Published for SISSA by Springer

Received: October 17, 2023
Accepted: January 3, 2024
Published: January 25, 2024

Schwinger-Keldysh effective field theory for stable and causal relativistic hydrodynamics

Akash Jain [a,b] **and Pavel Kovtun** [c]

[a] *Institute for Theoretical Physics, University of Amsterdam, 1090 GL Amsterdam, The Netherlands*
[b] *Dutch Institute for Emergent Phenomena, 1090 GL Amsterdam, The Netherlands*
[c] *Department of Physics & Astronomy, University of Victoria, 3800 Finnerty Road, Victoria, British Columbia V8P 5C2, Canada*

E-mail: ajain@uva.nl, pkovtun@uvic.ca

Abstract: We construct stable and causal effective field theories (EFTs) for describing statistical fluctuations in relativistic diffusion and relativistic hydrodynamics. These EFTs are fully non-linear, including couplings to background sources, and enable us to compute n-point time-ordered correlation functions including the effects of statistical fluctuations. The EFTs we construct are inspired by the Maxwell-Cattaneo model of relativistic diffusion and Müller-Israel-Stewart model of relativistic hydrodynamics respectively, and have been derived using both the Martin-Siggia-Rose and Schwinger-Keldysh formalisms. The EFTs non-linearly realise the dynamical Kubo-Martin-Schwinger (KMS) symmetry, which ensures that n-point correlation functions and interactions in the theory satisfy the appropriate fluctuation-dissipation theorems. Since these EFTs typically admit ultraviolet sectors that are not fixed by the low-energy infrared symmetries, we find that they simultaneously admit multiple realisations of the dynamical KMS symmetry. We also comment on certain obstructions to including statistical fluctuations in the recently-proposed stable and causal Bemfica-Disconzi-Noronha-Kovtun model of relativistic hydrodynamics.

Keywords: Field Theory Hydrodynamics, Non-Equilibrium Field Theory, Global Symmetries, Stochastic Processes

ArXiv ePrint: 2309.00511

https://doi.org/10.1007/JHEP01(2024)162

JHEP01(2024)162

Contents

1 Introduction

Hydrodynamics has been an immensely successful framework for describing near-equilibrium dissipative many-body systems, without relying on the precise intricate knowledge of their microscopic constituents and mutual interactions. The fundamental requirement for hydrodynamics to apply is a separation of "microscopic" vs. "macroscopic" scales, i.e. any characteristic spacetime scales of the microscopic theory, e.g. the mean free path or mean free time, must be much smaller than the spacetime scales at which the system is being probed. If so, most excitations that operate at the microscopic scales, dubbed the "fast" degrees of freedom, decay by the time we reach the macroscopic scales. The only remaining "slow" degrees of freedom are the collective excitations of conserved charges that are protected by the global symmetries of the system under consideration and thus cannot decay locally. In this hydrodynamic regime, we expect that the effective low-energy description of the system can be universally captured by its conserved charges and the respective conservation equations. All the intricate microscopic information about the system coalesces into the hydrodynamic constitutive relations: how the fluxes of conserved charges are related to the conserved densities and their derivatives.

JHEP01(2024)162

Like every theoretical model in physics, the validity of the assertions above crucially depends on the type and precision of the questions being asked. Even with a clear separation of scales, interactions between the "slow" and the ignored "fast" degrees of freedom can add up over macroscopic scales and lead to significant and qualitative deviations from the naive hydrodynamic predictions [1–6]. As an example, n-point retarded correlation functions or response functions of hydrodynamic observables (i.e. conserved densities and respective fluxes), which measure how the system responds to external perturbations, are known to admit "long-time tails" as a consequence of stochastic interactions that are not predicted by classical hydrodynamics [7, 8]. Furthermore, n-point symmetric correlation functions, which measure how the system is correlated across space and time, cannot be accessed using the classical theory of hydrodynamics at all and require a systematic effective theory including stochastic noise.

The state-of-the-art solution to these problems is presented by the Martin-Siggia-Rose (MSR) formalism [2], where the hydrodynamic constitutive relations are supplemented with random stochastic noise and physical observables are computed by averaging over all possible noise configurations sampled from a Gaussian distribution. The ambiguity in the spread of the Gaussian distribution is fixed by invoking the fluctuation-dissipation theorem (FDT) from thermal field theory [9]. It requires that the Fourier-space symmetric and retarded 2-point correlation functions of arbitrary observables $\mathcal{O}$ and $\mathcal{O}'$, computed in a thermal state with inverse temperature $\beta_0 = 1/T_0$, must satisfy

$$G^S_{\mathcal{O}\mathcal{O}'}(\omega, k) = \frac{2}{\beta_0 \omega} \operatorname{Im} G^R_{\mathcal{O}\mathcal{O}'}(\omega, k), \tag{1.1}$$

where ω and k_i denote the frequency and wavevector in Fourier-space. Similar FDTs also exist for higher-point time-ordered (i.e. symmetric, retarded, and partially retarded) correlation functions [10], which can be used to fix how the spread of the Gaussian distribution functionally depends on the hydrodynamic fields. In recent years, a more systematic Schwinger-Keldysh (SK) effective field theory (EFT) framework has been developed [11–16], where one can start from symmetry principles to directly construct an effective action for stochastic hydrodynamics and use this to obtain n-point time-ordered correlation functions of hydrodynamic observables. In particular, the FDT requirement in eq. (1.1) and its higher-point generalisations are not imposed by hand in this formalism, instead they are conveniently realised via a dynamical Kubo-Martin-Schwinger (KMS) symmetry of the effective action. SK-EFT enables us to compute stochastic, as well as classical non-linear, corrections to n-point correlation functions perturbatively as "loop corrections" coming from non-quadratic interactions in the effective action. As it turns out, assuming the distribution of stochastic noise to be Gaussian,[1] the MSR and SK formalisms yield the same EFTs for stochastic hydrodynamics up to redefinitions of the stochastic noise fields. We will thoroughly explore this connection during the course of this work.

[1] Gaussian noise here means that every term in the effective action is at most quadratic in noise fields. Interactions involving one or two noise fields and multiple physical hydrodynamic fields can be described by the MSR formalism, as we illustrate in the forthcoming discussion. However, interactions involving three or more noise fields require the full SK-EFT formalism.

JHEP01(2024)162

In this work, we are particularly interested in the EFT framework for relativistic hydrodynamics. In fact, the SK-EFT formalism itself was originally developed for relativistic hydrodynamics [17], thanks to the enormous control provided the spacetime Poincaré symmetries, and was only later generalised to non-relativistic contexts [18, 19]. A systematic study of stochastic noise in relativistic hydrodynamics can help us better describe a number of physical systems where stochastic effects are naturally pronounced. Potential applications include relativistic heavy-ion collisions [20–23] and the putative critical point in Quantum Chromodynamics [4, 24–26]. While developing SK-EFT for relativistic hydrodynamics, one is forced to confront the unfortunate fact that the conventional textbook formulation of dissipative relativistic hydrodynamics is unstable and acausal [27, 28]. In the linear regime, stability and causality can be investigated through the linearised mode spectrum of the theory, $\{\omega(k)\}$, defined as poles of the Fourier-space retarded 2-point correlation function $G^R_{\mathcal{O}\mathcal{O}'}(\omega, k)$. We must have

$$\operatorname{Im}\omega(k) \leq 0, \qquad \left|\operatorname{Re}\frac{\omega(k)}{k}\right| \leq 1\,, \tag{1.2}$$

at real k. The first condition ensures that a small external perturbation does not cause an exponential growth in the hydrodynamic response. It also ensures the hierarchy between cause and effect: the retarded correlation function vanishes when the measurement precedes the perturbation. The second condition ensures that an initial perturbation does not induce a response faster than the speed of light. Both the conditions in eq. (1.2) are violated in the textbook formulation of relativistic hydrodynamics, even without loop corrections, when viewed by a boosted inertial observer.

The problems typically arise at timescales $t \sim \tau_*$ set by the magnitude of dissipative transport coefficients, like conductivities and viscosities. Since hydrodynamics is only supposed to be reliable at spacetime scales much longer than all other scales in the system, the stability and causality issues lie well beyond the hydrodynamic regime of applicability and do not really signal an inconsistency of the hydrodynamic framework itself, see e.g. [29]. Nonetheless, they render the theory inadequate for any practical application that involves actually solving a Cauchy problem for the relativistic hydrodynamic equations to determine the fluid flow. To circumvent these issues, we need to appropriately modify the hydrodynamic equations near or before the timescales of instabilities τ_* in a way that the physics in the hydrodynamic regime remains unchanged. There are two popular proposals for this in the literature: the Müller-Israel-Stewart (MIS) formalism [30–37] and more recently the Bemfica-Disconzi-Noronha-Kovtun (BDNK) formalism [38–41]. The MIS formalism allows the fluxes of conserved charges to evolve independently of the respective densities, such that they relax to their hydrodynamic values after some characteristic relaxation timescale $t \sim \tau$. By appropriately tuning $\tau \gtrsim \tau_*$, we can mend the stability and causality issues that would otherwise appear near τ_*. Closely related to these are the so-called "divergence-type theories" where the evolution equations for fluxes take the form of sourced total-divergence equations [42]; see the review by [43]. On the other hand, the BDNK formalism uses clever redefinitions of hydrodynamic fields to introduce the relaxation timescale τ into the hydrodynamic equations, so that they are perfectly causal and stable when expressed in terms of the new variables.

JHEP01(2024)162

The above procedure of modifying the physics at short timescales has a natural analogue in the language of EFTs. Physical observables in an EFT are obtained by averaging over all off-shell field configurations with arbitrary frequencies. Any low-energy EFT, which is not intended to describe high-energy ultraviolet (UV) phenomena, needs an appropriate UV-regularisation prescription to handle high-frequency field configurations that lie beyond its regime of applicability. In this sense, the MIS and BDNK formalisms outlined above can be thought of as potential stable and causal UV-regularisations of relativistic hydrodynamics. In the context of SK-EFTs, an important requirement for a regularisation prescription is that the regularised n-point correlation functions must also satisfy the respective FDTs. As shown in [44], this ensures that stochastic loop corrections do not alter the analyticity properties of n-point correlation functions.[2] In particular, provided that retarded 2-point correlation functions have no unstable or acausal poles at tree-level (i.e. in the classical linear theory), an FDT-compatible UV-regularisation prescription ensures that stability and causality remains intact perturbatively at arbitrary loop orders in the SK-EFT.

The goal of this work is to develop UV-regularised SK-EFTs for relativistic hydrodynamics that are stable and causal by construction, at least perturbatively in interactions, and realise the dynamical KMS symmetry to ensure compatibility with FDTs. In particular, since the UV sector of these SK-EFTs is not fixed by any global symmetry requirements, we find multiple consistent realisations of dynamical KMS symmetry that leave the effective action invariant simultaneously. We will first consider a toy model with single diffusive conserved current J^μ in section 2, the MIS-like version of which is known as the Maxwell-Cattaneo (MC) model of relativistic diffusion [46]. We will construct a stable and causal SK-EFT for relativistic diffusion using the MSR formalism and then present a first-principle derivation of the same using the SK formalism. In section 3, we will generalise these results to construct a stable and causal SK-EFT for relativistic hydrodynamics based on the MIS formalism. In particular, we shall see that the right choice of variables is quintessential for non-linearly realising the dynamical KMS symmetry in the effective theory. Working with stress as a relaxed degree of freedom directly, as is done in the conventional treatments of the MIS-hydrodynamics [30–34], is not compatible with the dynamical KMS symmetry. We will end the main part of the paper with some discussion and future directions in section 4.

As it turns out, the BDNK formalism of relativistic hydrodynamics, while appropriate for making the classical hydrodynamic equations stable and causal beyond their regime of applicability, does not generalise to a consistent SK-EFT.[3] We comment on these issues in section A. In the same appendix, we also compare with the recent work of [47], where authors used a BDNK-like EFT for diffusion to compute UV-regulated one-loop stochastic corrections to 2-point correlation functions of conserved charge. Though this EFT agrees with 2-point FDT at tree-level, we argue that compatibility with higher-point FDTs or with 2-point FDT at loop-levels requires the dissipative coefficients, i.e. charge conductivity and relaxation time, to be constants and not depend on the conserved charge. In section B, we

[2] If one wishes to use a regularisation prescription that is not compatible with FDT, such as dimensional-regularisation or zeta-function regularisation, one is forced to add additional BRST ghost fields into the framework to preserve the analyticity properties of n-point correlation functions [13, 45].

[3] We thank J. Noronha and M. Hippert for bringing these issues to our attention and for various useful discussions in this regard.

JHEP01(2024)162

outline a perturbative diagrammatic procedure for computing various time-ordered correlation functions in the UV-regulated EFT for diffusion. We explicitly compute the tree-level 2- and 3-point correlation functions and investigate their consistency with FDTs. The generalisations of this appendix to include stochastic loop corrections and extensions to the complete theory of relativistic hydrodynamics are left for future work. Finally, in appendix C, we discuss alternate prescriptions of dynamical KMS symmetry in the SK-EFTs for MC-diffusion and MIS-hydrodynamics.

Notation. We work in $d+1$ spacetime dimensions. The spacetime indices "$\mu, \nu, \ldots$" collectively run over the time coordinate "t" and the spatial indices "$i, j, \ldots$". The spacetime indices are raised/lowered using the background spacetime metric $g_{\mu\nu}$, which in flat spacetime is given by the Minkowski metric $\eta_{\mu\nu} = \text{diag}(-1, 1, 1, \ldots)$. The frequency ω and wavevector k_i in Fourier space is collectively denoted by $p_\mu = (-\omega, k_i)$. We will assume natural units where the speed of light c and the Boltzmann constant k_B are set to 1, however we will keep the reduced Planck's constant $\hbar$ explicit.

Note added. While this paper was nearing completion, we received a preliminary draft of [48] that offers a different perspective on the problems explored here and appears on the same day on arXiv.

2 Maxwell-Cattaneo model of diffusion

In order to transparently highlight the conceptual aspects of our construction, let us start with a simple diffusion model with single relativistic conserved charge. We will revisit the Maxwell-Cattaneo (MC) model of relativistic diffusion from the perspective of the second law of thermodynamics and review how it circumvents the issues with causality and stability that arise in the ordinary theory of relativistic diffusion; see e.g. [46, 49]. We will explore how to incorporate stochastic fluctuations in the theory of MC-diffusion using the MSR framework and make our way to the associated SK-EFT. We will pay close attention to the dynamical KMS symmetry in the EFT, which ensures the conformity of stochastic fluctuations with FDTs. The goal of this section is to setup the groundwork for the construction of a causal and stable EFT of charged relativistic hydrodynamics, which we will undertake in section 3.

2.1 Theory of diffusion

Consider a physical system invariant under certain internal U(1) global symmetry, with the associated Noether conserved current J^μ satisfying the conservation equation

$$\partial_\mu J^\mu = 0. \tag{2.1}$$

We will also find it helpful to introduce a background gauge field A_μ coupled to the current J^μ. The conservation of J^μ means that the theory must be invariant under background gauge transformations of the gauge field $A_\mu \to A_\mu + \partial_\mu \Lambda$. Given an inertial relativistic observer with constant four-velocity u_0^μ (normalised as $u_0^\mu u_\mu^0 = -1$), the current J^μ can generically be parametrised as

$$J^\mu = n u_0^\mu + \mathcal{J}^\mu, \tag{2.2}$$

JHEP01(2024)162

where n and $\mathcal{J}^\mu$ (normalised as $u_0^\mu \mathcal{J}_\mu = 0$) respectively denote the charge density and flux in the local rest frame of the inertial observer. The charge conservation (2.1) is a scalar equation and, given the flux $\mathcal{J}^\mu$, can be thought of as determining the dynamics of the density n. In the long-distance late-time regime, where all other non-conserved "fast" degrees of freedom have relaxed, the constitutive relations for the $\mathcal{J}^\mu$ can effectively be expressed in terms of n, A_μ, and their spacetime derivatives, known as the *constitutive relations*. To leading order in derivatives, we have the well-known Fick's law for diffusion

$$\mathcal{J}^\mu = -\sigma\, \Delta^{\mu\nu}\Big(\partial_\nu \mu - F_{\nu\rho} u_0^\rho\Big), \tag{2.3}$$

where $\Delta^{\mu\nu} = \eta^{\mu\nu} + u_0^\mu u_0^\nu$ is the spatial projection operator, $\mu(n)$ is the chemical potential thermodynamically conjugate to n, $\sigma(n)$ is the conductivity transport coefficient, and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the external electromagnetic field.

Note that the response of charge propagation to electric fields $F_{\mu\nu} u_0^\nu$ and gradients of chemical potential $\partial_\mu \mu$ is controlled by the same transport coefficient σ. A physical implication of this is that the "hydrostatic equilibrium configuration" $\mu = \mu_0 + u_0^\mu A_\mu$ identically solves the diffusion equation when coupled to background gauge fields that are time-independent in the rest frame of the inertial observer $u_0^\mu \partial_\mu A_\nu = 0$.

The particular form of the constitutive relations can also be understood as a consequence of the local second law of thermodynamics, i.e. there exists an entropy current S^μ whose divergence is non-negative for all solutions of the conservation equation. To see this, let us start with the internal energy density $\epsilon(s, n)$ as a function of the thermodynamic entropy density s and charge density n, defining the thermodynamic equation of state of the system. The first law of thermodynamics can be parametrised as

$$\mathrm{d}\epsilon = T_0 \mathrm{d}s + \mu \mathrm{d}n, \tag{2.4}$$

which defines the chemical potential μ. The thermodynamic temperature T_0 is taken to be constant; to allow for a spacetime-varying profile of temperature, we will need to bring in another equation for the conservation of energy; see section 3. Since we are only considering systems with conserved charge, we take the rate of change of energy density to be given by the associated Lorentz-power term $\partial_\mu(\epsilon u_0^\mu) = J^\mu F_{\mu\nu} u_0^\nu$. Using these, we can obtain the expression for entropy production

$$\partial_\mu S^\mu = -\frac{1}{T_0} \mathcal{J}^\mu \Big(\partial_\mu \mu - F_{\mu\nu} u_0^\nu\Big) \geq 0, \tag{2.5}$$

where $S^\mu = s u_0^\mu - \mu/T_0\, \mathcal{J}^\mu$. We require that the constitutive relations for $\mathcal{J}^\mu$ must be such that entropy is strictly produced for all onshell configurations of μ. This allows us to identify the constitutive relations in eq. (2.3) for an arbitrary non-negative conductivity $\sigma \geq 0$. Having done this, the entropy production rate is given as

$$\partial_\mu S^\mu = \frac{1}{T_0 \sigma} \mathcal{J}_\mu \mathcal{J}^\mu \geq 0. \tag{2.6}$$

The utility of coupling the theory to background gauge fields A_μ is that we can use these to obtain the response functions (retarded correlation functions) of the conserved current

JHEP01(2024)162

J^μ. With the forthcoming generalisation to hydrodynamics in mind, let us use a collective notation for the operators $\mathcal{O} = (J^\mu)$ and the respective sources $\phi = (A_\mu)$. We define the classical expectation values of operators in the presence of sources

$$\langle \mathcal{O} \rangle = \mathcal{O}\,\big|_{\text{onshell}}, \tag{2.7}$$

formally evaluated on the solutions of the conservation equation (2.1). The retarded 2-point retarded correlation functions can then be defined as simply

$$G^R_{\mathcal{O}\mathcal{O}'} = \frac{\delta\langle \mathcal{O}\rangle}{\delta\phi'}\bigg|_{\phi_0}, \tag{2.8}$$

where $\phi_0 = (A_\mu = 0)$ denote flat equilibrium sources. We can similarly obtain the higher-point retarded correlation functions by taking higher variations with respect to ϕ. In Fourier-space, with four-momentum $p_\mu = (-\omega, k_i)$, and assuming the inertial observer to be at rest $u^\mu_0 = \delta^\mu_t$, we find that the retarded 2-point functions of the charge density J^t and the transverse flux $J^i_\perp = k^i_j J^j$ (where $k_{ij} = \delta_{ij} - k_i k_j/k^2$) take the form

$$\begin{aligned} G^R_{J^t J^t}(\omega, k) &= \frac{-\sigma k^2}{i\omega - Dk^2}, \\ G^R_{J^i_\perp J^j_\perp}(\omega, k) &= i\omega\sigma\, k^{ij}, \end{aligned} \tag{2.9}$$

where $D = \sigma/\chi$ is the diffusion constant with $\chi = \partial n/\partial\mu$ being the charge susceptibility. The response functions involving the longitudinal flux $J_\parallel = J^i k_i/k$ can be obtained using the conservation law: $-i\omega J^t + ikJ_\parallel = 0$. The pole structure of the response functions yield the mode spectrum of the theory. There is a single diffusive mode carried by the conserved charge fluctuations, with the dispersion relations

$$\omega = -iDk^2 + \ldots, \tag{2.10}$$

where ellipses denote the higher-k corrections arising from the possible higher-derivative generalisations of the diffusion model. Thermodynamic stability of the equilibrium state requires that $\chi \geq 0$. Combined with the second law requirement $\sigma \geq 0$, this implies that $D \geq 0$, ensuring the stability of the diffusive mode.

We can obtain the retarded correlation functions and mode spectrum of relativistic diffusion in a boosted frame of reference $u^\mu_0 = (1, v^i_0)/\sqrt{1-v_0^2}$ by performing a Lorentz boost on the dispersion relations in the local rest frame, i.e.

$$\begin{aligned} &\omega \to \frac{1}{\sqrt{1-v_0^2}}\left(\omega - kv_0\cos\theta\right), \qquad k_i \to \frac{1}{\sqrt{1-v_0^2}}\left(k\cos\theta - \omega v_0\right)\frac{v^0_i}{v_0} + k_j\left(\delta^j_i - \frac{v^0_i v^j_0}{v_0^2}\right), \\ &k^2 \to \frac{1}{1-v_0^2}\left(k\cos\theta - \omega v_0\right)^2 + k^2\sin^2\theta, \end{aligned} \tag{2.11}$$

where $k_i v^i_0 = kv_0\cos\theta$. In a boosted reference frame, the dispersion relations are determined by a quadratic polynomial in ω, giving rise to an additional mode. We find a boosted diffusive mode and a gapped mode

$$\omega = kv_0\cos\theta - iDk^2\sqrt{1-v_0^2}\left(1 - v_0^2\cos^2\theta\right) + \ldots, \qquad \omega = i\frac{\sqrt{1-v_0^2}}{v_0^2 D} + \ldots. \tag{2.12}$$

JHEP01(2024)162

While the diffusive mode is still stable, disturbingly, the new gapped mode has a positive imaginary part and hence is unstable. This is a generic feature of relativistic dissipative hydrodynamic theories: the rest-frame dispersion relations have higher powers of k than ω due to the presence of dissipation, whereas after a boost transformation, we generically have as many powers of ω as k. This generates new unphysical unstable gapped modes in the boosted frame of reference, making the theory unusable beyond the perturbative regime. The theory also features modes propagating faster than the speed of light for large wavevectors and small boost parameter

$$\omega = \left(\frac{\cos\theta}{v_0} \pm i\sqrt{\frac{1-v_0^2}{v_0^2}\sin^2\theta} + \mathcal{O}(1/k) \right) k, \tag{2.13}$$

whose real part is clearly not bounded from above.

These are not really conceptual problems with the relativistic diffusion model itself: the formalism is only strictly valid in the low-frequency regime. Since the unphysical modes typically show up at scales $|\omega| \gtrsim 1/\tau_* \equiv 1/D$, they lie outside the regime of applicability of the model. However, as noted in the introduction, they do pose practical problems when solving the Cauchy problem in a relativistic context, in particular the diffusion equation can not be coupled to Einstein's equations of General Relativity. To overcome these issues, one popular solution is to introduce additional degrees of freedom in the hydrodynamic theory. The extra degrees of freedom are gapped, so that the small-k prediction in eq. (2.10) remains untouched; their purpose is to serve as a consistent UV regularisation of the relativistic diffusion model.

2.2 Maxwell-Cattaneo model

While setting up the model of diffusion above, we assumed that all non-conserved "fast" degrees of freedom in the system have relaxed and thus the flux of conserved charge $\mathcal{J}^\mu$ is entirely given in terms of the density n. We can go slightly beyond this "hydrodynamic regime" by allowing the flux $\mathcal{J}^\mu$ to evolve independently of the density, with certain characteristic relaxation timescale τ. The resulting theory is sometimes referred to as the Maxwell-Cattaneo (MC) model of diffusion [46]. The inclusion of extra degrees of freedom can be thought of as conservatively including the "slowest" gapped non-hydrodynamic modes (one each in longitudinal and transverse sectors) into the framework. More formally, we should think of this procedure as introducing an ultraviolet regularisation to the diffusion model at the frequency scales $1/\tau$. If we tune $\tau \gtrsim \tau_*$, the new flux degrees of freedom will modify the physics before/near the onset of instabilities in eq. (2.12) in frequency space, and will enable us to construct a diffusion model that is well-behaved for all Lorentz observers [30–34].

