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Lorentzian Robin Universe

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Abstract: In this paper, we delve into the gravitational path integral of Gauss-Bonnet gravity in four spacetime dimensions, in the mini-superspace approximation. Our primary focus lies in investigating the transition amplitude between distinct boundary configurations. Of particular interest is the case of Robin boundary conditions, known to lead to a stable Universe in Einstein-Hilbert gravity, alongside Neumann boundary conditions. To ensure a consistent variational problem, we supplement the bulk action with suitable surface terms. This study leads us to compute the necessary surface terms required for Gauss-Bonnet gravity with the Robin boundary condition, which wasn't known earlier. Thereafter, we perform an exact computation of the transition amplitude. Through $\hbar \to 0$ analysis, we discover that the Gauss-Bonnet gravity inherently favors the initial configuration, aligning with the Hartle-Hawking no-boundary proposal. Remarkably, as the Universe expands, it undergoes a transition from the Euclidean (imaginary time) to the Lorentzian signature (real time). To further reinforce our findings, we employ a saddle point analysis utilizing the Picard-Lefschetz methods. The saddle point analysis allows us to find the initial configurations which lead to Hartle-Hawking no-boundary Universe that agrees with the exact computations. Our study concludes that for positive Gauss-Bonnet coupling, initial configurations corresponding to the Hartle-Hawking no-boundary Universe gives dominant contribution in the gravitational path-integral.

Keywords: Cosmology of Theories BSM, Models of Quantum Gravity, Nonperturbative Effects, Space-Time Symmetries

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1 Introduction

Transition amplitudes give us information about whether a particular transition is allowed. A change of endpoints (also known as boundary) will affect the transition amplitude. Methods of quantum field theory (QFT) developed over the decades allow us to compute such transition amplitudes systematically, which can be matched with relevant experiments for verification. In fact, several experimental verifications of Standard Model predictions further strengthen our reliability on QFT and methods used to formulate it. Path integral is one way to reliably study QFT and compute transition amplitudes systematically.

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However, to properly define path integral in QFT one needs to address the issue of infinities that arise while computing transition amplitudes. These are taken care of by proper regularization and renormalization.[1] Besides these, one still has to specify an integration contour as the standard Lorentzian path integrals are highly oscillatory and not absolutely convergent. In standard flat spacetime QFT (non-gravitational) this is achieved by Wick rotation: a process of going from real-time to imaginary time. This transforms the original flat spacetime Lorentzian path integral into a convergent Euclidean path integral.

These successes are hard to replicate when gravity is involved, where besides dealing with issues of divergences, renormalizability and gauge-fixing, one also has to address the issue of the contour of integration. For the gravitational system, this need not be standard Wick-rotation! Moreover, Euclidean gravitational path integral suffers from the conformal factor problem (the path integral over the conformal factor is unbounded from below [1]), motivating to study the gravitational path integral directly in the Lorentzian signature. A gravitational path integral with metric as the degree of freedom, on a manifold with boundaries can be written as

$$G = \int_{\mathcal{M}+\partial\mathcal{M}} \mathcal{D}g_{\mu\nu} e^{iS[g_{\mu\nu}]/\hbar}\,, \tag{1.1}$$

where $g_{\mu\nu}$ is the metric and $S[g_{\mu\nu}]$ is the corresponding action. The path integral is defined on manifold $\mathcal{M}$ with boundary $\partial\mathcal{M}$. This abstract mathematical expression represents summation over all possible metrics with specified boundary conditions, where each geometry comes with a corresponding 'weight-factor'.

The gravitational action that we will focus on in this paper is given by following (see [2, 3] for earlier works investigating the path integral of such gravitational theories)

$$S = \frac{1}{16\pi G}\int \mathrm{d}^D x\sqrt{-g}\Big[-2\Lambda + R + \alpha\Big(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2\Big)\Big]\,. \tag{1.2}$$

This is the Gauss-Bonnet gravity action where G is the Newton's gravitational constant, Λ is the cosmological constant term, α is the Gauss-Bonnet (GB) coupling and D is spacetime dimensionality. The mass dimensions of various couplings are: $[G] = M^{2-D}$, $[\Lambda] = M^2$ and $[\alpha] = M^{-2}$.

The action in eq. (1.2) falls in the class of lovelock gravity theories [4–6], and is a sub-class of higher-derivative gravity. In this, the dynamical evolution equation of the metric field remains second order in time always. Interestingly, the GB term also arises in the low-energy effective action of the heterotic string theory with $\alpha > 0$ [7–9]. It should also be emphasized that for the first time, the GB-coupling α has received observational constraints [10]. These constraints arise from the analysis of the gravitational wave (GW) data of the event GW150914 which also offered the first observational confirmation of the area theorem [11].

Our interest in this paper is to study the path integral given in the eq. (1.1) for the gravitational action specified by eq. (1.2), and to analyze the effects of the boundary conditions on the transition amplitude (see [12–14, 25, 26] for the role played by boundary terms). We start by considering a spatially homogeneous and isotropic metric in D spacetime dimensions.

[1] In gauge theories, one also has to do gauge-fixing to prevent over-counting of gauge degree of freedom and also suitably introduce relevant ghosts which ensures the gauge-fixing process is systematically done.

It is the FLRW metric in arbitrary spacetime dimension with dimensionality D. In polar coordinates $\{t_p, r, \theta, \cdots\}$ it is given by

$$\mathrm{d}s^2 = -N_p^2(t_p)\mathrm{d}t_p^2 + a^2(t_p)\left[\frac{\mathrm{d}r^2}{1-kr^2} + r^2\mathrm{d}\Omega_{D-2}^2\right]. \tag{1.3}$$

It has two unknown time-dependent functions: lapse $N_p(t_p)$ and scale-factor $a(t_p)$, $k = (0, \pm 1)$ is the curvature, and $\mathrm{d}\Omega_{D-2}$ is the metric for the unit sphere in $D-2$ spatial dimensions. This is the *mini-superspace* approximation of the metric. In this approximation, though we don't have gravitational waves we still retain diffeomorphism invariance of the time co-ordinate t_p and the dynamical scale-factor $a(t_p)$. This simple setting is enough to highlight the importance of boundary conditions and/or Gauss-Bonnet gravitational terms that might become relevant at the boundary.

The Feynman path integral with reduced degree of freedom becomes

$$G[\mathrm{Bd}_f, \mathrm{Bd}_i] = \int_{\mathrm{Bd}_i}^{\mathrm{Bd}_f} \mathcal{D}N_p \mathcal{D}\pi \mathcal{D}a(t_p)\mathcal{D}p\mathcal{D}\mathcal{C}\mathcal{D}\bar{P}\exp\left[\frac{i}{\hbar}\int_0^1 \mathrm{d}t_p\left(N_p'\pi + a'p + \mathcal{C}'\bar{P} - N_pH\right)\right], \tag{1.4}$$

where beside the integration over the scale-factor $a(t_p)$, lapse $N_p(t_p)$ and the Fermionic ghost $\mathcal{C}$, we also have an integration over their corresponding conjugate momenta given by p, π, and $\bar{P}$ respectively. ($'$) here denotes derivative with respect to t_p, while the time t_p co-ordinate is chosen to range from $0 \le t_p \le 1$. Bd_i and Bd_f refers to the field configuration at initial ($t_p = 0$) and final ($t_p = 1$) boundaries respectively. The Hamiltonian constraint H consists of two parts

$$H = H_{\mathrm{grav}}[a, p] + H_{\mathrm{gh}}[N, \pi, \mathcal{C}, \bar{P}], \tag{1.5}$$

where H_{grav} refers to the Hamiltonian for the gravitational action and the Batalin-Fradkin-Vilkovisky (BFV) [15] ghost Hamiltonian is denoted by H_{gh}.[2] Although the degrees of freedom are quite reduced in mini-superspace approximation, the theory still retains time reparametrization invariance which need to be gauge-fixed (we choose proper-time gauge $N_p' = 0$). For more elaborate discussion on BFV quantization process and ghost see [16–18].

Most of the path integral in eq. (1.4) in the mini-superspace approximation can be performed exactly (the path integral over ghosts ($\mathcal{C}$ and $\bar{P}$) and the conjugate momenta (π and p)) leaving behind the following path integral

$$G[\mathrm{Bd}_f, \mathrm{Bd}_i] = \int_{0^+}^{\infty} \mathrm{d}N_p \int_{\mathrm{Bd}_i}^{\mathrm{Bd}_f} \mathcal{D}a(t_p)\ e^{iS[a,N_p]/\hbar}. \tag{1.6}$$

The path integral $\int \mathcal{D}a(t_p)\ e^{iS[a,N_p]/\hbar}$ gives the transition amplitude for the Universe to evolve from one boundary configuration to another in the proper time N_p, while the lapse integration

[2]The BFV ghost is a generalization of the standard Fadeev-Popov ghost based on BRST symmetry. Usual gauge theories constraint algebra forms a Lie algebra. However, for gravitational systems respecting diffeomorphism invariance the constraint algebra doesn't close, requiring BFV quantization. In mini-superspace approximation, although the algebra trivially closes as there is only one constraint (Hamiltonian H) but still, BFV quantization is preferable.

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implies that one needs to consider paths of every proper duration $0 < N_p < \infty$. This choice leads to causal evolution from one boundary configuration Bd_i to another Bd_f [19].

The purpose of this paper is to study the path integral in eq. (1.6) for the gravitational action given in eq. (1.2) focusing mainly on the case of Robin boundary condition (RBC) imposed at the initial boundary. Imposing RBC at the initial time is like specifying a combination of field and its corresponding conjugate momenta. This is a broad class of RBC. In this paper, we focus on a subclass where RBC is imposed as a linear combination of field and its conjugate momenta. This translates into a one-parameter family of all allowed possible boundary conditions for a given specific value of the combination of field and its conjugate momenta. Such a boundary condition interpolates between Dirichlet BC and Neumann BC. Past studies dealing with Dirichlet BC showed that imposition of DBC at initial times leads to unsuppressed behavior of the gravitational fluctuations in the no-boundary proposal of the Universe [20–24], which, with the DBC, corresponds to defining path-integral starting with zero size (see [34] for review on the no-boundary proposal). This study motivates one to investigate the situation in the case of Neumann and Robin BC at the initial boundary, besides also indicating that Dirichlet BC is perhaps not the right boundary condition for gravity [25, 26].

Neumann boundary condition (NBC) has been studied in [2, 3, 30–32, 35] using Picard-Lefschetz methods, where it was seen that gravitational fluctuations are well-behaved in the no-boundary proposal of the Universe, which for Neumann BC is defined as the path-integral over regular geometries. Exact computations are done for the NBC case for the Gauss-Bonnet gravity further showed some surprises mentioned in [3]. For Einstein-Hilbert gravity, Robin boundary condition has been studied using the Picard-Lefschetz methods [30, 33, 35–37]. These perturbative studies showed that fluctuations are well-behaved for the no-boundary proposal of the Universe.

In this paper, we take a fresh look at the implications of imposing RBC at the initial time and studying the path integral for the Gauss-Bonnet gravity. We study the Gauss-Bonnet gravity path integral using both perturbative and non-perturbative methods to gain a proper understanding of the effects of RBCs on the behavior of the transition amplitude. In the process, we also construct the suitable surface term for the Robin case needed for the Gauss-Bonnet gravity to have a consistent variational problem. We use time-slicing method of evaluating the gravitational path integral to compute the transition amplitude exactly. This is then compared with the results obtained via Picard-Lefschetz (PL) methods to gain a better understanding of the role played by saddle point geometries which could be complex.

PL methodology generalizes the notion of 'Wick-rotation' by adapting it to tackle highly oscillatory integral in a systematic manner.[3] By providing a framework that takes into account contributions from all the possibly relevant complex saddle points of the path integral, this framework uniquely determines contours of integration along which the oscillatory integrals (like the ones appearing in eq. (1.6)) becomes well-behaved. Such contours termed *Lefschetz thimbles* constitute the generalized Wick-rotated contour. In the context of Lorentzian

[3]Some studies involving Wick-rotation in curved spacetime have been initiated in [38–41]. However, more work needs to be done on this.

quantum cosmology, these methods have been extensively used to analyze the nature of path integral [20, 21, 24, 27–29] and the role played by boundaries [2, 3, 30, 31, 33].[4]

PL-methods being inherently based on saddle-point approximation, fall short of analyzing situations dealing with degenerate case i.e. when saddles become degenerate. In such cases, one has to go beyond saddle-point approximation, as the approximation breaks down in such limiting cases. Non-perturbative and exact results whenever available become useful in investigating these situations. This was particularly noticed in the study of gravitational path integral for the NBC case [3], where it was seen that our Universe undergoes a transition from an Euclidean to a Lorentzian signature as the size of the Universe increases. However, saddle-point analysis broke down at the turning point, a clear understanding of which came from the exact results. In the RBC case, it is expected that a similar situation could arise. It is therefore, best to approach the investigations of the gravitational path integral in the RBC case via both perturbative and non-perturbative manner.

The outline of the paper is as follows: section 1 gives introduction and motivation to the problem being studied. Section 2 studies path integral of a particle in potential in one dimension potential with various boundary conditions. This not only helps us understand path integrals with non-trivial boundary conditions but also give rise to relations between them, which becomes useful later in the paper. Section 3 discusses the mini-superspace approximation and writes the gravitational action in the mini-superspace approximation. Section 4 studies the variational problem and computes the surface terms that are needed to have a consistent variational problem. Section 5 studies the transition amplitude for the NBC and RBC case, and compute the exact expression for it by exploiting the results of section 2. Section 6 studies the classical limit ($\hbar \to 0$) of the exact transition amplitude which shows the boundary configuration giving a dominant contribution in the path integral. Section 7 does the Picard-Lefschetz analysis of the contour integral. We conclude the paper in section 8 along with an outlook.

2 One-dimensional quantum mechanics

Before we proceed further with our study of gravitational path integral, we do a quick review of the path integral of one particle system to which our gravitational path integral reduces in the mini-superspace approximation.

We look at the path integral of one-particle system whose initial and final boundary conditions are specified. As the dynamical equation of motion involves two-derivatives of time, it needs only two boundary conditions. In the following, we will consider the cases where the final is always Dirichlet boundary condition and initial could be either Dirichlet or Neumann or Robin boundary condition. We will study a free-particle system and particle living in a linear potential which is of relevance to our studies in quantum cosmology.

[4]Earlier work using complex analysis methods to analyze Euclidean gravitational path integral was done in [42, 43]. Exploration of boundary effects in the Euclidean quantum cosmology was done in *tunnelling* proposal [44–46] and *no-boundary* proposal [42, 43, 47]. Furthermore, the choice of a contour of integration via usage of complex analysis methods was also done in [48–50].

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Our purpose is to compute the following path integral

$$\bar{G}[\mathrm{Bd_f}, \mathrm{Bd_i}] = \int_{\mathrm{Bd_i}}^{\mathrm{Bd_f}} \mathcal{D}q(t)\ e^{iS_{\mathrm{tot}}[q]/\hbar}\,, \tag{2.1}$$

where $S_{\mathrm{tot}}[q]$ is the action for the one particle system

$$S_{\mathrm{tot}}[q] = S[q] + S_{\mathrm{bd}} = \int_0^1 \mathrm{d}t \left[\frac{m}{2}\dot{q}^2 - V(q)\right] + S_{\mathrm{bd}}\,, \tag{2.2}$$

m is a t-independent parameter, 'dot' denotes t-derivative, $V(q)$ is the potential, S_{bd} is the surface term added to have a consistent variational problem.[5] The 'bar' over G in eq. (2.1) is added to prevent confusion later on with gravitational transition amplitude.

