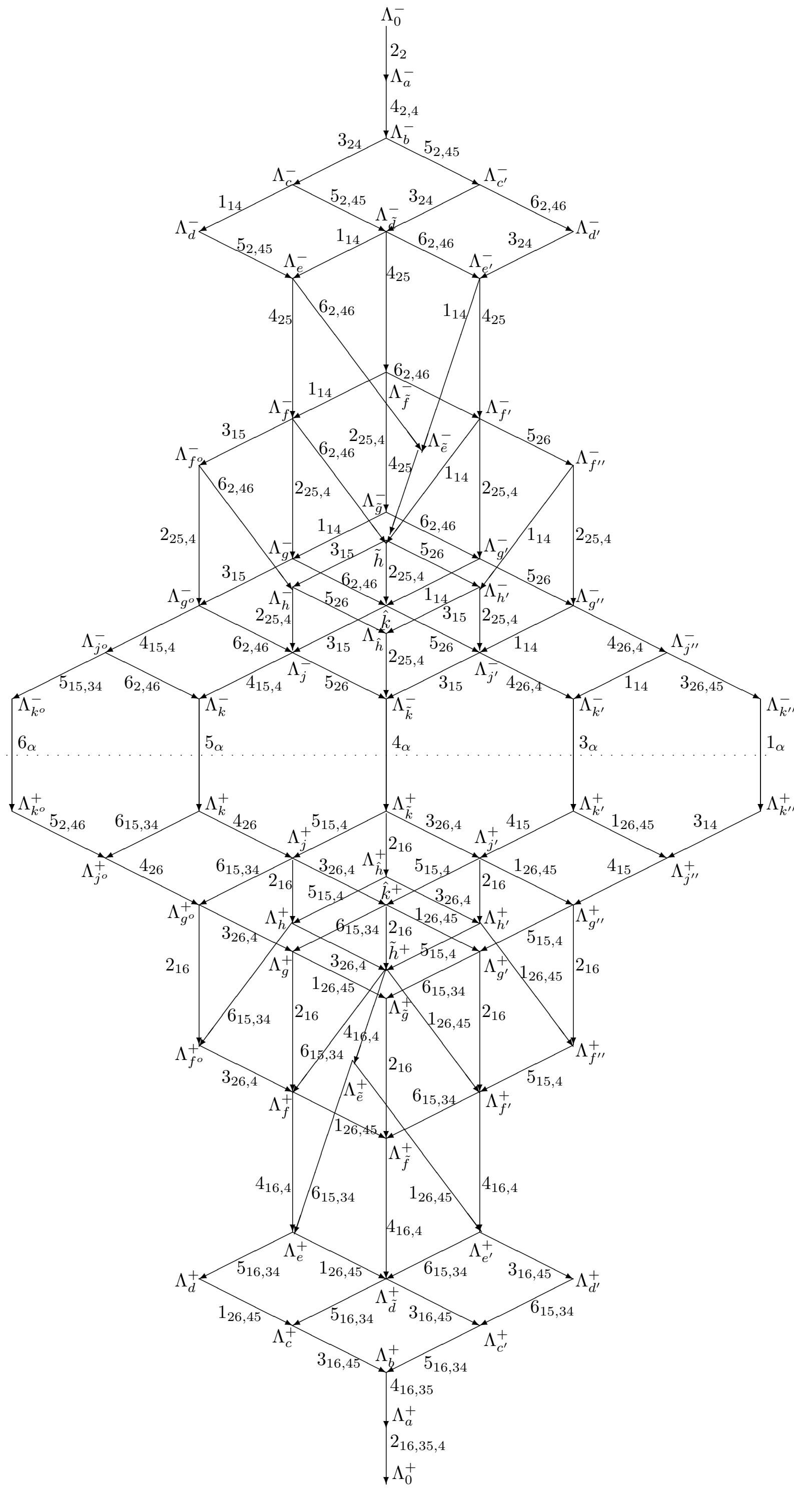


Figure 10. Main type of multiplets for $E_{6(6)}$, $E_{6(2)}$ and $E_{6(-14)}$ with parabolic factor $\mathcal{M}^{\mathbb{C}} = sl(6, \mathbb{C})$.

to extend our considerations beyond the class of algebras, which we call 'conformal Lie algebras' (CLA). For this we introduce the new notion of *parabolic relation* between two non-compact semisimple Lie algebras $\mathcal{G}$ and $\mathcal{G}'$ that have the same complexification and possess maximal parabolic subalgebras with the same complexification. Thus, we considered the algebras $so(p,q)$ all of which are parabolically related to the conformal algebra $so(n,2)$ with $p+q=n+2$, then the algebras $su^*(4k)$ and $sl(4k,\mathbb{R})$ parabolically related to the CLA $su(2k,2k)$, then $sp(r,r)$ as parabolically related to the CLA $sp(2r)$ (of rank $2r$), then the exceptional Lie algebra $E_{7(7)}$ which is parabolically related to the CLA $E_{7(-25)}$, finally the exceptional Lie algebras $E_{6(6)}$ and $E_{6(2)}$ parabolically related to the hermitian symmetric case $E_{6(-14)}$.

We have given a formula for the number of representations in the main multiplets valid for CLAs and all algebras that are parabolically related to them. In all considered cases we have given the main multiplets of indecomposable elementary representations including the necessary data for all relevant invariant differential operators. In the case of $so(p,q)$ we have given also the reduced multiplets. We note that the multiplets are given in the most economic way in pairs of *shadow fields* related by the Knapp-Stein restricted Weyl symmetry (and the corresponding integral operators).

Finally, we should stress that the classification of all invariant differential operators includes as special cases all possible *conservation laws* and *conserved currents*, unitary or not.

We plan also to extend these considerations to the supersymmetric cases and also to the quantum group setting. Such considerations are expected to be very useful for applications to string theory and integrable models. It is interesting to note that almost all of the algebras that appear in table 1 of [80] are treated in the present paper, though our motivations and approach are different (see also [186–188]).

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