We will construct the model of MC-diffusion using an additional vector degree of freedom υ_μ (satisfying $\upsilon_\mu u_0^\mu = 0$). At this point, the new degree of freedom υ_μ is unrelated to the flux $\mathcal{J}^\mu$ defined in eq. (2.2). We promote the equation of state to $\epsilon = \epsilon(s, n, \upsilon^2)$, so that the energy density can now also depend on the scalar $\upsilon^2 = \upsilon^\mu \upsilon_\mu$, and the first law of thermodynamics in eq. (2.4) accordingly gets modified to

$$\mathrm{d}\epsilon = T_0 \mathrm{d}s + \mu \mathrm{d}n + \frac{\chi_\upsilon}{2}\mathrm{d}\upsilon^2, \tag{2.14}$$

which defines χ_υ, a new "thermodynamic" coefficient. To get some intuition about the new term, let us look at the associated canonical free energy density $\mathcal{F}_{\text{can}} = \epsilon - Ts$, whose variation

JHEP01(2024)162

at fixed temperature is given as $\mathrm{d}\mathcal{F}_{\text{can}} = \mu \mathrm{d}n + \chi_\upsilon \upsilon^\mu \mathrm{d}\upsilon_\mu$. The field υ_μ is akin to a vector order parameter in Landau's theory of phase transitions. Here we are studying the phase with $\upsilon_\mu = 0$ in equilibrium, and neglecting the kinetic term for υ_μ, as it is irrelevant compared to the mass term. The model essentially corresponds to model C of the Hohenberg-Halperin classification [4],[4] with a conserved scalar density n and a non-conserved vector density υ_μ.[5] In this way of thinking, υ_μ is an arbitrary field we introduced in the effective theory without inherent microscopic definition and we are free to arbitrarily redefine it to suit our needs. We shall fix this redefinition freedom by choosing the flux $\mathcal{J}^\mu$ to be parametrised as

$$\mathcal{J}^\mu = \alpha_\upsilon \upsilon^\mu, \tag{2.15}$$

for some arbitrary coefficient α_υ. We can, of course, fix α_υ as well by rescaling the definition of υ_μ. For example, we can choose $\alpha_\upsilon = 1$ to give υ^μ the interpretation of flux or $\alpha_\upsilon = \chi_\upsilon$ to give it the interpretation of the thermodynamic conjugate of flux. With the former choice, the relation in eq. (2.14) corresponds to extended irreversible thermodynamics [50]. For now, we will keep α_υ arbitrary as it will prove to be useful later. We can obtain the dynamical equations for υ_μ using the second law of thermodynamics as our guiding principle. The entropy production equation (2.5) accordingly modifies to

$$\partial_\mu S^\mu = -\frac{\alpha_\upsilon}{T_0} \upsilon^\mu \Big(\partial_\mu \mu - F_{\mu\nu} u_0^\nu + \frac{\chi_\upsilon}{\alpha_\upsilon} u_0^\lambda \partial_\lambda \upsilon_\mu \Big) \geq 0, \tag{2.16}$$

again with $S^\mu = s u_0^\mu - \mu/T_0\, \mathcal{J}^\mu$. Inspecting this equation, we can read out the constitutive equations for υ_μ to be

$$\alpha_\upsilon \upsilon^\mu = -\sigma\, \Delta^{\mu\nu} \Big(\partial_\nu \mu - F_{\nu\rho} u_0^\rho + \frac{\chi_\upsilon}{\alpha_\upsilon} u_0^\lambda \partial_\lambda \upsilon_\nu \Big). \tag{2.17}$$

The net rate of entropy production is still given by eq. (2.6), however σ can now be a function of both n and υ^2. We note that the constitutive equations (2.17) are only meant to be representative and do not characterise the most general constitutive relations within a derivative-order scheme.[6] We can eliminate υ_μ in favour of $\mathcal{J}^\mu$ to write down an equation for the relaxation of flux

$$\mathcal{J}^\mu + \tau\, \Delta^{\mu\nu} u_0^\lambda \partial_\lambda \mathcal{J}_\nu = -\sigma\, \Delta^{\mu\nu} \Big(\partial_\nu \mu - F_{\nu\rho} u_0^\rho \Big) + \tau u_0^\lambda \partial_\lambda \ln \alpha_\upsilon\, \mathcal{J}^\mu, \tag{2.18}$$

[4]The free energy of model C of Hohenberg-Halperin also contains a term proportional to $\partial_\mu n \partial^\mu n$, which is second order in derivatives. Technically, we should count $\upsilon_\mu \sim \mathcal{O}(\partial)$ and thus this term is as important as the υ^2 term in the free energy. However, given that our goal is not to rigorously describe the second order corrections to the diffusion model, but instead to probe the effects of a gapped mode, we will drop this and other similar second-derivative terms here for simplicity.

[5]The additional contribution to the free energy is reminiscent of the kinetic term in fluid dynamics, with υ_i playing the role of momentum density that is relaxed (i.e. not conserved), possibly due to the presence of impurities, and $1/\chi_\upsilon$ that of mass density. The term might also be familiar from superfluid dynamics, with υ_i playing the role of superfluid velocity and χ_υ that of the superfluid density. However, since the curl of υ_i is unconstrained, this model actually corresponds to the theory of superfluidity with vortices, the presence of which causes the superfluid velocity to relax.

[6]For example, noting that the flux is counted as $\upsilon^i \sim \mathcal{O}(\partial)$ and the time-derivatives are counted as $\partial_t \sim \mathcal{O}(\partial^2)$ in the diffusion model, we can also imagine additional terms like $\partial^i \partial_t n$, $\partial_k \partial^k \upsilon^i$, $\partial_k \partial^k \partial^i n$ in the constitutive equations, which we are going to omit here for simplicity. As we mentioned in footnote 4, we are only interested in probing the effects of a massive mode and not rigorously describe higher-derivative corrections to the diffusion model.

JHEP01(2024)162

where we have identified the relaxation time-scale $\tau = \sigma\chi_v/\alpha_v^2$. For larger time-scales $t \gg \tau$, the description effectively reduces to the one with just the conserved hydrodynamic mode.

We should emphasise that the theory described above is more general than the original Maxwell-Cattaneo theory of diffusion, because all coefficients μ, τ, σ, α_v can be functions of both n and v^2 (or $\mathcal{J}^2$). We can construct a simpler model where all these are taken to be just functions of n. However, using the thermodynamic relation (2.14), we know that $\partial\chi_v/\partial n = 2\partial\mu/\partial v^2$, implying that $\chi_v = \alpha_v^2\tau/\sigma$ must be constant when μ just depends on n. Defining the charge susceptibility $\chi = (\partial\mu/\partial n)^{-1}$ and the diffusion coefficient $D = \sigma/\chi$, both generic functions of n, the constitutive relations reduce to

$$\mathcal{J}^\mu + \tau(n)\,\Delta^{\mu\nu}u_0^\lambda\partial_\lambda\mathcal{J}_\nu = -D(n)\,\Delta^{\mu\nu}\partial_\nu n + \sigma(n)\,F^{\mu\rho}u_\rho^0 + \frac{\sigma}{2}\frac{\partial(\tau/\sigma)}{\partial n}\mathcal{J}^\mu\partial_\rho\mathcal{J}^\rho, \tag{2.19}$$

where we have eliminated $2\partial_\mu \ln\alpha_v = \partial_\mu \ln(\sigma/\tau_{\mathcal{J}})$ for constant χ_v together with the charge conservation equation (2.1). If we wish to get rid of the last term, as in the original Maxwell-Cattaneo theory, we must impose a condition on the derivatives of relaxation time $\tau'(n) = (\tau(n)/\sigma(n))\sigma'(n)$. Since the bare σ is only visible when coupling to background sources, the constraint between $\tau'(n)$ and $\sigma'(n)$ is not relevant for the classical dynamical equations of n in the absence of sources. Note that the constraint on $\sigma'(n)$ does not constrain $D'(n) = \sigma'(n)/\chi(n) - (\sigma(n)/\chi(n)^2)\chi'(n)$, because it has independent functional dependence coming from $\chi'(n)$.

We can use the variational formula in eq. (2.8) to obtain the retarded correlation functions in the presence of the new relaxed vector degrees of freedom. For the theory (2.19), the expressions in eq. (2.9) for the retarded correlation functions of density and transverse-flux computed in the local rest frame modify to

$$\begin{aligned} G^R_{J^tJ^t}(\omega,k) &= \frac{-\sigma k^2}{i\omega(1-i\omega\tau) - Dk^2}, \\ G^R_{J^i_\perp J^j_\perp}(\omega,k) &= \frac{i\omega\sigma}{1-i\omega\tau}k^{ij}. \end{aligned} \tag{2.20}$$

The form of the response functions is almost identical to the ones we saw before in eq. (2.9), but with a modified pole structure. The correlation functions involving v_μ can also be obtained via the variations with respect to A_μ using the relation in eq. (2.15). In a generically boosted frame of reference, obtained by implementing the Fourier-space boost transformation in eq. (2.11), we find the boosted diffusion mode and a gapped mode in the longitudinal sector

$$\omega = kv_0\cos\theta - iDk^2\sqrt{1-v_0^2}\left(1 - v_0^2\cos^2\theta\right) + \ldots, \qquad \omega = -i\frac{\sqrt{1-v_0^2}}{\tau - v_0^2 D} + \ldots, \tag{2.21a}$$

and another gapped mode in the transverse sector

$$\omega = -i\frac{\sqrt{1-v_0^2}}{\tau} + \ldots. \tag{2.21b}$$

For the gapped poles to be stable for all inertial frames of references $0 \le |v_0| \le 1$, we need to bound τ from below

$$\tau > D > 0. \tag{2.22}$$

JHEP01(2024)162

Given the second law constraint $\sigma \geq 0$, this also implies an inequality for thermodynamic susceptibilities $0 < \alpha_\upsilon^2/\chi_\upsilon < \chi$. With the stability bounds in place, the theory (2.19) is also causal and the modes at large wavevector propagating as

$$\omega = \left(c_\infty v_0 \cos\theta \pm \sqrt{(1-c_\infty)(1-c_\infty v_0^2\cos^2\theta)} + \mathcal{O}(1/k)\right) k, \tag{2.23a}$$

with

$$c_\infty = \frac{\tau - D}{\tau - v_0^2 D}. \tag{2.23b}$$

Note that $0 < c_\infty \leq 1$ due to the stability bounds, which implies that the speed of modes at large wavevector is bounded from above by 1 for all θ.

2.3 Stochastic fluctuations and effective action

The Maxwell-Cattaneo diffusion model is deterministic. Even in the strict long-distance late-time hydrodynamic regime, where the only relevant macroscopic degrees of freedom are the conserved charges, the ignored microscopic degrees of freedom, colloquially called "stochastic noise", can still leave qualitatively distinct signatures through random interactions. The standard method to include stochastic noise into classical deterministic equations was outlined by Martin-Siggia-Rose (MSR) [2]. The starting point of this formalism is to modify the constitutive equations for υ_μ with an arbitrary random noise term θ^μ, i.e.

$$\alpha_\upsilon \upsilon^\mu = -\sigma\, \Delta^{\mu\nu}\Big(\partial_\nu \mu - F_{r\nu\rho} u_0^\rho + \frac{\chi_\upsilon}{\alpha_\upsilon} u_0^\lambda \partial_\lambda \upsilon_\nu\Big) + \theta^\mu. \tag{2.24}$$

In anticipation of our later Schwinger-Keldysh discussion, we will label the background electromagnetic and gauge fields with a subscript "r" from this section onward. We can imagine solving the stochastic equations (2.24) to obtain the noisy expectation values $\langle \ldots \rangle_\theta$ in the presence of background fields $\phi_r = (A_{r\mu})$ and noise fields $\theta = (\theta^\mu)$. The physical expectation values can be obtained by performing a weighted average over arbitrary random noise configurations

$$\langle \ldots \rangle = \int \mathcal{D}\theta\, \langle \ldots \rangle_\theta \exp\Big(-\frac{1}{4}\int \mathrm{d}^{d+1}x \left\langle \frac{\Delta_{\mu\nu}}{T_0 \tilde{\sigma}} \right\rangle_\theta \theta^\mu \theta^\nu\Big). \tag{2.25}$$

Here $\tilde{\sigma}(n, \upsilon^2)$ is a new coefficient introduced to model the strength of stochastic fluctuations. Note the $\langle \ldots \rangle_\theta$ around the noise strength in the above expression, which means that the $\tilde{\sigma}(n, \upsilon^2)$ also needs to be evaluated on the solutions of the equations of motion (2.24). This is a slight generalisation of the original MSR formulation where the noise strength is taken to be a constant and is needed for conformity with non-linear fluctuation-dissipation theorems; see the following. In practice, we can convert the onshell objects $\langle \ldots \rangle_\theta$ inside the path integral into the respective offshell versions by introducing Lagrange multipliers $\psi_a = (\varphi_a, V_{a\mu})$ (normalised as $u_0^\mu V_{a\mu} = 0$) for the equations of motion, leading to

$$\begin{aligned}\langle \ldots \rangle = \int \mathcal{D}\theta\, \mathcal{D}\psi_r\, \mathcal{D}\psi_a\, (\ldots) \exp\Big(-\frac{1}{4}\int \mathrm{d}^{d+1}x \frac{\Delta_{\mu\nu}}{T_0\tilde{\sigma}} \theta^\mu\theta^\nu\Big) \\ \times \exp\Big(-i\int \mathrm{d}^{d+1}x\, \varphi_a \partial_\mu (n u_0^\mu + \alpha_\upsilon \upsilon^\mu)\Big) \\ \times \exp\Big(-i\int \mathrm{d}^{d+1}x\, V_{a\mu}\Big(\alpha_\upsilon \upsilon^\mu + \sigma\, \Delta^{\mu\nu}\Big(\partial_\nu \mu - F_{r\nu\rho} u_0^\rho + \frac{\chi_\upsilon}{\alpha_\upsilon} u_0^\lambda \partial_\lambda \upsilon_\nu\Big) - \theta^\mu\Big)\Big),\end{aligned} \tag{2.26}$$

JHEP01(2024)162

where $\psi_r = (\mu, \upsilon_\mu)$ collectively denote the dynamical hydrodynamic degrees of freedom. In the limit $\chi_\upsilon \to 0$, the field υ_μ becomes a simple Lagrange multiplier to impose $V_{a\mu} = \partial_\mu \varphi_a$, and we recover the Schwinger-Keldysh effective action for the ordinary diffusion model reported in [51]. To compute stochastic correlation functions of operators $\mathcal{O} = (J^\mu)$ using the path integral described above, it is convenient to introduce another set of background fields $\phi_a = (A_{a\mu})$ and consider the *Schwinger-Keldysh generating functional*

$$\mathcal{Z}[\phi_r, \phi_a] = \left\langle \exp\left(i \int \mathrm{d}^{d+1}x \, \phi_a \mathcal{O} \right) \right\rangle . \tag{2.27}$$

In terms of this, the retarded 2-point Green's function given in eq. (2.8) and the symmetric 2-point Green's function $G^S_{\mathcal{O}\mathcal{O}'} = \langle \mathcal{O}\mathcal{O}' \rangle - \langle \mathcal{O} \rangle \langle \mathcal{O}' \rangle \big|_{\phi_0}$ can be computed using the variational formulae

$$\begin{aligned} G^R_{\mathcal{O}\mathcal{O}'} &= \left(\frac{\delta}{\delta \phi'_r} \right) \left(\frac{-i\delta}{\delta \phi_a} \right) \ln \mathcal{Z}[\phi_r, \phi_a] \Big|_{\phi_0} , \\ G^S_{\mathcal{O}\mathcal{O}'} &= \left(\frac{-i\delta}{\delta \phi'_a} \right) \left(\frac{-i\delta}{\delta \phi_a} \right) \ln \mathcal{Z}[\phi_r, \phi_a] \Big|_{\phi_0} , \end{aligned} \tag{2.28}$$

and similarly for higher-point functions. To simplify the generating functional, we note that path integral over the noise fields θ^μ in (2.27) is Gaussian and can be performed analytically to yield

$$\mathcal{Z}[\phi_r, \phi_a] = \int \mathcal{D}\psi_r \, \mathcal{D}\psi_a \exp(iS[\psi_r, \psi_a; \phi_r, \phi_a]), \tag{2.29}$$

where the *Schwinger-Keldysh effective action* is given as

$$\begin{aligned} S = \int \mathrm{d}^{d+1}x \Big[& B_{a\mu} n u_0^\mu + \alpha_\upsilon (B_{a\mu} - V_{a\mu}) \upsilon^\mu \\ & - \sigma V_a^\mu \left(\partial_\mu \mu - F_{r\mu\rho} u_0^\rho + \frac{\chi_\upsilon}{\alpha_\upsilon} u_0^\lambda \partial_\lambda \upsilon_\mu \right) + i T_0 \tilde{\sigma} V_a^\mu V_{a\mu} \Big], \end{aligned} \tag{2.30}$$

and we have defined $B_{a\mu} = \partial_\mu \varphi_a + A_{a\mu}$. In the limit $\chi_\upsilon \to 0$, the field υ_μ becomes a simple Lagrange multiplier to impose $V_{a\mu} = B_{a\mu}$ and we recover the Schwinger-Keldysh effective action for the ordinary diffusion model reported in [12].

FDT and dynamical KMS symmetry. Using this action, it is straight-forward to verify that we recover the 2-point retarded correlation functions as given in eq. (2.20). We can also obtain the symmetric 2-point correlation functions

$$\begin{aligned} G^S_{J^t J^t}(\omega, k) &= \frac{2k^2 T_0 \tilde{\sigma}}{|i\omega(1 - i\omega\tau) - Dk^2|^2}, \\ G^S_{J^i_\perp J^i_\perp}(\omega, k) &= \frac{2T_0 \tilde{\sigma}}{|1 - i\omega\tau|^2}, \end{aligned} \tag{2.31}$$

which are proportional to the strength of stochastic noise in the model. A characteristic feature of finite-temperature field theories are fluctuation-dissipation theorems. They posit that retarded and symmetric 2-point Green's functions of any operator, when computed in a

JHEP01(2024)162

thermal state with global temperature T_0 and velocity $u_0^\mu = \delta_t^\mu$, must satisfy the fluctuation-dissipation theorem (FDT) in eq. (1.1). The boosted version of eq. (1.1) can be obtained by replacing ω with $-u_0^\mu p_\mu$. Applying this to the symmetric functions in eq. (2.31) fixes the strength of noise to be given by the dissipative conductivity transport coefficient

$$\tilde{\sigma} = \sigma. \tag{2.32}$$

Technically, the 2-point FDT only imposes this equality in thermodynamic equilibrium at $\mu = \mu_0$ and $\upsilon_\mu = 0$. However, similar non-linear FDTs exist for higher-point functions, which impose the relation (2.32) for arbitrary thermodynamic arguments. Of course, checking this order-by-order for all higher-point correlation functions is a tedious exercise and it would be great if we could realise it as a symmetry directly at the level of the effective action. This is precisely the dynamical Kubo-Martin-Schwinger (KMS) symmetry as we discuss in the next subsection.

To avoid getting lost into technicalities, let us already summarise the dynamical KMS symmetry of the effective action (2.30). First, we need to identify certain discrete symmetry Θ of the underlying microscopic theory including the time-reversal transformation T. Depending on the physical system under consideration, this could be just the time-reversal $\Theta = \mathrm{T}$, spacetime-parity $\Theta = \mathrm{PT}$, time-reversal with charge conjugation $\Theta = \mathrm{CT}$, or spacetime-parity together with charge conjugation $\Theta = \mathrm{CPT}$. The eigenvalues of various fields under different Θ symmetries is given in table 1. The KMS symmetry is defined as the invariance of the Schwinger-Keldysh generating functional $\mathcal{Z}[\phi_r, \phi_a]$ under a transformation of the background fields $\phi_{r,a} = (A_{r,a\mu})$ given by[7]

$$\phi_r \to \Theta\phi_r, \qquad \phi_a \to \Theta\left(\phi_a + i\beta_0^\mu \partial_\mu \phi_r\right). \tag{2.33}$$

Note that the auxiliary field φ_a appears in the effective action via the combination $B_{a\mu} = \partial_\mu \varphi_a + A_{a\mu}$, so its Θ-eigenvalue is forced upon us by the Θ-eigenvalue of the background fields, which turns out to be opposite the Θ-eigenvalue of μ. However, there is no such constraint for the auxiliary field $V_{a\mu}$, resulting in two independent KMS prescriptions for the dynamical fields. If we take the Θ-eigenvalue of $V_{a\mu}$ to be the same as υ_μ, we find the "standard" prescription of KMS symmetry

$$\begin{aligned} \mu &\to \Theta\mu, & \varphi_a &\to \Theta\left(\varphi_a + i\left(\frac{\mu-\mu_0}{T_0} - \beta_0^\mu A_{r\mu}\right)\right), \\ \upsilon_\mu &\to \Theta\upsilon_\mu, & V_{a\mu} &\to \Theta\left(V_{a\mu} + \frac{i}{T_0}\Delta_\mu^\nu\left(\partial_\nu\mu + u_0^\lambda F_{r\lambda\nu} + \frac{\chi_\upsilon}{\alpha_\upsilon} u_0^\lambda \partial_\lambda \upsilon_\nu\right)\right). \end{aligned} \tag{2.34}$$

[7] Given a field $f(x)$ with Θ-eigenvalue $\eta_f^\Theta = \pm 1$, defined for complex spacetime coordinates, its Θ-transformation is given as $\Theta f(x) = \eta_f^\Theta f(\Theta x)$, where $\Theta t = -t$ and $\Theta\vec{x} = \pm\vec{x}$ depends on whether Θ includes the spatial parity operator P. For example, taking $\Theta = \mathrm{PT}$ and f to be PT-even, we have $\Theta f(x) = f(-x)$. Another field $g(x) = f(x + iV)$ is an eigenfunction of Θ if and only if the vector V^μ is real, with the Θ-eigenvalue $\eta_g^\Theta = \eta_f^\Theta$. The Θ-transformation of this equation is $\Theta g(x^*) = \Theta f(x^* - iV)$. In particular, the Θ-transformation of an equation also implements a complex conjugation $\Theta : i \to -i$. The C,P,T eigenvalues of the derivatives $F_t = \partial_t f$ and $F_i = \partial_i f$ are defined as $(\eta_f^{\mathrm{C}}, \eta_f^{\mathrm{P}}, -\eta_f^{\mathrm{T}})$ and $(\eta_f^{\mathrm{C}}, -\eta_f^{\mathrm{P}}, \eta_f^{\mathrm{T}})$ respectively. We will define $\Theta\partial_\mu f(x) = \eta_{\partial_\mu f}^\Theta (\partial_\mu f)(\Theta x)$; this convention is different from [17] which uses η_f^Θ instead of $\eta_{\partial_\mu f}^\Theta$. The benefit of this convention is that we simply have $\Theta F_\mu = \Theta\partial_\mu f$, unlike the conventions of [17] that require $\Theta F_\mu = \eta_{\partial_\mu}^\Theta \Theta\partial_\mu f$.

JHEP01(2024)162

On the other hand, if we take the Θ-eigenvalue of $V_{a\mu}$ to be the opposite that of υ_μ, its KMS transformation has to be replaced with the "alternate" prescription

$$V_{a\mu} \to \Theta\left(V_{a\mu} + \frac{i}{T_0}\frac{\alpha_\upsilon}{\sigma}\upsilon_\mu\right). \tag{2.35}$$

It can be explicitly checked that both these prescriptions leave the effective action (2.30) invariant. We will discuss the physical principles underlying the standard prescription in detail in appendix 2.4, while the details of the alternate prescription can be found in appendix C.[8]

2.4 Schwinger-Keldysh formalism

In the previous subsection, we used the MSR formalism to phenomenologically derive the effective action for MC-diffusion, by integrating over random noise configurations dictated by FDT. In recent years, a more systematic SK-EFT framework has been formalised, which can be used to derive such effective actions starting from symmetry principles [13, 15, 16]. Furthermore, while the MSR formalism is only apt for describing Gaussian stochastic noise, the symmetry principles of the SK framework can also be used to systematically include higher non-Gaussian stochastic interactions into the hydrodynamic framework; see e.g. [8]. In this subsection, we review the SK formalism for relativistic diffusion, with appropriate modifications to account for the gapped MC degrees of freedom.

Dynamical fields and global symmetries. SK field theories are set-up on a closed-time contour, with leg "1" going forward in time and leg "2" returning backward in time to the initial state. While the conventional single-time contour from textbook quantum field theory can only access correlation functions at zero temperature, this exotic closed-time contour can be used to compute symmetric, retarded, and advanced correlators at finite temperature. Each leg of the contour is equipped with its own set of degrees of freedom and background fields. For the present case of interest, the dynamical field content is a pair of phase fields $\varphi_{1,2}$, with the subscripts labelling the respective legs of the contour. The theory is required to obey independent global U(1) symmetries on the two legs, i.e.

$$\varphi_{1,2} \to \varphi_{1,2} - \Lambda_{1,2}. \tag{2.36a}$$

We can gauge these symmetries by introducing a set of background gauge fields $A_{1,2\mu}$, with transformation rules

$$A_{1,2\mu} \to A_{1,2\mu} + \partial_\mu \Lambda_{1,2}. \tag{2.36b}$$

[8]The authors in [48] arrived at the alternate prescription of KMS transformations by invoking the principle of detailed balance in the MSR effective action. To make contact with their language, let us define $\delta n = n(\mu) - n(\mu_0)$, $\delta\mathcal{J}_\mu = \alpha_\upsilon \upsilon_\mu$, $\delta\bar{n} = T_0\chi\varphi_a$, and $\delta\bar{\mathcal{J}}_\mu = T_0\sigma V_{a\mu}$. Choosing $\Theta = \mathrm{PT}$ and turning off background fields, the alternate KMS transformations act as $\delta n \to \delta n$, $\delta\mathcal{J}_\mu \to \delta\mathcal{J}_\mu$, $\delta\bar{n} \to -\delta\bar{n} - i\delta n$, and $\delta\bar{\mathcal{J}}_\mu \to -\delta\bar{\mathcal{J}}_\mu - i\delta\mathcal{J}_\mu$, with the respective right-hand sides evaluated at $-x^\mu$, where we have ignored higher-order terms in fluctuations. These are precisely the KMS transformations obtained in [48], although we note that this simple form only applies at linear order in fluctuations and in the absence of background fields. On the other hand, under the standard KMS prescription, the transformation of $\delta\bar{\mathcal{J}}_\mu$ is modified to $\delta\bar{\mathcal{J}}_\mu \to \delta\bar{\mathcal{J}}_\mu + iD\Delta^\nu_\mu\partial_\nu n + i\tau u_0^\lambda\partial_\lambda\delta\mathcal{J}_\mu$, which is also a symmetry of the effective action derived in [48].

JHEP01(2024)162

Using these, we can define the global symmetry invariants

$$B_{1,2\mu} = \partial_\mu \varphi_{1,2} + A_{1,2\mu}, \tag{2.37}$$

which are manifestly invariant under the global U(1) symmetries. The theory also carries a fixed thermal vector $\beta_0^\mu = u_0^\mu / T_0$, characterising the four-velocity and temperature of the inertial observer with respect to which the global thermal equilibrium state is defined.

In addition, to implement the gapped degrees of freedom present in MC-diffusion, we also need to add a pair of dynamical spatial vector fields $\upsilon_{1,2\mu}$ satisfying

$$\beta_0^\mu \upsilon_{1,2\mu} = 0. \tag{2.38}$$

The new degrees of freedom are taken to be invariant under the pair of global U(1) symmetries.

It is useful to introduce an average-difference basis $f_r = (f_1 + f_2)/2$, $f_a = (f_1 - f_2)/\hbar$ for various dynamical and background fields. The average "r" combinations are understood as the "physical" macroscopic fields, while the difference "a" combinations as the stochastic noise associated with them. The Schwinger-Keldysh effective action $S[\Phi_r, \Phi_a; \beta_0^\mu]$ of the theory can be expressed in terms of the global symmetry invariants $\Phi_{r,a} = (B_{r,a\mu}, \upsilon_{r,a\mu})$ and the thermal vector β_0^μ.