The path integral in eq. (2.2) can be computed by time-slicing method (first principles). The only subtlety arises at the end points around which the computation has to be done with care. To achieve this, we approach the problem in the *Hamiltonian* language as it is easier to incorporate the boundary conditions. The classical Hamiltonian for the above system is given by

$$H(p, q) = p\dot{q} - L(q, \dot{q}) = \frac{p^2}{2m} + V(q)\,, \tag{2.3}$$

where $p = m\dot{q}$. The quantum transition amplitude from one state to another in the Hamiltonian picture (which is equivalent to computing the above path integral) is given by,

$$\langle \mathrm{Bd_f}, t = 1 | \mathrm{Bd_i}, t = 0\rangle = \langle \mathrm{Bd_f}| e^{-iH(\hat{p},\hat{q})/\hbar} |\mathrm{Bd_i}\rangle\,. \tag{2.4}$$

Here we have lifted p and q to operators $\hat{p}$ and $\hat{q}$ respectively, which then obey the commutation relation

$$[\hat{q}, \hat{p}] = i\hbar\,. \tag{2.5}$$

It should be mentioned that the effect of surface terms S_{bd} gets taken care in the Hamiltonian style of doing the computation from one boundary state to another. The Hamiltonian follows from the Lagrangian by Legendre transform.

2.1 Dirichlet boundary condition at $t = 0$

Let's first consider the case of a free particle where $V(q) = 0$. The quantum Hamiltonian is given by

$$H_{\mathrm{free}} = \frac{\hat{p}^2}{2m}\,. \tag{2.6}$$

One is interested in computing the transition amplitude from one state to another. However, these states are given by specifying the initial and final position: $q(t = 1) = q_f$ and $q(t = 0) = q_i$. The path integral in eq. (2.1) becomes

$$\bar{G}^{\mathrm{free}}_{\mathrm{DBC}}(q_f, q_i) = \langle q_f | e^{-iH_{\mathrm{free}}/\hbar} | q_i\rangle\,, \tag{2.7}$$

[5]Variation of the action with respect to $q(t)$ to first order leads to terms proportional to $\delta q(t)$ and its derivatives. This will give rise to the equation of motion for $q(t)$ (terms proportional to $\delta q(t)$), while any residual term will either vanish due to choice of boundary condition or has to be canceled by a suitable addition of a surface term.

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This can be evaluated from first principles by the time-slicing method, which discretizes the problem. Breaking the time-length into M segments gives the time duration of each segment to be $\epsilon = 1/M$. This means that $q(t) \ :\Rightarrow \ q_0 = q_i, q_1, q_2, \cdots, q_k, \cdot, q_{M-1}, q_M = q_f$, with the end points $q_0 = q_i$ and $q_M = q_f$ are fixed. This implies that we can write the transition amplitude given in eq. (2.7) in the following discretized form

$$\begin{aligned} \bar{G}^{\rm free}_{\rm DBC}(q_f, q_i) = & \lim_{M\to\infty,\epsilon\to 0}\left(\prod_{k=1}^{M-1}\int_{-\infty}^{\infty} {\rm d}q_k\right)\langle q_M|U_{M,M-1}|q_{M-1}\rangle\langle q_{M-1}|U_{M-1,M-2}|q_{M-2}\rangle\cdots \\ & \times\langle q_k|U_{k,k-1}|q_{k-1}\rangle\cdots\langle q_2|U_{2,1}|q_1\rangle\langle q_1|U_{1,0}|q_0\rangle\,, \end{aligned} \tag{2.8}$$

where

$$U_{k,k-1} = e^{-iH_{\rm free}\epsilon/\hbar}. \tag{2.9}$$

Consider the following matrix element $\langle q_k|e^{-iH_{\rm free}\epsilon/\hbar}|q_{k-1}\rangle$. In this, one can plug completeness-relation in momentum space and perform the momentum integral

$$\begin{aligned} \langle q_k|e^{-iH_{\rm free}\epsilon/\hbar}|q_{k-1}\rangle &= \int_{-\infty}^{\infty}\frac{{\rm d}p_k}{2\pi\hbar}\langle q_k|e^{-iH_{\rm free}\epsilon/\hbar}|p_k\rangle\langle p_k|q_{k-1}\rangle\,, \\ &= \int_{-\infty}^{\infty}\frac{{\rm d}p_k}{2\pi\hbar}e^{-i\epsilon p_k^2/(2m\hbar)+ip_k(q_k-q_{k-1})/\hbar} = \sqrt{\frac{m}{2\pi i\epsilon\hbar}}e^{im(q_k-q_{k-1})^2/(2\epsilon\hbar)}, \end{aligned} \tag{2.10}$$

where we performed the Gaussian integral in p_k to obtain the last expression. Eventually, we are left with the following expression for $\bar{G}^{\rm free}_{\rm DBC}$

$$\bar{G}^{\rm free}_{\rm DBC}(q_f, q_i) = \lim_{M\to\infty,\epsilon\to 0}\left(\prod_{k=1}^{M-1}\int_{-\infty}^{\infty}{\rm d}q_k\right)\left(\frac{m}{2\pi i\epsilon\hbar}\right)^{M/2}\exp\left[\frac{im}{2\epsilon\hbar}\sum_{k=1}^{M}(q_k-q_{k-1})^2\right]. \tag{2.11}$$

The series of Gaussian q-integrals can be carried out one-by-one starting from q_1 till we reach integration over q_{M-1} which is the final integration that one has to perform. This will lead to the following expression for $\bar{G}^{\rm free}_{\rm DBC}(q_f, t=1; q_i, t=0)$

$$\bar{G}^{\rm free}_{\rm DBC}(q_f, t=1; q_i, t=0) = \sqrt{\frac{m}{2\pi i\hbar}}e^{im(q_f-q_i)^2/2\hbar}\,, \tag{2.12}$$

where we have used $M\epsilon = 1$, when taking the limit $M\to\infty$ and $\epsilon\to 0$. In the case interaction term $V(q)$ is present, then eq. (2.11) gets modified into the following

$$\begin{aligned} \bar{G}_{\rm DBC}(q_f, q_i) = & \lim_{M\to\infty,\epsilon\to 0}\left(\prod_{k=1}^{M-1}\int_{-\infty}^{\infty}{\rm d}q_k\right)\left(\frac{m}{2\pi i\epsilon\hbar}\right)^{M/2} \\ & \times\exp\left[\frac{im}{2\epsilon\hbar}\sum_{k=1}^{M}(q_k-q_{k-1})^2 - \frac{i\epsilon}{\hbar}V(q_{k-1})\right]. \end{aligned} \tag{2.13}$$

This is nothing but a discretized version of the path integral given in eq. (2.1) where $\dot{q}_k = (q_k - q_{k-1})/\epsilon$. However, for a generic potential $V(q)$ this can't be computed exactly. In the case for linear potential $V(q) = \lambda q$ (which will also appear in quantum cosmology, as we will later see), one can easily compute the above-discretized version of the path integral exactly. The end result after the q-integrations is the following,

$$\begin{aligned}\bar{G}_{\rm DBC}(q_f, q_i) &= \sqrt{\frac{m}{2\pi i\hbar}} \exp\left[\frac{i}{\hbar}\left\{\frac{m(q_f-q_i)^2}{2} - \frac{\lambda(q_f+q_i)}{2} - \frac{\lambda^2}{24m}\right\}\right], \\ &= \bar{G}^{\rm free}_{\rm DBC}(q_f, q_i)\ e^{-i\lambda\left\{\lambda+12m(q_f+q_i)\right\}/(24m\hbar)}\,. \end{aligned} \tag{2.14}$$

Notice the quadratic structure of the λ-dependent terms. The part coming from interaction vanishes for two values of λ: $\lambda = 0$ (free theory) and $\lambda = -12m(q_f + q_i)$ (interacting).

2.2 Neumann boundary condition at $t = 0$

The path integral for the Neumann boundary condition is easier to deal with when handled in the Hamiltonian framework. The quantity we are interested in computing is

$$\bar{G}_{\rm NBC}(q_f, t=1; p_i, t=0) = \langle q_f | e^{-iH/\hbar} | p_i\rangle\,, \tag{2.15}$$

where at the initial time, we fix the momentum. We note that the initial momentum state can be written as the following

$$|p_i\rangle = \int_{-\infty}^{\infty} {\rm d}q_0\ e^{ip_i q_0/\hbar}\ |q_0\rangle\,. \tag{2.16}$$

This allows one to express $\bar{G}_{\rm NBC}$ as a Fourier transform of another transition amplitude

$$\bar{G}_{\rm NBC}(q_f, t=1; p_i, t=0) = \int_{-\infty}^{\infty} {\rm d}q_0\ e^{ip_i q_0/\hbar}\ \underbrace{\langle q_f | e^{-iH/\hbar} | q_0\rangle}_{\bar{G}_{\rm DBC}(q_f,t=1;q_0,t=0)}\,. \tag{2.17}$$

This shows that $\bar{G}_{\rm NBC}(q_f, t = 1; p_i, t = 0)$ is related to $\bar{G}_{\rm DBC}(q_f, t = 1; q_0, t = 0)$ by a Fourier transform thereby implying that one can be obtained from another by transformation (if $\bar{G}_{\rm NBC}(q_f, t = 1; p_i, t = 0)$ is known then $\bar{G}_{\rm DBC}(q_f, t = 1; q_0, t = 0)$ can be obtained from it by an inverse Fourier transform). In the present situation as we have already worked out the transition amplitude $\bar{G}_{\rm DBC}(q_f, t = 1; q_0, t = 0)$, the above Fourier transform can be used to compute the expression for the Neumann transition amplitude. This is given by

$$\bar{G}^{\rm free}_{\rm NBC}(q_f, t=1; p_i, t=0) = e^{i(p_i q_f - p_i^2/2m)/\hbar}\,. \tag{2.18}$$

Similarly, one can exploit the above technique to work out the path integral for a particle in the linear-potential with Neumann boundary condition at the initial time. Here if we plug eq. (2.13) on the r.h.s. of eq. (2.17) and writing $q_i \to q_0$, then on integration with respect to q_0 we get the $\bar{G}_{\rm NBC}(q_f, p_i)$.

$$\begin{aligned}\bar{G}_{\rm NBC}(q_f, p_i) &= \exp\left[\frac{i}{\hbar}\left\{p_i q_f - \frac{p_i^2}{2m} - \frac{\lambda\,(\lambda - 3p_i + 6mq_f)}{6m}\right\}\right], \\ &= \bar{G}^{\rm free}_{\rm NBC}(q_f, p_i) \times e^{-i\lambda\left(\lambda-3p_i+6mq_f\right)/(6m\hbar)}\,. \end{aligned} \tag{2.19}$$

The interaction dependent term is quadratic in λ and vanishes for two values of λ: $\lambda = 0$ and $\lambda = 3(p_i - 2mq_f)$.

2.3 Robin boundary condition at $t = 0$

Now we are interested in computing the path integral for the case when Robin boundary condition (RBC) is specified at the initial time $t = 0$. An RBC at $t = 0$ means that a given linear combination of position and momentum is fixed at that time.

The boundary value problem we are interested in is the following

$$\mathrm{Bd_f} : q(t = 1) = q_f \quad \text{and} \quad \mathrm{Bd_i} : p_i + \beta\, q_i = P_i\,, \tag{2.20}$$

where $q_i \equiv q(t = 0)$ and $p_i \equiv p(t = 0)$. The path integral we are interested in computing is

$$\bar{G}_{\mathrm{RBC}}(q_f, t = 1; p_i + \beta\, q_i, t = 0) = \langle q_f | e^{-iH/\hbar} | p_i + \beta\, q_i \rangle\,. \tag{2.21}$$

An interesting way to evaluate this is to do a canonical transformation and define a new momentum variable P and position variable Q as

$$P(t) = p(t) + \beta q(t) \quad \text{and} \quad Q(t) = q(t)\,. \tag{2.22}$$

Under this transformation, the commutation relation is preserved, i.e.,

$$[\hat{Q}, \hat{P}] = [\hat{q}, \hat{P}] = [\hat{q}, \hat{p} + \beta\hat{q}] = i\hbar\,. \tag{2.23}$$

This also means that the Jacobian of transformation is unity. However, in terms of new variables, the Hamiltonian acquires a new form $H(\hat{p}, \hat{q}) = H(\hat{P} - \beta\hat{Q}, \hat{Q}) \equiv H_\beta(\hat{P}, \hat{Q})$ (where subscript 'β' symbolizes β-dependence of the Hamiltonian). In terms of new variables, our transition amplitude acquires the following form

$$\bar{G}_{\mathrm{RBC}}(q_f, t = 1; p_i + \beta\, q_i, t = 0) = \bar{G}_{\mathrm{NBC}}(Q_f, t = 1; P_i, t = 0) = \langle Q_f | e^{-iH_\beta/\hbar} | P_i \rangle\,. \tag{2.24}$$

In terms of new variables the Robin boundary condition problem transforms into an NBC problem. One can write the initial state as follows

$$|P_i\rangle = \int_{-\infty}^{\infty} \mathrm{d}Q_0\; e^{iP_iQ_0/\hbar}\; |Q_0\rangle\,. \tag{2.25}$$

This means that for the Robin boundary condition the path integral becomes the following

$$\bar{G}_{\mathrm{RBC}}(q_f, t = 1; p_i + \beta\, q_i, t = 0) = \int_{-\infty}^{\infty} \mathrm{d}Q_0\; e^{iP_iQ_0/\hbar}\; \underbrace{\langle Q_f | e^{-iH_\beta(\hat{P},\hat{Q})/\hbar} | Q_0 \rangle}_{\bar{G}^{\beta}_{\mathrm{DBC}}(Q_f, t=1; Q_0, t=0)}\,. \tag{2.26}$$

The object of interest here is $\langle Q_f | e^{-iH_\beta(\hat{P},\hat{Q})/\hbar} | Q_0 \rangle$ which is the DBC path integral for the transformed Hamiltonian $H_\beta(P, Q)$. It should be pointed out that this process of evaluating RBC path integrals remain valid even for interacting theories, but to gain clarity on the methods, we first consider free theory. The free Hamiltonian H^{free}_β is given by

$$H^{\mathrm{free}}_\beta(\hat{P}, \hat{Q}) = \frac{(\hat{P} - \beta\hat{Q})^2}{2m} = \frac{\hat{P}^2}{2m} + \frac{\beta^2\hat{Q}^2}{2m} - \frac{\beta}{2m}(\hat{Q}\hat{P} + \hat{P}\hat{Q})\,, \tag{2.27}$$

and in transformed variables the boundary conditions become

$$Q(t = 1) = Q_f = q_f \quad \text{and} \quad P(t = 0) = P_i\,. \tag{2.28}$$

Therefore, our original interests in computing path integral with RBC shifts to computing path integral in new variables but with NBC. However, the Hamiltonian in new variables is different which means that one needs to be careful while computing path integral via time-slicing method. We are interested in computing something like eq. (2.8) with the Hamiltonian given in eq. (2.27). Let us consider the following matrix

$$\begin{aligned}\langle Q_{k+1}|U_{k+1,k}|Q_k\rangle &= \int_{-\infty}^{\infty} \frac{\mathrm{d}P_k}{2\pi\hbar}\langle Q_{k+1}|e^{-i\epsilon H_\beta^{\text{free}}(\hat{P},\hat{Q})/\hbar}|P_k\rangle\langle P_k|Q_k\rangle\,, \\ &= \left(\frac{m}{2\pi i\epsilon\hbar}\right)^{1/2} e^{\beta\epsilon/2m}\ \exp\left[\frac{i}{\hbar}\left\{\frac{m(Q_{k+1}-Q_k)^2}{2\epsilon}+\beta Q_{k+1}(Q_{k+1}-Q_k)\right\}\right].\end{aligned} \tag{2.29}$$

This means that

$$\begin{aligned}\bar{G}^{\text{free}}_{\text{RBC}}(Q_f, t=1; P_i, t=0) = &\lim_{M\to\infty,\epsilon\to 0}\left(\prod_{k=0}^{M-1}\int_{-\infty}^{\infty}\mathrm{d}Q_k\left(\frac{m}{2\pi i\epsilon\hbar}\right)^{1/2} e^{\beta\epsilon/2m}\right) \\ &\times e^{iP_iQ_0/\hbar}e^{i\beta(Q_M^2-Q_0^2)/(2\hbar)}e^{im\sum_{k=0}^{M-1}(Q_{k+1}-Q_k)^2/(2\epsilon\hbar)}\,.\end{aligned} \tag{2.30}$$

The term in the exponent $\beta(Q_M^2-Q_0^2)/2$ is obtained by noticing that $\beta\sum_k Q_{k+1}(Q_{k+1}-Q_k)$ is actually a discretized version of $\beta\int_0^1 \mathrm{d}t\, Q\dot{Q}$, which can be written as a total derivative $(\beta/2)\int_0^1 \mathrm{d}t\,\mathrm{d}Q^2/\mathrm{d}t$. This eventually gives a simplified expression for the Robin-boundary condition transition amplitude.

$$\bar{G}^{\text{free}}_{\text{RBC}}(Q_f, t=1; P_i, t=0) = \int_{-\infty}^{\infty}\mathrm{d}Q_0\ e^{iP_iQ_0/\hbar}e^{i\beta(Q_M^2-Q_0^2)/(2\hbar)}\ \bar{G}^{\text{free}}_{\text{DBC}}[Q_M, t=1; Q_0, t=0]\,. \tag{2.31}$$

This can be easily computed after plugging the value of $\bar{G}^{\text{free}}_{\text{DBC}}$ from eq. (2.12) which gives

$$\bar{G}^{\text{free}}_{\text{RBC}}(Q_f, t=1; P_i, t=0) = \sqrt{\frac{m}{m-\beta}}\exp\left[\frac{i}{2\hbar}\left\{\beta Q_f^2 - \frac{\beta m Q_f^2 + P_i^2 - 2mP_iQ_f}{m-\beta}\right\}\right], \tag{2.32}$$

which in limit $\beta\to 0$ reproduces the expression for the Neumann boundary condition result given. in eq. (2.18).