Diagonal shift symmetry. In addition to the global symmetries above, we impose a local diagonal shift symmetry between the two phase fields

$$\varphi_{1,2} \to \varphi_{1,2} + \lambda, \tag{2.39}$$

that acts uniformly on the two legs of the Schwinger-Keldysh contour. This symmetry is required to be time-independent, i.e.

$$\beta_0^\mu \partial_\mu \lambda = 0, \tag{2.40}$$

Among the global symmetry invariants defined above, it is easy to see that the difference combination $B_{a\mu}$ is invariant under the diagonal shift symmetry, while average combination $B_{r\mu}$ transforms as a gauge field

$$B_{r\mu} \to B_{r\mu} + \partial_\mu \lambda. \tag{2.41}$$

Due to the time-independent nature of the symmetry, the time-component $u^\mu B_{r\mu}$ is invariant, which is identified as the chemical potential μ in the theory of diffusion.

Schwinger-Keldysh generating functional. The fundamental object of interest in non-equilibrium field theory is the SK generating functional $\mathcal{Z}[\phi_r, \phi_a]$, prescribed as a functional of the two sets of background fields $\phi_{r,a} = (A_{r,a\mu})$, which can be used to compute retarded and symmetric correlation functions of operators in the theory using the variational formulae (2.28). In the SK-EFT formalism, the generating functional is obtained by performing a path integral of the effective action $S[\Phi_r, \Phi_a; \beta^\mu]$ over the two sets of dynamical fields $\psi_{r,a} = (\varphi_{r,a}, \upsilon_{r,a\mu})$, as in

$$\mathcal{Z}[\phi_r, \phi_a] = \int \mathcal{D}\psi_r \, \mathcal{D}\psi_a \exp(iS[\Phi_r, \Phi_a; \beta_0^\mu]) \tag{2.42}$$

JHEP01(2024)162

The generating functional is required to satisfy the following three conditions

$$\mathcal{Z}[\phi_r, \phi_a = 0] = 1, \qquad \mathcal{Z}[\phi_r, -\phi_a] = \mathcal{Z}^*[\phi_r, \phi_a], \qquad \mathrm{Re}\,\mathcal{Z}[\phi_r, \phi_a] \leq 0, \tag{2.43}$$

arising from generic properties of quantum field theories defined on a closed-time contour. More details regarding the underlying physics can be found in the review of [17]. These conditions can naturally be extended to the effective action as

$$S[\Phi_r, \Phi_a = 0; \beta_0^\mu] = 0, \qquad S[\Phi_r, -\Phi_a; \beta_0^\mu] = -S^*[\Phi_r, \Phi_a; \beta_0^\mu], \qquad \mathrm{Im}\, S[\Phi_r, \Phi_a; \beta_0^\mu] \geq 0. \tag{2.44}$$

We can arrange S as a series in powers of Φ_a, in which case the three conditions mean that: S must at least be linear in Φ_a, the terms with even-powers of Φ_a must be imaginary, and these imaginary terms must be arranged into a quadratic form with non-negative coefficients.

Dynamical KMS symmetry. While the SK conditions (2.43) are satisfied for arbitrary non-equilibrium field theories defined on a closed-time contour, to describe correlation functions in a thermal state, the SK generating functional $\mathcal{Z}[\phi_r, \phi_a]$ is also required to satisfy a discrete dynamical KMS (Kubo-Martin-Schwinger) symmetry that is responsible for implementing the FDT (1.1) and its higher-point generalisations. Using the discrete microscopic Θ symmetry identified at the end of the previous subsection, the KMS symmetry is defined as a transformation of the background fields

$$\phi_1(x) \to \Theta\phi_1(x), \qquad \phi_2(x) \to \Theta\phi_2(x + i\hbar\,\Theta\beta_0), \tag{2.45}$$

where the $\Theta\phi_2$ is evaluated on the complex spacetime arguments $x^\mu + i\hbar\,\Theta\beta_0^\mu$, with $\beta_0^\mu = u_0^\mu/T_0$ being the thermal vector associated with the inertial equilibrium observer. It can be checked that KMS is a $\mathbb{Z}_2$ transformation: repeating it returns to the original background field configuration. More details can be found in textbook treatments of thermal field theory, e.g. in [52], or in the more recent review of [17]. Since the KMS condition acts differently on the two copies of background fields $\phi_{1,2}$, its action on the average-different fields $\phi_{r,a}$ is highly non-local. However, given that the background fields are sufficiently smooth functions, we can derive a simple expression in the classical ($\hbar \to 0$) limit, given in eq. (2.33). This classical version of the dynamical KMS symmetry is more relevant for classical stochastic field theories valid at small frequencies $\omega \ll T_0/\hbar$, where quantum effects are suppressed. At quadratic order in background fields, this condition precisely yields the 2-point fluctuation-dissipation theorem in eq. (1.1). Similar higher-point relations can be derived by using this symmetry at higher non-linear orders in background fields.

In our SK-EFT for diffusion, the dynamical KMS symmetry is naturally realised on the dynamical fields as

$$\begin{aligned} \varphi_1(x) &\to \Theta\varphi_1(x), & \varphi_2(t, \vec{x}) &\to \Theta\varphi_2(x + i\hbar\,\Theta\beta_0), \\ \upsilon_{1\mu}(x) &\to \Theta\upsilon_{1\mu}(x), & \upsilon_{2\mu}(x) &\to \Theta\upsilon_{2\mu}(x + i\hbar\,\Theta\beta_0). \end{aligned} \tag{2.46}$$

In the classical limit, these transformations are realised on the average-difference basis as

$$\begin{aligned} \varphi_r &\to \Theta\varphi_r, & \varphi_a &\to \Theta\left(\varphi_a + i\beta_0^\lambda\partial_\lambda\varphi_r\right), \\ \upsilon_{r\mu} &\to \Theta\upsilon_{r\mu}, & \upsilon_{a\mu} &\to \Theta\left(\upsilon_{a\mu} + i\beta_0^\lambda\partial_\lambda\upsilon_{r\mu}\right). \end{aligned} \tag{2.47}$$

JHEP01(2024)162

	C	P	T	PT	CT	CPT
X^0, $t = x^0$, σ^0	$+$	$+$	$-$	$-$	$-$	$-$
X^i, x^i, σ^i	$+$	$-$	$+$	$-$	$+$	$-$
φ	$-$	$+$	$-$	$-$	$+$	$+$
u^i	$+$	$-$	$-$	$+$	$-$	$+$
T	$+$	$+$	$+$	$+$	$+$	$+$
μ	$-$	$+$	$+$	$+$	$-$	$-$
κ_{ij}	$+$	$+$	$+$	$+$	$+$	$+$
υ_i	$-$	$-$	$-$	$+$	$+$	$-$
T^{00}, g_{00}	$+$	$+$	$+$	$+$	$+$	$+$
T^{0i}, g_{0i}	$+$	$-$	$-$	$+$	$-$	$+$
T^{ij}, g_{ij}	$+$	$+$	$+$	$+$	$+$	$+$
J^0, A_0	$-$	$+$	$+$	$+$	$-$	$-$
J^i, A_i	$-$	$-$	$-$	$+$	$+$	$-$

Table 1. Action of parity (P), time-reversal (T), and charge conjugation (C) of various quantities in classical relativistic hydrodynamics and effective field theory. Schwinger-Keldysh double copies of various quantities in the effective theory, "1/2" or "r/a", behave the same as their unlabelled versions.

The global symmetry invariants $\Phi_{r,a}$ and the thermal vector β^μ comprising the Schwinger-Keldysh effective action $S[\Phi_r, \Phi_a; \beta_0^\mu]$ transform under the classical KMS symmetry as

$$\Phi_r \to \Theta\Phi_r, \qquad \Phi_a \to \Theta\left(\Phi_a + i\beta_0^\lambda \partial_\lambda \Phi_r\right), \qquad \beta_0^\mu \to \Theta\beta_0^\mu. \tag{2.48}$$

Given that KMS is a $\mathbb{Z}_2$ transformation, is also follows that $\Phi_a + i\beta_0^\lambda \partial_\lambda \Phi_r \to \Theta\Phi_a$.

Effective action and hydrodynamic frames. As an example, truncating the theory to at most quadratic order in Φ_a noise fields and dropping any explicit spatial derivatives, the most general effective action for MC-diffusion is given as

$$\begin{aligned} S = \int \mathrm{d}^{d+1}x \Big[& B_{a\mu} n u_0^\mu - \chi_\upsilon \upsilon_{a\mu} \upsilon_r^\mu \\ & + iT_0 \left(-\lambda\, u_0^\mu u_0^\nu + \sigma\, \Delta^{\mu\nu}\right) B_{a\mu} \left(B_{a\nu} + i\beta_0^\lambda \partial_\lambda B_{r\nu}\right) \\ & + iT_0 \sigma_\upsilon\, \Delta^{\mu\nu} \upsilon_{a\mu} \left(\upsilon_{a\nu} + i\beta_0^\lambda \partial_\lambda \upsilon_{r\nu}\right) \\ & + iT_0 \gamma_\times\, \Delta^{\mu\nu} \Big(B_{a\mu} \left(\upsilon_{a\nu} + i\beta_0^\lambda \partial_\lambda \upsilon_{r\nu}\right) + \upsilon_{a\mu} \left(B_{a\nu} + i\beta_0^\lambda \partial_\lambda B_{r\nu}\right) \Big) \\ & + iT_0 \bar{\gamma}_\times\, \Delta^{\mu\nu} \Big(B_{a\mu} \left(\upsilon_{a\nu} + i\beta_0^\lambda \partial_\lambda \upsilon_{r\nu}\right) - \upsilon_{a\mu} \left(B_{a\nu} + i\beta_0^\lambda \partial_\lambda B_{r\nu}\right) \Big) \Big], \end{aligned} \tag{2.49}$$

where we have identified $\mu \equiv B_{rt}$ and $\upsilon_\mu \equiv \upsilon_{r\mu}$, and various coefficients n, χ_υ, λ, σ, σ_υ, $\gamma_\times$, and $\bar{\gamma}_\times$ are functions of μ and υ^2. The action is manifestly invariant under the global U(1) and spatially-local diagonal shift symmetries. It trivially obeys the first two SK conditions;

JHEP01(2024)162

for the third condition, we can arrange the imaginary part of the action into a sum of squares

$$-T_0\lambda\,(u_0^\mu B_{a\mu})^2 + iT_0\sigma\left(B_{a\mu}+\frac{\gamma_\times}{\sigma}v_{a\mu}\right)\left(B_a^\mu+\frac{\gamma_\times}{\sigma}v_a^\mu\right)+T_0\left(\sigma_v-\frac{\gamma_\times^2}{\sigma}\right)v_{a\mu}v_a^\mu \geq 0, \tag{2.50}$$

leading to the inequality constraints

$$\lambda \leq 0, \qquad \sigma \geq 0, \qquad \sigma_v \geq \frac{\gamma_\times^2}{\sigma}. \tag{2.51}$$

Finally, let us look at the dynamical KMS symmetry. Using $\Theta = \mathrm{T}, \mathrm{PT}$, the terms in the first line map to themselves (up to a coordinate flip in the integral $x^\mu \to \Theta x^\mu$), with the residual terms that sum to a total derivative and drop out from the effective action, i.e.

$$i\beta_0^\mu\left(n\,\partial_\mu\mu - \chi_v v^\nu\partial_\mu v_\nu\right) = \partial_\mu(ip\beta_0^\mu)\,, \tag{2.52}$$

Here we have used the thermodynamic relations in eq. (2.14) and identified the thermodynamic pressure $p = T_0 s + n\mu - \epsilon$. Since Φ_a and $\Phi_a + i\beta_0^\lambda\partial_\lambda\Phi_r$ map to each other under KMS transformations (up to $x^\mu \to \Theta x^\mu$), the terms in the next three lines are manifestly invariant. Finally, the final $\bar{\gamma}_\times$ term is disallowed by KMS for $\Theta = \mathrm{T}, \mathrm{PT}$. However, for $\Theta = \mathrm{CT}, \mathrm{CPT}$, this term is allowed as long as $\bar{\gamma}_\times$ is an odd function of μ. Imposing the KMS symmetry with $\Theta = \mathrm{CT}, \mathrm{CPT}$ also requires n to be an odd function of μ, while the remaining coefficients χ_v, λ, σ, σ_v, and $\gamma_\times$ must be even functions of μ.

The SK effective action (2.49) has more parameters compared to the action (2.30) derived using the MSR formalism, so is clearly more general. However, this is merely a manifestation of choice of hydrodynamic frame. To see this, let us look at the vector equation of motion obtained by varying with respect to v_{ai}; in the absence of noise fields we get

$$\chi_v v^\mu = -(\gamma_\times - \bar{\gamma}_\times)\Delta^{\mu\nu}(\partial_\nu\mu - F_{r\nu\rho}u_0^\rho) - \sigma_v\Delta^{\mu\nu}u_0^\lambda\partial_\lambda v_\nu, \tag{2.53}$$

where we have used $u_0^\lambda\partial_\lambda B_{r\mu} = \partial_\mu\mu - F_{r\mu\nu}u_0^\nu$. On the other hand, we can find the physical current J^μ by varying the action with respect to $A_{a\mu}$; using the above equations and in the absence of noise fields, we find

$$\begin{aligned} J^\mu &= \left(n + \lambda\,u_0^\lambda\partial_\lambda\mu\right)u_0^\mu - \sigma\,\Delta^{\mu\nu}\left(\partial_\nu\mu - F_{r\nu\rho}u_0^\rho\right) - (\gamma_\times + \bar{\gamma}_\times)\Delta^{\mu\nu}u_0^\lambda\partial_\lambda v_\nu \\ &= \left(n + \lambda\,u_0^\lambda\partial_\lambda\mu\right)u_0^\mu + \frac{\sigma\chi_v}{\gamma_\times - \bar{\gamma}_\times}v^\mu + \frac{\sigma\sigma_v - \gamma_\times^2 + \bar{\gamma}_\times^2}{\gamma_\times - \bar{\gamma}_\times}\Delta^{\mu\nu}u_0^\lambda\partial_\lambda v_\nu. \end{aligned} \tag{2.54}$$

Reconciling this with the frame choice we made in eqs. (2.2) and (2.15), and noting that the $\bar{\gamma}_\times$ term is not KMS-invariant for $\Theta = \mathrm{T}, \mathrm{PT}$, we are required to set

$$\lambda = 0, \qquad \sigma_v = \frac{\gamma_\times^2}{\sigma}, \qquad \gamma_\times = \frac{\sigma\chi_v}{\alpha_v}, \qquad \bar{\gamma}_\times = 0. \tag{2.55}$$

On the other hand, for $\Theta = \mathrm{CPT}, \mathrm{CT}$, since $\bar{\gamma}_\times$ is an odd function of μ, requring the coefficient of v_r^μ in eq. (2.54) to be α_v for all μ ultimately yields the same constraints as above. The final effective action is given as

$$\begin{aligned} S = \int \mathrm{d}^{d+1}x\Big[&B_{a\mu}nu_0^\mu - \chi_v v_{a\mu}v_r^\mu \\ &+ iT_0\sigma\,\Delta^{\mu\nu}\left(B_{a\mu}+\frac{\chi_v}{\alpha_v}v_{a\mu}\right)\left(B_{a\nu}+i\beta_0^\lambda\partial_\lambda B_{r\nu}+\frac{\chi_v}{\alpha_v}\left(v_{a\nu}+i\beta_0^\lambda\partial_\lambda v_{r\nu}\right)\right)\Big]. \end{aligned} \tag{2.56}$$

JHEP01(2024)162

This is precisely the effective action (2.30) derived using the MSR formalism, with the KMS condition (2.32) already imposed, with the field identifications

$$v_\mu = v_{r\mu}, \qquad V_{a\mu} = \Delta^\nu_\mu B_{a\nu} + \frac{\chi_v}{\alpha_v} v_{a\mu}. \tag{2.57}$$

One can check that this results in the standard prescription of KMS symmetry given in eq. (2.34). To realise the alternate prescription in eq. (2.35), we need to modify the SK-EFT by imposing a diagonal shift symmetry between the vector degrees of freedom; see appendix C.

3 Müller-Israel-Stewart model of hydrodynamics

In this section, we will implement the lessons learnt from the diffusion model and setup an EFT framework for the Müller-Israel-Stewart (MIS) theory of relativistic hydrodynamics. Unlike the internal U(1) symmetry in the diffusion model, we will now need to account for the spacetime Poincaré transformations in relativistic hydrodynamics. Spacetime symmetries are highly non-linear and give rise to certain subtleties in the MIS framework, causing the subsequent effective field theory framework to be considerably more involved. Nonetheless, on a conceptual front, the construction here will be a straightforward generalisation of our discussion in section 2.

3.1 Relativistic hydrodynamics

The starting point of relativistic hydrodynamics are a set of conservation equations for the energy-momentum tensor $T^{\mu\nu}$ and the charge current J^μ, associated with spacetime Poincaré and internal U(1) symmetries, i.e.

$$\begin{aligned} \nabla_\mu T^{\mu\nu} &= F^{\nu\rho} J_\rho, \\ \nabla_\mu J^\mu &= 0. \end{aligned} \tag{3.1}$$

We have implicitly introduced a background spacetime metric $g_{\mu\nu}$ coupled to the energy-momentum tensor $T^{\mu\nu}$ via the associated covariant derivative operator ∇_μ. Note that energy-momentum tensor is sourced by the Lorentz force term in the presence of background electromagnetic field strength $F_{\mu\nu} = 2\partial_{[\mu} A_{\nu]}$. The energy-momentum conservation equation can be used to determine the dynamics of the energy density ϵ and fluid velocity u^μ, defined as the time-like eigenvalue and eigenvector of the energy-momentum tensor $T^{\mu\nu} u_\nu = -\epsilon\, u^\mu$ (with $u^\mu u_\mu = -1$). This definition of ϵ and u^μ is usually referred to as the "Landau frame" in the literature, following [53]. Similarly, the charge conservation equation can be used to determine the dynamics of the charge density n, defined via $J^\mu u_\mu = -n$. However, to complete these equations, we need to specify a set of constitutive relations for the remaining components of the $T^{\mu\nu}$ and J^μ.

This is where the second law of thermodynamics comes in: we require that there exists an entropy current S^μ, whose divergence is locally non-negative for all solutions of the conservation equations. We start, as with the diffusion model, with the equation of state $\epsilon(s, n)$. However, this time around, energy conservation is included within our set of

JHEP01(2024)162

conservation equations, so we do not need to make a separate ansatz for $\partial_t \epsilon$ like we did in the diffusion model. We can parametrise the first law of thermodynamics as

$$\mathrm{d}\epsilon = T\mathrm{d}s + \mu \mathrm{d}n, \tag{3.2}$$

where T and μ are the local thermodynamic temperature and chemical potential. Note that the temperature is not a constant and we have allowed it to have a nontrivial spacetime profile. Let us also parametrise the constitutive relations as

$$\begin{aligned} T^{\mu\nu} &= (\epsilon + p)u^\mu u^\nu + p\, g^{\mu\nu} + \Pi^{\mu\nu}, \\ J^\mu &= n\, u^\mu + \mathcal{J}^\mu, \end{aligned} \tag{3.3}$$

where $p(\epsilon, n) = Ts + \mu n - \epsilon$ denotes the thermodynamic pressure of the fluid and $\Delta^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu$ is the spatial projection operator. The transverse tensor $\Pi^{\mu\nu}$ and vector $\mathcal{J}^\mu$ (satisfying $\Pi^{\mu\nu} u_\nu = \mathcal{J}^\mu u_\mu = 0$ and $\Pi^{\mu\nu} = \Pi^{\nu\mu}$) parametrise the to-be-determined information in the stress tensor (spatial components of the energy-mometum tensor) and charge flux. The utility of this parametrisation is that the second law statement reduces to a simple form

$$\nabla_\mu S^\mu = -\Pi^{\mu\nu} \nabla_{(\mu}\beta_{\nu)} - \mathcal{J}^\mu \left(\partial_\mu \frac{\mu}{T} + \beta^\lambda F_{\lambda\mu} \right) \geq 0, \tag{3.4}$$

where $\beta^\mu = u^\mu/T$ and the entropy current is given as $S^\mu = s\, u^\mu - \mu/T\, \mathcal{J}^\mu$. A simple entropy-conserving solution of this equation is $\Pi^{\mu\nu} = \mathcal{J}^\mu = 0$; these are nothing but the ideal non-dissipative relativistic fluids. The leading order dissipative corrections are given as

$$\begin{aligned} \Pi^{\mu\nu} &= -\zeta\, \Delta^{\mu\nu} \nabla_\lambda u^\lambda - 2\eta\, \Delta^{\rho\langle\mu} \Delta^{\nu\rangle\sigma} \nabla_\rho u_\sigma, \\ \mathcal{J}^\mu &= -\sigma\, \Delta^{\mu\nu} \left(T \nabla_\nu \frac{\mu}{T} + u^\lambda F_{\lambda\nu} \right), \end{aligned} \tag{3.5}$$

where $\zeta, \eta, \sigma \geq 0$ are non-negative transport coefficients identified as fluid bulk viscosity, shear viscosity, and conductivity respectively. The angular brackets denote the symmetric transverse-traceless combination of indices: $X_{\langle\mu\nu\rangle} = 1/2\,(X_{\mu\nu} + X_{\nu\mu}) - 1/d\, \Delta_{\mu\nu}\Delta^{\rho\sigma} X_{\rho\sigma}$. Above, we have used the fact that $\Delta^{\mu\nu}\nabla_\mu \beta_\nu = 1/T\, \nabla_\mu u^\mu$ and $\Delta^\rho_{\langle\mu}\Delta^\sigma_{\nu\rangle} \nabla_\rho \beta_\sigma = 1/T\, \Delta^\rho_{\langle\mu}\Delta^\sigma_{\nu\rangle} \nabla_\rho u_\sigma$.

Consider the equilibrium configuration $\beta^\mu = \delta^\mu_t / T_0$ and $\mu/T = (\mu_0 + A_t)/T_0$ in the presence of time-independent background sources $g_{\mu\nu}(\vec{x})$, $A_\mu(\vec{x})$. It is easy to see that in this configuration

$$2\nabla_{(\mu}\beta_{\nu)} = \frac{1}{T_0} \partial_t g_{\mu\nu} = 0, \qquad \nabla_\mu \frac{\mu}{T} + \beta^\lambda F_{\lambda\mu} = \frac{1}{T_0} \partial_t A_\mu = 0. \tag{3.6}$$

As a consequence, entropy is identically conserved in an equilibrium configuration. It is also easy to see that the dissipative corrections $\Pi^{\mu\nu}$, $\mathcal{J}^\mu$ identically vanish in equilibrium and the ideal fluid constitutive relations trivially satisfy the conservation equations. Correspondingly, the equilibrium configuration is a trivial solution of the dissipative hydrodynamic equations.

Having set up the hydrodynamic model, we can use the variational formulae in eq. (2.8) to compute the retarded correlation functions of the operators $\mathcal{O} = (\sqrt{-g}\, T^{\mu\nu}, \sqrt{-g}\, J^\mu)$ by varying with respect to the associated background sources $\psi = (\frac{1}{2} g_{\mu\nu}, A_\mu)$, with the flat values $\psi_0 = (\frac{1}{2} g_{\mu\nu} = \frac{1}{2}\eta_{\mu\nu}, A_\mu = 0)$. As an example, let us re-evaluate the 2-point retarded

JHEP01(2024)162

Green's functions for the density J^t and transverse flux $J^i_\perp$ using the hydrodynamic model in a state with equilibrium velocity $u^\mu_0 = \delta^\mu_t$, i.e.

$$G^R_{J^t J^t}(\omega, k) = -\frac{\sigma k^2}{i\omega - D_n k^2} - \frac{\frac{n^2}{\epsilon+p} k^2}{\omega^2 - v_s^2 k^2 + i\omega D^\parallel_\pi k^2},$$
$$G^R_{J^i_\perp J^j_\perp}(\omega, k) = \left(i\omega\sigma - \frac{\frac{n^2}{\epsilon+p} i\omega}{i\omega - D^\perp_\pi k^2} \right) k^{ij}, \tag{3.7}$$

where we have defined

$$v_s^2 = \left.\frac{\partial p}{\partial \epsilon}\right|_{s/n}, \qquad \frac{1}{\chi} = \left.\frac{T\partial(\mu/T)}{\partial n}\right|_\epsilon, \qquad D_n = \frac{\sigma}{\chi}, \qquad D^\perp_\pi = \frac{\eta}{\epsilon + p}, \qquad D^\parallel_\pi = \frac{\zeta + 2\frac{d-1}{d}\eta}{\epsilon + p}. \tag{3.8}$$

For clarity of presentation, we have taken $\partial p/\partial n|_\epsilon = 0$, which causes the mode spectrum in the charge and energy-momentum sectors to neatly decouple: in the longitudinal sector we find the familiar charge diffusion mode and a new fluid sound mode

$$\omega = -iD_n k^2 + \ldots, \qquad \omega = \pm v_s k - \frac{i}{2} D^\parallel_\pi k^2 + \ldots. \tag{3.9a}$$

In the transverse sector, we find the shear diffusion mode

$$\omega = -iD^\perp_\pi k^2 + \ldots. \tag{3.9b}$$

Given $\zeta, \eta, \sigma \geq 0$, all these modes are stable, provided that we also take $\chi, \epsilon + p \geq 0$. More details on the retarded correlation functions and mode spectrum of relativistic hydrodynamics can be found in the lecture notes of [54]. The instabilities we encountered in a boosted frame of reference in the relativistic diffusion model near eq. (2.12) also plague dissipative relativistic hydrodynamics. In a boosted frame of reference, obtained using the transformation (2.11), the sound and diffusion modes above get appropriately boosted, but the dissipative terms in the poles of eq. (3.7) also generate an unstable gapped mode in each of sound and diffusion channels, i.e.

$$\omega = i\frac{\sqrt{1 - v_0^2}}{v_0^2 D_n} + \ldots, \qquad \omega = i\frac{\sqrt{1 - v_0^2}(1 - v_0^2 v_s^2)}{v_0^2 D^\parallel_\pi} + \ldots, \qquad \omega = i\frac{\sqrt{1 - v_0^2}}{v_0^2 D^\perp_\pi} + \ldots. \tag{3.10}$$

Similarly at large wavevectors, we also find a pair of acausal modes in each of sound and diffusion channels with the dispersion relations (2.13) that propagate faster than the speed of light. To cure these acausalities and instabilities, we will need to introduce new gapped modes near the frequency scales of instabilities, which will be the goal of MIS-hydrodynamics in the next subsection.