In the case when interactions are present, the modified Hamiltonian gets amended by potential term $V(Q)$

$$H_\beta(\hat{P},\hat{Q}) = \frac{(\hat{P}-\beta\hat{Q})^2}{2m} + V(\hat{Q})\,. \tag{2.33}$$

Then following the same steps as above one arrives at an analogous expression for $\langle Q_{k+1}|U_{k+1,k}|Q_k\rangle$ which is given by

$$\begin{aligned}\langle Q_{k+1}|U_{k+1,k}|Q_k\rangle = &\left(\frac{m}{2\pi i\epsilon\hbar}\right)^{1/2} e^{\frac{\beta\epsilon}{2m}}\exp\left[\frac{i}{\hbar}\left\{\frac{m(Q_{k+1}-Q_k)^2}{2\epsilon}\right.\right. \\ &\left.\left. +\beta Q_{k+1}(Q_{k+1}-Q_k) - \epsilon V(Q_k)\right\}\right].\end{aligned} \tag{2.34}$$

Once again the term $\beta \sum_k Q_{k+1}(Q_{k+1} - Q_k)$ is discretized version of $\beta \int_0^1 \mathrm{d}t\, Q\dot{Q}$ and can be written as a total derivative $(\beta/2) \int_0^1 \mathrm{d}t\, \mathrm{d}Q^2/\mathrm{d}t$. Then the expression for the $\bar{G}_{\rm RBC}$ with the potential is given by

$$\begin{aligned}
&\bar{G}_{\rm RBC}(Q_f, t=1; P_i, t=0) \\
&= \lim_{M\to\infty, \epsilon\to 0} \left(\prod_{k=0}^{M-1} \int_{-\infty}^{\infty} \mathrm{d}Q_k \left(\frac{m}{2\pi i \epsilon \hbar}\right)^{1/2} e^{\beta\epsilon/2m} \right) \\
&\quad \times \exp\left[\frac{i}{\hbar} \left\{ P_i Q_0 + \frac{\beta(Q_M^2 - Q_0^2)}{2} + \frac{m}{2\epsilon} \sum_{k=0}^{M-1} (Q_{k+1} - Q_k)^2 - \epsilon \sum_{k=0}^{M-1} V(Q_k) \right\} \right], \\
&= \int_{-\infty}^{\infty} \mathrm{d}Q_0\, e^{iP_iQ_0/\hbar} e^{i\beta(Q_M^2 - Q_0^2)/(2\hbar)}\, \bar{G}_{\rm DBC}[Q_M, t=1; Q_0, t=0]\,.
\end{aligned} \tag{2.35}$$

It should be highlighted that this last line is valid for any arbitrary potential $V(Q)$. For the case of linear potential one can plug the expression for $\bar{G}_{\rm DBC}[Q_M, t=1; Q_0, t=0]$ from the eq. (2.13) in the above to obtain $\bar{G}_{\rm RBC}(q_f, t=1; p_i + \beta\, q_i, t=0)$

$$\begin{aligned}
\bar{G}_{\rm RBC}(Q_f, t=1; P_i, t=0) &= \bar{G}^{\rm free}_{\rm RBC}(Q_f, t=1; P_i, t=0) \\
&\quad \times \exp\left[\frac{i\lambda \left\{ \beta q_f + (P_i - 2mq_f) \right\}}{2(m-\beta)\hbar} - \frac{i\lambda^2(4m-\beta)}{24m(m-\beta)\hbar} \right].
\end{aligned} \tag{2.36}$$

Notice that in the limit $\beta \to 0$ this RBC expression goes to the NBC expression given in eq. (2.19). The full Robin path integral for interacting theory is a product of Robin path integral for free theory and a coupling dependent piece.

The expression given in eq. (2.35) can be converted into a relation between $\bar{G}_{\rm RBC}(Q_f, t = 1; P_i, t=0)$ and $\bar{G}_{\rm NBC}(q_f, t=1; p_i, t=0)$ by exploiting the inverse Fourier transform. After a bit of algebra and computation of Q_0-Gaussian integral, it is seen that

$$\bar{G}_{\rm RBC}(Q_f, t=1; P_i, t=0) = \sqrt{\frac{e^{i\beta Q_f^2}}{i\beta}} \int_{-\infty}^{\infty} \frac{\mathrm{d}\tilde{p}}{\sqrt{2\pi\hbar}} e^{i\frac{(P_i - \tilde{p})^2}{2\beta\hbar}} \bar{G}_{\rm NBC}(q_f, t=1; \tilde{p}, t=0)\,, \tag{2.37}$$

where $P_i = p_i + \beta\, q_i$. This is an exact relation connecting the two quantities. In the following, we will use this relation to compute the exact expression of gravitational path integral in mini-superspace approximation for Robin BC and investigate it in detail to study the implications of Robin BC on the evolution of the Universe.

3 Mini-superspace action

We start by considering the FLRW metric given in eq. (1.3) which is conformally-flat and hence has a vanishing Weyl-tensor ($C_{\mu\nu\rho\sigma} = 0$). For a generic d-dimensional FLRW metric the non-zero entries of the Riemann tensor are [51–53]

$$\begin{aligned}
R_{0i0j} &= -\left(\frac{a''}{a} - \frac{a' N_p'}{a N_p} \right) g_{ij}\,, \\
R_{ijkl} &= \left(\frac{k}{a^2} + \frac{a'^2}{N_p^2 a^2} \right) (g_{ik} g_{jl} - g_{il} g_{jk})\,.
\end{aligned} \tag{3.1}$$

Here g_{ij} is the spatial part of the FLRW metric while ($'$) denotes the derivative with respect to t_p. The non-zero components of Ricci-tensor are

$$\begin{aligned} R_{00} &= -(D-1)\left(\frac{a''}{a} - \frac{a'N_p'}{aN_p}\right), \\ R_{ij} &= \left[\frac{(D-2)(kN_p^2 + a'^2)}{N_p^2 a^2} + \frac{a''N_p - a'N_p'}{aN_p^3}\right] g_{ij}, \end{aligned} \tag{3.2}$$

while the Ricci-scalar for FLRW is given by

$$R = 2(D-1)\left[\frac{a''N_p - a'N_p'}{aN_p^3} + \frac{(D-2)(kN_p^2 + a'^2)}{2N_p^2 a^2}\right]. \tag{3.3}$$

Weyl-flat metrics also have a wonderful property that allows one to express the Riemann tensor in terms of Ricci-tensor and Ricci scalar as follows

$$R_{\mu\nu\rho\sigma} = \frac{R_{\mu\rho}g_{\nu\sigma} - R_{\mu\sigma}g_{\nu\rho} + R_{\nu\sigma}g_{\mu\rho} - R_{\nu\rho}g_{\mu\sigma}}{D-2} - \frac{R(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho})}{(D-1)(D-2)}. \tag{3.4}$$

This furthermore helps one to express the square of Riemann-tensor as a linear combination of Ricci-tensor-square and Ricci-scalar-square. This is given by

$$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = \frac{4}{D-2}R_{\mu\nu}R^{\mu\nu} - \frac{2R^2}{(D-1)(D-2)}. \tag{3.5}$$

This identity is often used to simplify the expression for the Gauss-Bonnet gravity action for the case of Weyl-flat metrics.

$$\begin{aligned} &\int \mathrm{d}^D x\sqrt{-g}\left(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2\right) \\ &= \frac{D-3}{D-2}\int \mathrm{d}^D x\sqrt{-g}\left(-4R_{\mu\nu}R^{\mu\nu} + \frac{DR^2}{D-1}\right). \end{aligned} \tag{3.6}$$

On plugging the FLRW metric of eq. (1.3) in the gravitational action stated in eq. (1.2), we get an action for scale-factor $a(t_p)$ and lapse $N_p(t_p)$ in D-dimensions

$$\begin{aligned} S[a, N_p] = \frac{V_{D-1}}{16\pi G}\int \mathrm{d}t_p\Bigg[&\frac{a^{D-3}}{N_p^2}\Big\{(D-1)(D-2)kN_p^3 - 2\Lambda a^2 N_p^3 - 2(D-1)aa'N_p' \\ &+(D-1)(D-2)a'^2 N_p + 2(D-1)N_p a a''\Big\} + (D-1)(D-2)(D-3)\alpha\Big\{\frac{a^{D-5}(D-4)}{N_p^3} \\ &\times(kN_p^2 + a'^2)^2 + 4a^{D-4}\frac{\mathrm{d}}{\mathrm{d}t_p}\left(\frac{ka'}{N_p} + \frac{a'^3}{3N_p^3}\right)\Big\}\Bigg]. \end{aligned} \tag{3.7}$$

V_{D-1} here refers to the volume of the $D-1$ dimensional spatial hyper-surface which is given by,

$$V_{D-1} = \frac{\Gamma(1/2)}{\Gamma(D/2)}\left(\frac{\pi}{k}\right)^{(D-1)/2}. \tag{3.8}$$

In four spacetime dimensions ($D = 4$) the terms proportional to α (Gauss-Bonnet coupling parameter) becomes a total time derivative. In $D = 4$ the mini-superspace gravitational action then becomes the following

$$S[a, N_p] = \frac{V_3}{16\pi G}\int \mathrm{d}t_p \left[6kaN_p - 2\Lambda a^3 N_p - \frac{6a^2 a' N_p'}{N_p} + \frac{6aa'^2}{N_p} + \frac{6a'' a^2}{N_p} + 24\alpha \frac{\mathrm{d}}{\mathrm{d}t_p}\left(\frac{ka'}{N_p} + \frac{a'^3}{3N_p^3}\right)\right]. \tag{3.9}$$

At this point, one can introduce rescaling of lapse N_p and scale factor to bring out a more appealing form of the mini-superspace action by doing the following transformation

$$N_p(t_p)\mathrm{d}t_p = \frac{N(t)}{a(t)}\mathrm{d}t\,, \qquad q(t) = a^2(t)\,. \tag{3.10}$$

This co-ordinate transformation re-expresses our original FLRW metric in eq. (1.3) in a different form.

$$\mathrm{d}s^2 = -\frac{N^2}{q(t)}\mathrm{d}t^2 + q(t)\left[\frac{\mathrm{d}r^2}{1 - kr^2} + r^2 \mathrm{d}\Omega_{D-2}^2\right]. \tag{3.11}$$

Notice also that in the new coordinate frame, the new 'time' is denoted by t. Under this transformation, the original gravitational action in $D = 4$ given in eq. (3.9) acquires the following simple form

$$S[q, N] = \frac{V_3}{16\pi G}\int_0^1 \mathrm{d}t \left[(6k - 2\Lambda q)N + \frac{3\dot{q}^2}{2N} + 3q\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\dot{q}}{N}\right) + 24\alpha\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{k\dot{q}}{2N} + \frac{\dot{q}^3}{24N^3}\right)\right]. \tag{3.12}$$

Here $\dot{()}$ represent time t derivative. Note that the GB-term is a total time derivative. Integration by parts of the terms in the action re-expresses the action in a more recognizable form along with some surface terms.

$$\begin{aligned} S[q, N] = {}& \frac{V_3}{16\pi G}\int_0^1 \mathrm{d}t \left[(6k - 2\Lambda q)N - \frac{3\dot{q}^2}{2N}\right] + \frac{V_3}{16\pi G}\left[\frac{3q_f \dot{q}_f}{N} - \frac{3q_i \dot{q}_i}{N}\right. \\ & \left. + 24\alpha\left(\frac{k\dot{q}_f}{2N} + \frac{\dot{q_f}^3}{24N^3} - \frac{k\dot{q}_i}{2N} - \frac{\dot{q_i}^3}{24N^3}\right)\right]. \end{aligned} \tag{3.13}$$

The surface terms consist of two parts: one coming from EH-part of gravitational action while the other is from GB sector. The bulk term is like an action of a one-particle system in a linear potential. From now onwards we will work with the convention that $V_3 = 8\pi G$, and in the rest of the paper we will study the path integral of this action.

4 Action variation and boundary terms

We start by constructing the variational problem by considering the variation of the action in eq. (3.12) with respect to $q(t)$. This will not only yield terms that will eventually lead to the equation of motion but also generate a collection of boundary terms.

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In the rest of the paper, we will work with the ADM gauge $\dot{N} = 0$, which implies setting $N(t) = N_c$ (constant). To set up the variational problem consistently and avoid missing of any terms we write

$$q(t) = \bar{q}(t) + \epsilon \delta q(t)\,, \tag{4.1}$$

where $\bar{q}(t)$ is some background $q(t)$ and $\delta q(t)$ is the fluctuation around this. The parameter ϵ is used to keep track of the order of fluctuation terms. On plugging this in the action given in eq. (3.12) and on expanding it to first order in ϵ, we notice that the terms proportional to ϵ are given by following

$$\delta S = \frac{\epsilon}{2} \int_0^1 \mathrm{d}t \left[\left(-2\Lambda N_c + \frac{3\ddot{\bar{q}}}{N_c} \right) \delta q + \frac{3}{N_c} \frac{\mathrm{d}}{\mathrm{d}t} \left(\bar{q} \delta \dot{q} \right) + 24\alpha \frac{\mathrm{d}}{\mathrm{d}t} \left\{ \left(\frac{k}{2N_c} + \frac{\dot{\bar{q}}^2}{8N_c^3} \right) \delta \dot{q} \right\} \right]. \tag{4.2}$$

We also notice that in the above expression, there are two total time-derivative pieces which contribute at the boundaries. These boundary pieces need to be canceled appropriately by supplementing the action with suitable surface terms in order to have a consistent boundary value problem for a particular choice of boundary condition. The terms proportional to δq on the other hand will lead to the equation of motion if one demands that the full expression multiplying it vanishes. This gives the dynamical equation obeyed by $\bar{q}(t)$

$$\ddot{\bar{q}} = \frac{2}{3} \Lambda N_c^2\,. \tag{4.3}$$

This process of construing surface action is based on analyzing the behavior of fluctuations around the classical trajectory (on-shell geometries), and may sometime miss to properly capture the features of topological effects as by definition topological terms don't play a role in the equation of motion. This will become more clear as we proceed further in the paper.

The second-order ODE given in eq. (4.3) is easy to solve and its general solution is given by

$$\bar{q}(t) = \frac{\Lambda N_c^2}{3} t^2 + c_1 t + c_2\,. \tag{4.4}$$

The constants $c_{1,2}$ are determined by requiring the solution to satisfy the boundary conditions. The total-derivative terms in eq. (4.2) lead to a collection of boundary terms

$$S_{\mathrm{bdy}} = \frac{\epsilon}{2} \left[\frac{3}{N_c} \left(\bar{q}_f \delta \dot{q}_f - \bar{q}_i \delta \dot{q}_i \right) + 24\alpha \left\{ \left(\frac{k \delta \dot{q}_f}{2N_c} + \frac{\dot{\bar{q}}_f^2 \delta \dot{q}_f}{8N_c^3} \right) - \left(\frac{k \delta \dot{q}_i}{2N_c} + \frac{\dot{\bar{q}}_i^2 \delta \dot{q}_i}{8N_c^3} \right) \right\} \right], \tag{4.5}$$

where

$$\bar{q}_i = \bar{q}(t=0)\,, \qquad \bar{q}_f = \bar{q}(t=1)\,, \qquad \dot{\bar{q}}_i = \dot{\bar{q}}(t=0)\,, \qquad \dot{\bar{q}}_f = \dot{\bar{q}}(t=1)\,. \tag{4.6}$$

Note that the boundary terms are function of $\bar{q}_{(i,f)}$ and $\dot{\bar{q}}_{(i,f)}$ which are on-shell quantities. These are different from $q_{(i,f)}$ and $\dot{q}_{(i,f)}$ which also includes boundary values of the off-shell trajectories. The bulk action given in eq. (3.13) is used to determine the momentum conjugate to the field $q(t)$

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot{q}} = -\frac{3\dot{q}}{2N_c}\,. \tag{4.7}$$

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In terms of conjugate momentum the boundary terms can be expressed as follows

$$S_{\rm bdy} = -\epsilon\left[(\bar{q}_f\delta\pi_f - \bar{q}_i\delta\pi_i) + 4\alpha\left\{\left(k\delta\pi_f + \frac{\bar{\pi}_f^2\delta\pi_f}{9}\right) - \left(k\delta\pi_i + \frac{\bar{\pi}_i^2\delta\pi_i}{9}\right)\right\}\right]. \tag{4.8}$$

For the consistent variational problem these terms either have to be appropriately canceled by supplementing the original action given in eq. (3.12) with suitable surface terms or they vanish identically for the choice of boundary conditions.

In the following, we will always impose Dirichlet boundary condition at $t = 1$, while at $t = 0$ we will either have Neumann or Robin boundary conditions (as imposing Dirichlet at $t = 0$ leads to unsuppressed perturbations, so it won't be addressed [20, 21]). However, as the path integral in two cases (NBC and RBC) are related by eq. (2.37) therefore one can first compute the path integral in NBC case and use the relation in eq. (2.37) to compute the results for the Robin BC.

4.1 Neumann Boundary condition (NBC) at $t = 0$

Imposing Neumann boundary condition [25, 33] at $t = 0$ and a Dirichlet boundary condition at $t = 1$ gives a consistent variational problem. This means that the initial momentum π_i and final position q_f of all the trajectories are fixed. This will mean

$$\pi_i \;\&\; q_f = \text{fixed} \qquad \Rightarrow \qquad \delta\pi_i = 0 \quad \& \quad \delta q_f = 0\,. \tag{4.9}$$

Such a boundary condition imposition also has an additional merit that they lead to a well-behaved path integral where the perturbations are suppressed [2, 30–32]. For this case, the boundary terms given in eq. (4.8) arising during the variation of action will reduce to the following

$$S_{\rm bdy}\Big|_{\rm NBC} = -\epsilon\left[\bar{q}_f\delta\pi_f + 4\alpha\left(k\delta\pi_f + \frac{\bar{\pi}_f^2\delta\pi_f}{9}\right)\right]. \tag{4.10}$$

These residual boundary terms which don't vanish after imposing boundary condition need to be canceled by a suitable addition of surface terms so that the variational problem is consistent. It is seen that if one adds the following terms at the boundary

$$S_{\rm surface}\Big|_{\rm NBC} = \frac{1}{2}\left[-\frac{3q_f\dot{q}_f}{N_c} - 24\alpha\left(\frac{k\dot{q}_f}{2N_c} + \frac{\dot{q}_f^3}{24N_c^3}\right)\right] = q_f\pi_f + 4\alpha\left(k\pi_f + \frac{\pi_f^3}{27}\right), \tag{4.11}$$

then its variation (as suggested in eq. (4.1) will precisely cancel the terms given in eq. (4.10). The first term in eq. (4.11) is a Gibbon-Hawking-York (GHY)-term and the second one is a Chern-Simon like term at the final boundary.

The constants $c_{1,2}$ that appear in the solution to the equation of motion eq. (4.4) can now be determined for the choice of boundary conditions. This will imply

$$\bar{q}(t) = \frac{\Lambda N_c^2}{3}(t^2 - 1) - \frac{2N_c\pi_i}{3}(t-1) + q_f\,, \tag{4.12}$$

where 'bar' over q is added as it is the solution to equation of motion. In this setting at $t = 0$, we notice that the on-shell value of q_i is given by

$$\bar{q}_i = q_f + \frac{2N_c\pi_i}{3} - \frac{\Lambda N_c^2}{3} \,. \tag{4.13}$$

For off-shell trajectories, q_i can be anything as that is not fixed at $t = 0$. The surface terms obtained in eq. (4.11) when added to the action in eq. (3.13) leads to the full action of the system. This is given by

$$S_{\rm tot}[q, N_c] = \frac{1}{2}\int_0^1 {\rm d}t\left[(6k - 2\Lambda q)N_c - \frac{3\dot{q}^2}{2N_c}\right] + q_i\pi_i + 4\alpha\left(k\pi_i + \frac{\pi_i^3}{27}\right)\,, \tag{4.14}$$

Furthermore, on substituting the solution to the equation of motion eq. (4.12) and using eq. (4.13) in the above, we will obtain the on-shell action which is given by

$$S_{\rm tot}^{\rm on-shell}[\bar{q}, N_c] = \frac{\Lambda^2}{9}N_c^3 - \frac{\Lambda\pi_i}{3}N_c^2 + \left(3k - \Lambda q_f + \frac{\pi_i^2}{3}\right)N_c + q_f\pi_i + 4\alpha\left(k\pi_i + \frac{\pi_i^3}{27}\right)\,. \tag{4.15}$$

This is also the action for the lapse N_c. Compare this action with the action computed when Dirichlet boundary condition is imposed at $t = 0$ [2, 20, 21, 24, 30–32] and notice the lack of singularity at $N_c = 0$. The lack of singularity at $N_c = 0$ can be physically understood as in the NBC path integral we sum over all possible transitions from q_i (fixed π_i) to fixed q_f. This will also include the transition from q_f to q_f which can occur instantaneously i.e. with $N_c = 0$. Thereby implying that there is nothing singular happening at $N_c = 0$.

4.2 Robin boundary condition (RBC) at $t = 0$

We now consider the situation when we have Robin boundary condition at $t = 0$ and Dirichlet boundary condition imposed at $t = 1$. An example of imposing Robin BC is fixing the Hubble parameter at a given time in the cosmological evolution. This boundary value problem poses a consistent variational problem. In this case, we fix a linear combination of the initial scale factor q_i and the corresponding initial conjugate momentum π_i, while the final scale factor q_f is fixed at $t = 1$. This means

$$\pi_i + \beta q_i = P_i = \text{fixed} \quad \& \quad q_f = \text{fixed} \quad\Rightarrow\quad \delta P_i = 0 \quad \& \quad \delta q_f = 0\,. \tag{4.16}$$

For these boundary conditions, it has also been noted that perturbations are suppressed and in path integral their effects are bounded [2, 30–32]. For the boundary conditions mentioned in eq. (4.16), the boundary terms generated during the variation of action given in eq. (4.8) reduces to the following

$$S_{\rm bdy}\Big|_{\rm RBC} = -\epsilon\left[\bar{q}_f\delta\pi_f + \beta\bar{q}_i\delta q_i + 4\alpha\left\{k\delta\pi_f + \frac{\bar{\pi}_f^2\delta\pi_f}{9} + \beta\left(k\delta q_i + \frac{(\bar{P}_i - \beta\bar{q}_i)^2\delta q_i}{9}\right)\right\}\right]. \tag{4.17}$$

For the consistent variational problem, these terms need to be removed by supplementing the original gravity action with suitable surface terms. The additional suitable surface terms when varied will precisely cancel the above terms leading to a consistent variational problem. It is seen that if one adds the following terms at the boundary

$$S_{\text{surface}}\Big|_{\text{RBC}} = q_f\pi_f + 4\alpha\left(k\pi_f + \frac{\pi_f^3}{27}\right) + \frac{\beta}{2}(q_f^2+q_i^2)$$
$$+4\alpha\beta\left[kq_i + \frac{q_i}{9}\left(P_i^2 - \beta P_i q_i + \frac{\beta^2 q_i^2}{3}\right)\right], \tag{4.18}$$

Here the first two terms are the same as for the NBC case, while the rest of the terms arise due to the imposition of RBC at $t=0$. In the limit of $\beta \to 0$ these terms will vanish and we get back the surface terms for the NBC which gives us a check of consistency. The variation of term $\beta q_f^2/2$ is although zero (due to imposition of DBC at $t=1$), however its addition facilitates comparison with the results given in section 2.3. The surface term $\beta q_i^2/2$ in eq. (4.18) comes due to the RBC at $t=0$ from the Einstein-Hilbert part of the gravitational action and was known from before [33], The term proportional to $\alpha\beta$ is new and corresponds to surface terms that need to be added for the Gauss-Bonnet gravity part.

The surface terms obtained in eq. (4.18) when added to the action in eq. (3.13) leads to the full action of the system. This is given by

$$S_{\text{tot}}[q,N_c] = \frac{1}{2}\int_0^1 \mathrm{d}t\left[(6k-2\Lambda q)N_c - \frac{3\dot{q}^2}{2N_c}\right] + q_i P_i + \frac{\beta}{2}(q_f^2 - q_i^2) + 4\alpha\left(kP_i + \frac{P_i^3}{27}\right). \tag{4.19}$$

Comparing this action with the action for the NBC case given in eq. (4.14), we get in the limit $\beta \to 0$

$$S_{\text{tot}}\Big|_{\text{RBC}} \xRightarrow{\beta\to 0} S_{\text{tot}}\Big|_{\text{NBC}}, \tag{4.20}$$

as $P_i \to \pi_i$. It should be mentioned that these boundary terms (which are called canonical boundary terms) don't have a covariant expression (at least so far, it is not known) [30, 33]. As gravity is a covariant theory, one would expect the boundary action to be correspondingly covariant. Indeed there exist other covariant boundary terms in literature satisfying the variational problem, which have been studied in [26, 37, 58]. These have been investigated in the mini-superspace path integral under certain situations in [30]. However, the canonical terms when viewed from the quantum mechanical perspective [33] have an interpretation of a complex coherent state, which becomes a plane wave in the limit $\beta \to 0$. Do note that such boundary terms also arise when one studies the one-dimensional quantum mechanical problem of a particle in an arbitrary potential with RBC as shown in section 2.3. We will discuss more on the choice of β and the usefulness of these boundary terms in section 6.3.

The constants $c_{1,2}$ that appear in the solution to the equation of motion given in eq. (4.4) can now be determined for the case of RBC. This will imply

$$\bar{q}(t) = \frac{\Lambda N_c^2}{3}t^2 + \frac{P_i}{\beta} + \left(1+\frac{3}{2\beta N_c}\right)^{-1}\left(t + \frac{3}{2\beta N_c}\right)\left(q_f - \frac{P_i}{\beta} - \frac{\Lambda N_c^2}{3}\right), \tag{4.21}$$

where 'bar' over q indicates the solution of the equation of motion. Setting $t=0$ in this gives the on-shell value of the q_i which is given by

$$\bar{q}_i = \frac{P_i}{\beta} + \left(\frac{3}{3+2\beta N_c}\right)\left(q_f - \frac{P_i}{\beta} - \frac{\Lambda N_c^2}{3}\right). \tag{4.22}$$

Off-shell q_i could be anything as it is not fixed by the boundary condition imposed in RBC. The on-shell action can be computed by substituting the solution of the equation of motion and $\bar{q}_i$ into the action for RBC given in eq. (4.19). This is given by

$$S_{\rm tot}^{\rm on-shell}[\bar{q}, N_c] = \frac{1}{18(3+2N_c\beta)}\Big[\beta\Lambda^2 N_c^4 + 6\Lambda^2 N_c^3 + N_c^2\{108\beta k - 18\Lambda(P_i + \beta q_f)\} \\ +18N_c\left\{9k + P_i^2 + q_f\left(\beta^2 q_f - 3\Lambda\right)\right\} + 54P_i q_f\Big] + 4\alpha\left(kP_i + \frac{P_i^3}{27}\right). \quad (4.23)$$

This is the action for the lapse N_c in the case of RBC at $t = 0$ and DBC at $t = 1$. In the limit $\beta \to 0$, this reduces to the NBC on-shell action given in eq. (4.15).

5 Transition amplitude

We now move forward to compute the transition amplitude from one 3-geometry to another (for both NBC and RBC case). The quantity that is of relevance here in the mini-superspace approximation is as follows (see [20, 48] for the Euclidean gravitational path integral in mini-superspace approximation)

$$G[\mathrm{Bd}_f, \mathrm{Bd}_i] = \int_{0^+}^{\infty} \mathrm{d}N_c \int_{\mathrm{Bd}_i}^{\mathrm{Bd}_f} \mathcal{D}q(t)\ \exp\left(\frac{i}{\hbar} S_{\rm tot}[q, N_c]\right), \quad (5.1)$$

where Bd_i and Bd_f are the initial and final boundary configurations respectively, and $S_{\rm tot}$ refers to the mini-superspace action, which for the NBC case is given by eq. (4.14). We will first compute the expression for $G_{\rm NBC}[\mathrm{Bd}_f, \mathrm{Bd}_i]$ using eq. (5.1) and (2.19). We will then use the relation given in eq. (2.37) to compute the expression for $G_{\rm RBC}[\mathrm{Bd}_f, \mathrm{Bd}_i]$.

5.1 NBC at $t = 0$

To compute the expression for the transition amplitude in the case of Neumann boundary condition we make use of the results given in. eq. (2.19). However, to make this connection we first compare the one-particle action written in eq. (2.2) with the mini-superspace action for the NBC case given in eq. (4.14). This shows that the following substitution needs to be made

$$m \to -\frac{3}{2N_c}, \qquad V(q) = \lambda q \to \Lambda N_c q \quad \Rightarrow \quad \lambda \to \Lambda N_c, \qquad p_i \to \pi_i. \quad (5.2)$$

At this point, we are interested in computing

$$\int_{\mathrm{Bd}_i}^{\mathrm{Bd}_f} \mathcal{D}q(t)\ \exp\left(\frac{i}{\hbar} S_{\rm tot}[q, N_c]\right) = \exp\left[\frac{i}{\hbar}\left\{3kN_c + 4\alpha\left(k\pi_i + \frac{\pi_i^3}{27}\right)\right\}\right] \\ \times \int_{\mathrm{Bd}_i}^{\mathrm{Bd}_f} \mathcal{D}q(t) \exp\left[\frac{i}{2\hbar}\int_0^1 \mathrm{d}t\left\{-2\Lambda q N_c - \frac{3\dot{q}^2}{2N_c}\right\} + \frac{i}{\hbar} q_i \pi_i\right]. \quad (5.3)$$

This second line can be computed by exploiting the results given in eq. (2.19) and making use of the substitutions mentioned in eq. (5.2). Note that the presence of term $q_i\pi_i$ with an

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integration over the initial q_i leads to Fourier transform as expected in the NBC case. This gives the following expression for the Neumann boundary condition

$$\begin{aligned}\int_{\mathrm{Bd}_i}^{\mathrm{Bd}_f} \mathcal{D}q(t) \exp\left(\frac{i}{\hbar} S_{\mathrm{tot}}[q, N_c]\right) &= \exp\left(\frac{i}{\hbar} S_{\mathrm{tot}}^{\mathrm{on-shell}}[\bar{q}, N_c]\right) \\ &= \exp\left[\frac{i}{\hbar}\left\{3kN_c + 4\alpha\left(k\pi_i + \frac{\pi_i^3}{27}\right)\right\}\right] \bar{G}_{\mathrm{NBC}}[q_f, t=1; \pi_i, t=0]\,, \end{aligned} \tag{5.4}$$

where $S_{\mathrm{tot}}^{\mathrm{on-shell}}[\bar{q}, N_c]$ is given in eq. (4.15) and $\bar{G}_{\mathrm{NBC}}[q_f, t=1; \pi_i, t=0]$ is. given by

$$\bar{G}_{\mathrm{NBC}}[q_f, t=1; \pi_i, t=0] = \exp\left[\frac{i}{\hbar}\left\{\frac{\Lambda^2 N_c^3}{9} - \frac{\Lambda \pi_i N_c^2}{3} + \left(\frac{\pi_i^2}{3} - \Lambda q_f\right) N_c + \pi_i q_f\right\}\right]. \tag{5.5}$$

Notice that eq. (5.4) this differs from the result obtained in [3] by a numerical prefactor. This difference in normalization is attributed to different styles of doing computation of path integral involving zeta-functions. The process of computing path integral via time-slicing is more reliable and correctly fixes the normalization.