3.2 Müller-Israel-Stewart model

To construct the Müller-Israel-Stewart (MIS) model of hydrodynamics, let us introduce new tensor $\kappa_{\mu\nu}$ and vector υ_μ degrees of freedom (satisfying $\kappa_{\mu\nu}u^\nu = \upsilon_\mu u^\mu = 0$ and $\kappa_{\mu\nu} = \kappa_{\nu\mu}$)

JHEP01(2024)162

to allow for all the components in the energy-momentum tensor $T^{\mu\nu}$ and J^μ to fluctuate independently. The first law of thermodynamics from eq. (3.2) needs to be appropriately modified to include the additional dependence on the scalars $\operatorname{tr}\kappa = \kappa^\mu{}_\mu$, $\kappa^2 = \kappa^{\mu\nu}\kappa_{\mu\nu}$, and $\upsilon^2 = \upsilon^\mu \upsilon_\mu$, i.e.

$$\mathrm{d}\epsilon = T\mathrm{d}s + \mu\mathrm{d}n + \frac{\chi_\kappa^{\mathsf{S}}}{8}\mathrm{d}(\operatorname{tr}\kappa)^2 + \frac{\chi_\kappa^{\mathsf{T}}}{4}\mathrm{d}\left(\kappa^2 - \frac{1}{d}(\operatorname{tr}\kappa)^2\right) + \frac{\chi_\upsilon}{2}\mathrm{d}\upsilon^2, \tag{3.11}$$

where χ_κ^{S}, χ_κ^{T}, and χ_υ are new thermodynamic coefficients and d is the number of spatial dimensions.[9] A priori, $\kappa_{\mu\nu}$ and υ_μ are introduced as arbitrary degrees of freedom unrelated to $T^{\mu\nu}$ and J^μ. To fix this, let us decompose the grand-canonical free-energy density $\mathcal{F}(\epsilon, n, \operatorname{tr}\kappa, \kappa^2, \upsilon^2) = \epsilon - Ts - \mu n$ into a fluid part $-p(\epsilon, n)$ and the remainder $-\mathcal{P}(\epsilon, n, \operatorname{tr}\kappa, \kappa^2, \upsilon^2)$ that vanishes when both $\kappa_{\mu\nu}$ and υ_μ are taken to zero. Having done this, we choose the parametrisation for the constitutive relations in eq. (3.3) and choose $\Pi^{\mu\nu}$ and $\mathcal{J}^\mu$ to be related to $\kappa_{\mu\nu}$ and υ_μ as

$$\begin{aligned}\Pi^{\mu\nu} &= \frac{\alpha_\kappa^{\mathsf{S}}}{2}\Delta^{\mu\nu}\operatorname{tr}\kappa + \alpha_\kappa^{\mathsf{T}}\kappa^{\langle\mu\nu\rangle} + \left(\mathcal{P}\Delta^{\mu\nu} + \frac{\chi_\kappa^{\mathsf{S}}}{2}\kappa^{\mu\nu}\operatorname{tr}\kappa + \chi_\kappa^{\mathsf{T}}\left(\kappa^{\mu\rho}\kappa^\nu{}_\rho - \frac{1}{d}\kappa^{\mu\nu}\operatorname{tr}\kappa\right) + \chi_\upsilon \upsilon^\mu\upsilon^\nu\right),\\ \mathcal{J}^\mu &= \alpha_\upsilon \upsilon^\mu,\end{aligned} \tag{3.12}$$

where we have introduced arbitrary coefficients $\alpha_\kappa^{\mathsf{S}}$, $\alpha_\kappa^{\mathsf{T}}$, and α_υ akin to our choice in eq. (2.15). Similar to our discussion below eq. (2.15), we can always arbitrarily redefine the fields $\kappa_{\mu\nu}$ and υ_μ to set these coefficients to 1, which, linearly, would give them the physical interpretation of spatial stress tensor and flux directly, or to χ_κ^{S}, χ_κ^{T}, and χ_υ respectively, giving them the physical interpretation that of the respective thermodynamic conjugates. The additional terms in the parenthesis in eq. (3.12) have been included for later compatibility with the SK-EFT.[10] For the time being, the utility of this rather non-trivial choice is that

[9] This setup is reminiscent of the theory of plasticity, with $\frac{1}{2}\kappa_{\mu\nu}$ serving as the strain tensor, and the coefficients χ_κ^{S} and χ_κ^{T} as the bulk and shear moduli respectively. However, note that here $\kappa_{\mu\nu}$ is not required to be given by the symmetric derivative of a displacement field, hence this setup applies to a plastic material, not elastic, leading to a relaxed plastic stress tensor; see e.g. [55, 56]. Correspondingly, the dynamical equation for $\Pi_{\mu\nu}$ in (3.17) that we will soon derive, correspond to the rheology equations of a Maxwell material.

[10] This is the so-called "thermodynamic frame", where the conserved currents are decomposed into a "hydrostatic part" obtained from the grand-canonical free energy density $\mathcal{F} = -p - \mathcal{P}$ and an arbitrary "non-hydrostatic part" transverse to the fluid velocity, i.e.

$$T^{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\,\mathcal{F})}{\delta g_{\mu\nu}} + T^{\mu\nu}_{\text{nhs}}, \qquad J^\mu = -\frac{1}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\,\mathcal{F})}{\delta A_\mu} + J^\mu_{\text{nhs}}. \tag{3.13}$$

The variations above are evaluated at fixed $\beta^\mu = u^\mu/T$, $\Lambda_\beta = \mu/T - \beta^\mu A_\mu$, $\kappa_{\mu\nu}$, and υ_μ. The fields $\kappa_{\mu\nu}$ and υ_μ are related to $T^{\mu\nu}_{\text{nhs}}$ and J^μ_{nhs} as

$$T^{\mu\nu}_{\text{nhs}} = \frac{\alpha_\kappa^{\mathsf{S}}}{2}\Delta^{\mu\nu}\operatorname{tr}\kappa + \alpha_\kappa^{\mathsf{T}}\kappa^{\langle\mu\nu\rangle}, \qquad J^\mu_{\text{nhs}} = \alpha_\upsilon\,\upsilon^\mu. \tag{3.14}$$

The SK-EFT for hydrodynamics, realising the non-linear discrete KMS symmetry, naturally give rise to the hydrodynamic equations in the thermodynamic Landau frame, as we shall review in the following.

Note that with our particular choice of thermodynamics in eq. (3.11), the thermodynamic frame agrees with the definition of the Landau frame $T^{\mu\nu}u_\nu = -\epsilon\,u^\mu$ and $J^\mu u_\mu = -n$. However, this does not hold more generally. In particular, as explained in [57], if we allow the thermodynamic equation of state in eq. (3.11) to also depend on the spacetime curvature (as is the case even in non-interacting field theories), the thermodynamic

JHEP01(2024)162

the expression for entropy production only modifies in terms of Lie derivatives of $\kappa_{\mu\nu}$ and υ_μ along $\beta^\mu = u^\mu/T$, denoted by $\pounds_\beta$, i.e.

$$\nabla_\mu S^\mu = -\frac{\alpha_\kappa^{\mathsf{S}}}{4}\,\mathrm{tr}\,\kappa\left(2\nabla_\lambda\beta^\lambda + \frac{\chi_\kappa^{\mathsf{S}}}{\alpha_\kappa^{\mathsf{S}}}\Delta^{\rho\sigma}\pounds_\beta\kappa_{\rho\sigma}\right) - \frac{1}{2}\alpha_\kappa^{\mathsf{T}}\kappa^{\langle\mu\nu\rangle}\left(2\nabla_\mu\beta_\nu + \frac{\chi_\kappa^{\mathsf{T}}}{\alpha_\kappa^{\mathsf{T}}}\pounds_\beta\kappa_{\mu\nu}\right)$$
$$- \alpha_\upsilon\upsilon^\mu\left(\partial_\mu\frac{\mu}{T} + \beta^\lambda F_{\lambda\mu} + \frac{\chi_\upsilon}{\alpha_\upsilon}\pounds_\beta\upsilon_\mu\right) \geq 0. \tag{3.15}$$

The entropy current is again given as $S^\mu = s\,u^\mu - \mu/T\,\mathcal{J}^\mu$. Physically, the Lie derivatives in this expression mean that the new degrees of freedom only modify the entropy production of relativistic hydrodynamics if they evolve in the local rest frame of the fluid. Requiring that entropy is strictly produced for all hydrodynamic configurations immediately allows us identify the dynamical equations for $\kappa_{\mu\nu}$ and υ_μ, i.e.

$$\frac{\alpha_\kappa^{\mathsf{S}}}{2}\Delta^{\mu\nu}\,\mathrm{tr}\,\kappa + \alpha_\kappa^{\mathsf{T}}\kappa^{\langle\mu\nu\rangle} = -\zeta\,\Delta^{\mu\nu}\left(\nabla_\lambda u^\lambda + \frac{T\chi_\kappa^{\mathsf{S}}}{2\alpha_\kappa^{\mathsf{S}}}\Delta^{\rho\sigma}\pounds_\beta\kappa_{\rho\sigma}\right) - 2\eta\,\Delta^{\rho\langle\mu}\Delta^{\nu\rangle\sigma}\left(\nabla_\rho u_\sigma + \frac{T\chi_\kappa^{\mathsf{T}}}{2\alpha_\kappa^{\mathsf{T}}}\pounds_\beta\kappa_{\rho\sigma}\right),$$
$$\alpha_\upsilon\upsilon^\mu = -\sigma\,\Delta^{\mu\nu}\left(T\nabla_\nu\frac{\mu}{T} + u^\lambda F_{\lambda\nu} + \frac{T\chi_\upsilon}{\alpha_\upsilon}\pounds_\beta\upsilon_\nu\right). \tag{3.16}$$

Due to the non-linear relation in eq. (3.12), the relaxation equations for $\Pi^{\mu\nu}$ and $\mathcal{J}^\mu$ are quite involved. At the linearised level, we find

$$\Pi^{\mu\nu} + \left(\frac{1}{d}\tau_\Pi^{\mathsf{S}}\Delta^{\mu\nu}\Delta^{\rho\sigma} + \tau_\Pi^{\mathsf{T}}\Delta^{\rho\langle\mu}\Delta^{\nu\rangle\sigma}\right)T\pounds_\beta\Pi_{\rho\sigma} = -\zeta\,\Delta^{\mu\nu}\nabla_\lambda u^\lambda - 2\eta\,\Delta^{\rho\langle\mu}\Delta^{\nu\rangle\sigma}\nabla_\rho u_\sigma + \ldots,$$
$$\mathcal{J}^\mu + \tau_{\mathcal{J}}\,\Delta^{\mu\nu}T\pounds_\beta\mathcal{J}_\nu = -\sigma\,\Delta^{\mu\nu}\left(T\nabla_\nu\frac{\mu}{T} + u^\lambda F_{\lambda\nu}\right) + \ldots, \tag{3.17}$$

where we have identified the three distinct relaxation time-scales in the scalar, tensor, and vector sectors of the theory, i.e. $\tau_\Pi^{\mathsf{S}} = \zeta\chi_\kappa^{\mathsf{S}}/(\alpha_\kappa^{\mathsf{S}})^2$, $\tau_\Pi^{\mathsf{T}} = \eta\chi_\kappa^{\mathsf{T}}/(\alpha_\kappa^{\mathsf{T}})^2$, and $\tau_{\mathcal{J}} = \sigma\chi_\upsilon/\alpha_\upsilon^2$. More complete expressions for conformal fluids are provided later in section 3.5.

We can use the modified theory MIS-hydrodynamics to obtain the retarded Green's functions of hydrodynamic observables. In practice, this amounts to replacing $\zeta \to \zeta/(1 - i\omega\tau_\Pi^{\mathsf{S}})$, $\eta \to \eta/(1 - i\omega\tau_\Pi^{\mathsf{T}})$, and $\sigma \to \sigma/(1 - i\omega\tau_{\mathcal{J}})$ in the response functions we found in eq. (3.7). Choosing $\tau_\Pi^{\mathsf{S}} = \tau_\Pi^{\mathsf{T}} = \tau_\Pi$ for simplicity, we are led to

$$G^R_{J^tJ^t}(\omega,k) = -\frac{\sigma k^2}{i\omega(1-i\omega\tau_{\mathcal{J}}) - D_n k^2} - \frac{\frac{n^2}{\epsilon+p}k^2(1-i\omega\tau_\Pi)}{(\omega^2 - v_s^2k^2)(1-i\omega\tau_\Pi) + i\omega D_\pi^{\parallel}k^2},$$
$$G^R_{J^i_\perp J^j_\perp}(\omega,k) = \left(\frac{i\omega\sigma}{1-i\omega\tau_{\mathcal{J}}} - \frac{\frac{n^2}{\epsilon+p}i\omega(1-i\omega\tau_\Pi^{\mathsf{T}})}{i\omega(1-i\omega\tau_\Pi) - D_\pi^{\perp}k^2}\right)k^{ij}. \tag{3.18}$$

The pole structure of these correlators still admits the hydrodynamic modes presented in eq. (3.9). We also find new gapped modes, which in a generic boosted frame of reference take the form

$$\omega = -i\frac{\sqrt{1-v_0^2}}{\tau_{\mathcal{J}} - v_0^2 D_n} + \ldots, \qquad \omega = -i\frac{\sqrt{1-v_0^2}}{\tau_\Pi - v_0^2/(1-v_0^2v_s^2)\,D_\pi^{\parallel}} + \ldots, \tag{3.19}$$

frame would no longer satisfy the Landau frame condition. This means that the definition of thermodynamic variables, such as temperature T, in the thermodynamic frame are generically different from their Landau frame definitions.

JHEP01(2024)162

in the longitudinal sector and

$$\omega = -i\frac{\sqrt{1-v_0^2}}{\tau_{\mathcal{J}}} + \ldots, \qquad \omega = -i\frac{\sqrt{1-v_0^2}}{\tau_\Pi - v_0^2 D_\pi^\perp} + \ldots, \tag{3.20}$$

in the transverse sector. For stability of these modes, we need to take the relaxation times to be bounded from below as

$$\tau_{\mathcal{J}} \geq D_n \geq 0, \qquad \tau_\Pi \geq \frac{1}{1-v_s^2} D_\pi^\parallel \geq 0, \qquad \tau_\Pi \geq D_\pi^\perp \geq 0. \tag{3.21}$$

More intricate bounds exist when we allow for $\tau_\Pi^{\mathsf{S}} \neq \tau_\Pi^{\mathsf{T}}$ and $\partial p/\partial n|_\epsilon \neq 0$, which we will not explore here. These bounds also ensure that the mode spectrum is causal. The potentially acausal modes get regulated to the same form as in eq. (2.23), but with different values of c_∞ for the charge diffusion, sound, and shear diffusion sectors

$$c_\infty = \frac{\tau_{\mathcal{J}} - D_n}{\tau_{\mathcal{J}} - v_0^2 D_n}\,, \quad \frac{(1-v_s^2)\tau_\Pi - D_\pi^\parallel}{(1-v_0^2 v_s^2)\tau_\Pi - v_0^2 D_\pi^\parallel}\,, \quad \frac{\tau_\Pi - D_\pi^\perp}{\tau_\Pi - v_0^2 D_\pi^\perp}. \tag{3.22}$$

In all three of these cases, one can check that $0 < c_\infty \leq 1$ due to the stability bounds, ensuring that the speeds of modes at large wavevectors remain bounded from above by 1. Generically, given a theory of dissipative relativistic hydrodynamics, we can always setup a theory of MIS-hydrodynamics with sufficiently large relaxation time-scales τ_Π^{S}, τ_Π^{T}, and $\tau_{\mathcal{J}}$, so that the theory remains linearly causal and stable in all boosted frames of references.

Had we used a more traditional parametrisation of the constitutive relations in terms of $\kappa_{\mu\nu}$ and υ_μ, i.e. absorbed $\mathcal{P}$ in eq. (3.12) into the thermodynamic pressure p in eq. (3.3) and dropped the remaining terms in the parenthesis in eq. (3.12), the Lie derivatives $\pounds_\beta$ in eqs. (3.15), (3.16) and (3.17) above would have been replaced with $\beta^\mu \nabla_\mu$. This would result in a more familiar version of MIS-hydrodynamics as found in [30–37]. However, as we will see later in appendix 3.4, the associated EFT derived using the MSR formalism would not agree with the dynamical KMS symmetry non-linearly and thus would violate higher-point FDTs. Note that the two choices of parametrisations only differ from each-other non-linearly and yield the same results for mode spectrum within the regime of linearised hydrodynamics.

3.3 Stochastic fluctuations and effective action

We can follow the MSR procedure in section 2.3 to promote the MIS theory of hydrodynamics presented above into a effective field theory. We start by introducing random noise terms $\theta_{\mathcal{T}}^{\mu\nu}$ and $\theta_{\mathcal{J}}^\mu$ in the constitutive relations (3.16) respectively, i.e.

$$\begin{aligned}
\frac{\alpha_\kappa^{\mathsf{S}}}{2}\Delta^{\mu\nu}\,\mathrm{tr}\,\kappa + \alpha_\kappa^{\mathsf{T}}\kappa^{\langle\mu\nu\rangle} &= -\zeta\,\Delta^{\mu\nu}\left(\nabla_\lambda u^\lambda + \frac{T\chi_\kappa^{\mathsf{S}}}{2\alpha_\kappa^{\mathsf{S}}}\Delta^{\rho\sigma}\pounds_\beta\kappa_{\rho\sigma}\right) \\
&\quad - 2\eta\,\Delta^{\rho\langle\mu}\Delta^{\nu\rangle\sigma}\left(\nabla_\rho u_\sigma + \frac{T\chi_\kappa^{\mathsf{T}}}{2\alpha_\kappa^{\mathsf{T}}}\pounds_\beta\kappa_{\rho\sigma}\right) + \theta_\Pi^{\mu\nu}, \\
\alpha_\upsilon \upsilon^\mu &= -\sigma\,\Delta^{\mu\nu}\left(T\nabla_\nu\frac{\mu}{T} + u^\lambda F_{\lambda\nu} + \frac{T\chi_\upsilon}{\alpha_\upsilon}\pounds_\beta\upsilon_\nu\right) + \theta_{\mathcal{J}}^\mu,
\end{aligned} \tag{3.23}$$

Physical correlations are computed similar to eq. (2.25) by integrating the noisy expectation values of operators $\langle\ldots\rangle_\theta$, evaluated on the solutions of the stochastic equations in the

JHEP01(2024)162

presence of the background fields $\phi_r = (\frac{1}{2}g_{r\mu\nu}, A_{r\mu})$ and noise fields $\theta = (\theta^{\mu\nu}_{\Pi}, \theta^{\mu}_{\mathcal{J}})$. The noise configurations are sampled from the weight distribution

$$\exp\left(-\frac{1}{4}\int \mathrm{d}^{d+1}x\sqrt{-g_r}\left[\left\langle \frac{\Delta_{\mu\nu}\Delta_{\rho\sigma}}{d^2T\tilde{\zeta}} + \frac{\Delta_{\rho\langle\mu}\Delta_{\nu\rangle\sigma}}{2T\tilde{\eta}}\right\rangle_\theta \theta^{\mu\nu}_{\Pi}\theta^{\rho\sigma}_{\Pi} + \left\langle \frac{\Delta_{\mu\nu}}{T\tilde{\sigma}}\right\rangle_\theta \theta^{\mu}_{\mathcal{J}}\theta^{\nu}_{\mathcal{J}}\right]\right), \tag{3.24}$$

where the new coefficients $\tilde{\zeta}$, $\tilde{\eta}$, and $\tilde{\sigma}$ control the strength of stochastic fluctuations. Next, similar to eq. (2.26), we can convert the noisy expectation values $\langle\ldots\rangle_\theta$ into offshell fields by integrating over the hydrodynamic fields $\psi_r = (\beta^\mu, \mu, \kappa_{\mu\nu}, \upsilon_\mu)$ and a set of Lagrange multipliers $\psi_a = (X^\mu_a, \varphi_a + X^\mu_a A_{r\mu}, \frac{1}{2}W_{a\mu\nu}, V_{a\mu})$ for the two conservation equations in eq. (3.1) and the two sets of constitutive relations in eq. (3.23) respectively. To conveniently compute correlation functions, we can define the Schwinger-Keldysh path integral as eq. (2.27), by introducing "a"-type background sources $\phi_a = (\frac{1}{2}g_{a\mu\nu}, A_{a\mu})$ to probe the operators $\mathcal{O} = (\sqrt{-g}\,T^{\mu\nu}, \sqrt{-g}\,J^\mu)$. Finally, we can analytically perform the Gaussian path integral over the noise fields $\theta = (\theta^{\mu\nu}_{\Pi}, \theta^{\mu}_{\mathcal{J}})$ and arrive at the generating functional in eq. (2.29), with the associated effective action

$$\begin{aligned}
S = \int \mathrm{d}^{d+1}x\sqrt{-g_r}\Big[&\frac{1}{2}G_{a\mu\nu}\left(\epsilon\, u^\mu u^\nu - \mathcal{F}\Delta^{\mu\nu}\right) + B_{a\mu}nu^\mu \\
&+ \frac{1}{2}(G_{a\mu\nu} - W_{a\mu\nu})\left(\frac{\alpha^{\mathsf{S}}_\kappa}{2}\Delta^{\mu\nu}\operatorname{tr}\kappa + \alpha^{\mathsf{T}}_\kappa\kappa^{\langle\mu\nu\rangle}\right) + (B_{a\mu} - V_{a\mu})\alpha_\upsilon\upsilon^\mu \\
&+ \frac{1}{2}G_{a\mu\nu}\left(\frac{\chi^{\mathsf{S}}_\kappa}{2}\kappa^{\mu\nu}\operatorname{tr}\kappa + \chi^{\mathsf{T}}_\kappa\left(\kappa^{\mu\rho}\kappa^\nu_\rho - \frac{1}{d}\kappa^{\mu\nu}\operatorname{tr}\kappa\right) + \chi_\upsilon\upsilon^\mu\upsilon^\nu\right) \\
&- \frac{1}{2}\zeta\,\Delta^{\mu\nu}\Delta^{\rho\sigma}W_{a\mu\nu}\left(\nabla_\rho u_\sigma + \frac{T\chi^{\mathsf{S}}_\kappa}{2\alpha^{\mathsf{S}}_\kappa}\pounds_\beta\kappa_{\rho\sigma}\right) + \frac{iT}{4}\tilde{\zeta}\,\Delta^{\mu\nu}\Delta^{\rho\sigma}W_{a\mu\nu}W_{a\rho\sigma} \\
&- \eta\,\Delta^{\rho\langle\mu}\Delta^{\nu\rangle\sigma}W_{a\mu\nu}\left(\nabla_\rho u_\sigma + \frac{T\chi^{\mathsf{T}}_\kappa}{2\alpha^{\mathsf{T}}_\kappa}\pounds_\beta\kappa_{\rho\sigma}\right) + \frac{iT}{2}\tilde{\eta}\,\Delta^{\rho\langle\mu}\Delta^{\nu\rangle\sigma}W_{a\mu\nu}W_{a\rho\sigma} \\
&- \sigma\,\Delta^{\mu\nu}V_{a\mu}\left(T\nabla_\nu\frac{\mu}{T} + u^\lambda F_{r\lambda\nu} + \frac{T\chi_\upsilon}{\alpha_\upsilon}\pounds_\beta\upsilon_\nu\right) + iT\tilde{\sigma}\,\Delta^{\mu\nu}V_{a\mu}V_{a\nu}\Big],
\end{aligned} \tag{3.25}$$

where $G_{a\mu\nu} = g_{a\mu\nu} + \pounds_{X_a}g_{r\mu\nu}$ and $B_{a\mu} = A_{a\mu} + \partial_\mu\varphi_a + \pounds_{X_a}A_{r\mu}$. In the limit $\chi^{\mathsf{S}}_\kappa, \chi^{\mathsf{T}}_\kappa, \chi_\upsilon \to 0$, the fields $\kappa_{\mu\nu}$ and υ_μ become Lagrange multipliers to impose the conditions $W_{a\mu\nu} = G_{a\mu\nu}$ and $V_{a\mu} = B_{a\mu}$ respectively, and we recover the effective field theory for ordinary relativistic hydrodynamics developed in [13, 15, 16].

FDT and KMS symmetry. We can use the effective action derived above to compute the retarded and symmetric correlation functions of relativistic MIS-hydrodynamics. Requiring the 2-point correlation functions to satisfy FDT imposes

$$\tilde{\zeta} = \zeta, \qquad \tilde{\eta} = \eta, \qquad \tilde{\sigma} = \sigma. \tag{3.26}$$

As we will see in the next subsection in detail, this choice also guarantees that the effective action realises the non-linear dynamical KMS symmetry and thus is compatible with all higher-point FDTs as well. The KMS symmetry is realised on the background fields $\phi_{r,a} = (\frac{1}{2}g_{r,a\mu\nu}, A_{r,a\mu})$ the same as eq. (2.33). However, similar to our discussion around eq. (2.34), the dynamical fields can realise four different prescriptions of KMS transformations depending on the choice of Θ-eigenvalues of the auxiliary fields $W_{a\mu\nu}$ and $V_{a\mu}$. If we take these to

JHEP01(2024)162

transform the same way as $\kappa_{\mu\nu}$ and υ_μ respectively, the effective action realises the "standard" prescription of the dynamical KMS symmetry

$$\begin{aligned}
\beta^\mu &\to \Theta\beta^\mu, & X^\mu_a &\to \Theta\Big(X^\mu_a + i(\beta^\mu - \beta^\mu_0)\Big),\\
\mu &\to \Theta\mu, & \varphi_a &\to \Theta\left(\varphi_a + i\left(\frac{\mu}{T} - \frac{\mu_0}{T_0} - \beta^\mu A_{r\mu}\right)\right),\\
\kappa_{\mu\nu} &\to \Theta\kappa_{\mu\nu}, & W_{a\mu\nu} &\to \Theta\left(W_{a\mu\nu} + \frac{i}{T}\left(2\Delta^\rho_{\langle\mu}\Delta^\sigma_{\nu\rangle}\nabla_\rho u_\sigma + \left(\frac{1}{d}\frac{\chi^{\mathsf{S}}_\kappa}{\alpha^{\mathsf{S}}_\kappa}\Delta_{\mu\nu}\Delta^{\rho\sigma} + \frac{\chi^{\mathsf{T}}_\kappa}{\alpha^{\mathsf{T}}_\kappa}\Delta^\rho_{\langle\mu}\Delta^\sigma_{\nu\rangle}\right)T\pounds_\beta\kappa_{\rho\sigma}\right)\right),\\
\upsilon_\mu &\to \Theta\upsilon_\mu, & V_{a\mu} &\to \Theta\left(V_{a\mu} + \frac{i}{T}\Delta^\nu_\mu\left(T\partial_\nu\frac{\mu}{T} + u^\lambda F_{r\lambda\nu} + \frac{T\chi_\upsilon}{\alpha_\upsilon}\pounds_\beta\upsilon_\nu\right)\right).
\end{aligned} \tag{3.27}$$

On the other hand, if $W_{a\mu\nu}$ and/or $V_{a\mu}$ transform oppositely to $\kappa_{\mu\nu}$ and/or υ_μ, their KMS transformations are given by the alternate prescriptions

$$W_{a\mu\nu} \to \Theta\left(W_{a\mu\nu} + \frac{i}{T}\left(\frac{1}{d}\frac{\alpha^{\mathsf{S}}_\kappa}{\zeta}\Delta_{\mu\nu}\Delta^{\rho\sigma} + \frac{\alpha^{\mathsf{T}}_\kappa}{\eta}\Delta^\rho_{\langle\mu}\Delta^\rho_{\nu\rangle}\right)\kappa_{\rho\sigma}\right), \tag{3.28a}$$

and/or

$$V_{a\mu} \to \Theta\left(V_{a\mu} + \frac{i}{T}\frac{\alpha_\upsilon}{\sigma}\upsilon_\mu\right). \tag{3.28b}$$

respectively. The derivation of these transformation rules in presented in appendices 3.4 and C.