Note $S_{\mathrm{tot}}^{\mathrm{on-shell}}[\bar{q}, N_c]$ doesn't become singular at $N_c = 0$ and computation of the path integral over $q(t)$ doesn't give rise to additional functional dependence on N_c which can be singular. This allows us to extend the limits of N_c-integration all the way upto $-\infty$. This means that the transition amplitude is given by

$$G_{\mathrm{NBC}}[\mathrm{Bd}_f, \mathrm{Bd}_i] = \frac{1}{2}\int_{-\infty}^{\infty} \mathrm{d}N_c \ \exp\left(\frac{i}{\hbar} S_{\mathrm{tot}}^{\mathrm{on-shell}}[\bar{q}, N_c]\right)\,, \tag{5.6}$$

where $S_{\mathrm{tot}}^{\mathrm{on-shell}}[\bar{q}, N_c]$ is given in eq. (4.15).

To deal with the lapse integration in the NBC case we first make a change of variables. This is done by shifting the lapse N_c by a constant

$$N_c = \bar{N} + \frac{\pi_i}{\Lambda} \qquad \Rightarrow \qquad \mathrm{d}N_c \quad \to \quad \mathrm{d}\bar{N}\,. \tag{5.7}$$

This change of variable correspondingly implies that the action for the lapse $S_{\mathrm{tot}}^{\mathrm{on-shell}}[\bar{q}, N_c]$ becomes the following

$$S_{\mathrm{tot}}^{\mathrm{on-shell}}[\bar{q}, \bar{N}] = \frac{\Lambda^2}{9}\bar{N}^3 + (3k - \Lambda q_f)\bar{N} + \left(\frac{3}{\Lambda} + 4\alpha\right)\left(k\pi_i + \frac{\pi_i^3}{27}\right)\,. \tag{5.8}$$

An interesting outcome of this is that after the change of variables the π_i (initial momentum) dependence only appears in the constant term. After the change of variables, the transition amplitude is given by

$$\begin{aligned} G[\mathrm{Bd}_f, \mathrm{Bd}_i] &= \frac{1}{2}\exp\left[\frac{i}{\hbar}\left(\frac{3}{\Lambda} + 4\alpha\right)\left(k\pi_i + \frac{\pi_i^3}{27}\right)\right] \int_{-\infty}^{\infty} \mathrm{d}\bar{N} \ \exp\left[\frac{i}{\hbar}\left\{\frac{\Lambda^2}{9}\bar{N}^3 + (3k - \Lambda q_f)\bar{N}\right\}\right] \\ &= \Psi_1(\pi_i)\Psi_2(q_f)\,, \end{aligned} \tag{5.9}$$

where

$$\Psi_1(\pi_i) = \frac{1}{2}\exp\left[\frac{i}{\hbar}\left(\frac{3}{\Lambda} + 4\alpha\right)\left(k\pi_i + \frac{\pi_i^3}{27}\right)\right]\,, \tag{5.10}$$

$$\Psi_2(q_f) = \int_{-\infty}^{\infty} \mathrm{d}\bar{N} \ \exp\left[\frac{i}{\hbar}\left\{\frac{\Lambda^2}{9}\bar{N}^3 + (3k - \Lambda q_f)\bar{N}\right\}\right]\,. \tag{5.11}$$

The transition amplitude for the Neumann BC at $t = 0$ and Dirichlet BC at the $t = 1$ is a product of two parts: $\Psi_1(\pi_i)\Psi_2(q_f)$. $\Psi_1(\pi_i)$ is entirely dependent on initial momentum π_i and other $\Psi_2(q_f)$ is function of q_f is related to the final size of the Universe. The dependence on two boundaries gets separated, a factorization also noticed in [31] (and also in [3]) where the authors studied the Wheeler-DeWitt (WdW) equation in mini-superspace approximation of Einstein-Hilbert gravity.

5.2 RBC at $t = 0$

We now come to the task of computing eq. (5.1) for the case of RBC at $t = 0$, where $S_{\rm tot}[q, N_c]$ for the RBC case is given in eq. (4.19). This mini-superspace RBC action can be compared with the quantum mechanical RBC problem discussed in subsection 2.3. This comparison leads to the substitution mentioned in eq. (5.2). At this point, we are interested in computing

$$
\begin{aligned}
&\int_{\mathrm{Bd}_i}^{\mathrm{Bd}_f} \mathcal{D}q(t)\ \exp\left(\frac{i}{\hbar}S_{\rm tot}[q,N_c]\right)\\
&= \exp\left[\frac{i}{\hbar}\left\{3kN_c + 4\alpha\left(kP_i + \frac{P_i^3}{27}\right)\right\}\right]\\
&\quad\times \int_{\mathrm{Bd}_i}^{\mathrm{Bd}_f} \mathcal{D}q(t) \exp\left[\frac{i}{2\hbar}\int_0^1 \mathrm{d}t\left\{-2\Lambda q N_c - \frac{3\dot{q}^2}{2N_c}\right\} + \frac{i}{\hbar}q_i P_i + \frac{\beta}{2\hbar}(q_f^2 - q_i^2)\right]\\
&= \exp\left[\frac{i}{\hbar}\left\{3kN_c + 4\alpha\left(kP_i + \frac{P_i^3}{27}\right)\right\}\right] \bar{G}_{\rm RBC}[q_f, t=1; P_i, t=0]\,,
\end{aligned}
\tag{5.12}
$$

where

$$
\begin{aligned}
\bar{G}_{\rm RBC}[q_f, t=1; P_i, t=0] = \mathcal{N}\int_{\mathrm{Bd}_i}^{\mathrm{Bd}_f} \mathcal{D}q(t) \exp\Bigg[&\frac{i}{2\hbar}\int_0^1 \mathrm{d}t\left\{-2\Lambda q N_c - \frac{3\dot{q}^2}{2N_c}\right\}\\
&+\frac{i}{\hbar}q_i P_i + \frac{\beta(q_f^2 - q_i^2)}{2\hbar}\Bigg]\,.
\end{aligned}
\tag{5.13}
$$

It is noted that the expression for $\bar{G}_{\rm RBC}[q_f, t=1; P_i, t=0]$ is exactly referring to the RBC path integral in terms of DBC path integral with the substitution given in eq. (5.2). Except that $\mathcal{N}$ has been added to get rid of $e^{i\beta q_f^2/2\hbar}$ due to requirement of classicality, where in a WKB sense, the amplitude is expected to become more classical as the Universe expands [20, 30]. However, using eq. (2.37) one can express $\bar{G}_{\rm RBC}[q_f, t=1; P_i, t=0]$ in terms of $\bar{G}_{\rm NBC}$ via the integral transform. This means we have

$$
\begin{aligned}
\int_{\mathrm{Bd}_i}^{\mathrm{Bd}_f} \mathcal{D}q(t)\ \exp\left(\frac{i}{\hbar}S_{\rm tot}[q,N_c]\right) = &\exp\left[\frac{i}{\hbar}\left\{3kN_c + 4\alpha\left(kP_i + \frac{P_i^3}{27}\right)\right\}\right]\\
&\times\left(\frac{2\pi\hbar}{i\beta}\right)^{1/2}\int_{-\infty}^{\infty}\frac{\mathrm{d}\tilde{p}}{2\pi\hbar}e^{i(P_i-\tilde{p})^2/2\hbar\beta}\bar{G}_{\rm NBC}[q_f, t=1; \tilde{p}, t=0]\,.
\end{aligned}
\tag{5.14}
$$

The full path integral in the RBC case also involves integration over lapse N_c with limits $(0, \infty)$. This means

$$
\begin{aligned}
G[\mathrm{Bd}_f, \mathrm{Bd}_i] = &\left(\frac{2\pi\hbar}{i\beta}\right)^{1/2} \exp\left[\frac{4i\alpha}{\hbar}\left(kP_i + \frac{P_i^3}{27}\right)\right]\int_{-\infty}^{\infty}\frac{\mathrm{d}\tilde{p}}{2\pi\hbar}e^{i(P_i-\tilde{p})^2/2\hbar\beta}\\
&\times\int_{0^+}^{\infty}\mathrm{d}N_c \exp\left(\frac{3kN_c i}{\hbar}\right)\bar{G}_{\rm NBC}[q_f, t=1; \tilde{p}, t=0]\,.
\end{aligned}
\tag{5.15}
$$

As the N_c integrand is not singular at $N_c = 0$ so one can extend the integration limit all the way up to $-\infty$. The N_c-integration thereafter becomes similar to the integral studied in the case of NBC except the lack of Gauss-Bonnet term dependent on α. This means one can write

$$\int_{0^+}^{\infty} \mathrm{d}N_c \exp\left(\frac{3kN_c i}{\hbar}\right) \bar{G}_{\mathrm{NBC}}[q_f, t = 1; \tilde{p}, t = 0] = \Psi_1(\tilde{p})\Big|_{\alpha=0} \times \Psi_2(q_f)\,, \tag{5.16}$$

where Ψ_1 and Ψ_2 are given in eq. (5.10) and (5.11) respectively. Here again, as the integrand doesn't have $N_c = 0$ singularity, the integral can be extended all to way to $-\infty$ and including an extra $1/2$ factor which is absorbed in the definition of Ψ_1. The leftover integral is the integral over $\tilde{p}$. This integral can be cast into a more familiar form by re-definition of $\tilde{p}$ as

$$\tilde{p} \to \bar{p} - \frac{3\Lambda}{2\beta}\,. \tag{5.17}$$

Such a transformation allows us to rewrite the $\tilde{p}$ integral as an Airy integral. This is given by

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}\tilde{p}}{2\pi\hbar} e^{i(P_i-\tilde{p})^2/2\hbar\beta} \Psi_1(\tilde{p})\Big|_{\alpha=0} = \frac{1}{2} \exp\left[\frac{i(-18k\beta^2 + 2P_i^2\beta^2 + 6P_i\beta\Lambda + 3\Lambda^2)}{4\hbar\beta^3}\right] \Phi(P_i, \beta)\,. \tag{5.18}$$

where

$$\Phi(P_i, \beta) = \int_{-\infty}^{\infty} \frac{\mathrm{d}\bar{p}}{2\pi\hbar} \exp\left[\frac{i}{\hbar}\left\{\frac{\bar{p}^3}{9\Lambda} + \left(\frac{3k}{\Lambda} - \frac{P_i}{\beta} - \frac{3\Lambda}{4\beta^2}\right)\bar{p}\right\}\right]. \tag{5.19}$$

Putting all the pieces together give

$$\begin{aligned} G[\mathrm{Bd}_f, \mathrm{Bd}_i] = &\frac{1}{2}\left(\frac{2\pi\hbar}{i\beta}\right)^{1/2} \exp\left[\frac{4i\alpha}{\hbar}\left(kP_i + \frac{P_i^3}{27}\right)\right] \exp\left[\frac{i(-18k\beta^2 + 2P_i^2\beta^2 + 6P_i\beta\Lambda + 3\Lambda^2)}{4\hbar\beta^3}\right] \\ &\times \Phi(P_i, \beta)\ \Psi_2(q_f)\,. \end{aligned} \tag{5.20}$$

It is noticed that for the RBC case too the transition amplitude gets factorized in two parts: one dependent on the final boundary and one dependent on the initial boundary. In the next sub-section, we will compute these integrals in terms of Airy functions. It is expected that the RBC transition amplitude will be a product of two Airy functions along with exponential prefactors.

In order to see the $\beta \to 0$ limit clearly it is more cleaner to write the β-dependent terms in eq. (5.14) as follows

$$\left(\frac{2\pi\hbar}{i\beta}\right)^{1/2} e^{i(P_i-\tilde{p})^2/2\hbar\beta} = \int_{-\infty}^{\infty} \mathrm{d}\xi \exp\left[-\frac{i\beta}{2\hbar}\xi^2 + \frac{i}{\hbar}(P_i - \tilde{p})\xi\right]. \tag{5.21}$$

In this way, the limit $\beta \to 0$ gives a δ-function $\delta(P_i - \tilde{p})$. This limit is more harder to see from the end result of transition amplitude as one has to work with asymptotic forms of the Airy-functions.

5.3 Airy function

In this sub-section, we will compute the integrals $\Psi_2(q_f)$ and $\Phi(P_i, \beta)$ given in eq. (5.11) and (5.19) respectively. It should be noted that these functions can be identified with the *Airy*-integrals. These integrals are sensitive to the contour of integration. In the case of

Airy-integrals, the regions of convergence are within the following phase angles $\theta \equiv \arg(\bar{N})$: $0 \leq \theta \leq \pi/3$ (region 1), $2\pi/3 \leq \theta \leq \pi$ (region 0), and $4\pi/3 \leq \theta \leq 5\pi/3$ (region 2). One can define the following contours: $\mathcal{C}_0$ the contour running from region 0 to region 1, $\mathcal{C}_1$ the contour running from region 1 to region 2, and $\mathcal{C}_2$ the contour running from region 2 to region 0. By making use of the above contours of integration one can define the following integrals

$$Ai(z) = \frac{1}{2\pi}\int_{\mathcal{C}_0} \mathrm{d}x \exp\left[i\left(\frac{x^3}{3}+zx\right)\right], \tag{5.22}$$

$$Bi(z) = \frac{i}{2\pi}\int_{\mathcal{C}_2-\mathcal{C}_1} \mathrm{d}x \exp\left[i\left(\frac{x^3}{3}+zx\right)\right]. \tag{5.23}$$

$\Psi_2(q_f)$ can be computed as discussed in detail in the paper [3]. It is given by

$$\Psi_2(q_f) = \sqrt{3}\left(\frac{3\hbar}{\Lambda^2}\right)^{1/3} Ai\left[\left(\frac{\sqrt{3}}{\hbar\Lambda}\right)^{2/3}(3k-\Lambda q_f)\right], \tag{5.24}$$

The computation for $\Phi(P_i, \beta)$ can be done in an analogous manner. It is given by,

$$\Phi(P_i,\beta) = \left(\frac{3\Lambda}{\hbar^2}\right)^{1/3} Ai\bigg[\left(\frac{3\Lambda}{\hbar^2}\right)^{1/3}\left(\frac{3k}{\Lambda}-\frac{P_i}{\beta}-\frac{3\Lambda}{4\beta^2}\right)\bigg]. \tag{5.25}$$

Putting these expressions together gives the exact expression for the transition amplitude for the NBC and RBC case. These are given by,

$$\begin{aligned} G_{\mathrm{NBC}}[\mathrm{Bd}_i,\mathrm{Bd}_f] &= \frac{\sqrt{3}}{2}\left(\frac{3\hbar}{\Lambda^2}\right)^{\frac{1}{3}}\exp\left[\frac{i}{\hbar}\left(\frac{3}{\Lambda}+4\alpha\right)\left(k\pi_i+\frac{\pi_i^3}{27}\right)\right] \\ &\quad\times Ai\left[\left(\frac{3}{\hbar^2\Lambda^2}\right)^{\frac{1}{3}}(3k-\Lambda q_f)\right], \qquad (5.26) \\ G_{\mathrm{RBC}}[\mathrm{Bd}_i,\mathrm{Bd}_f] &= \sqrt{\frac{3\pi\hbar}{2i\beta}}\exp\left[\frac{4i\alpha}{\hbar}\left(kP_i+\frac{P_i^3}{27}\right)\right]\exp\bigg[\frac{i(-18k\beta^2+2P_i^2\beta^2+6P_i\beta\Lambda+3\Lambda^2)}{4\hbar\beta^3}\bigg] \\ &\quad\times\left(\frac{9}{\Lambda\hbar}\right)^{\frac{1}{3}}Ai\left[\left(\frac{3}{\hbar^2\Lambda^2}\right)^{\frac{1}{3}}(3k-\Lambda q_f)\right]Ai\bigg[\left(\frac{3\Lambda}{\hbar^2}\right)^{\frac{1}{3}}\left(\frac{3k}{\Lambda}-\frac{P_i}{\beta}-\frac{3\Lambda}{4\beta^2}\right)\bigg]. \quad (5.27) \end{aligned}$$

This is an exact result for the case when NBC and RBC are imposed at initial time $t = 0$ respectively. Notice also the correction coming from the Gauss-Bonnet sector of the gravitational action which appears as an exponential prefactor. It is also crucial to notice that this GB correction doesn't appear in the Airy functions which only depend on the boundary conditions.