Once again, had we used a more traditional parametrisation of the constitutive relations in terms of $\kappa_{\mu\nu}$ and υ_μ, as described towards the end of section 3.2, the occurrences of $\pounds_\beta$ in the effective action (3.25) would be replaced with $\beta^\mu\nabla_\mu$. However, as we discuss in the next subsection, the resulting effective action cannot be made non-linearly compatible with the dynamical KMS symmetry. In other words, the simple choice of Gaussian weight distribution for noise configurations in eq. (3.24) would not be sufficient to guarantee agreement with all higher-point FDTs.

3.4 Schwinger-Keldysh formalism

We will now derive the effective action (3.25) for MIS-hydrodynamics using the Schwinger-Keldysh formalism and illustrate how it non-linearly realises the dynamical KMS symmetry. The following discussion is a straightforward generalisation of the original construction of SK-EFTs for relativistic hydrodynamics [13, 15, 16], extended to include the gapped MIS fields.

Fluid worldvolume, dynamical fields, and global symmetries. The presence of global spacetime symmetries makes the effective field theory for hydrodynamics a little more subtle than the linear diffusion model. Just like the doubled phase fields in the diffusion model on which the internal U(1) symmetries act, the spacetime on which the spacetime symmetries act also needs to be doubled. The elegant insight of [13, 15, 16] is to setup the effective field theory for hydrodynamics as a sigma-model on an auxiliary "fluid worldvolume" with coordinates σ^α. Dynamical fields live on the fluid worldvolume: a pair of coordinate fields $X^\mu_{1,2}(\sigma)$ defining the two copies of spacetime and a pair of U(1) phase fields $\varphi_{1,2}(\sigma)$ akin to the diffusion model.

The hydrodynamic effective field theory realises the global spacetime and internal symmetries independently on the two spacetimes. We will directly start with the gauged

JHEP01(2024)162

versions of these symmetries: spacetime diffeomorphisms and U(1) gauge transformations

$$\begin{aligned} X^{\mu}_{1,2} &\to X'^{\mu}_{1,2}(X_{1,2}), \\ \varphi_{1,2} &\to \varphi_{1,2} - \Lambda_{1,2}(X_{1,2}). \end{aligned} \tag{3.29}$$

with the associated background fields: a pair of background spacetime metrics $g_{1,2\mu\nu}(X_{1,2})$ and U(1) gauge fields $A_{1,2\mu}(X_{1,2})$. Background U(1) gauge transformations act on $A_{1,2\mu}$ as

$$A_{1,2\mu}(X_{1,2}) \to A_{1,2\mu}(X_{1,2}) + \frac{\partial}{\partial X^{\mu}_{1,2}} \Lambda_{1,2}(X_{1,2}), \tag{3.30}$$

whereas background diffeomorphisms act on $g_{1,2\mu\nu}$ and $A_{1,2\mu}$ as usual. When coupled to a flat background, $g_{1,2\mu\nu} = \eta_{\mu\nu}$, $A_{1,2\mu} = 0$, these symmetries reduce the global symmetries: spacetime Poincaré transformations and constant U(1) transformations. In practice, we can pullback these fields onto the fluid worldvolume into objects invariant under the global symmetries

$$\begin{aligned} \mathfrak{g}_{1,2\alpha\beta} &= g_{1,2\mu\nu}(X_{1,2})\, \partial_\alpha X^{\mu}_{1,2}\, \partial_\beta X^{\nu}_{1,2}, \\ \mathbb{A}_{1,2\alpha} &= A_{1,2\mu}(X_{1,2})\, \partial_\alpha X^{\mu}_{1,2} + \partial_\alpha \varphi_{1,2}. \end{aligned} \tag{3.31}$$

We also equip the fluid worldvolume with a fixed thermal vector $\mathfrak{\beta}^{\alpha} = \delta^{\alpha}_{0}/T_0$, which characterise the equilibrium thermal state around which the hydrodynamic theory is being setup.

To model the additional gapped modes present in the MIS theory, we will also introduce a pair of tensor fields $\mathfrak{k}_{1,2\alpha\beta}(\sigma)$ and a pair vector fields $\mathfrak{u}_{1,2\alpha}(\sigma)$ on the fluid worldvolume. Since we only intend to introduce new degrees of freedom corresponding to stresses and fluxes and not for energy-momentum and charge densities, we will choose these new fields to be transverse to the thermal vector, i.e.

$$\mathfrak{\beta}^{\alpha}\mathfrak{k}_{1,2\alpha\beta} = 0, \qquad \mathfrak{\beta}^{\alpha}\mathfrak{u}_{1,2\alpha} = 0. \tag{3.32}$$

Both these fields are taken to be invariant under the global spacetime and U(1) symmetries.

We can define the average-difference basis $f_r = (f_1 + f_2)/2$, $f_a = (f_1 - f_2)/\hbar$ for various quantities, which will be useful later. Due to the non-linear nature of the theory, $\mathfrak{g}_{r,a\alpha\beta}$ and $\mathbb{A}_{r,a\mu}$ are non-trivially related to $X^{\mu}_{r,a}$, $\varphi_{r,a}$, $g_{r,a\mu\nu}$, $A_{r,a\mu}$. For example, in the classical limit we find

$$\begin{aligned} \mathfrak{g}_{r\alpha\beta} &= g_{r\mu\nu}(X_r)\, \partial_\alpha X^{\mu}_{r}\, \partial_\beta X^{\nu}_{r} + \mathcal{O}(\hbar), \\ \mathfrak{g}_{a\alpha\beta} &= \Big(g_{a\mu\nu}(X_r) + \pounds_{X_a} g_{r\mu\nu}(X_r)\Big) \partial_\alpha X^{\mu}_{r}\, \partial_\beta X^{\nu}_{r} + \mathcal{O}(\hbar), \\ \mathbb{A}_{r\alpha} &= A_{r\mu}(X_r)\, \partial_\alpha X^{\mu}_{r} + \partial_\alpha \varphi_r + \mathcal{O}(\hbar), \\ \mathbb{A}_{a\alpha} &= \Big(A_{a\mu}(X_r) + \pounds_{X_a} A_{r\mu}(X_r)\Big) \partial_\alpha X^{\mu}_{r} + \partial_\alpha \varphi_a + \mathcal{O}(\hbar), \end{aligned} \tag{3.33}$$

up to quantum corrections. The Schwinger-Keldysh effective action of the theory can be expressed as $S[\mathfrak{\Phi}_r, \mathfrak{\Phi}_a; \mathfrak{\beta}^{\alpha}]$, in terms of the global symmetry invariants $\mathfrak{\Phi}_{r,a} = (\frac{1}{2}\mathfrak{g}_{r,a\alpha\beta}, \mathbb{A}_{r,a\alpha}, \frac{1}{2}\mathfrak{k}_{r,a\alpha\beta}, \mathfrak{u}_{r,a\alpha})$ and the fixed thermal vector $\mathfrak{\beta}^{\alpha}$, integrated over the fluid worldvolume coordinates σ^{α}.

JHEP01(2024)162

Fluid worldvolume symmetries. We also impose local diffeomorphisms $\sigma'^{\alpha}(\sigma)$ and U(1) gauge transformations $\lambda(\sigma)$ of the fluid worldvolume acting on the dynamical fields as

$$\begin{aligned} X^{\mu}_{1,2}(\sigma) &\to X'^{\mu}_{1,2}(\sigma'(\sigma)) = X^{\mu}_{1,2}(\sigma), \\ \phi_{1,2}(\sigma) &\to \phi'_{1,2}(\sigma'(\sigma)) = \phi_{1,2}(\sigma) + \lambda(\sigma). \end{aligned} \tag{3.34}$$

The MIS fields $\mathbb{k}_{1,2\alpha\beta}$ and $\mathbb{v}_{1,2\alpha}$ are taken to invariant under worldvolume U(1) gauge transformations and transform as appropriately ranked tensors under worldvolume diffeomorphisms. These transformations are taken to be time-independent, satisfying

$$\mathbb{\beta}^{\beta}\partial_{\beta}\sigma'^{\alpha}(\sigma) = \mathbb{\beta}^{\beta}, \qquad \mathbb{\beta}^{\alpha}\partial_{\alpha}\lambda(\sigma) = 0. \tag{3.35}$$

One consequence of this is that all global symmetry invariants in eq. (3.31) are invariant under worldvolume gauge transformations, except the spatial components of $\mathbb{\beta}^{\alpha}\mathbb{A}_{r\alpha}$, similar to the diffusion model. Worldvolume diffeomorphisms act on all global symmetry invariants as usual.

Physical spacetime formulation. The fluid worldvolume picture of the hydrodynamic effective field theory with two copies of spacetimes is theoretically neat and appealing. However, for practical purposes, it is more transparent to move to a single physical spacetime formulation, defined via $x^{\mu} = X^{\mu}_{r}(\sigma)$. We can use pullbacks with respect to this map to define objects that are invariant under the fluid worldvolume diffeomorphis.

$$\begin{aligned} G_{r\mu\nu} &= \frac{\partial\sigma^{\alpha}}{\partial x^{\mu}}\frac{\partial\sigma^{\beta}}{\partial x^{\nu}}\mathbb{g}_{r\alpha\beta} = g_{r\mu\nu} + \mathcal{O}(\hbar), \\ G_{a\mu\nu} &= \frac{\partial\sigma^{\alpha}}{\partial x^{\mu}}\frac{\partial\sigma^{\beta}}{\partial x^{\nu}}\mathbb{g}_{a\alpha\beta} = g_{a\mu\nu} + \pounds_{X_a} g_{r\mu\nu} + \mathcal{O}(\hbar), \\ B_{r\mu} &= \frac{\partial\sigma^{\alpha}}{\partial x^{\mu}}\mathbb{A}_{r\alpha} = A_{r\mu} + \partial_{\mu}\varphi_r + \mathcal{O}(\hbar), \\ B_{a\mu} &= \frac{\partial\sigma^{\alpha}}{\partial x^{\mu}}\mathbb{A}_{a\alpha} = A_{a\mu} + \partial_{\mu}\varphi_a + \pounds_{X_a} A_{r\mu} + \mathcal{O}(\hbar), \end{aligned} \tag{3.36}$$

together with the MIS fields

$$\kappa_{r,a\mu\nu} = \frac{\partial\sigma^{\alpha}}{\partial x^{\mu}}\frac{\partial\sigma^{\beta}}{\partial x^{\nu}}\mathbb{k}_{r\alpha\beta}, \qquad \upsilon_{r,a\mu} = \frac{\partial\sigma^{\alpha}}{\partial x^{\mu}}\mathbb{v}_{r,a\alpha}. \tag{3.37}$$

Similarly, we can define the physical spacetime thermal vector by pushing forward the fluid worldvolume thermal vector

$$\beta^{\mu}(x) = \mathbb{\beta}^{\alpha}\frac{\partial x^{\mu}}{\partial\sigma^{\alpha}(x)} = \frac{1}{T_0}\frac{\partial x^{\mu}}{\partial\sigma^{0}(x)}, \tag{3.38}$$

which can be used to define the fluid velocity and temperature as $\beta^{\mu} = u^{\mu}/T$. The fluid velocity is normalised as $u^{\mu}u^{\nu}G_{r\mu\nu} = -1$. The fluid worldvolume gauge transformations become the diagonal shift symmetry on the physical spacetime, acting on $B_{r\mu}$ as

$$B_{r\mu} \to B_{r\mu} + \partial_{\mu}\lambda, \qquad \beta^{\mu}\partial_{\mu}\lambda = 0. \tag{3.39}$$

The temporal components of $B_{r\mu}$ can be used to define the chemical potential via

$$\frac{\mu}{T} = \beta^{\mu}B_{r\mu} + \frac{\mu_0}{T_0}, \tag{3.40}$$

JHEP01(2024)162

which are invariant under the diagonal shift symmetry. The compromise for going to the physical spacetime formulation is that a diagonal part of the global physical spacetime symmetries becomes non-manifest

$$x^\mu \to x'^\mu(x), \tag{3.41}$$

which acts on various fields on the physical spacetime as expected according to their tensor structure. In the physical spacetime formulation, the Schwinger-Keldysh effective action can be expressed as $S[\Phi_r, \Phi_a; \beta^\mu]$, in terms of the global symmetry invariants $\Phi_{r,a} = (\frac{1}{2}G_{r,a\mu\nu}, B_{r,a\mu}, \frac{1}{2}\kappa_{r,a\mu\nu}, \upsilon_{r,a\mu})$ and the thermal vector β^μ. Note that, unlike the diffusion model, the thermal vector β^μ here is no longer a constant.

Schwinger-Keldysh generating functional. The Schwinger-Keldysh generating functional for hydrodynamics can be defined using the effective action as

$$\mathcal{Z}[\phi_r, \phi_a] = \int \mathcal{D}\psi_r\, \mathcal{D}\psi_a \exp(iS[\Phi_r, \Phi_a; \beta^\mu]), \tag{3.42}$$

where the path integral is performed over the set of dynamical fields $\psi_{r,a} = (X^\mu_{r,a}, \varphi_{r,a}, \mathbb{k}_{r,a\mu\nu}, \mathbb{u}_{r,a\mu})$ and depends on the background fields $\phi_{r,a} = (\frac{1}{2}g_{r,a\mu\nu}, A_{r,a\mu})$. The generating functional is required to satisfy the conditions in eq. (2.43), which translate in terms of the effective action as

$$S[\Phi_r, \Phi_a = 0; \beta^\mu] = 0, \qquad S[\Phi_r, -\Phi_a; \beta^\mu] = -S^*[\Phi_r, \Phi_a; \beta^\mu], \qquad \operatorname{Im} S[\Phi_r, \Phi_a; \beta^\mu] \geq 0. \tag{3.43}$$

Dynamical KMS symmetry. The Schwinger-Keldysh generating functional is required to satisfy the dynamical KMS symmetry given by its action on the background fields in eq. (2.45), or the one in eq. (2.33) in the classical limit. The action of KMS symmetry on the dynamical fields is naturally defined on the fluid worldvolume as

$$\begin{aligned}
X^\mu_1(\sigma) &\to \Theta X^\mu_1(\sigma), & X^\mu_2(\sigma) &\to \Theta X^\mu_2(\sigma + i\hbar\,\Theta\beta) - i\hbar\,\Theta\beta^\mu_0,\\
\varphi_1(\sigma) &\to \Theta\varphi_1(\sigma), & \varphi_2(\sigma) &\to \Theta\varphi_2(\sigma + i\hbar\,\Theta\beta),\\
\mathbb{k}_{1\alpha\beta}(\sigma) &\to \Theta\mathbb{k}_{1\alpha\beta}(\sigma), & \mathbb{k}_{2\alpha\beta}(\sigma) &\to \Theta\mathbb{k}_{2\alpha\beta}(\sigma + i\hbar\,\Theta\beta),\\
\mathbb{u}_{1\alpha}(\sigma) &\to \Theta\mathbb{u}J_{1\alpha}(\sigma), & \mathbb{u}_{2\alpha}(\sigma) &\to \Theta\mathbb{u}_{2\alpha}(\sigma + i\hbar\,\Theta\beta).
\end{aligned} \tag{3.44}$$

The extra constant contribution in the transformation X^μ_2 is taken so that KMS symmetry preserve the equilibrium configuration $X^\mu_{1,2}(\sigma) = \delta^\mu_\alpha\sigma^\alpha$, $\varphi_{1,2}(\sigma) = 0$. In the physical spacetime formulation in the classical limit, these give rise to

$$\begin{aligned}
\beta^\mu &\to \Theta\beta^\mu, & X^\mu_a &\to \Theta\left(X^\mu_a + i(\beta^\mu - \beta^\mu_0)\right),\\
\varphi_r &\to \Theta\varphi_r, & \varphi_a &\to \Theta\left(\varphi_a + i\pounds_\beta\varphi_r\right),\\
\kappa_{r\mu\nu} &\to \Theta\kappa_{r\mu\nu}, & \kappa_{a\mu\nu} &\to \Theta\left(\kappa_{a\mu\nu} + i\pounds_\beta\kappa_{r\mu\nu}\right),\\
\upsilon_{r\mu} &\to \Theta\upsilon_{r\mu}, & \upsilon_{a\mu} &\to \Theta\left(\upsilon_{a\mu} + i\pounds_\beta\upsilon_{r\mu}\right).
\end{aligned} \tag{3.45}$$

These transformation properties induce the following dynamical KMS transformation on the building blocks of the effective action

$$\Phi_r \to \Theta\Phi_r, \qquad \Phi_a \to \Theta\left(\Phi_a + i\pounds_\beta\Phi_r\right), \qquad \beta^\mu \to \Theta\beta^\mu, \tag{3.46}$$

which is the appropriate generalisation of the transformations (2.48) from the diffusion model.

JHEP01(2024)162

Hydrodynamic effective action. Truncating the theory to at most quadratic order in Φ_a fields and dropping any explicit spatial derivatives, the most general hydrodynamic effective action for MIS-hydrodynamics model is given as

$$
\begin{aligned}
S = \int \mathrm{d}^{d+1}x\sqrt{-g_r}\Bigg[&\frac{1}{2}G_{a\mu\nu}\left(\epsilon\, u^\mu u^\nu - \mathcal{F}\Delta^{\mu\nu}\right) + B_{a\mu} n u^\mu \\
&- \frac{1}{2}\kappa_{a\mu\nu}\left(\frac{\chi_\kappa^{\mathsf{S}}}{2}\Delta^{\mu\nu}\,\mathrm{tr}\,\kappa_r + \chi_\kappa^{\mathsf{T}}\kappa_r^{\langle\mu\nu\rangle}\right) - \chi_\upsilon \upsilon_{a\mu}\upsilon_r^\mu \\
&+ \frac{1}{2}G_{a\mu\nu}\left(\frac{\chi_\kappa^{\mathsf{S}}}{2}\kappa_r^{\mu\nu}\,\mathrm{tr}\,\kappa_r + \chi_\kappa^{\mathsf{T}}\left(\kappa_r^{\mu\rho}\kappa_{r\rho}^{\nu} - \frac{1}{d}\kappa_r^{\mu\nu}\,\mathrm{tr}\,\kappa_r\right) + \chi_\upsilon \upsilon_r^\mu \upsilon_r^\nu\right) \\
&+ \frac{iT}{4}\zeta\,\Delta^{\mu\nu}\Delta^{\rho\sigma}\left(G_{a\mu\nu} + \frac{\chi_\kappa^{\mathsf{S}}}{\alpha_\kappa^{\mathsf{S}}}\kappa_{a\mu\nu}\right)\left(G_{a\rho\sigma} + i\pounds_\beta g_{r\rho\sigma} + \frac{\chi_\kappa^{\mathsf{S}}}{\alpha_\kappa^{\mathsf{S}}}\left(\kappa_{a\rho\sigma} + i\pounds_\beta \kappa_{r\rho\sigma}\right)\right) \\
&+ \frac{iT}{2}\eta\,\Delta^{\rho\langle\mu}\Delta^{\nu\rangle\sigma}\left(G_{a\mu\nu} + \frac{\chi_\kappa^{\mathsf{T}}}{\alpha_\kappa^{\mathsf{T}}}\kappa_{a\mu\nu}\right)\left(G_{a\rho\sigma} + i\pounds_\beta g_{r\rho\sigma} + \frac{\chi_\kappa^{\mathsf{T}}}{\alpha_\kappa^{\mathsf{T}}}\left(\kappa_{a\rho\sigma} + i\pounds_\beta \kappa_{r\rho\sigma}\right)\right) \\
&+ iT\sigma\,\Delta^{\mu\nu}\left(B_{a\mu} + \frac{\chi_\upsilon}{\alpha_\upsilon}\upsilon_{a\mu}\right)\left(B_{a\nu} + i\delta_{\mathcal{B}}A_\nu + \frac{\chi_\upsilon}{\alpha_\upsilon}\left(\upsilon_{a\nu} + i\pounds_\beta \upsilon_{r\nu}\right)\right)\Bigg],
\end{aligned}
\tag{3.47}
$$

where we have already imposed the hydrodynamic frame conditions as given in eq. (3.12) for clarity. The action is manifestly invariant under the spacetime global symmetries and worldvolume gauge symmetries. It also trivially obeys the first two Schwinger-Keldysh conditions, while the third one requires

$$
\zeta \geq 0, \qquad \eta \geq 0, \qquad \sigma \geq 0. \tag{3.48}
$$

Using $\Theta = \mathrm{T}$ or PT, the terms in the first three lines map to themselves (up to a coordinate flip in the integral $x^\mu \to \Theta x^\mu$), with the residual terms that sum to a total derivative and drop out from the effective action, i.e.

$$
\begin{aligned}
&i\sqrt{-g_r}\Bigg[\frac{1}{2}\left(\epsilon\, u^\mu u^\nu - \mathcal{F}\Delta^{\mu\nu}\right)\pounds_\beta G_{r\mu\nu} + n u^\mu \pounds_\beta B_{r\mu} - \frac{1}{2}\left(\frac{\chi_\kappa^{\mathsf{S}}}{2}\Delta^{\mu\nu}\,\mathrm{tr}\,\kappa + \chi_\kappa^{\mathsf{T}}\kappa^{\langle\mu\nu\rangle}\right)\pounds_\beta \kappa_{\mu\nu} \\
&\qquad - \chi_\upsilon \upsilon^\mu \pounds_\beta \upsilon_\mu + \frac{1}{2}\left(\frac{\chi_\kappa^{\mathsf{S}}}{2}\kappa^{\mu\nu}\,\mathrm{tr}\,\kappa + \chi_\kappa^{\mathsf{T}}\left(\kappa^{\mu\rho}\kappa_\rho^\nu - \frac{1}{d}\kappa^{\mu\nu}\,\mathrm{tr}\,\kappa\right) + \chi_\upsilon \upsilon^\mu \upsilon^\nu\right)\pounds_\beta G_{r\mu\nu}\Bigg] \\
&= i\sqrt{-g_r}\left(-\mathcal{F}\nabla_\mu\beta^\mu + s\pounds_\beta T + n\pounds_\beta \mu - \frac{\chi_\kappa^{\mathsf{S}}}{8}\pounds_\beta(\mathrm{tr}\,\kappa)^2 - \frac{\chi_\kappa^{\mathsf{T}}}{4}\pounds_\beta\left(\kappa^2 - \frac{1}{d}(\mathrm{tr}\,\kappa)^2\right) - \frac{\chi_\upsilon}{2}\pounds_\beta \upsilon^2\right), \\
&= -\partial_\mu\left(\sqrt{-g_r}\, i\beta^\mu\,\mathcal{F}\right).
\end{aligned}
\tag{3.49}
$$

We have identified $\kappa_{\mu\nu} \equiv \kappa_{r\mu\nu}$ and $\upsilon_\mu \equiv \upsilon_{r\mu}$ and used the thermodynamic relations (3.11). On account of eq. (3.46), the last three lines in the effective action are individually KMS-invariant. If $\Theta = \mathrm{CPT}$ or CT, we will also need to require n to be an odd function of μ, while all other coefficients are even functions of μ.

This action is precisely the one we derived in eq. (3.25), with the KMS conditions (3.26) already implemented, as we can see by performing the following field redefinition

$$
\begin{aligned}
&\kappa_{\mu\nu} = \kappa_{r\mu\nu}, && \upsilon_\mu = \upsilon_{r\mu}, \\
&W_{a\mu\nu} = \Delta_\mu^\rho\Delta_\nu^\sigma G_{a\rho\sigma} + \left(\frac{1}{d}\frac{\chi_\kappa^{\mathsf{S}}}{\alpha_\kappa^{\mathsf{S}}}\Delta_{\mu\nu}\Delta^{\rho\sigma} + \frac{\chi_\kappa^{\mathsf{T}}}{\alpha_\kappa^{\mathsf{T}}}\Delta_{\langle\mu}^\rho\Delta_{\nu\rangle}^\sigma\right)\kappa_{a\rho\sigma}, && V_{a\mu} = \Delta_\mu^\nu B_{a\nu} + \frac{\chi_\upsilon}{\alpha_\upsilon}\upsilon_{a\mu}.
\end{aligned}
\tag{3.50}
$$

JHEP01(2024)162

While making the identification, it is also useful to note that $\partial_\mu(\mu/T) + \beta^\lambda F_{r\lambda\mu} = \pounds_\beta B_{r\mu}$. This results in the standard prescription of KMS symmetry given in eq. (3.27). To realise the alternate prescriptions in eq. (3.28), we need to modify the SK-EFT by imposing a diagonal shift symmetries between the vector and tensor degrees of freedom respectively; see appendix C.

Note that the dynamical KMS transformation of $\Phi_a = (\frac{1}{2}G_{r,a\mu\nu}, B_{r,a\mu}, \frac{1}{2}\kappa_{r,a\mu\nu}, \upsilon_{r,a\mu})$ fields in eq. (3.46) involves a Lie derivative $\pounds_\beta$ along the thermal vector β^μ, which forces the occurrences of $\pounds_\beta$ in the dissipative terms in the last three lines of the effective action (3.49). In particular, if $\pounds_\beta \kappa_{r\mu\nu}$ and $\pounds_\beta \upsilon_{r\mu}$ were replaced with $\beta^\lambda \nabla_\lambda \kappa_{r\mu\nu}$ and $\beta^\lambda \nabla_\lambda \upsilon_{r\mu}$, these terms would no longer be non-linearly compatible with the dynamical KMS symmetry. This ties back to our comments towards the ends of sections 3.2 and 3.3, that the right choice of definitions of $\kappa_{\mu\nu}$ and υ_μ in the classical hydrodynamic equations is crucial for compatibility of the EFT with the dynamical KMS symmetry and thus guarantee the agreement with all higher-point FDT requirements.

3.5 Conformal hydrodynamics

The Müller-Israel-Stewart framework of relativistic hydrodynamics hugely simplifies in the conformal limit and was considered in the holographic model of [36]. Conformal symmetry is defined as the invariance of the hydrodynamic equations under a conformal rescaling of background metric, i.e.

$$g_{\mu\nu} \to \Omega^2 g_{\mu\nu}, \qquad A_\mu \to A_\mu, \tag{3.51}$$

for arbitrary function $\Omega(x)$. The hydrodynamic conservation equations (3.1) are left invariant under this transformation, provided that the conserved currents scale as

$$T^{\mu\nu} \to \Omega^{-d-3} T^{\mu\nu}, \qquad J^\mu \to \Omega^{-d-1} J^\mu, \tag{3.52}$$

and the energy-momentum tensor is traceless

$$T^\mu_{\ \mu} = 0. \tag{3.53}$$

Conformal transformations are realised on the hydrodynamic fields as

$$u^\mu \to \Omega^{-1} u^\mu, \qquad T \to \Omega^{-1} T, \qquad \mu \to \Omega^{-1}\mu, \qquad \kappa_{\mu\nu} \to \kappa_{\mu\nu}, \qquad \upsilon_\mu \to \upsilon_\mu. \tag{3.54}$$

Note that we are free to choose the conformal weight of the fields $\kappa_{\mu\nu}$ and υ_μ. However, had we rescaled these fields to fix $\alpha^{\mathsf{S}}_\kappa$, $\alpha^{\mathsf{T}}_\kappa$, and α_υ introduced in eqs. (2.15) and (3.12) to a particular value, the conformal weight of these fields would also be fixed by the conformal weights of the conserved currents. By scaling arguments, we can obtain the conformal weights of various coefficients in the hydrodynamic model

$$\begin{aligned}
&\epsilon \to \Omega^{-d-1}\epsilon, && p \to \Omega^{-d-1} p, && n \to \Omega^{-d} n, && s \to \Omega^{-d} s,\\
&\chi^{\mathsf{S}}_\kappa \to \Omega^{-d+3}\chi^{\mathsf{S}}_\kappa, && \chi^{\mathsf{T}}_\kappa \to \Omega^{-d+3}\chi^{\mathsf{T}}_\kappa, && \chi_\upsilon \to \Omega^{-d+1}\chi_\upsilon, && \\
&\alpha^{\mathsf{S}}_\kappa \to \Omega^{-d+1}\alpha^{\mathsf{S}}_\kappa, && \alpha^{\mathsf{T}}_\kappa \to \Omega^{-d+1}\alpha^{\mathsf{T}}_\kappa, && \alpha_\upsilon \to \Omega^{-d+1}\alpha_\upsilon, && \\
&\zeta \to \Omega^{-d}\zeta, && \eta \to \Omega^{-d}\eta, && \sigma \to \Omega^{-d+2}\sigma. &&
\end{aligned} \tag{3.55}$$

JHEP01(2024)162

Plugging these into the thermodynamic relation (3.11), we are led to the conformality constraints between thermodynamic coefficients

$$\epsilon = d\,p, \qquad d\,\mathcal{P} + \frac{\chi_\kappa^{\mathsf{S}}}{2}(\mathrm{tr}\,\kappa)^2 + \chi_\kappa^{\mathsf{T}}\left(\kappa^2 - \frac{1}{d}(\mathrm{tr}\,\kappa)^2\right) + \chi_\upsilon \upsilon^2 = 0. \tag{3.56}$$

Furthermore, requiring the constitutive relations in eq. (3.16) to transform homogeneously under non-constant conformal transformations, forces us to fix

$$\zeta = 0 \qquad \Longrightarrow \qquad \mathrm{tr}\,\kappa = 0. \tag{3.57}$$

These conditions together ensure that $T^\mu_{\ \mu} = 0$. The final constitutive relations are given as

$$\begin{aligned} T^{\mu\nu} &= \epsilon\left(u^\mu u^\nu + \frac{1}{d}\Delta^{\mu\nu}\right) + \Pi^{\mu\nu}, \\ J^\mu &= n\,u^\mu + \mathcal{J}^\mu, \end{aligned} \tag{3.58}$$

together with the non-linear relations between $\Pi^{\mu\nu}$, $\mathcal{J}^\mu$ and $\kappa_{\mu\nu}$, υ_μ, i.e.

$$\begin{aligned} \Pi^{\mu\nu} &= \alpha_\kappa^{\mathsf{T}}\kappa^{\mu\nu} + \chi_\kappa^{\mathsf{T}}\left(\kappa^{\mu\rho}\kappa^\nu_{\ \rho} - \frac{1}{d}\Delta^{\mu\nu}\kappa^2\right) + \chi_\upsilon\left(\upsilon^\mu\upsilon^\nu - \frac{1}{d}\Delta^{\mu\nu}\upsilon^2\right), \\ \mathcal{J}^\mu &= \alpha_\upsilon \upsilon^\mu. \end{aligned} \tag{3.59}$$

The dynamical equations for $\kappa_{\mu\nu}$ and υ_μ take the form

$$\begin{aligned} \alpha_\kappa^{\mathsf{T}}\kappa^{\mu\nu} &= -2\eta\,\Delta^{\rho\langle\mu}\Delta^{\nu\rangle\sigma}\left(\nabla_\rho u_\sigma + \frac{T\chi_\kappa^{\mathsf{T}}}{2\alpha_\kappa^{\mathsf{T}}}\pounds_\beta \kappa_{\rho\sigma}\right), \\ \alpha_\upsilon \upsilon^\mu &= -\sigma\,\Delta^{\mu\nu}\left(T\nabla_\nu\frac{\mu}{T} + u^\lambda F_{\lambda\nu} + \frac{T\chi_\upsilon}{\alpha_\upsilon}\pounds_\beta \upsilon_\nu\right). \end{aligned} \tag{3.60}$$

We can also obtain the relaxation equations for the dissipative stress tensor and flux. To this end, let us take a simple ansatz for $\alpha_\kappa^{\mathsf{T}} = c_\kappa^{\mathsf{T}}\epsilon^{(d-1)/(d+1)}$ and $\alpha_\upsilon = c_\upsilon \epsilon^{(d-1)/(d+1)}$, for constants c_κ^{T} and c_υ. With this choice, we find

$$\begin{aligned} \Pi^{\mu\nu} &= -2\eta\,\Delta^{\rho\langle\mu}\Delta^{\nu\rangle\sigma}\nabla_\rho u_\sigma - \tau_\Pi^{\mathsf{T}}\,\Delta^{\rho\langle\mu}\Delta^{\nu\rangle\sigma}\left(T\pounds_\beta \Pi_{\rho\sigma} + \frac{d-1}{d}\Pi_{\rho\sigma}\nabla_\lambda u^\lambda\right) \\ &\quad + \frac{\tau_\Pi^{\mathsf{T}}}{\eta}\left(\Pi^{\mu\rho}\Pi^\nu_{\ \rho} - \frac{1}{d}\Delta^{\mu\nu}\Pi^2\right) + \frac{\tau_{\mathcal{J}}}{\sigma}\left(\mathcal{J}^\mu\mathcal{J}^\nu - \frac{1}{d}\Delta^{\mu\nu}\mathcal{J}^2\right) + \mathcal{O}(\partial^3), \\ \mathcal{J}^\mu &= -\sigma\,\Delta^{\mu\nu}\left(T\nabla_\nu\frac{\mu}{T} + u^\lambda F_{\lambda\nu}\right) - \tau_{\mathcal{J}}\,\Delta^{\mu\nu}\left(T\pounds_\beta \mathcal{J}_\nu + \frac{d-1}{d}\mathcal{J}_\nu\nabla_\lambda u^\lambda\right) + \mathcal{O}(\partial^3), \end{aligned} \tag{3.61}$$

where we have ignored 3- and higher-derivative corrections. We have also used the leading order energy conservation equation $\pounds_\beta\epsilon + (\epsilon+p)/T\,\nabla_\mu u^\mu = 0$. These equations can be directly compared with the ones derived in [36]. However, unlike [36], our hydrodynamic equations are not designed to be exhaustive at second-order in derivatives. In particular, we have ignored any dependence of the constitutive relations on the fluid vorticity that is present in [36].

JHEP01(2024)162

The effective action for MIS-hydrodynamics in the conformal limit simplifies to

$$
\begin{aligned}
S = \int \mathrm{d}^{d+1}x\sqrt{-g_r}\Big[&\frac{1}{2}G_{a\mu\nu}\left(u^\mu u^\nu + \frac{1}{d}\Delta^{\mu\nu}\right)\epsilon + B_{a\mu}nu^\mu - \frac{\chi_\kappa^{\intercal}}{2}\kappa_{a\mu\nu}\kappa_r^{\mu\nu} - \chi_\upsilon \upsilon_{a\mu}\upsilon_r^\mu \\
&+ \frac{1}{2}G_{a\langle\mu\nu\rangle}\left(\chi_\kappa^{\intercal}\kappa_r^{\mu\rho}\kappa_{r\rho}^{\nu} + \chi_\upsilon \upsilon_r^\mu \upsilon_r^\nu\right) \\
&+ \frac{iT}{2}\eta\,\Delta^{\rho\langle\mu}\Delta^{\nu\rangle\sigma}\left(G_{a\mu\nu} + \frac{\chi_\kappa^{\intercal}}{\alpha_\kappa^{\intercal}}\kappa_{a\mu\nu}\right)\left(G_{a\rho\sigma} + i\pounds_\beta g_{r\rho\sigma} + \frac{\chi_\kappa^{\intercal}}{\alpha_\kappa^{\intercal}}\left(\kappa_{a\rho\sigma} + i\pounds_\beta \kappa_{r\rho\sigma}\right)\right) \\
&+ iT\sigma\,\Delta^{\mu\nu}\left(B_{a\mu} + \frac{\chi_\upsilon}{\alpha_\upsilon}\upsilon_{a\mu}\right)\left(B_{a\nu} + i\delta_{\mathcal{B}}A_\nu + \frac{\chi_\upsilon}{\alpha_\upsilon}\left(\upsilon_{a\nu} + i\pounds_\beta \upsilon_{r\nu}\right)\right)\Big],
\end{aligned}
\tag{3.62}
$$

which is better suited for potential holographic applications.

4 Discussion

In this work, we have constructed UV-regularised stable and causal Schwinger-Keldysh effective field theories for relativistic diffusion and hydrodynamics. The SK-EFTs are appropriately coupled to double-copy background sources and non-linearly realise the dynamical KMS symmetry, and thus can be used to compute n-point time-ordered correlation functions, including perturbative stochastic loop corrections, consistent with FDT, stability, and causality requirements. We derived the respective SK-EFTs using both the MSR and SK formalisms, although the latter is more systematic and general and can potentially be used to include non-Gaussian stochastic interactions into the framework.

We also find that it is possible to construct multiple models in the SK formalism that, up to certain field redefinitions, ultimately result in the same UV-regularised SK effective action; see appendix C. As a consequence, the SK-EFTs discussed in this work simultaneously realise multiple dynamical KMS symmetries that act differently on the UV degrees of freedom. This is not really surprising because the UV sectors of these models are not constrained by the global symmetries in the hydrodynamic regime.

To construct these causal and stable SK-EFTs, we used hydrodynamic models inspired from the Maxwell-Cattaneo model of relativistic diffusion and the Müller-Israel-Stewart model of relativistic hydrodynamics, wherein one promotes the charge flux and spatial stress tensor to independent dynamical degrees of freedom with respective characteristic relaxation times. However, it is interesting to note that the classical evolution equations of the models we constructed, given in eqs. (2.17) and (3.16) respectively, are non-linearly different from the conventional treatments of MC-diffusion and MIS-hydrodynamics respectively. The precise form of the equations depends on the particular choice of variables for the relaxed degrees of freedom and we found that using charge flux or spatial stress tensor directly for this purpose does not generically non-linearly agree with the dynamical KMS symmetry. We are forced to work in the so-called "thermodynamic frame", where the choice of variables is enforced upon us by the grand canonical free energy density of the system under consideration. In retrospect, this is not surprising since SK-EFTs are designed to describe thermal fluctuations, while the very notion of relativistic temperature is not treated consistently in the traditional Landau frame; see e.g. [57]. Thus, extensions of relativistic hydrodynamics based on Landau frame, such as [36], do not lend themselves naturally to producing EFTs that respect FDT

JHEP01(2024)162

for n-point functions. As a future direction, it will be interesting to explore this connection between the choice of variables and the SK formalism.

Generally speaking, relaxation times are second-order transport coefficients in the hydrodynamic derivative expansion. In this work, our goal was to develop the general framework for extending relativistic SK-EFTs to include UV sectors that are consistent with stability, causality, and FDT requirements. To this end, we have analysed minimal models that fit the bill and have not attempted to be complete or rigorous at second order in derivatives, which would be an interesting avenue to pursue in the future. Care must be taken, however, because certain second order terms might end up worsening the causality and stability issues. A simple example of this is allowing an arbitrary σ_υ coefficient in the MC-diffusion model as given in eq. (2.53), not fixed by the frame condition in eq. (2.55). This modifies the dispersion relations in the denominator of eq. (2.20) by a term $+iD_\upsilon\omega k^2$, where the coefficient $D_\upsilon = \left(\sigma_\upsilon - \sigma\chi^2/\alpha_\upsilon^2\right) D_n/\chi_\upsilon$ is non-negative due to the constraints in eq. (2.51). In a boosted frame of reference with small boost speed $v_0 \ll 1$, this yields an unstable mode that goes as $\omega = i\tau_{\mathcal{J}}/(v_0^2 D_\upsilon) + \ldots$. To fix these issues, we will need to raise the powers of ω in the dispersion relations further by introducing higher-derivative terms in the effective theory and thus continuing the vicious cycle.

We should also emphasise that these SK-EFTs are not meant to describe physics near the relaxation timescales with any level of generality. They should only be viewed as introducing a consistent UV-regularisation into the hydrodynamic framework, while all physical predictions are still only universally valid at the longest spacetime scales within the hydrodynamic regime. If we wish to reliably describe physics near the relaxation timescales, we will need more information regarding the relaxed degrees of freedom and the relevant symmetries applicable at these scales, generically leading to qualitatively distinct results. For example, as we discussed in footnote 5, the relaxed vector modes in MC-diffusion model can come from momentum density relaxed due to impurities or superfluid velocity relaxed due to vortices. Near the relaxation timescales, the former realises an approximate translation symmetry while the latter realises an approximate higher-form symmetry; see e.g. [58, 59], both of which will lead to drastically distinct physical signatures. Similarly, recalling footnote 9, the tensor modes in MIS-hydrodynamics might arise from a plastic crystal, which also realises an approximate higher-form symmetry near the relaxation timescales; see e.g. [55, 56]. Without the knowledge of the underlying small(er) scale physics, the predictability of these models is only limited to the universal hydrodynamic sector.

As we mentioned in the introduction, the recently proposed BDNK formalism also provides an algorithm to bypass stability and causality issues in relativistic hydrodynamics by invoking the freedom to arbitrarily redefine hydrodynamic fields. However, this formalism, as it stands, is not well-suited for being lifted to a SK-EFT and describing stochastic fluctuations. Relegating details to section A, one finds that tuning the transport coefficients in the BDNK formalism to ensure stability and causality at the classical level leads to potentially negative symmetric 2-point correlation functions and thus violations of unitarity. That being said, BDNK-hydrodynamics provides a certain advantage over MIS-hydrodynamics within the context of classical evolution: it is not straightforward to demonstrate causality and stability of the fully non-linear equations of MIS-hydrodynamics [60], while it is relatively simple

JHEP01(2024)162

to prove this for BDNK-hydrodynamics [40, 41]. It will be interesting to explore whether one can adjust the BDNK formalism so as to yield a consistent SK-EFT and yet preserve the inherent simplicity of the non-linear hydrodynamic equations. On the other hand, it will also be interesting to explore whether the SK-EFT framework for MIS-hydrodynamics developed in this work, and appropriate extensions to other MIS-inspired models in the literature [37, 43, 61, 62], can help us analyse the stability and causality of these models at the full non-linear level.

The primary practical objective of SK-EFTs is to provide a systematic platform for studying statistical fluctuation corrections to classical hydrodynamic observables such as the retarded correlation functions of conserved densities and the corresponding fluxes. This entails using the linearised SK-EFT with interactions, like the one presented in section B, to compute stochastic loop corrections to various correlation functions order-by-order in the loop-expansion. This analysis for non-UV-regularised models appeared in [8, 63, 64] and more recently for a UV-regularised model of diffusion in [47]. We comment on the model of [47] in the second half of section A. It will be interesting to undertake the computation of stochastic loop corrections using the KMS-compatible SK-EFTs developed in our work, which we leave for future explorations.

It will also be interesting to extend our results to more intricate dissipative relativistic hydrodynamic theories such as relativistic superfluid hydrodynamics [53, 65, 66] or relativistic magnetohydrodynamics [67–72], which are all plagued by similar stability and causality issues. It might also be interesting to extend these ideas of UV-regularisation to non-relativistic SK-EFTs, such as the ones discussed in [18, 19]. While an additional UV sector is not necessitated by any stability or causality requirements in non-relativistic contexts, the loop integrals in the respective SK-EFTs still need to be regulated with a KMS-compatible prescription and the results of this work might provide a viable and simple resolution to this end.

This work is closely related to the recent work of [73], where authors developed a general mechanism based on information current for introducing stochastic interactions into causal and stable models of linear relativistic hydrodynamics, like linear MC-diffusion and MIS-hydrodynamics. In a follow-up work that appears on arXiv on the same day as this paper, the authors have also utilised the information current perspective together with the MSR formalism to construct free EFTs for these linear hydrodynamic models consistent with 2-point FDT [48]. In particular, the authors have identified a modified-KMS symmetry that is responsible for implementing 2-point FDT in their formalism. As we discussed in footnote 8, this is nothing but an alternate prescription of KMS transformations admissible in UV-regularised SK-EFTs. By contrast, the SK-EFTs constructed here correspond to the fully non-linear theories inspired from MC-diffusion and MIS-hydrodynamics, and include arbitrarily non-linear interactions consistent with all n-point FDT requirements. It will be interesting to further explore the connections with the work of [73], in particular with the information current identified by the authors and how it is realised in the fully non-linear EFT.

Acknowledgments

We would like to thank N. Abbasi, G. Denicol, M. Hippert, M. Kaminski, and J. Noronha for various useful discussions. We are also thankful to the authors of [48] for sharing a draft of

JHEP01(2024)162

their paper prior to submission. AJ is funded by the European Union's Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement NonEqbSK No. 101027527. AJ are partly supported by the Netherlands Organization for Scientific Research (NWO) and by the Dutch Institute for Emergent Phenomena (DIEP) cluster at the University of Amsterdam. PK is supported in part by the NSERC of Canada. This project was initiated during "The Many Faces of Relativistic Fluid Dynamics" program at KITP, Santa Barbara, supported by the National Science Foundation under Grant No. NSF PHY-1748958.

A Comparison with BDNK formalism

In the introduction, we mentioned an alternate formalism for stabilising the gapped modes in a boosted frame of reference in relativistic hydrodynamic theories, without the need for introducing additional relaxed degrees of freedom, known as the Bemfica-Disconzi-Noronha-Kovtun (BDNK) formalism [38–41]. Let us look at the diffusion model for concreteness. The key idea is to untie the physical density $-J^\mu u^0_\mu$ from the thermodynamic density n that appears in the thermodynamic relations (2.4) and, just like the flux $\mathcal{J}^\mu$, allow for arbitrary constitutive relations for $J^\mu u^0_\mu$ as well. To this end, let us split the constitutive relations into the ideal and dissipative parts

$$J^\mu = (n + \mathcal{N})\, u_0^\mu + \mathcal{J}^\mu. \tag{A.1}$$

The expression for entropy production modifies from eq. (2.5) to

$$\partial_\mu S^\mu = -\frac{1}{T_0}\mathcal{N} u_0^\mu \partial_\mu \mu - \frac{1}{T_0}\mathcal{J}^\mu\Big(\partial_\mu \mu + u_0^\rho F_{\rho\nu}\Big), \tag{A.2}$$

where $S^\mu = (s - \mu/T_0\,\mathcal{N})u_0^\mu - \mu/T_0\,\mathcal{J}^\mu$. This allows us to read out the constitutive relations

$$\begin{aligned} \mathcal{N} &= \lambda\, u_0^\mu \partial_\mu \mu, \\ \mathcal{J}^\mu &= -\sigma\, \Delta^{\mu\nu}\left(\partial_\nu \mu + u_0^\rho F_{\rho\nu}\right), \end{aligned} \tag{A.3}$$

for some arbitrary dissipative transport coefficients λ and σ. Looking at the expression for entropy production, i.e.

$$\partial_\mu S^\mu = -\frac{1}{T_0\lambda}\mathcal{N}^2 + \frac{1}{T_0\sigma}\mathcal{J}_\mu \mathcal{J}^\mu, \tag{A.4}$$

we might be tempted to claim that σ is non-negative while λ is non-positive. However, note that entropy production is only required to be positive onshell. Therefore, before imposing the positivity of entropy production, we should replace

$$\mathcal{N} = \frac{\lambda}{\chi} u_0^\mu \partial_\mu n = -\frac{\lambda}{\chi}\left(u_0^\mu \partial_\mu \mathcal{N} + \partial_\mu \mathcal{J}^\mu\right). \tag{A.5}$$

In particular, we see that $\mathcal{N}$ is a 2-derivative onshell and thus the contribution from λ to the entropy production is two orders suppressed relative to the contribution from σ. It was observed in [74, 75] that the second law does not impose any constraints beyond the leading order in derivatives; we can always introduce even further derivative order corrections to the

JHEP01(2024)162

constitutive relations so that the positivity of entropy production is ensured with only the leading order dissipative coefficients being non-negative. As a consequence, the positivity of entropy production only imposes $\sigma \geq 0$, while λ is left unconstrained.

We can compute the retarded correlation functions for this theory using the variational formulae in eq. (2.8). We find

$$\begin{aligned} G^R_{J^t J^t}(\omega, k) &= \frac{-(1 - i\omega\tau)\sigma k^2}{i\omega(1 - i\omega\tau) - Dk^2}, \\ G^R_{J^i_\perp J^i_\perp}(\omega, k) &= i\omega\sigma\, k^{ij}, \end{aligned} \tag{A.6}$$

where $\tau = \lambda/\chi$. Note that these correlation functions are considerably different from the ones obtained in the Maxwell-Cattaneo theory of diffusion in eq. (2.20). Nonetheless, they get the job done: in arbitrary boosted frame of reference, we find a diffusive and a gapped mode

$$\omega = kv_0 \cos\theta - iDk^2\sqrt{1 - v_0^2}\left(1 - v_0^2 \cos^2\theta\right) + \ldots, \qquad \omega = -i\frac{\sqrt{1 - v_0^2}}{\tau - v_0^2 D} + \ldots, \tag{A.7}$$

and provided that we take $\tau \geq D \geq 0$, they are stable for any boost parameter. Given that $\chi \geq 0$, this essentially implies that stability requires $\lambda \geq \sigma \geq 0$. As discussed near eq. (A.5), despite the positive λ, these constraints are not at odds with the positivity of entropy production.

The positivity of λ does pose a different problem. We can revisit our discussion of stochastic fluctuations from section 2.3 for the BDNK model. We can introduce noise $\theta_{\mathcal{N}}$ and $\theta^\mu_{\mathcal{J}}$ in the constitutive relations (A.3) and set-up a path integral over noise configurations similar to eq. (A.3), but with the new weight factor

$$\exp\left(-\frac{1}{4}\int \mathrm{d}^{d+1}x \left[\left\langle \frac{-1}{T_0\tilde{\lambda}}\right\rangle_\theta \theta^2_{\mathcal{N}} + \left\langle \frac{\Delta_{\mu\nu}}{T_0\tilde{\sigma}}\right\rangle_\theta \theta^\mu_{\mathcal{J}}\theta^\nu_{\mathcal{J}}\right]\right). \tag{A.8}$$

Following through the procedure in section 2.3, we can obtain the symmetric 2-point Green's functions

$$\begin{aligned} G^S_{J^t J^t}(\omega, k) &= 2T_0 \frac{(1 + \omega^2\tau^2)\tilde{\sigma}k^2 - \tilde{\lambda}D^2k^4}{|i\omega(1 - i\omega\tau) - Dk^2|^2}, \\ G^S_{J^i_\perp J^i_\perp}(\omega, k) &= 2T_0\tilde{\sigma}\, k^{ij}. \end{aligned} \tag{A.9}$$

Requiring the symmetric and retarded Green's functions to satisfy the fluctuation-dissipation theorem will force us to take

$$\tilde{\sigma} = \sigma, \qquad \tilde{\lambda} = \lambda, \tag{A.10}$$

meaning that $\tilde{\lambda}$ is also positive. This, however, means that the $\theta^2_{\mathcal{N}}$ term in the noise weight factor (A.8) has the wrong sign: configurations with larger stochastic noise contribute more to the classical expectation values of operators. Also, the symmetric correlation function $G^S_{J^t J^t}(\omega, k)$ is nothing but the variance of a random stochastic variable J^t and thus is not allowed to be negative for any ω, k_i. Ignoring this problem for the moment, we will be led to the effective action for BDNK-diffusion

$$S = \int \mathrm{d}^{d+1}x \left[B_{a\mu} n u_0^\mu + iT_0\left(-\lambda\, u_0^\mu u_0^\nu + \sigma\, \Delta^{\mu\nu}\right) B_{a\mu}\left(B_{a\nu} + i\beta_0^\lambda \partial_\lambda B_{r\nu}\right)\right]. \tag{A.11}$$

JHEP01(2024)162

In fact, we already derived this action using the Schwinger-Keldysh formalism in appendix 2.4. It follows from the general Schwinger-Keldysh effective action in eq. (2.49) upon dropping all dependence on the gapped Maxwell-Cattaneo fields and skipping imposing any hydrodynamic frame requirement. However, the third SK-condition in eq. (2.44) is violated for positive λ, causing higher-frequency configurations of the noise field φ_a to contribute arbitrarily more to the path integral. In fact, in the MSR formalism, we needed to perform a Gaussian integral over noise fields $\theta_{\mathcal{N}}$, $\theta^{\mu}_{\mathcal{J}}$ to derive this effective action, similar to our discussion preceding eq. (2.30), which will be ill-defined with the wrong sign of $\theta^2_{\mathcal{N}}$ term. These problems can be avoided by taking λ to be negative, however this will cause the mode spectrum to be unstable and compromise the entire purpose of introducing gapped modes in the first place.

It is worth pointing out that the wrong sign of λ in the path integral smells similar to the apparent reduction of entropy in eq. (A.4) for positive λ. However, there, we could brush the problem under the rug by noting that we can always find some higher-derivative corrections so that entropy production remains positive semi-definite while leaving λ unconstrained; see the discussion in [74, 75]. It is, in principle, possible that a higher-derivative generalisation of BDNK-diffusion will circumvent the problems with the associated Schwinger-Keldysh path integral as well. However, in its current form, BDNK-diffusion model cannot be used to reliably compute UV-regulated Green's functions and stochastic fluctuations.