6 $\hbar \to 0$ limit

In this section, we will study the $\hbar \to 0$ limit of the exact wave-function computed in the previous section. It is relevant to look at this as in the $\hbar \to 0$ limit the exact computation should reproduce known features and should agree with the results coming from saddle-point approximation which will be discussed in more detail in section 7. The $\hbar \to 0$ limit also highlights the configuration space which gives the dominant contribution in the path integral. This ultimately translates into preferable values for the initial boundary parameters. One can then do the saddle point analysis for these values of initial parameters which gives the dominant contribution in the path integral.

We will start by analyzing the nature of the contribution coming from the Gauss-Bonnet sector and the constraints that come from it in the limit $\hbar \to 0$.

6.1 Gauss-Bonnet contribution

The first thing we will focus on is the exponential prefactor that includes the effects coming from the Gauss-Bonnet sector. This prefactor is the same in both NBC and RBC cases ($P_i \to \pi_i$ in limit $\beta \to 0$). This prefactor is the following

$$\exp\left[\frac{4i\alpha}{\hbar}\left(kP_i + \frac{P_i^3}{27}\right)\right] = e^{iA(P_i)/\hbar}\,, \tag{6.1}$$

where $A(P_i)$ is given by

$$A(P_i) = 4\alpha\left(kP_i + \frac{P_i^3}{27}\right)\,. \tag{6.2}$$

In the limit $\hbar \to 0$ the dominant contribution comes from configuration which extremizes $A(P_i)$, which corresponds to the

$$A'(P_i) = 0 \quad \Rightarrow \quad P_i = \pm 3i\sqrt{k}\,. \tag{6.3}$$

In the limit $\beta \to 0$ this corresponds to $\pi_i = \pm 3i\sqrt{k}$! Note that $\pi_i = -3i\sqrt{k}$ also corresponds to the required value of the initial momentum for regular geometries with well-behaved perturbations in the case of Hartle-Hawking (HH) wave-function for the NBC [3] ($3i\sqrt{k}$ corresponds to Vilenkin, where perturbations are unstable). These two signs of P_i (or, π_i) correspond to two different orientations of wick rotations.

It should be emphasised that the $\hbar \to 0$ limit imposes classicalization when the quantum system starts behaving classically. This same end state where the system behaves like a classical system can also be achieved when $\hbar$ is not small, but the Gauss-Bonnet coupling is large. In this situation, when the Gauss-Bonnet coupling α is large, the dominant contribution comes from the configuration whose P_i lies around $P_i = \pm 3i\sqrt{k}$. However, this situation is entirely quantum in nature, as $\hbar$ is not small while the system acquires the end state of "classicality" in the limit of large Gauss-Bonnet coupling. This can happen in the very early stages of the Universe.

Let us consider the rotated $P_i = iy$, which is like considering an Euclideanised version action A. Then we have the Euclidean action $A_E(y)$ given by

$$A_E(y) = 4\alpha\left(ky - \frac{y^3}{27}\right)\,, \qquad e^{iA(P_i)/\hbar} \quad \Rightarrow \quad e^{-A_E(y)/\hbar}\,. \tag{6.4}$$

On extremization of Euclidean action $A_E(y)$ it is seen that for $\alpha > 0$ the point $y = -3\sqrt{k}$ corresponds to a stable saddle point and leads to an exponential with a positive argument

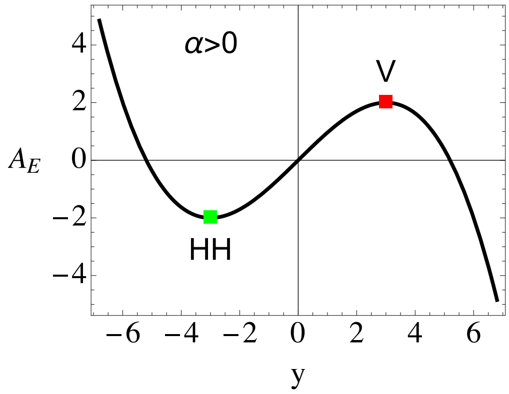

Figure 1. Plot of Euclidean action A_E given in eq. (6.4) vs y. The two saddle points are depicted by green and red squares. For $\alpha > 0$: green one corresponds to $P_i = -3i\sqrt{k}$ (stable) while the red one corresponds to $P_i = 3i\sqrt{k}$ (unstable). Interestingly, $P = -3i\sqrt{k}$ is also the value for which geometry become regularised at $t = 0$ and perturbations are well-behaved.

(Hartle-Hawking) while $y = 3\sqrt{k}$ correspond to unstable saddle point leading to exponential with a negative argument (Vilenkin Tunneling). The plot of $A_E(y)$ vs y is shown in figure 1. Coincidently, as was also mentioned in [3, 30] $P_i = -3i\sqrt{k}$ also correspond to initial condition for which geometry becomes regularised at $t = 0$ and perturbations are well-behaved within a certain regime of β, as discussed in subsequent sections.

6.2 Choice of β

The next question that one needs to address is that for the given $P_i = -3i\sqrt{k}$ (as required by stability), what are the allowed possibilities for the β in the case of RBC transition amplitude. This issue doesn't arise in the case of NBC as $\beta = 0$ for the NBC transition amplitude. To analyze the constraints (allowed values) that get imposed on β from the $\hbar \to 0$ limit, it is sufficient to study the nature of RBC wave-function at $t = 0$ for the fixed allowed value of $P_i = -3i\sqrt{k}$.

We start by considering the $\hbar \to 0$ limit of $\Phi(P_i, \beta)$ which for $P_i = -3i\sqrt{k}$ can be written as

$$\Phi(-3i\sqrt{k}, \beta) = (3\Lambda\hbar)^{1/3}\, Ai\left[\left(\frac{9\sqrt{\Lambda}}{\hbar}\right)^{2/3}\left(\sqrt{\frac{k}{\Lambda}} + \frac{i\sqrt{\Lambda}}{2\beta}\right)^2\right]. \tag{6.5}$$

For imaginary β the argument of Airy-function is always positive for $\Lambda > 0$. This observation allows us to obtain easily the $\hbar \to 0$ limit by exploiting the asymptotic structure of the Airy-functions with positive arguments

$$\Phi\left(-3i\sqrt{k}, \beta\right)\Big|_{\hbar\to 0} \sim (3\Lambda\hbar)^{\frac{1}{3}} \exp\left[-\frac{6\sqrt{\Lambda}}{\hbar}\left|\sqrt{\frac{k}{\Lambda}} + \frac{i\sqrt{\Lambda}}{2\beta}\right|^3\right] = (3\Lambda\hbar)^{\frac{1}{3}} \exp\left[-\frac{B_1(\beta)}{\hbar}\right], \tag{6.6}$$

where

$$B_1(\beta) = 6\sqrt{\Lambda}\left|\sqrt{\frac{k}{\Lambda}} + \frac{i\sqrt{\Lambda}}{2\beta}\right|^3, \tag{6.7}$$

and the $\sim$ sign implies that we are ignoring the numerical prefactor arising from the asymptotic form of Airy's function, which is not relevant for the following discussion. From the asymptotic structure, one could see that in limit $\hbar \to 0$ the dominant contribution comes from those configurations for which $B_1'(\beta) = 0$. This means we have

$$B_1'(\beta) = 18\sqrt{\Lambda}\left(\sqrt{\frac{k}{\Lambda}} + \frac{i\sqrt{\Lambda}}{2\beta}\right)^2 \left(\frac{-i\sqrt{\Lambda}}{2\beta^2}\right) = 0\,. \tag{6.8}$$

This means that the dominant contribution comes from configuration when β lies around $\beta_{\rm dom} = -i\Lambda/2\sqrt{k}$ (we don't consider the $\beta \to \infty$ case).

We next consider the exponential prefactor independent of α for $P_i = -3i\sqrt{k}$. This means that we have to investigate

$$\exp\left[\frac{i\left(3\Lambda^2 - 18i\beta\Lambda\sqrt{k} - 36k\beta^2\right)}{4\hbar\beta^3}\right] = \exp\left(\frac{iB_2(\beta)}{\hbar}\right)\,, \tag{6.9}$$

where

$$B_2(\beta) = \frac{\left(3\Lambda^2 - 18i\beta\Lambda\sqrt{k} - 36k\beta^2\right)}{4\beta^3}\,. \tag{6.10}$$

Once again in the $\hbar \to 0$ limit the dominant contribution comes from those configurations for which β lies around the extrema given by $B_2'(\beta) = 0$. This given $\beta_{\rm dom} = -i\Lambda/2\sqrt{k}$ same as before. This eventually give rise to two cases: $i\beta < i\beta_{\rm dom}$ and $i\beta > i\beta_{\rm dom}$ (note β and $\beta_{\rm dom}$ are imaginary). To study these cases carefully let us write

$$\beta = -\frac{i\Lambda x}{2\sqrt{k}} = \beta_{\rm dom} x \quad \text{where} \quad x \geq 0\,. \tag{6.11}$$

This gives

$$B_1(\beta) = 6\frac{k^{3/2}}{\Lambda}\left|1 - \frac{1}{x}\right|^3, \qquad B_2(\beta) = -\frac{6ik^{3/2}\left(3x^2 - 3x + 1\right)}{x^3\Lambda}\,. \tag{6.12}$$

At this point, it is worthwhile to note that according to the above analysis, purely (negative) imaginary initial momentum automatically enforces that the physically interesting range of β has to lie also in the purely (negative) imaginary direction. Imaginary β reminds the quantum nature of the initial condition. In the subsequent analysis, we will identify the specific sub-region that is physically relevant to our paper. Now, we consider cases when $x \leq 1$ and $x > 1$, respectively.

6.2.1 $0 \leq x \leq 1$

In this case, we will have

$$\exp\left(-\frac{B_1(\beta)}{\hbar}\right)\exp\left(\frac{iB_2(\beta)}{\hbar}\right) = \exp\left[\frac{6k^{3/2}}{\hbar\Lambda}\right]. \tag{6.13}$$

This can be combined with the exponential prefactor coming from the Gauss-Bonnet sector of gravity (α-dependent part) which for $P_i = -3i\sqrt{k}$ is given by $\exp(8k^{3/2}\alpha/\hbar)$. Combining the two exponential prefactors gives the following

$$\exp\left(\frac{6k^{3/2}}{\hbar\bar{\Lambda}}\right) \quad \text{where, } \bar{\Lambda} = \Lambda/\left(1+4\alpha\Lambda/3\right) . \tag{6.14}$$

This is precisely the Hartle-Hawking state with positive weighting. We note that the Gauss-Bonnet modification doesn't prevent the no-boundary Universe solution to exist whose contribution appears as a multiplicative factor (exponential weight). This higher-derivative gravity correction (which, although is topological) further supports the findings in the paper [59], where the authors found the no-boundary Universe even after the inclusion of generic higher-derivative terms. The topological nature of Gauss-Bonnet gravity, however allow us to go beyond the perturbative analysis done in [59]. It is worth emphasizing that the exponential prefactor is independent of β. The positive exponent says that lower Λ is more favorable, i.e., low values of potential are preferred. The analysis also includes the case $\beta = 0$ i.e., Neumann BC. For a special value of $\alpha = -3/4\Lambda$, this HH-factor becomes unity.

6.2.2 $x > 1$

In this case, we will have

$$\exp\left(-\frac{B_1(\beta)}{\hbar}\right)\exp\left(\frac{iB_2(\beta)}{\hbar}\right) = \exp\left[\frac{6k^{3/2}}{\hbar\Lambda}\left(-1+\frac{6}{x}-\frac{6}{x^2}+\frac{2}{x^3}\right)\right] . \tag{6.15}$$

This exponential factor has to be multiplied by the contribution coming from the Gauss-Bonnet sector $e^{8k^{3/2}\alpha/\hbar}$ to get the full overall exponential prefactor. In the large $x \to \infty$ limit, we get inverse Hartle-Hawking. This configuration represents the tunneling geometry, and perturbations are unstable [21]. However, for other values of $x > 1$, the nature of exponential depend on the behavior of the functional form of x. Let us call this function

$$f(x) = -1 + \frac{6}{x} - \frac{6}{x^2} + \frac{2}{x^3} . \tag{6.16}$$

We notice that $f(1) = 1$ and $f(\infty) = -1$. It should be mentioned that $f'(x) = -6(x-1)^2/x^4 < 0$ for all values x except $x = 1$ and $x = \infty$. This means $f(x)$ is a monotonically decreasing function of x within the range between 1 and -1. At some point

$$x_0 = 2 + 2^{1/3} + 2^{2/3} , \tag{6.17}$$

the function $f(x)$ changes sign and becomes negative for $x > x_0$ while it remains positive for $1 < x < x_0$ (although it is not equal to 1, thereby implying that it doesn't have the actual Hartle-Hawking factor $6k^{3/2}/\hbar\bar{\Lambda}$ as in eq. (6.14)).

6.3 Interpretation

Before proceeding further, let us mention a possible interpretation of the Robin boundary condition in the context of no boundary proposal that is immediate from the above choice of

β (purely imaginary). In this case, one can interpret the boundary terms as being complex coherent state i.e.,

$$\exp\left[i\left(-\frac{\beta q_i^2}{2}+P_i q_i\right)\right] \xrightarrow{\text{for, }\beta=-i|\beta|} \exp\left[-\frac{|\beta|q_i^2}{2}+iP_i q_i\right]. \tag{6.18}$$

This is a Gaussian state with an imaginary momentum P_i, peak at $q_i = 0$ and uncertainty determined by $\beta(\Lambda)$ [30]. It describes a state with shared uncertainty between the scale factor and the momentum of the universe. In $\beta \to 0$, this state becomes a plane wave, which is a momentum eigenstate. Taking this as an initial state for the Robin boundary condition [30, 33], one can express the final state as a path-integral in the following manner

$$\Psi[q_f,\beta,P_i] = \int dN\mathcal{D}q\, dq_i e^{iS_{DD}[q_f,N,q]/\hbar}\,\psi_0[P_i,\beta], \quad \psi_0[P_i,\beta] \propto e^{i\left(-\frac{\beta q_i^2}{2}+P_i q_i\right)}, \tag{6.19}$$

where S_{DD} is action with imposition of Dirichlet BC at the two endpoints. In the limit $|\beta| \to \infty$, $e^{-i\beta q_i^2/2} \to \delta(q_i)$. This is a sharp imposition of the boundary condition $q_i = 0$, which is the Dirichlet condition. The other limit $|\beta| \to 0$ gives Neumann BC. Finite values of β act as a kind of 'regulator' embedding a regularized version of δ-function in the path-integral. In a sense, RBC is a regularised DBC. This is also compatible with quantum uncertainty principle in the sense knowing the q_i arbitrarily accurate would render the initial momentum completely undetermined. Also, quantum uncertainty doesn't allow to determine the initial size with arbitrary accuracy [33]. In a way, studying Robin BC is the most appropriate scenario due to its compatibility with the quantum uncertainty principle and the regularized way of introducing Dirichlet BC overcoming technical complications associated with dealing with path integrals involving δ-function.