The arguments presented above also qualitatively apply to the full theory of relativistic BDNK-hydrodynamics [38–41]. We start by untying the physical energy current $-T^{\mu\nu}u_\nu$ and charge density $-J^\mu u_\mu$ from the respective thermodynamic quantities ϵu^μ and n, allowing for arbitrary derivative corrections $\mathcal{E}u^\mu + \mathcal{Q}^\mu$ and $\mathcal{N}$ relating the two. These corrections generically admit λ-like transport coefficients described above, which are required to be positive by the stability and causality and negative by the consistency of the EFT formalism, thus leading to a contradiction.

From MC-diffusion to BDNK-diffusion. Despite our comments above, there is still something to be understood. Note that, taking τ and α_υ to be constant, we can take a gradient of the constitutive relations of MC-diffusion in eq. (2.18) and precisely land on the equations of BDNK-diffusion

$$\partial_\mu\Big((n + \lambda\, u_0^\nu \partial_\nu \mu)\, u_0^\mu - \sigma\, \Delta^{\mu\nu}\Big(\partial_\nu \mu - F_{\nu\rho} u_0^\rho\Big)\Big) = 0, \tag{A.12}$$

where we have identified $\lambda = \chi\tau$. So, at least for constant τ and α_υ, the classical equations of motion of the two theories are precisely the same. The difference between the two is really the coupling to background fields: e.g. while $A_\mu u_0^\mu$ couples to n in MC-diffusion, it couples to $n - \lambda\partial_t\mu$ in BDNK-diffusion. Given so, it should be possible to derive a SK-EFT for BDNK-diffusion for constant τ by using its MC counterpart. To this end, let us start with the effective action for MC-diffusion in the form (2.30) and assume that all coefficients appearing therein are just functions of n and not υ^2. If so, υ_μ becomes a Lagrange multiplier to set

$$V_{a\mu} - \frac{1}{\alpha_\upsilon} u_0^\lambda \partial_\lambda(\alpha_\upsilon \tau V_{a\mu}) = B_{a\mu}. \tag{A.13}$$

Provided that we take τ and α_υ to be constants, this can be explicitly solved in terms of non-local relation $V_{a\mu} = (1 - \tau u_0^\lambda \partial_\lambda)^{-1} B_{a\mu}$. Plugging this back in and performing a

JHEP01(2024)162

field redefinition

$$\varphi_a \to \hat{\varphi}_a - \tau u_0^\mu \partial_\mu \hat{\varphi}_a, \tag{A.14}$$

we are led to the effective action

$$S = \int \mathrm{d}^{d+1}x \Big[\hat{B}_{a\mu} n u_0^\mu + T_0 \lambda u_0^\mu u_0^\nu \, \hat{B}_{a\mu} \, \beta_0^\lambda \partial_\lambda B_{r\nu} + i T_0 \sigma \, \Delta^{\mu\nu} \hat{B}_{a\mu} \left(\hat{B}_{a\nu} + i \beta_0^\lambda \partial_\lambda B_{r\nu} \right) \Big], \tag{A.15}$$

where we have defined

$$\hat{B}_{a\mu} = \partial_\mu \hat{\varphi}_a + \frac{1}{(1 - \tau u_0^\lambda \partial_\lambda)} A_{a\mu}. \tag{A.16}$$

It should also be noted that taking $\mu(n)$ to not depend on v^2, the thermodynamic relation (2.14) forces us to also take $\chi_v = \alpha_v^2 \tau / \sigma$ to be constant; see our previous discussion around eq. (2.19). Together with the requirement that τ and α_v are constants, this forces us to take the conductivity σ to also be a constant as well. Even though the two effective actions (A.11) and (A.15) lead to the same classical equations of motion, modulo the constraints on coefficients outlined above, they are quite different from each other. Firstly, the λ term does not appear in the imaginary part of the action at all, so the whole problem concerning the wrong sign of the path integral is avoided. Secondly, the coupling to background fields is different, so the correlation functions we obtain from this effective action are those given in eqs. (2.20) and (2.31) and not the ones from eqs. (A.6) and (A.9). The downside, however, is that there is no simple realisation of the dynamical KMS symmetry in the new effective action (A.15), even though it was derived from a KMS-invariant effective action (2.30). We refer to this new EFT as BDNK*-diffusion to distinguish it from the original (and inconsistent) EFT of BDNK-diffusion in (A.11). The effective action (A.15) of BDNK*-diffusion becomes simpler if we choose the local rest frame $u_0^\mu = \delta_t^\mu$ and only turn on the time-components of the background fields $A_{r,at}$. We get

$$S = \int \mathrm{d}^{d+1}x \Big[\partial_t \hat{\varphi}_a (n + \tau \partial_t n) - D \, \partial_i \hat{\varphi}_a \partial^i n + i T_0 \sigma \, \partial_i \hat{\varphi}_a \partial^i \hat{\varphi}_a + A_{at} n + \sigma \, \partial_i \hat{\varphi}_a \partial^i A_{rt} \Big], \tag{A.17}$$

where σ and τ are constants, while D can be an arbitrary function of n.

The EFT for BDNK*-diffusion in eq. (A.17) was recently analysed in the work of [47], however the authors allowed the conductivity σ to not be constant and depend arbitrarily on n. As we discussed above, σ and τ need to be constant for this effective action to be derived from the KMS-invariant effective action in eq. (2.30). Without this constraint, the tree-level 3- and higher-point functions derived using the EFT in eq. (A.17) will not satisfy FDTs; we look at these in detail in section B. This further implies that stochastic loop corrections to 2-point correlation functions computed using eq. (A.17) with non-constant σ and τ will also violate the FDT in eq. (1.1), however it would be interesting to verify this explicitly. In the context of [47], the coefficients λ_σ and λ'_σ in their work must be switched off for the results to be compatible with FDTs at full non-linear level. We note that the authors in [47] only computed the loop-corrected symmetric 2-point correlation function of density using the EFT in eq. (A.17) and assumed the 2-point FDT to define the associated retarded correlation function. Hence, the constraints arising from non-linear FDTs was invisible in

JHEP01(2024)162

their computation. If we wish to probe the effects of non-constant σ and τ in the EFT, we will need to invoke the effective action for Maxwell-Cattaneo model of diffusion given in eq. (2.30) with relaxed vector degrees of freedom.

Our comments above are not intended as a criticism of the analysis of [47], but rather as a cautionary note when using the MSR formalism to study statistical interactions in stochastic thermal systems. Ordinarily in the MSR formalism, one only imposes the 2-point FDT at tree-level to fix the noise term in the action. In ordinary theories of diffusion and hydrodynamics (without UV-regularisation), this luckily also ensures conformity with higher-point FDTs at tree-level and one can faithfully use the resulting EFT to compute statistical loop corrections. However, this does not work as neatly with the EFT employed in [47]. Although this EFT respects tree-level 2-point FDT, one finds that higher-point FDTs are violated by the correlation functions. Note that, if we had not used the correct thermodynamic frame for MIS-hydrodynamics in section 3, a similar problem would have occurred for us while constructing the respective EFT using the MSR formalism; see the comments at the end of section 3.3.

This construction has no obvious analogue for the full BDNK-hydrodynamics, even for a particular choice of transport coefficients. Because of the spacetime-dependent velocity field u^μ, the classical dynamical equations of MIS-hydrodynamics in eq. (3.16) cannot be recast into a BDNK-like format, as was possible for the simple diffusion model in eq. (A.12). Therefore, the two theories are quite different even at the classical level and there is no simple way of deriving a consistent effective action for BDNK-hydrodynamics, similar to the one in eq. (A.15), utilising its MIS counterpart.

B Linearised effective field theory and correlation functions

Given the Schwinger-Keldysh generating functional $\mathcal{Z}[\phi_r, \phi_a]$, we can compute various time-ordered n-point correlation functions as

$$G_{r\ldots a\ldots} = \left(\frac{-i\delta}{\delta\phi_a}\ldots\right)\left(\frac{\delta}{\delta\phi_r}\ldots\right)\ln\mathcal{Z}[\phi_r,\phi_a]. \tag{B.1}$$

For example, $G^R = G_{ra}$ and $G^S = G_{rr}$ are the retarded and symmetric 2-point correlation functions, while G_{raa}, G_{rra}, and G_{rrr} are the retarded, partially-retarded, and completely symmetric 3-point correlation functions. When computed in a thermal state with global constant temperature T_0 and local rest frame velocity $u_0^\mu = \delta_0^\mu$, the Fourier-space 2- and 3-point functions are required to satisfy the fluctuation-dissipation theorems

$$G_{rr}(p_1,p_2) = \frac{T_0}{i\omega}\Big(G_{ra}(p_1;p_2) - G^*_{ra}(p_1;p_2)\Big), \tag{B.2a}$$

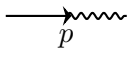
$$G_{rra}(p_1,p_2;p_3) = -\frac{T_0}{i\omega_2}\Big(G_{raa}(p_1;p_2,p_3) - G^*_{raa}(p_3;p_2,p_1)\Big) + (1\leftrightarrow 2), \tag{B.2b}$$

$$G_{rrr}(p_1,p_2,p_3) = -\frac{T_0^2}{\omega_2\omega_3}\Big(G_{raa}(p_1;p_2,p_3) + G^*_{raa}(p_1;p_2,p_3)\Big) + (1\leftrightarrow 2) + (1\leftrightarrow 3), \tag{B.2c}$$

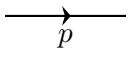
JHEP01(2024)162

where $\sum_n p_n^\mu = 0$, and so on for higher-point functions [10].[11] These theorems are guaranteed by dynamical KMS symmetry of the Schwinger-Keldysh effective field theory.

BDNK*-diffusion. As an illustration, let us consider the effective action for BDNK*-diffusion given in eq. (A.17). Let us also allow the relaxation time τ and conductivity σ in this action to depend on n, to see how this causes violations of higher-point FDTs. The discussion here is a straight-forward generalisation of the work in [8, 64]. Denoting n by solid lines and $\hat{\varphi}_a$ by wavy lines, we find the nonzero bare propagators

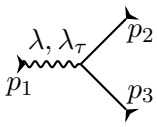


$$= \frac{1}{F(p)}, \qquad$$

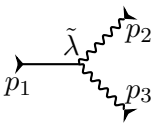


$$= \frac{2T_0\sigma k^2}{|F(p)|^2}, \tag{B.3}$$

where $F(p) = \omega(1 - i\tau\omega) + iD_n k^2$. There are also two 3-point interaction vertices

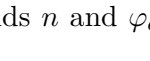


$$= -\frac{i}{2}\left(\lambda k_1^2 - \lambda_\tau \omega_1^2\right), \qquad$$

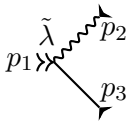


$$= T_0\chi\tilde{\lambda}(k_2 \cdot k_3), \tag{B.4}$$

where $\lambda = \partial D/\partial n$, $\tilde{\lambda} = \chi^{-1}\partial\sigma/\partial n$, and $\lambda_\tau = \partial\tau/\partial n$. We can similarly work out higher-point interaction vertices as needed. Finally, since the background field A_{rt} couples non-trivially to the dynamical fields n and φ_a, we have background coupling vertices



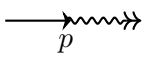
$$= i\sigma k^2, \qquad$$



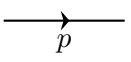
$$= -i\chi\tilde{\lambda}(k_1 \cdot k_2), \tag{B.5}$$

where we have denoted an A_{at} insertion with double arrow. We have not included a diagrammatic notation for A_{at} because it couples trivially to n.[12] Blending these ingredients together, the two-point symmetric and retarded functions at tree-level are given as

$G_{ra}(p) =$



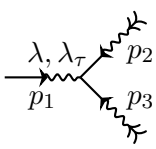
$$= \frac{i\sigma k^2}{F(p)}, \qquad G_{rr}(p) =$$



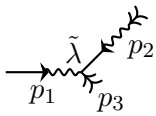
$$= \frac{2T_0\sigma k^2}{|F(p)|^2}, \tag{B.6}$$

It is easy to check that they satisfy the fluctuation-dissipation theorem in eq. (B.2). We can also compute the tree-level 3-point correlation functions: retarded

$G_{raa}(p_1, p_2, p_3) =$



$+ \Bigg($



$+ (2 \leftrightarrow 3)\Bigg)$

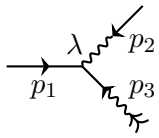
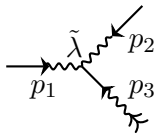
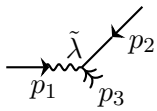
$$= \frac{i\sigma^2 k_2^2 k_3^2\left(\lambda k_1^2 - \lambda_\tau\omega_1^2\right)}{F(p_1)F(p_2)^* F(p_3)^*} - \chi\sigma\tilde{\lambda}\left(\frac{k_2^2(k_1 \cdot k_3)}{F(p_1)F(p_2)^*} + (2 \leftrightarrow 3)\right), \tag{B.7a}$$

[11] The convention for correlation functions in [10] is related to ours as $G^{\rm WH}_{r\ldots a\ldots} = i/2(-1)^{n_a}(-2i)^{n_r} G_{r\ldots a\ldots}$, where n_r and n_a are numbers of "r" and "a" type fields in the correlation function.

[12] The choice of variables here is slightly different from our previous work [8, 64]: here we are working with fluctuations in density $n(\mu) - n_0$, whereas in [8, 64] we worked with $\delta n = n(\mu - A_{rt}) - n_0$, resulting in different background coupling vertices. One consequence of this being that there are non-trivial interactions vertices among A_{at} and A_{rt} in [8, 64]. Since this is just a choice of variables, the final answers for the correlation functions remain unaffected.

JHEP01(2024)162

partially-retarded

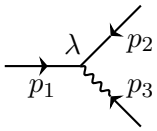
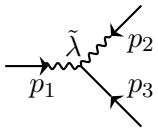
$$G_{rra}(p_1,p_2,p_3) = \Bigg($$  $$+ (1 \leftrightarrow 2)\Bigg) + $$  $$+ \Bigg($$  $$+ (1 \leftrightarrow 2)\Bigg)$$

$$= -2T_0\sigma^2 \left(\frac{k_1^2 k_3^2 \left(\lambda k_2^2 - \lambda_\tau \omega_2^2\right)}{|F(p_1)|^2 F(p_2) F(p_3)^*} + (1 \leftrightarrow 2) \right)$$

$$- 2iT_0\chi\sigma\tilde{\lambda} \left(\frac{k_3^2 (k_1 \cdot k_2)}{F(p_1)F(p_2)F(p_3)^*} + \left(\frac{k_2^2 (k_1 \cdot k_3)}{F(p_1)|F(p_2)|^2} + (1 \leftrightarrow 2) \right) \right), \quad \text{(B.7b)}$$

and symmetric

$$G_{rrr}(p_1,p_2,p_3)$$

$$= \Bigg($$  $$+ (1 \leftrightarrow 3) + (2 \leftrightarrow 3)\Bigg) + \Bigg($$  $$+ (1 \leftrightarrow 3) + (2 \leftrightarrow 3)\Bigg)$$

$$= -4iT_0^2\sigma^2 \left(\frac{k_1^2 k_2^2 \left(\lambda k_3^2 - \lambda_\tau \omega_3^2\right)}{|F(p_1)|^2 |F(p_2)|^2 F(p_3)} + (1 \leftrightarrow 3) + (2 \leftrightarrow 3) \right)$$

$$+ 4T_0^2\chi\sigma\tilde{\lambda} \left(\frac{k_3^2 (k_1 \cdot k_2)}{F(p_1)F(p_2)|F(p_3)|^2} + (1 \leftrightarrow 3) + (2 \leftrightarrow 3) \right). \quad \text{(B.7c)}$$

We find that these correlation functions do not satisfy the fluctuation-dissipation theorems in eq. (B.2) for non-constant σ and τ, i.e. nonzero $\tilde{\lambda}$ and λ_τ. The G_{rra} and G_{rrr} fluctuation-dissipation theorems have residual contributions

$$\left(\frac{\lambda_\tau}{\omega_1\omega_2\omega_3} \left(\omega_1^3 k_2^2 k_3^2 + \omega_2^3 k_1^2 k_3^2 + \omega_3^3 k_1^2 k_2^2 \right) - \frac{\tau\chi\tilde{\lambda}}{\sigma} \left(k_1^2 (k_2 \cdot k_3) + k_2^2 (k_1 \cdot k_3) + k_3^2 (k_1 \cdot k_2) \right) \right)$$

$$\times \frac{T_0 \sigma^2 \omega_3}{F(p_1)F(p_2)F(p_3)^*}, \quad \text{(B.8a)}$$

and

$$\left(\frac{\lambda_\tau}{\omega_1\omega_2\omega_3} \left(\omega_1^3 k_2^2 k_3^2 + \omega_3^3 k_1^2 k_2^2 + \omega_2^3 k_1^2 k_3^2 \right) - \frac{\tau\chi\tilde{\lambda}}{\sigma} \left(k_1^2 (k_2 \cdot k_3) + k_2^2 (k_1 \cdot k_3) + k_3^2 (k_1 \cdot k_2) \right) \right) \times$$

$$\times 2iT_0^2\sigma^2 \left(\frac{-1}{F(p_1)F(p_2)F(p_3)} + \left(\frac{\omega_1}{|F(p_1)|^2 F(p_2) F(p_3)} + (1 \leftrightarrow 2) + (1 \leftrightarrow 3) \right) \right), \quad \text{(B.8b)}$$

on the right-hand sides respectively. For non-zero relaxation time $\tau \neq 0$, the FDTs are only satisfied $\chi\tilde{\lambda} = \partial\sigma/\partial n = 0$ and $\lambda_\tau = \partial\tau/\partial n = 0$, i.e. σ and τ are taken to be a constants. Note that σ is allowed to be non-constant in the ordinary diffusion model when $\tau = 0$. Since higher-point interaction vertices also contribute to 2-point correlation functions via stochastic interactions, the loop-corrected 2-point correlation functions will also violate the respective FDT in eq. (B.2) for non-constant σ or τ, when $\tau \neq 0$.

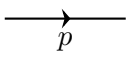
JHEP01(2024)162

MC-diffusion. To include the effects of non-constant σ and τ while respecting FDTs, we will need to invoke the full EFT for MC-diffusion derived in eq. (2.30) with additional vector degrees of freedom. We could, of course, work with the form of the effective action in eq. (2.56) derived using the SK formalism, but the interactions are simpler in the MSR incarnation of the effective action in eq. (2.30). Recall that the two are related via field redefinitions in eq. (2.57). For simplicity, we will assume μ to just depend on n and choose $\alpha_v = 1$, which, owing to our discussion around eq. (2.19), will imply that τ/σ is still a constant. Nonetheless, we only really care about the generality of the "hydrodynamic" transport coefficient σ; the additional relaxed sector included for stability and causality reasons is anyway not universal, so we might as well work with the simplest such extension. For completeness, let us rewrite the effective action (2.19) in the local rest frame $u_0^\mu = \delta_t^\mu$ and with only the time-components of the background gauge fields $A_{r,at}$ turned on, leading to

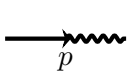
$$S = \int \mathrm{d}^{d+1}x \Big[\partial_t\varphi_a n + \partial_i\varphi_a v^i - V_a^i(v_i + \tau\partial_t v_i) - D_n V_a^i \partial_i n + iT_0\sigma\, V_a^i V_{ai} + A_{at} n + \sigma\, V_a^i \partial_i A_{rt}\Big]. \tag{B.9}$$

As opposed to the simple propagators in eq. (B.3), we have a more intricate set of propagators in this case due the additional vector degrees of freedom. Denoting n and φ_a still by solid and wavy lines, and v^i and V_a^i by the respective thicker versions, we find

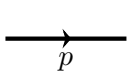
$$= \frac{1 - i\tau\omega}{F(p)},$$



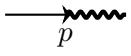
$$= \frac{2T_0\sigma k^2}{|F(p)|^2},$$



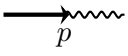
$$= \frac{-i\omega}{F(p)}\frac{k^i k^j}{k^2} - \frac{i}{1 - i\tau\omega}k^{ij},$$



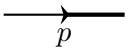
$$= \frac{2T_0\sigma\omega^2}{|F(p)|^2}\frac{k^i k^j}{k^2} + \frac{2T_0\sigma}{|1 - i\tau\omega|^2}k^{ij},$$



$$= \frac{-ik^i}{F(p)},$$

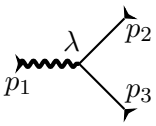


$$= \frac{-iD_n k^i}{F(p)},$$

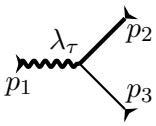


$$= \frac{2T_0\sigma\omega\, k^i}{|F(p)|^2}. \tag{B.10}$$

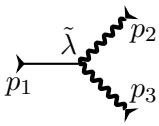
We have three 3-point interaction vertices



$$= \frac{\lambda}{2}k_1^i,$$



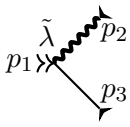
$$= \lambda_\tau\omega_2\delta^{ij},$$



$$= -T_0\chi\tilde{\lambda}\delta^{ij}. \tag{B.11}$$

Note that there are no interactions involving the noise field φ_a. We can similarly work out higher-point interaction vertices as needed. Finally, we have the background coupling vertices

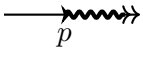
$$= \sigma k^i,$$



$$= \chi\tilde{\lambda}k_1^i, \tag{B.12}$$

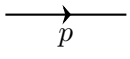
where we have still denoted the insertion of A_{rt} with a double arrow. Using this EFT, the 2-point correlation functions of density at tree-level can be computed simply as

$$G_{ra}(p) =$$



$$= \frac{i\sigma k^2}{F(p)},$$

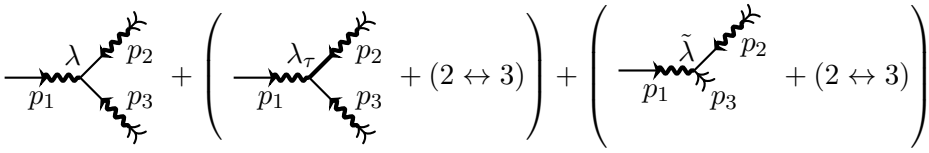
$$G_{rr}(p) =$$



$$= \frac{2T_0\sigma k^2}{|F(p)|^2}, \tag{B.13}$$

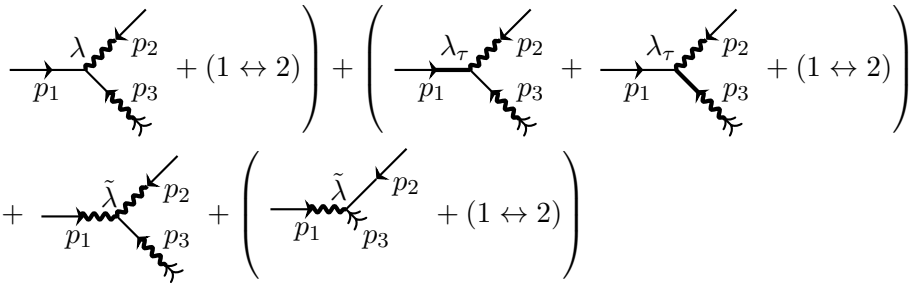
JHEP01(2024)162

and yield the same results as in eq. (B.6), consistent with FDT. The computation for tree-level 3-point correlation functions is a little more involved and we find: retarded

$G_{raa}(p_1,p_2,p_3) =$ 

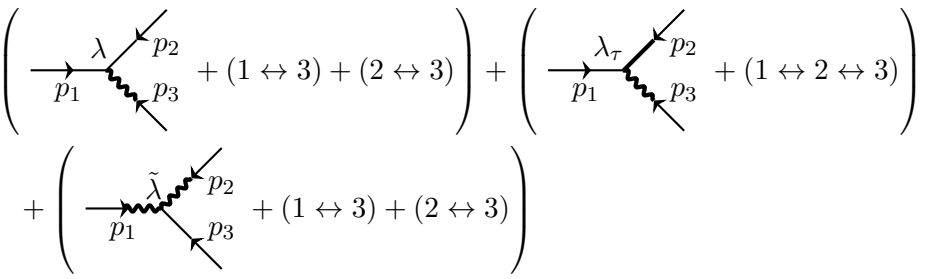
$$
\begin{aligned}
&= \frac{i\sigma^2\lambda k_1^2 k_2^2 k_3^2}{F(p_1)F(p_2)^*F(p_3)^*} + i\sigma^2\lambda_\tau \frac{\omega_2^2 k_3^2 (k_1\cdot k_2) + \omega_3^2 k_2^2 (k_1\cdot k_3)}{F(p_1)F(p_2)^*F(p_3)^*} \\
&\quad - \chi\sigma\tilde{\lambda}\left(\frac{k_2^2(k_1\cdot k_3)}{F(p_1)F(p_2)^*} + (2\leftrightarrow 3)\right),
\end{aligned}
\tag{B.14a}
$$

partially-retarded

$G_{rra}(p_1,p_2,p_3) =$ 

$$
\begin{aligned}
&= -2T_0\sigma^2\lambda\left(\frac{k_1^2k_2^2k_3^2}{|F(p_1)|^2F(p_2)F(p_3)^*} + (1\leftrightarrow 2)\right) \\
&\quad - 2T_0\sigma^2\lambda_\tau\left(\frac{\omega_1^2k_3^2(k_1\cdot k_2) + k_1^2\omega_3^2(k_2\cdot k_3)}{|F(p_1)|^2F(p_2)F(p_3)^*} + (1\leftrightarrow 2)\right) \\
&\quad - 2iT_0\chi\sigma\tilde{\lambda}\left(\frac{k_3^2(k_1\cdot k_2)}{F(p_1)F(p_2)F(p_3)^*} + \left(\frac{k_2^2(k_1\cdot k_3)}{F(p_1)|F(p_2)|^2} + (1\leftrightarrow 2)\right)\right),
\end{aligned}
\tag{B.14b}
$$

and symmetric

$G_{rrr}(p_1,p_2,p_3) =$ 

$$
\begin{aligned}
&= -4iT_0^2\sigma^2\lambda\left(\frac{k_1^2k_2^2k_3^2}{|F(p_1)|^2|F(p_2)|^2F(p_3)} + (1\leftrightarrow 3) + (2\leftrightarrow 3)\right) \\
&\quad - 4iT_0^2\sigma^2\lambda_\tau\left(\frac{\omega_2^2k_1^2(k_2\cdot k_3) + \omega_1^2k_2^2(k_1\cdot k_3)}{|F(p_1)|^2|F(p_2)|^2F(p_3)} + (1\leftrightarrow 3) + (2\leftrightarrow 3)\right) \\
&\quad + 4T_0^2\chi\sigma\tilde{\lambda}\left(\frac{k_3^2(k_1\cdot k_2)}{F(p_1)F(p_2)|F(p_3)|^2} + (1\leftrightarrow 3) + (2\leftrightarrow 3)\right).
\end{aligned}
\tag{B.14c}
$$

JHEP01(2024)162

Plugging these into the respective FDTs in eq. (B.2), we find that the residual terms on the right-hand side are given as

$$\left(\lambda_\tau - \frac{\chi\tau\tilde{\lambda}}{\sigma}\right) T_0\sigma^2\omega_3 \left(\frac{k_1^2(k_2\cdot k_3) + k_2^2(k_1\cdot k_3) + k_3^2(k_1\cdot k_2)}{F(p_1)F(p_2)F(p_3)^*}\right), \tag{B.15a}$$

and

$$2iT_0^2\sigma^2 \left(\lambda_\tau - \frac{\chi\tau\tilde{\lambda}}{\sigma}\right) \left(k_1^2(k_2\cdot k_3) + k_2^2(k_1\cdot k_3) + k_3^2(k_1\cdot k_2)\right)\times \\ \left(\frac{-1}{F(p_1)F(p_2)F(p_3)} + \left(\frac{\omega_1}{|F(p_1)|^2F(p_2)F(p_3)} + (1\leftrightarrow 2) + (1\leftrightarrow 3)\right)\right), \tag{B.15b}$$

respectively. As mentioned before, FDTs are only satisfied when $\lambda_\tau - \chi\tau\tilde{\lambda}/\sigma = \sigma\,\partial(\tau/\sigma)/\partial n = 0$, i.e. τ/σ is taken to be a constant.