7 Saddle-point approximation

The analysis of $\hbar \to 0$ has shown us that certain initial configurations are favorable and give the dominant contribution in the path integral. The $\hbar \to 0$ limit of the Gauss-Bonnet contribution shows that the dominant contribution comes from $P_i = \pm 3i\sqrt{k}$. Of which only the $P_i = -3i\sqrt{k}$ correspond to a stable configuration. Coincidentally, this is also the same required value of P_i which leads to stable well-behaved fluctuations in the Hartle-Hawking no-boundary proposal of the Universe [2, 3, 30–33]. In a sense, the Gauss-Bonnet presence favors the stable Hartle-Hawking no-boundary Universe.

In this section, we will do the saddle point analysis of the gravitational path integral in the mini-superspace approximation. We will do this for the case of Robin boundary condition as the case of Neumann boundary condition has been already investigated in [3]. However, the saddle-point study of the path integral will be done in a slightly different manner. The gravitational path integral in the mini-superspace approximation in the RBC case is given in eq. (5.1) where the path integral over the $q(t)$ is given in eq. (5.12). We use the methods developed in analyzing the quantum-mechanical RBC problem to convert eq. (5.12) into eq. (5.14), which relates the RBC path integral of $q(t)$ to the NBC path integral of the same gravitational theory. Doing lapse N_c integration over this gives the full transition amplitude, relating the RBC transition amplitude with the NBC one as given in eq. (5.15). This allows us to write the full RBC transition amplitude as a product of two integrals after we make use of eq. (5.16). One is integral over $\tilde{p}$ and other integral over $\bar{N}$, where $\bar{N}$-integral is given by eq. (5.11). Our strategy in doing the saddle-point analysis is to analyze each of these integrals (integral over N_c and integral over $\tilde{p}$) separately.

We first take a look at the integral over N_c given in eq. (5.16). If one shifts N_c as in eq. (5.7) then this integral becomes an integral over $\bar{N}$ and an exponential prefactor dependent on $\tilde{p}$. The $\bar{N}$-integral depends on q_f written as $\Psi_2(q_f)$. The saddle point analysis of this has already been done in [3] and won't be repeated here. The results from the saddle analysis revealed an interesting feature. For all values of q_f there are two saddle points, however, for $q_f < 3k/\Lambda$ only one saddle point which lies on the imaginary axis is relevant, while for $q_f > 3k/\Lambda$ there are two relevant saddle points lying on the real axis. This means that for $q_f < 3k/\Lambda$ Universe is Euclidean as can be seen from the exponential form of the transition amplitude indicating imaginary 'time', while for $q_f > 3k/\Lambda$ Universe is Lorentzian as is noticed from the oscillatory nature of transition amplitude indicating real time [3].

Our task then shifts to analyzing the integral over $\tilde{p}$ given in eq. (5.18).

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}\tilde{p}}{2\pi\hbar} e^{i(P_i-\tilde{p})^2/2\hbar\beta} \Psi_1(\tilde{p}) \bigg|_{\alpha=0} = \int_{-\infty}^{\infty} \frac{\mathrm{d}\tilde{p}}{2\pi\hbar} e^{iB(\tilde{p})/\hbar} = \int_{-\infty}^{\infty} \frac{\mathrm{d}\tilde{p}}{2\pi\hbar} e^{\{h(\tilde{p})+iH(\tilde{p})\}/\hbar}, \tag{7.1}$$

where

$$B(\tilde{p}) = \frac{(P_i-\tilde{p})^2}{2\beta} + \frac{3}{\Lambda}\left(k\tilde{p} + \frac{\tilde{p}^3}{27}\right). \tag{7.2}$$

and $h(\tilde{p})$ is the corresponding Morse-function while the $H(\tilde{p})$ corresponds to real part of the B-function. By a shift of variable as stated in eq. (5.17), the $\tilde{p}$-integral can be cast into an Airy integral along with an exponential prefactor, as discussed in section 5.2. Here, we will study integral in eq. (7.1) using Picard-Lefschetz methods (see [2, 20, 54–57] for review on Picard-Lefschtez and analytic continuation).

The saddle-points of $\tilde{p}$ can be obtained by computing the expression $\mathrm{d}B(\tilde{p})/\mathrm{d}\tilde{p}$. The saddle points are computed from the equation

$$\frac{\mathrm{d}B(\tilde{p})}{\mathrm{d}\tilde{p}} = \frac{\tilde{p}^2}{3\Lambda} + \frac{\tilde{p}}{\beta} + \left(\frac{3k}{\Lambda} - \frac{P_i}{\beta}\right) = 0. \tag{7.3}$$

This is a quadratic equation in $\tilde{p}$ resulting in two saddle points. The discriminant Δ of the above quadratic equation is given by

$$\Delta = \frac{1}{\beta^2} - \frac{4k}{\Lambda^2} + \frac{4P_i}{3\beta\Lambda}. \tag{7.4}$$

For stable configuration referring to Hartle-Hawking no-boundary Universe ($P_i = -3i\sqrt{k}$) and β given by eq. (6.11), the discriminant Δ becomes the following

$$\Delta\bigg|_{\mathrm{Hartle-Hawking}} = -\frac{4k(x-1)^2}{x^2\Lambda^2}. \tag{7.5}$$

For all values of $x > 0$, the discriminant $\Delta < 0$ implying that both the saddle points are complex, which for stable Hartle-Hawking no-boundary Universe is seen to be both imaginary.

These are given by

$$\tilde{p}_1 = -3i\sqrt{k}\,, \quad \tilde{p}_2 = \frac{3i\sqrt{k}(x-2)}{x}\,. \tag{7.6}$$

This shows that while saddle point $\tilde{p}_1$ remains fixed at the same position for all x, the saddle point $\tilde{p}_2$ moves from negative imaginary axis to positive imaginary axis as x increases. It becomes zero for $x = 2$. At the saddle point $\tilde{p}_1$, we have a vanishing initial on-shell geometry, i.e., $\bar{q}_i = 0$, while at the other saddle point $\tilde{p}_2$, we have a non-vanishing initial geometry $\bar{q}_i = 12k(x-1)/\Lambda x^2$. The 'on-shell' value of $B(\tilde{p})$ ($B(\tilde{p})$ computed at the saddle points) is given by

$$B(\tilde{p}_1) = -\frac{6ik^{3/2}}{\Lambda}\,, \quad B(\tilde{p}_2) = -\frac{6ik^{3/2}}{\Lambda}\left(\frac{2}{x^3} - \frac{6}{x^2} + \frac{6}{x} - 1\right)\,. \tag{7.7}$$

It should be highlighted that for stable Hartle-Hawking no-boundary Universe referring to $P_i = -3i\sqrt{k}$ the 'action' $B(\tilde{p})$ given in eq. (7.2) is complex. The morse function $h(\tilde{p})$ at the saddle points for all values of x is given by

$$h(\tilde{p}_1) = \frac{6k^{3/2}}{\Lambda}\,, \quad h(\tilde{p}_2) = \frac{6k^{3/2}}{\Lambda}\left(\frac{2}{x^3} - \frac{6}{x^2} + \frac{6}{x} - 1\right)\,. \tag{7.8}$$

Higher derivatives of $B(\tilde{p})$ with respect to $\tilde{p}$ at the saddle point is given by,

$$\left.\frac{\mathrm{d}^2 B}{\mathrm{d}\tilde{p}^2}\right|_{\tilde{p}=\tilde{p}_1} = -\frac{2i(x-1)\sqrt{k}}{x\Lambda}\,, \quad \left.\frac{\mathrm{d}^2 B}{\mathrm{d}\tilde{p}^2}\right|_{\tilde{p}=\tilde{p}_2} = \frac{2i(x-1)\sqrt{k}}{x\Lambda}\,, \quad \left.\frac{\mathrm{d}^3 B}{\mathrm{d}\tilde{p}^3}\right|_{\tilde{p}=\tilde{p}_{1,2}} = \frac{2}{3\Lambda}\,. \tag{7.9}$$

One can expand the function $B(\tilde{p})$ around the saddle point $\tilde{p}_\sigma$ where $\sigma = \{1, 2\}$. This gives

$$B(\tilde{p}) = B(\tilde{p}_\sigma) + \left.\frac{\mathrm{d}B(\tilde{p})}{\mathrm{d}\tilde{p}}\right|_{\tilde{p}_\sigma}\delta\tilde{p} + \frac{1}{2}\left.\frac{\mathrm{d}^2 B(\tilde{p})}{\mathrm{d}\tilde{p}^2}\right|_{\tilde{p}_\sigma}(\delta\tilde{p})^2 + \frac{1}{6}\left.\frac{\mathrm{d}^3 B(\tilde{p})}{\mathrm{d}\tilde{p}^3}\right|_{\tilde{p}_\sigma}(\delta\tilde{p})^3\,, \tag{7.10}$$

where $\delta\tilde{p} = \tilde{p} - \tilde{p}_\sigma$. The series stops at cubic order as the highest power of $\tilde{p}$ in $B(\tilde{p})$ is three. The second variation at the saddle-point can be written as $B''(\tilde{p}_\sigma) = r_\sigma e^{i\rho_\sigma}$, where r_σ and ρ_σ depends on boundary conditions. Near the saddle point the change in iH will go like

$$\delta(iH) \propto i\left(B''(\tilde{p}_\sigma)\right)(\delta\tilde{p})^2 \sim v_\sigma^2 e^{i(\pi/2 + 2\theta_\sigma + \rho_\sigma)}\,, \tag{7.11}$$

where we write $\delta\tilde{p} = v_\sigma e^{i\theta_\sigma}$ and θ_σ is the direction of flow lines at the corresponding saddle point. Given that the imaginary part H remains constant along the flow lines, so this means

$$\theta_\sigma = \frac{(2k-1)\pi}{4} - \frac{\rho_\sigma}{2}\,, \tag{7.12}$$

where $k \in \mathbb{Z}$.

For the steepest descent and ascent flow lines, their corresponding $\theta_\sigma^{\mathrm{des/aes}}$ is such that the phase for δH correspond to $e^{i(\pi/2+2\theta_\sigma+\rho_\sigma)} = \mp 1$. This implies

$$\theta_\sigma^{\mathrm{des}} = k\pi + \frac{\pi}{4} - \frac{\rho_\sigma}{2}\,, \quad \theta_\sigma^{\mathrm{aes}} = k\pi - \frac{\pi}{4} - \frac{\rho_\sigma}{2}\,. \tag{7.13}$$

These angles can be computed numerically in our case.

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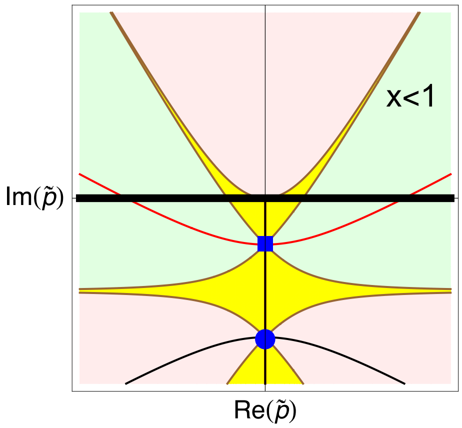

Figure 2. The gravitational path integral with the Gauss-Bonnet contribution favours $P_i = -3i\sqrt{k}$ as the stable configuration which also happens to correspond to Hartle-Hawking no-boundary Universe. For this value of P_i the action for B given in eq. (7.2) is complex. This figure arises in association with the Picard-Lefschetz analysis of the contour integral given in eq. (7.1). The red lines correspond to the steepest descent lines (thimbles $\mathcal{J}_\sigma$), while the thin black lines are the steepest ascent lines and are denoted by $\mathcal{K}_\sigma$. Here we choose parameter values: $k = 1$, $\Lambda = 3$, and $x = 1/2$. For this, the saddle point $\tilde{p}_1$ corresponds to blue-square, while saddle point $\tilde{p}_2$ corresponds to blue-circle. Only the saddle point $\tilde{p}_1$ (blue square) is relevant. The steepest ascent lines emanating from it intersect the original integration contour $(-\infty, +\infty)$, which is shown by thick-black line. The Morse-function h is positive for both saddle points: $h(\tilde{p}_{1,2}) > 0$. The light-green region is the allowed region with $h < h(\tilde{p}_\sigma)$ for all values of σ. The light-pink region (forbidden region) has $h > h(\tilde{p}_\sigma)$ for all σ. The intermediate region is depicted in yellow. The boundary of these regions is depicted in brown lines. Along these lines, we have $h = h(\tilde{p}_\sigma)$.

Under the saddle point approximation, the contour integral given in eq. (7.1) can be computed using Picard-Lefschetz methods (see [2] for details of PL methodology). This gives

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{\mathrm{d}\tilde{p}}{2\pi\hbar} e^{iB(\tilde{p})/\hbar} &= \sum_\sigma \frac{n_\sigma}{2\pi\hbar} e^{iB(\tilde{p}_\sigma)/\hbar} \int_{\mathcal{J}_\sigma} \mathrm{d}\delta\tilde{p} \exp\left[i \frac{B''(\tilde{p}_\sigma)}{2\hbar} (\delta\tilde{p})^2\right], \\ &= \sum_\sigma n_\sigma \sqrt{\frac{1}{2\pi\hbar |B''(\tilde{p}_\sigma)|}} e^{i\theta_\sigma} e^{iB(\tilde{p}_\sigma)/\hbar}. \end{aligned} \tag{7.14}$$

where $\mathcal{J}_\sigma$ refers to steepest descent line and n_σ is the intersection number which will take values $(0, \pm 1)$ accounting for the orientation of contour over each thimble. In the following we will be using this expression to compute the contour integral for various values of x.

7.1 $0 < x < 1$

For $0 < x < 1$, both the saddle points $\tilde{p}_1$ and $\tilde{p}_2$ lie on the negative imaginary axis with $|\tilde{p}_2| < |\tilde{p}_1|$, thereby implying that $\tilde{p}_2$ lie below $\tilde{p}_1$ on the negative imaginary axis.

For both the saddle points, the morse function is positive: $h(\tilde{p}_{1,2}) > 0$. In figure 2, we plot the various flow-line, saddle points, and forbidden/allowed regions. From the graph, we notice that only the steepest ascent line from $\tilde{p}_1$ intersects the original integration contour, thereby making it relevant. At this saddle point, the initial geometry is observed to be vanishing, thereby satisfying the "compactness" and "regularity" criterion of Hartle-Hawking

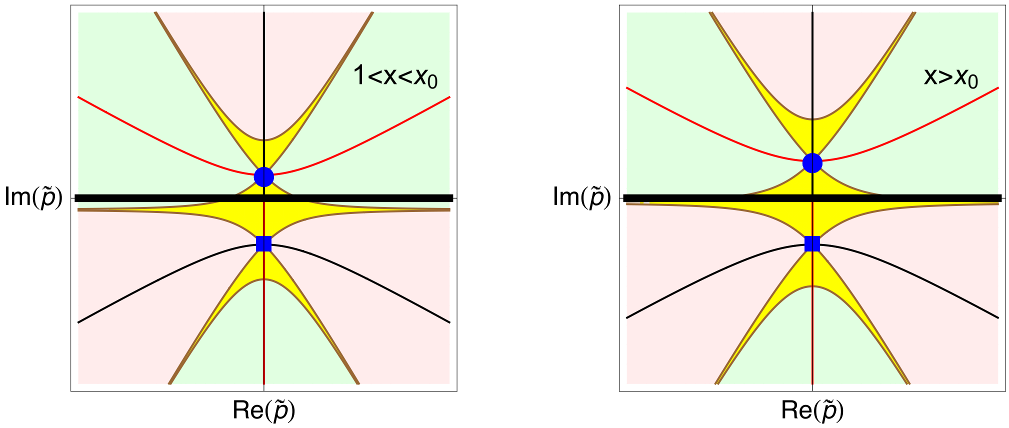

Figure 3. Here we choose parameter values: $k = 1$, $\Lambda = 3$, and $x = 4$ and $x = 10$ respectively. For this, the saddle point $\tilde{p}_2$ corresponds to blue-circle, while saddle point $\tilde{p}_1$ correspond to blue-square. Only the saddle point $\tilde{p}_2$ (blue square) is relevant. The Morse-function h for $\tilde{p}_1$ is always positive, while $h(\tilde{p}_2)$ goes from positive to negative as $1 < x < \infty$. The crossover happens at $x = x_0$ given by eq. (6.17).

no-boundary proposal. The thimbles passing through this saddle constitute the deformed contour of integration. It should also be specified that for $x < 1$, the second-derivative of $B(\tilde{p})$ computed at saddle point $\tilde{p}_1$ is proportional to $+i$, which in saddle point approximation gives appropriate Gaussian weight allowing to do Gaussian integration.

The Picard-Lefschetz theory then gives the following in the saddle point approximation as

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}\tilde{p}}{2\pi\hbar} e^{i(P_i-\tilde{p})^2/2\hbar\beta}\Psi_1(\tilde{p})\bigg|_{\alpha=0} \approx \sqrt{\frac{\Lambda x k^{-1/2}}{4\hbar\pi(1-x)}}\; e^{\frac{6k^{3/2}}{\hbar\Lambda}} \,. \tag{7.15}$$

This regime reproduces the same exponential factor as in Hartle-Hawking no-boundary Universe. This regime leads to stable perturbations for the Einstein-Hilbert gravity studied in [33]. It is expected that Gauss-Bonnet gravitational corrections which are topological in nature in 4D won't alter this as long as perturbations are small.

7.2 $x > 1$

For $x > 1$ both saddles still lie on the imaginary axis. However, only saddle point $\tilde{p}_1$ remains fixed on the negative imaginary axis for all values of x, while the saddle point $\tilde{p}_2$ moves from negative imaginary axis to positive imaginary axis. It crosses the origin at $x = 2$ in the case of boundary condition corresponding to Hartle-Hawking no-boundary Universe.

The Morse function corresponding to saddle point $\tilde{p}_1$ is always positive, while the Morse function corresponding to $\tilde{p}_2$ goes from positive to negative. The crossover takes place at $x = x_0$ where x_0 is given in eq. (6.17). After the cross-over the overall exponential weight becomes negative, thereby implying an inverse Hartle-Hawking regime. For $x > 1$, it is seen from figure 3 that only the steepest ascent curves emanating from $\tilde{p}_2$ intersect the original integration contour, implying that it is the only relevant saddle point. At this saddle, the initial geometry of the universe is non-vanishing, given by $\bar{q}_i = 12k(x-1)/\Lambda x^2$. The thimbles passing through this saddle will constitute the deformed contour of integration. The

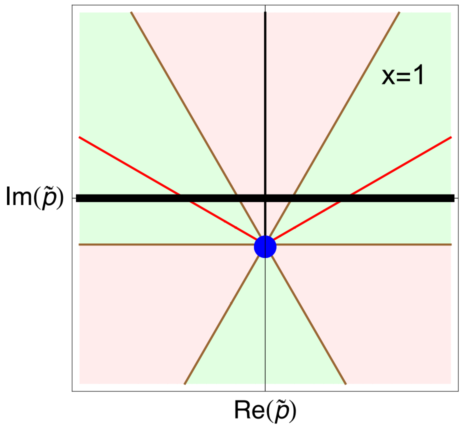

Figure 4. Here we choose parameter values: $k = 1$, $\Lambda = 3$, and $x = 1$. This is the degenerate situation for which the saddle point $\tilde{p}_1$ and $\tilde{p}_2$ coincide. The Morse-function h is positive for both saddle points: $h(\tilde{p}_{1,2}) = 6k^{3/2}/\Lambda = 2$.

Picard-Lefschetz analysis then gives the following in the saddle point approximation as

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}\tilde{p}}{2\pi\hbar} e^{i(P_i-\tilde{p})^2/2\hbar\beta}\Psi_1(\tilde{p})\bigg|_{\alpha=0} \approx \sqrt{\frac{\Lambda x k^{-1/2}}{4\pi\hbar(x-1)}}\; e^{\frac{6k^{3/2}}{\hbar\Lambda}\left(\frac{2}{x^3}-\frac{6}{x^2}+\frac{6}{x}-1\right)} . \tag{7.16}$$

However, the perturbations in this regime are unstable for Einstein-Hilbert gravity and the Gauss-Bonnet is expected to not change this.

7.3 $x = 1$

This is the degenerate case. The discriminant $\Delta = 0$ for $x = 1$ and both saddle points coincide: $\tilde{p}_1 = \tilde{p}_2 = -3i\sqrt{k}$, with vanishing initial geometry of the universe at the relevant saddle. In this degenerate case both the on-shell action ($B(\tilde{p}_1) = B(\tilde{p}_2)$) and the Morse-function become equal ($h(\tilde{p}_1) = h(\tilde{p}_2) = 6k^{3/2}/\Lambda$). In this situation, the saddle-point approximation breaks down as the second derivative $B''(\tilde{p}_\sigma) = 0$. One needs to go beyond the second order in the series expansion in eq. (7.10). In this series, the third order term is non-zero. This situation is depicted in figure 4. In this case, the contour integration gives the following

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}\tilde{p}}{2\pi\hbar} e^{i(P_i-\tilde{p})^2/2\hbar\beta}\Psi_1(\tilde{p})\bigg|_{\alpha=0} \approx \frac{\sqrt{3}}{2\pi\hbar} e^{iB(\tilde{p}_\sigma)/\hbar}\int_0^{\infty} \mathrm{d}v_\sigma e^{-v_\sigma^3/9\Lambda\hbar} = \frac{(\Lambda/3\hbar^2)^{1/3}}{\Gamma(2/3)} e^{\frac{6k^{3/2}}{\hbar\Lambda}} . \tag{7.17}$$

8 Conclusions and outlook

In this paper we consider the gravitational path integral of Gauss-Bonnet gravity and study it directly in the Lorentzian signature in four spacetime dimensions. Gauss-Bonnet sector of gravity being topological in nature in 4D doesn't contribute to the bulk dynamics of the field but has an active role to play at the boundaries or in situations where boundaries play an important role. One such situation is the path integral which is sensitive to boundary conditions. Past studies have investigated the effects of Gauss-Bonnet sector of gravity on the transition amplitude which is given by the gravitational path integral [2, 3]. These studies focussed on exploring the consequence of imposing Neumann boundary condition at the initial time. In this paper we study extensively the effects of imposing Robin boundary condition at the initial time and investigate the role played by the Gauss-Bonnet sector of gravity systematically in the Lorentzian gravitational path integral. We setup the platform for raising and addressing these issues in the reduced setup of the mini-superspace approximation.

We start by first considering path integrals of a particle in one-dimensional potential for various boundary conditions. The path integral is evaluated for three different choices of boundary condition: Dirichlet boundary condition (DBC) at both initial and final time (fixing position of particle at two end points), Neumann BC (fixing conjugate momenta denoted in paper by p_i) and Robin BC (fixing linear combination of conjugate momenta and position at the initial time: $p_i + \beta q_i = P_i$, where β is some parameter) with Dirichlet at final time. The transition amplitudes are shown to be inter-related with each other by integral transforms. By exploiting these inter-relations via integral transforms one can compute the path integral with NBC or RBC at initial time (and DBC at final time) from the path integral of DBC at both end points. These inter-relations can be further manipulated to express the RBC path integral as an integral transform of the NBC path integral. These results get later utilized in the paper for the computation of gravitational path integral in the mini-superspace.

We take a fresh look at the gravitational path integral in the mini-superspace approximation in the Lorentzian signature with Robin boundary condition at the initial boundary (that is fixing linear combination of conjugate momenta and field at the initial boundary). Past works have shown that NBC and RBC at the initial time leads to stable Universe [2, 3, 30–33], while DBC at initial time leads to unsuppressed perturbations [20, 21, 24]. These motivate us to study RBC gravitational path integral more carefully for the Gauss-Bonnet gravity (the case for Neumann BC was investigated in [3]). The transition amplitude from one 3-geometry to another is given by a path integral over $q(t)$ and a contour integration over lapse N_c.

The paper systematically studies the path integral in the mini-superspace approximation for the Gauss-Bonnet gravity. The gravitational action is varied and carefully analysed to setup a consistent variational problem with Robin BC at the initial boundary. This process leads to dynamical equation of motion and a set of surface terms that need to be supplemented to the action to make the gravitational system a consistent variational problem. In this process we construct the surface term needed for the Gauss-bonnet gravity with the Robin boundary condition. This surface action is proportional to β and smoothly reduces to the surface term for the case of Neumann BC problem [3] in the limit of $\beta \to 0$. To our knowledge this wasn't known in literature earlier. The path integral is then studied for the total action which has been supplemented by these surface terms.

The path integral to be computed is given in eq. (5.1), where S_{tot} for the NBC and RBC case are given in eq. (4.14) and (4.19) respectively. To compute the path integral over $q(t)$ we make use of the results derived in one-dimensional quantum mechanical problem discussed in section 2. In the NBC case: the lapse N_c integration and path integral over $q(t)$ can be performed exactly as it was also shown in [3]. The numerical prefactor has been correctly computed as we do the computation of path integral via first principles (method of 'time-slicing'). In the RBC case: results of sub-section 2.3 and algebraic manipulations allow us to reduce the problem into the product of two Airy-integrals (modulo exponential prefactors).

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One is just the lapse N_c integration and gives rise to $\Psi_2(q_f)$ (Airy-integral dependent only on the final boundary). The other is an integration over $\tilde{p}$: doing a Gaussian integral transform of $\Psi_1(\tilde{p})$ at $\alpha = 0$. With a change of integration variable, it is easy to convert this into an Airy integral defined as $\Phi(P_i, \beta)$ given in eq. (5.19), and an exponential prefactor dependent on P_i and β. All this when put together gives the exact transition amplitude in the RBC case which is given in eq. (5.27). In the limit $\beta \to 0$ this correctly reproduces the exact NBC transition amplitude given in eq. (5.26). This exact transition amplitude for the RBC including the Gauss-Bonnet effects is new and hasn't been known in literature earlier.

We then consider the $\hbar \to 0$ limit of the exact transition amplitude. This allows us to single out and focus on configurations giving dominant contribution to the path integral. The Gauss-Bonnet contribution which appears as an exponential prefactor only, shows that in the $\hbar \to 0$ limit two configurations will give dominant contribution: $P_i = -3i\sqrt{k}$ and $P_i = 3i\sqrt{k}$. The first one corresponds to stable configuration while the later is unstable. Coincidentally, $P_i = -3i\sqrt{k}$ also is the configuration for the Hartle-Hawking no-boundary Universe where the perturbation are well-behaved. In a sense the Gauss-Bonnet contribution naturally picks and favors the Hartle-Hawking no-boundary condition while the other boundary condition is disfavoured. Same thing happens for large Gauss-Bonnet coupling with $\hbar$ not small. In this case, we are still in deep quantum regime however the large Gauss-Bonnet coupling favors the Hartle-Hawking no-boundary Universe. This is truly a non-perturbative feature.

We next consider the $\hbar \to 0$ limit of the other terms in the RBC transition amplitude for the case of $P_i = -3i\sqrt{k}$ (Hartle-Hawking no-boundary Universe). This allows us to find the boundary configuration characterised by β which will give dominant contribution to the RBC transition amplitude. It is seen that $\beta = \beta_{\text{dom}} = -i\Lambda/2\sqrt{k}$ gives the dominant contribution. On scaling our $\beta = \beta_{\text{dom}} x$ we see that the domain of x gets separated into various regimes. For $x \leq 1$ we get the same Hartle-Hawking exponential prefactor $e^{6k^{3/2}/\hbar\bar{\Lambda}}$ where $\bar{\Lambda}$ is given in eq. (6.14). For $1 < x < x_0$ (where x_0 is given in eq. (6.17)), the argument of the exponential prefactor remains positive but monotonically decreases. At $x = x_0$ crossover happens and the argument becomes negative entering in inverse Hartle-Hawking regime. For $x_0 < x < \infty$, the argument of exponential decreases but asymptotically approaches to $e^{\frac{-6k^{3/2}}{\hbar\Lambda}\left(1-\frac{4\alpha\Lambda}{3}\right)}$ which is the exponential prefactor obtained with Dirichlet boundary condition at $t = 0$ and is known to be unstable as perturbations are unsuppressed [20, 21, 24]. This analysis shows that the allowed region where we correctly reproduce the Hartle-Hawking exponential prefactor is $0 \leq x \leq 1$. This was noticed in [30, 33] via a different route.

We then study the transition amplitude in the saddle-point approximation and make use of Picard-Lefschetz methods to compute the contour integrals. However, we address the problem in a different manner. Past studies in this direction applied Picard-Lefschetz methodology to the lapse N_c-integration after the $q(t)$ path integral has been worked out [30, 33]. Here as we managed to express RBC transition amplitude as NBC transition amplitude via eq. (5.15), so it gave us the flexibility of approaching the problem in another way. Instead of direct lapse N_c integration on the full on-shell action, we do saddle point analysis of only the momentum integration given in eq. (7.1). Saddle analysis of the N_c-integration acting only on the NBC part was done in [3]. It showed that there are two saddle points for all values of q_f. For $q_f < 3k/\Lambda$ the two saddle points lie on the imaginary axis in the complex N_c plane (only one is relevant), while for $q_f > 3k/\Lambda$ the two saddles lie on real axis in the complex N_c plane (both are relevant). As q_f increases Universe undergoes a transition from Euclidean to Lorentzian phase [3]. In this paper, we just study the $\tilde{p}$-integral via Picard-Lefschetz methods.

The saddle analysis of the $\tilde{p}$-integration shows that in the complex $\tilde{p}$-plane there are always two saddle points, both lying on the imaginary $\tilde{p}$ axis. The saddle point $\tilde{p}_1$ is independent of x (remains fixed), with vanishing initial geometry, while the saddle point $\tilde{p}_2$ varies with respect to x, with non-vanishing initial geometry, and gets pushed to infinity (on the negative imaginary axis) as $x \to 0$ (Neumann limit). For all values of x, only one of the two saddle point is relevant. For $0 < x < 1$ the saddle point $\tilde{p}_1$ relevant, while the irrelevant saddle point $\tilde{p}_2$ lies below $\tilde{p}_1$ on the negative imaginary axis in the complex $\tilde{p}$ plane (in the limit $x \to 0$ this saddle point is pushed to infinity and remains irrelevant). In this regime, we get the correct Hartle-Hawking exponential prefactor $e^{6k^{3/2}/\hbar\bar{\Lambda}}$. For $x > 1$, the saddle $\tilde{p}_2$ becomes relevant. $x = 1$ is the degenerate situation when $\tilde{p}_1 = \tilde{p}_2$ (relevant) giving $e^{6k^{3/2}/\hbar\bar{\Lambda}}$ (Hartle-Hawking) beside the numerical pre-factors. For $1 < x < 2$ the saddle point $\tilde{p}_2$ still lies on the negative imaginary axis. However, for $x > 2$ it crosses over and lies on the positive imaginary axis. For $1 < x < x_0$, the argument of the exponential prefactor decreases in magnitude and becomes negative for $x > x_0$. The range of x for which only the saddle point $\tilde{p}_1$ is relevant and produces the exact exponential prefactor $e^{6k^{3/2}/\hbar\bar{\Lambda}}$ of Hartle-Hawking no-boundary Universe is $0 < x < 1$.

Our investigations show the important non-trivial role played by the Gauss-Bonnet sector of the gravitational action in favoring the initial configurations which lead to the Hartle-Hawking no-boundary Universe. Gauss-Bonnet term arises in the low Energy-effective action of the heterotic string theory [7–9] with $\alpha > 0$. Although it is topological in nature in four spacetime dimensions and is expected to not play any role in the dynamical evolution, but our analysis clearly shows its contribution in the path integral. When Gauss-Bonnet term is not ignored in path-integral studies, then for $\alpha > 0$ (which is also the same sign appearing in low energy effective action of the heterotic string theory) it naturally picks and favors initial configurations which correspond to Hartle-Hawking no-boundary Universe. Moreover, it is expected that Gauss-Bonnet modifications won't alter the stability analysis done for Einstein-Hilbert gravity [33], where the perturbative stable regime was found to be $0 \leq x \leq 1$. This is because Gauss-Bonnet in four spacetime dimensions is topological. Perturbative analysis like the one done in [33] is for small perturbations which are expected to not change the background topology, and hence will be unaffected by the Gauss-Bonnet gravity.

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