We can also use the diagrammatic representation and Feynman rules described above to compute loop corrections to 2-point correlation functions consistent with FDT requirements. We leave this exercise for future work. We also leave the generalisation of the discussion in this appendix to MIS-hydrodynamics for future explorations.

C Alternate KMS prescriptions

Since the additional relaxed degrees of freedom in MC-diffusion and MIS-hydrodynamics are not associated with any symmetries, we can actually construct different models in the SK formalism that ultimately give rise to the same effective action, but with different realisations of the dynamical KMS symmetry. As an example, let us consider the SK model for MC-diffusion from appendix 2.4, but with vector fields $\hat{\upsilon}_{1,2\mu}$ (normalised as $\beta_0^\mu\hat{\upsilon}_{1,2\mu} = 0$) instead of $\upsilon_{1,2\mu}$ that respect a "spatial diagonal shift symmetry", i.e.

$$\hat{\upsilon}_{1,2\mu} \to \hat{\upsilon}_{1,2\mu} + \lambda_\mu, \quad \text{such that} \quad \beta_0^\mu\lambda_\mu = 0,\ \beta_0^\lambda\partial_\lambda\lambda_\mu = 0. \tag{C.1}$$

This means that all dependence on the average combination $\hat{\upsilon}_{r\mu}$ in the EFT must arise via its time-derivative $\upsilon_\mu \equiv u_0^\lambda\partial_\lambda\hat{\upsilon}_{r\mu}$, which we will use as the definition of the classical vector degree of freedom υ_μ is MC-diffusion. We take the C,P,T eigenvalues of the spatial-components of the new fields $\upsilon_{1,2i}$, $\upsilon_{r,ai}$ to be $(-,-,+)$, while those of the time-components can be obtained using the normalisation conditions. With the identification of υ_μ above, these lead to the correct C,P,T eigenvalues of υ_μ as given in table 1. The KMS transformation properties of $\hat{\upsilon}_{1,2\mu}$ and $\hat{\upsilon}_{r,a\mu}$ are given similar to eqs. (2.46) and (2.47), which imply

$$\upsilon_\mu \to \Theta\upsilon_\mu, \qquad \hat{\upsilon}_{a\mu} \to \Theta\left(\hat{\upsilon}_{a\mu} + \frac{i}{T_0}\upsilon_\mu\right). \tag{C.2}$$

JHEP01(2024)162

A simple EFT that realises this and all other requirements in appendix 2.4 can be written down similar to eq. (2.49) as

$$
\begin{aligned}
S = \int \mathrm{d}^{d+1}x \Big[& B_{a\mu} n u_0^\mu - \chi_\upsilon \hat{\upsilon}_a^\mu u_0^\lambda \partial_\lambda \upsilon_\mu \\
& + iT_0 \left(-\hat{\lambda}\, u_0^\mu u_0^\nu + \hat{\sigma}\, \Delta^{\mu\nu} \right) B_{a\mu} \left(B_{a\nu} + i\beta_0^\lambda \partial_\lambda B_{r\nu} \right) \\
& + iT_0 \hat{\sigma}_\upsilon\, \Delta^{\mu\nu} \hat{\upsilon}_{a\mu} \left(\hat{\upsilon}_{a\nu} + i\beta_0 \upsilon_\nu \right) \\
& + iT_0 \hat{\gamma}_\times\, \Delta^{\mu\nu} \Big(B_{a\mu} \left(\hat{\upsilon}_{a\nu} + i\beta_0 \upsilon_\nu \right) + \hat{\upsilon}_{a\mu} \left(B_{a\nu} + i\beta_0^\lambda \partial_\lambda B_{r\nu} \right) \Big) \\
& + iT_0 \hat{\tilde{\gamma}}_\times\, \Delta^{\mu\nu} \Big(B_{a\mu} \left(\hat{\upsilon}_{a\nu} + i\beta_0 \upsilon_\nu \right) - \hat{\upsilon}_{a\mu} \left(B_{a\nu} + i\beta_0^\lambda \partial_\lambda B_{r\nu} \right) \Big) \Big],
\end{aligned}
\tag{C.3}
$$

where all coefficients are functions of $\mu \equiv u_0^\mu B_{r\mu}$ and υ^2, and satisfy the inequality constraints similar to eq. (2.51). The first two terms satisfy the KMS condition up to a total-derivative term similar to eq. (2.52). The remaining terms are manifestly KMS-invariant. However, since $\hat{\upsilon}_{r,a\mu}$ and $B_{r,a\mu}$ have opposite eigenvalues under T this time around, the $\hat{\tilde{\gamma}}_\times$ term is disallowed by $\Theta = \mathrm{T}, \mathrm{PT}$. Whereas for $\Theta = \mathrm{CT}, \mathrm{CPT}$, all terms are allowed, provided that $\hat{\tilde{\gamma}}_\times$ and n is odd functions of μ, while all other coefficients are even functions of μ. Using this EFT, one can work out the vector equation of motion

$$
\chi_\upsilon \Delta^{\mu\nu} u_0^\lambda \partial_\lambda \upsilon_\nu = -(\hat{\gamma}_\times - \hat{\tilde{\gamma}}_\times) \Delta^{\mu\nu} (\partial_\nu \mu - F_{r\nu\rho} u_0^\rho) - \hat{\sigma}_\upsilon \upsilon^\mu,
\tag{C.4}
$$

and the current

$$
J_r^\mu = \left(n + \hat{\lambda}\, u_0^\lambda \partial_\lambda \mu \right) u_0^\mu - \hat{\sigma}\, \Delta^{\mu\nu} \left(\partial_\nu \mu - F_{r\nu\rho} u_0^\rho \right) - (\hat{\gamma}_\times + \hat{\tilde{\gamma}}_\times) \upsilon^\mu,
\tag{C.5}
$$

analogous to eqs. (2.53) and (2.54). Comparing these to the frame conditions in eqs. (2.2) and (2.15), we are led to set

$$
\hat{\lambda} = 0, \qquad \hat{\sigma} = 0, \qquad \hat{\sigma}_\upsilon = \frac{\alpha_\upsilon^2}{\sigma}, \qquad \hat{\gamma}_\times = 0, \qquad \hat{\tilde{\gamma}}_\times = -\alpha_\upsilon,
\tag{C.6}
$$

instead of the analogous expressions (2.55) in the original SK model. The effective action, in this case, simplifies to

$$
\begin{aligned}
S = \int \mathrm{d}^{d+1}x \Big[& B_{a\mu} n u_0^\mu - \chi_\upsilon \hat{\upsilon}_a^\mu u_0^\lambda \partial_\lambda \upsilon_\mu + \alpha_\upsilon\, \Delta^{\mu\nu} \Big(B_{a\mu} \upsilon_\nu - \hat{\upsilon}_{a\mu} u_0^\lambda \partial_\lambda B_{r\nu} \Big) \\
& + \frac{iT_0 \alpha_\upsilon^2}{\sigma} \Delta^{\mu\nu} \hat{\upsilon}_{a\mu} \left(\hat{\upsilon}_{a\nu} + i\beta_0 \upsilon_\nu \right) \Big].
\end{aligned}
\tag{C.7}
$$

We again recover the effective action (2.30) we derived using the MSR formalism, with the KMS condition (2.32) already imposed, but with the field identifications

$$
\upsilon_\mu = u_0^\lambda \partial_\lambda \hat{\upsilon}_{r\mu}, \qquad V_{a\mu} = \frac{\alpha_\upsilon}{\sigma} \hat{\upsilon}_{a\mu}.
\tag{C.8}
$$

Note that υ_μ and $V_{a\mu}$ have opposite Θ-eigenvalues under this prescription, precisely leading to the alternate prescription of KMS symmetry given in eq. (2.35).

JHEP01(2024)162

In the full theory of MIS-hydrodynamics, we can actually write down three distinct SK models with different realisations of the dynamical KMS symmetry, depending on if we allow a spatial diagonal shift symmetry in the tensor degrees of freedom, in vector degrees of freedom, or both, taking the form

$$\begin{aligned} \hat{\kappa}_{\mu\nu} &\to \hat{\kappa}_{\mu\nu} + \lambda_{\mu\nu}, \qquad \text{such that} \qquad \beta^\mu \lambda_{\mu\nu} = 0, \qquad \lambda_{\mu\nu} = \lambda_{\nu\mu}, \quad \pounds_\beta \lambda_{\mu\nu} = 0, \\ \hat{\upsilon}_\mu &\to \hat{\upsilon}_\mu + \lambda_\mu, \qquad \text{such that} \qquad \beta^\mu \lambda_\mu = 0, \qquad \pounds_\beta \lambda_\mu = 0. \end{aligned} \tag{C.9}$$

Following through the procedure outlined above, we find the same effective action as the MSR formalism in eq. (3.25), with the KMS conditions (3.26) already implemented, together with the field identifications of the tensor degrees of freedom

$$\kappa_{\mu\nu} = T \pounds_\beta \hat{\kappa}_{\mu\nu}, \qquad W_{a\mu\nu} = \left(\frac{1}{d} \frac{\alpha_\kappa^{\mathsf{S}}}{\zeta} \Delta_{\mu\nu} \Delta^{\rho\sigma} + \frac{\alpha_\kappa^{\mathsf{T}}}{\eta} \Delta^\rho_{\langle\mu} \Delta^\rho_{\nu\rangle} \right) \hat{\kappa}_{a\rho\sigma}, \tag{C.10}$$

and/or the vector degrees of freedom

$$\upsilon_\mu = T \pounds_\beta \hat{\upsilon}_\mu, \qquad V_{a\mu} = \frac{\alpha_\upsilon}{\sigma} \hat{\upsilon}_{a\mu}. \tag{C.11}$$

As expected, the physical and auxiliary fields have opposite Θ-eigenvalues under these identifications. These precisely lead to the alternate prescriptions of KMS transformations mentioned in eq. (3.28).

References

[1] L.P. Kadanoff and P.C. Martin, *Hydrodynamic equations and correlation functions*, *Annals Phys.* **24** (1963) 419 [INSPIRE].

[2] P.C. Martin, E.D. Siggia and H.A. Rose, *Statistical Dynamics of Classical Systems*, *Phys. Rev. A* **8** (1973) 423 [INSPIRE].

[3] Y. Pomeau and P. Resibois, *Time Dependent Correlation Functions and Mode-Mode Coupling Theories*, SACLAY-DPH-T-74-87, *Phys. Rept.* **19** (1974) 63 [INSPIRE].

[4] P.C. Hohenberg and B.I. Halperin, *Theory of Dynamic Critical Phenomena*, *Rev. Mod. Phys.* **49** (1977) 435 [INSPIRE].

[5] C. De Dominicis and L. Peliti, *Field Theory Renormalization and Critical Dynamics Above t(c): Helium, Antiferromagnets and Liquid Gas Systems*, *Phys. Rev. B* **18** (1978) 353 [INSPIRE].

[6] I. Khalatnikov, V. Lebedev and A. Sukhorukov, *Diagram technique for calculating long wave fluctuation effects*, *Phys. Lett. A* **94** (1983) 271 [INSPIRE].

[7] D. Forster, D.R. Nelson and M.J. Stephen, *Long-Time Tails and the Large-Eddy Behavior of a Randomly Stirred Fluid*, *Phys. Rev. Lett.* **36** (1976) 867 [INSPIRE].

[8] A. Jain and P. Kovtun, *Late Time Correlations in Hydrodynamics: Beyond Constitutive Relations*, *Phys. Rev. Lett.* **128** (2022) 071601 [arXiv:2009.01356] [INSPIRE].

JHEP01(2024)162

[9] H.B. Callen and T.A. Welton, *Irreversibility and generalized noise*, *Phys. Rev.* **83** (1951) 34 [INSPIRE].

[10] E. Wang and U.W. Heinz, *A generalized fluctuation dissipation theorem for nonlinear response functions*, *Phys. Rev. D* **66** (2002) 025008 [hep-th/9809016] [INSPIRE].

[11] S. Grozdanov and J. Polonyi, *Viscosity and dissipative hydrodynamics from effective field theory*, *Phys. Rev. D* **91** (2015) 105031 [arXiv:1305.3670] [INSPIRE].

[12] M. Harder, P. Kovtun and A. Ritz, *On thermal fluctuations and the generating functional in relativistic hydrodynamics*, *JHEP* **07** (2015) 025 [arXiv:1502.03076] [INSPIRE].

[13] M. Crossley, P. Glorioso and H. Liu, *Effective field theory of dissipative fluids*, *JHEP* **09** (2017) 095 [arXiv:1511.03646] [INSPIRE].

[14] F.M. Haehl, R. Loganayagam and M. Rangamani, *Topological sigma models & dissipative hydrodynamics*, *JHEP* **04** (2016) 039 [arXiv:1511.07809] [INSPIRE].

[15] F.M. Haehl, R. Loganayagam and M. Rangamani, *Effective Action for Relativistic Hydrodynamics: Fluctuations, Dissipation, and Entropy Inflow*, *JHEP* **10** (2018) 194 [arXiv:1803.11155] [INSPIRE].

[16] K. Jensen, N. Pinzani-Fokeeva and A. Yarom, *Dissipative hydrodynamics in superspace*, *JHEP* **09** (2018) 127 [arXiv:1701.07436] [INSPIRE].

[17] H. Liu and P. Glorioso, *Lectures on non-equilibrium effective field theories and fluctuating hydrodynamics*, *PoS* **TASI2017** (2018) 008 [arXiv:1805.09331] [INSPIRE].

[18] A. Jain, *Effective field theory for non-relativistic hydrodynamics*, *JHEP* **10** (2020) 208 [arXiv:2008.03994] [INSPIRE].

[19] J. Armas and A. Jain, *Effective field theory for hydrodynamics without boosts*, *SciPost Phys.* **11** (2021) 054 [arXiv:2010.15782] [INSPIRE].

[20] P. Romatschke and U. Romatschke, *Relativistic Fluid Dynamics In and Out of Equilibrium*, Cambridge University Press (2019) [DOI:10.1017/9781108651998] [INSPIRE].

[21] C. Gale, S. Jeon and B. Schenke, *Hydrodynamic Modeling of Heavy-Ion Collisions*, *Int. J. Mod. Phys. A* **28** (2013) 1340011 [arXiv:1301.5893] [INSPIRE].

[22] K. Murase and T. Hirano, *Relativistic fluctuating hydrodynamics with memory functions and colored noises*, arXiv:1304.3243 [INSPIRE].

[23] C. Young et al., *Thermally Fluctuating Second-Order Viscous Hydrodynamics and Heavy-Ion Collisions*, *Phys. Rev. C* **91** (2015) 044901 [arXiv:1407.1077] [INSPIRE].

[24] D.T. Son and M.A. Stephanov, *Dynamic universality class of the QCD critical point*, *Phys. Rev. D* **70** (2004) 056001 [hep-ph/0401052] [INSPIRE].

[25] X. An, G. Başar, M. Stephanov and H.-U. Yee, *Fluctuation dynamics in a relativistic fluid with a critical point*, *Phys. Rev. C* **102** (2020) 034901 [arXiv:1912.13456] [INSPIRE].

[26] M. Martinez, T. Schäfer and V. Skokov, *Critical behavior of the bulk viscosity in QCD*, *Phys. Rev. D* **100** (2019) 074017 [arXiv:1906.11306] [INSPIRE].

[27] W.A. Hiscock and L. Lindblom, *Generic instabilities in first-order dissipative relativistic fluid theories*, *Phys. Rev. D* **31** (1985) 725 [INSPIRE].

[28] W.A. Hiscock and L. Lindblom, *Linear plane waves in dissipative relativistic fluids*, *Phys. Rev. D* **35** (1987) 3723 [INSPIRE].

[29] R.P. Geroch, *Relativistic theories of dissipative fluids*, *J. Math. Phys.* **36** (1995) 4226 [INSPIRE].

JHEP01(2024)162

[30] I. Muller, *Zum Paradoxon der Warmeleitungstheorie*, *Z. Phys.* **198** (1967) 329 [INSPIRE].

[31] W. Israel, *Nonstationary irreversible thermodynamics: A causal relativistic theory*, *Annals Phys.* **100** (1976) 310 [INSPIRE].

[32] W. Israel and J.M. Stewart, *Thermodynamics of nonstationary and transient effects in a relativistic gas*, *Phys. Lett. A* **58** (1976) 213 [INSPIRE].

[33] W. Israel and J.M. Stewart, *Transient relativistic thermodynamics and kinetic theory*, *Annals Phys.* **118** (1979) 341 [INSPIRE].

[34] W. Israel and J.M. Stewart, *On transient relativistic thermodynamics and kinetic theory. II*, *Proc. Roy. Soc. London A* **365** (1979) 43.

[35] W.A. Hiscock and L. Lindblom, *Stability and causality in dissipative relativistic fluids*, *Annals Phys.* **151** (1983) 466 [INSPIRE].

[36] R. Baier et al., *Relativistic viscous hydrodynamics, conformal invariance, and holography*, *JHEP* **04** (2008) 100 [arXiv:0712.2451] [INSPIRE].

[37] G.S. Denicol, H. Niemi, E. Molnar and D.H. Rischke, *Derivation of transient relativistic fluid dynamics from the Boltzmann equation*, *Phys. Rev. D* **85** (2012) 114047 [*Erratum ibid.* **91** (2015) 039902] [arXiv:1202.4551] [INSPIRE].

[38] F.S. Bemfica, M.M. Disconzi and J. Noronha, *Causality and existence of solutions of relativistic viscous fluid dynamics with gravity*, *Phys. Rev. D* **98** (2018) 104064 [arXiv:1708.06255] [INSPIRE].

[39] P. Kovtun, *First-order relativistic hydrodynamics is stable*, *JHEP* **10** (2019) 034 [arXiv:1907.08191] [INSPIRE].

[40] F.S. Bemfica et al., *Nonlinear Causality of General First-Order Relativistic Viscous Hydrodynamics*, *Phys. Rev. D* **100** (2019) 104020 [*Erratum ibid.* **105** (2022) 069902] [arXiv:1907.12695] [INSPIRE].

[41] R.E. Hoult and P. Kovtun, *Stable and causal relativistic Navier-Stokes equations*, *JHEP* **06** (2020) 067 [arXiv:2004.04102] [INSPIRE].

[42] I.S. Liu, I. Müller and T. Ruggeri, *Relativistic thermodynamics of gases*, *Ann. Phys.* **169** (1986) 191.

[43] R.P. Geroch and L. Lindblom, *Dissipative relativistic fluid theories of divergence type*, *Phys. Rev. D* **41** (1990) 1855 [INSPIRE].

[44] P. Gao, P. Glorioso and H. Liu, *Ghostbusters: Unitarity and Causality of Non-equilibrium Effective Field Theories*, *JHEP* **03** (2020) 040 [arXiv:1803.10778] [INSPIRE].

[45] P. Gao and H. Liu, *Emergent Supersymmetry in Local Equilibrium Systems*, *JHEP* **01** (2018) 040 [arXiv:1701.07445] [INSPIRE].

[46] D.D. Joseph and L. Preziosi, *Heat waves*, *Rev. Mod. Phys.* **61** (1989) 41 [INSPIRE].

[47] N. Abbasi, M. Kaminski and O. Tavakol, *Ultraviolet-regulated theory of non-linear diffusion*, arXiv:2212.11499 [INSPIRE].

[48] N. Mullins, M. Hippert, L. Gavassino and J. Noronha, *Relativistic hydrodynamic fluctuations from an effective action: Causality, stability, and the information current*, *Phys. Rev. D* **108** (2023) 116019 [arXiv:2309.00512] [INSPIRE].

[49] P. Kostadt and M. Liu, *Causality and stability of the relativistic diffusion equation*, *Phys. Rev. D* **62** (2000) 023003 [cond-mat/0010276] [INSPIRE].

JHEP01(2024)162

[50] D. Jou and J. Casas-Vazquez, *Extended irreversible thermodynamics of heat conduction*, *Eur. J. Phys.* **9** (1988) 329.

[51] P. Kovtun, G.D. Moore and P. Romatschke, *Towards an effective action for relativistic dissipative hydrodynamics*, *JHEP* **07** (2014) 123 [arXiv:1405.3967] [INSPIRE].

[52] M.L. Bellac, *Thermal Field Theory*, Cambridge University Press (2011) [DOI:10.1017/CBO9780511721700] [INSPIRE].

[53] L. Landau and E. Lifshitz, *Fluid Mechanics*, Teoreticheskaia fizika, Pergamon Press (1959).

[54] P. Kovtun, *Lectures on hydrodynamic fluctuations in relativistic theories*, *J. Phys. A* **45** (2012) 473001 [arXiv:1205.5040] [INSPIRE].

[55] M. Fukuma and Y. Sakatani, *Relativistic viscoelastic fluid mechanics*, *Phys. Rev. E* **84** (2011) 026316 [arXiv:1104.1416] [INSPIRE].

[56] J. Armas, E. van Heumen, A. Jain and R. Lier, *Hydrodynamics of plastic deformations in electronic crystals*, *Phys. Rev. B* **107** (2023) 155108 [arXiv:2211.02117] [INSPIRE].

[57] P. Kovtun, *Temperature in relativistic fluids*, *Phys. Rev. D* **107** (2023) 086012 [arXiv:2210.15605] [INSPIRE].

[58] J. Armas, A. Jain and R. Lier, *Approximate symmetries, pseudo-Goldstones, and the second law of thermodynamics*, *Phys. Rev. D* **108** (2023) 086011 [arXiv:2112.14373] [INSPIRE].

[59] J. Armas and A. Jain, *Approximate higher-form symmetries, topological defects, and dynamical phase transitions*, arXiv:2301.09628 [INSPIRE].

[60] F.S. Bemfica et al., *Nonlinear Constraints on Relativistic Fluids Far from Equilibrium*, *Phys. Rev. Lett.* **126** (2021) 222301 [arXiv:2005.11632] [INSPIRE].

[61] L. Gavassino, M.M. Disconzi and J. Noronha, *Universality Classes of Relativistic Fluid Dynamics I: Foundations*, arXiv:2302.03478 [INSPIRE].

[62] L. Gavassino, M.M. Disconzi and J. Noronha, *Universality Classes of Relativistic Fluid Dynamics II: Applications*, arXiv:2302.05332 [INSPIRE].

[63] X. Chen-Lin, L.V. Delacrétaz and S.A. Hartnoll, *Theory of diffusive fluctuations*, *Phys. Rev. Lett.* **122** (2019) 091602 [arXiv:1811.12540] [INSPIRE].

[64] A. Jain, P. Kovtun, A. Ritz and A. Shukla, *Hydrodynamic effective field theory and the analyticity of hydrostatic correlators*, *JHEP* **02** (2021) 200 [arXiv:2011.03691] [INSPIRE].

[65] J. Bhattacharya, S. Bhattacharyya, S. Minwalla and A. Yarom, *A theory of first order dissipative superfluid dynamics*, *JHEP* **05** (2014) 147 [arXiv:1105.3733] [INSPIRE].

[66] S. Bhattacharyya, S. Jain, S. Minwalla and T. Sharma, *Constraints on Superfluid Hydrodynamics from Equilibrium Partition Functions*, *JHEP* **01** (2013) 040 [arXiv:1206.6106] [INSPIRE].

[67] S. Grozdanov, D.M. Hofman and N. Iqbal, *Generalized global symmetries and dissipative magnetohydrodynamics*, *Phys. Rev. D* **95** (2017) 096003 [arXiv:1610.07392] [INSPIRE].

[68] J. Hernandez and P. Kovtun, *Relativistic magnetohydrodynamics*, *JHEP* **05** (2017) 001 [arXiv:1703.08757] [INSPIRE].

[69] J. Armas and A. Jain, *Magnetohydrodynamics as superfluidity*, *Phys. Rev. Lett.* **122** (2019) 141603 [arXiv:1808.01939] [INSPIRE].

[70] J. Armas and A. Jain, *One-form superfluids & magnetohydrodynamics*, *JHEP* **01** (2020) 041 [arXiv:1811.04913] [INSPIRE].

JHEP01(2024)162

[71] P. Glorioso and D.T. Son, *Effective field theory of magnetohydrodynamics from generalized global symmetries*, arXiv:1811.04879 [INSPIRE].

[72] J. Armas and F. Camilloni, *A stable and causal model of magnetohydrodynamics*, *JCAP* **10** (2022) 039 [arXiv:2201.06847] [INSPIRE].

[73] N. Mullins, M. Hippert and J. Noronha, *Stochastic fluctuations in relativistic fluids: Causality, stability, and the information current*, *Phys. Rev. D* **108** (2023) 076013 [arXiv:2306.08635] [INSPIRE].

[74] S. Bhattacharyya, *Entropy current and equilibrium partition function in fluid dynamics*, *JHEP* **08** (2014) 165 [arXiv:1312.0220] [INSPIRE].

[75] S. Bhattacharyya, *Entropy Current from Partition Function: One Example*, *JHEP* **07** (2014) 139 [arXiv:1403.7639] [INSPIRE].

JHEP01(2024)162