

Non-invertible symmetries, brane dynamics, and tachyon condensation

Ibrahima Bah , Enoch Leung  and Thomas Waddleton 

*William H. Miller III Department of Physics and Astronomy, Johns Hopkins University,
3400 North Charles Street, Baltimore, MD 21218, U.S.A.*

E-mail: iboubah@jhu.edu, yleung5@jhu.edu, twaddle1@jhu.edu

ABSTRACT: We study the Symmetry Topological Field Theory in holography associated with 4d $\mathcal{N} = 1$ Super Yang-Mills theory with gauge algebra $\mathfrak{su}(M)$. From this, all the bulk symmetry operators are computed and matched to various D-brane configurations. The fusion algebra of the operators emerges from brane dynamics. In particular, we show that the symmetry operators are purely determined from the center-of-mass modes of the branes. We identify the TQFT fusion coefficients with the relative motion of the branes. We also establish the origin of condensation defects, arising from fusion of non-invertible operators, as the consequence of tachyon condensation in brane-anti-brane pairs.

KEYWORDS: D-Branes, Discrete Symmetries, Brane Dynamics in Gauge Theories

ARXIV EPRINT: [2306.15783](https://arxiv.org/abs/2306.15783)

Contents

1	Introduction	1
2	Bulk supergravity gauge symmetries	4
2.1	Symmetry topological field theory	4
2.2	5d bulk gauge symmetries	6
2.3	Gauss Laws and symmetry generators	7
3	Symmetries from branes	9
3.1	Bulk SymTFT vs. brane worldvolume action	9
3.2	D-brane anomaly polynomial	10
4	$\mathbb{Z}_{2M}^{(0)}$ and $\widehat{\mathbb{Z}}_M^{(1)}$ defect fusion	11
4.1	$\mathcal{N}_p \otimes \mathcal{N}_{p'}$ fusion and dielectric branes	11
4.2	$\mathcal{N}_p \otimes \mathcal{N}_{-p}$ fusion and Tachyon Condensation	15
4.3	Nature of the $\widehat{\mathbb{Z}}_M^{(1)}$ Defect	21
5	$\mathbb{Z}_M^{(1)}$ and $\widehat{\mathbb{Z}}_{2M}^{(2)}$ defect fusion	22
5.1	$\mathcal{O}_q \otimes \mathcal{O}_{q'}$ fusion	24
5.2	$\mathcal{O}_q \otimes \mathcal{O}_{-q}$ fusion	26
5.3	Nature of the $\widehat{\mathbb{Z}}_{2M}^{(2)}$ defect	27
6	Cross-symmetry defect fusion and braiding	28
6.1	Dissolved branes	28
6.2	Commutation relations	29
7	Boundary conditions and field theory	31
7.1	Allowed choices from the SymTFT	31
7.2	Boundary conditions and branes	32
8	Discussion and outlook	34
A	SymTFT computation	37
B	Brief review of Bockstein homomorphisms	39
C	Derivation of fusion rules	41

1 Introduction

In recent times, the study of Quantum Field Theory (QFT) has localized to its *surprisingly* rich kinematic sector. This follows from the novel perspective, due to [1], that such aspects of QFT can be characterized by various topological structures acting on them. It has led to

several generalizations of the notion of symmetry such as higher-form symmetries, higher groups, non-invertible symmetries and inevitably to categorical symmetries. In all of these cases there continues to be rapid development on the nature of topological symmetries, and the many ways they provide a fundamental scaffolding of QFT. *A priori* these emerging symmetries seem exotic, however in the last two years they have been discovered in many “bread-and-butter” QFTs such as Yang-Mills theories in four dimensions, and even for the Standard Model of particle physics.

Two central themes of research now are to understand the scope of topological symmetries, broadly, and the mathematical framework needed to characterize them. It has come to focus that a useful way to describe topological symmetries associated to a QFT in a d -dimensional space, M_d , is to consider a topological field theory on a $(d + 1)$ -dimensional space X_{d+1} with $\partial X_{d+1} = M_d$ [1–3]. Such TQFTs are referred to as Symmetry Topological Field Theories (SymTFTs) [4]; they naturally emerge via anomaly inflow methods for QFTs engineered from string theory and holography [4–7]. From this point of view, string theory provides an important avenue for studying topological symmetries and their role in the dynamics of QFTs.

An important result appeared recently in the context of non-invertible symmetries [8, 9]. In these works, it was demonstrated how to realize them in string theory and holography from supergravity and brane dynamics. Such developments seeded many other interesting results where non-invertible symmetries are constructed and analyzed from various D-brane setups [8–17].

In this paper, we will explore the SymTFT associated to 4d $\mathcal{N} = 1$ Super Yang-Mills (SYM) theory with gauge algebra $\mathfrak{su}(M)$. This QFT emerges in the IR limit of a stack of N $D3$ branes probing the conifold in IIB string theory, with M units of G_3 flux where M is a divisor of N . This is famously described by the Klebanov-Tseytlin [18] and Klebanov-Strassler [19] (KT-KS) backgrounds in IIB supergravity; these are holographically dual to cascading field theories. A relevant review for these backgrounds is provided in the appendix A. Our main interest is to obtain the full SymTFT as initiated in [8], and then construct all the symmetry operators and their fusion rules. We will identify all the brane configurations that are associated to the operators, and then characterize their fusion rules from brane dynamics. Another objective will be to sharply understand what data captures the topological operators and their associated physics derived from the SymTFT.

We start with a brief discussion of our main results. First we complete the derivation of the SymTFT of the KT-KS background discussed in [8]. Here we identify all the couplings relevant to the TQFT and its associated discrete gauge symmetries. In particular we emphasize how continuum fields in supergravity can be used to construct and describe the discrete gauge fields in a TQFT. By using the (classical) Gauss Law constraints, we derive all the topological operators which generate a bulk gauge symmetry from the SymTFT [8, 20]. We then match each symmetry generator with a specific Dp -brane configuration by considering the reduction of the Wess-Zumino action for the branes. The details are the focus of sections 2, 4, and 5, but a quick summary can be found in table 1.

Once we have the symmetry generators, the next question is to understand their fusion algebra. When considering the parallel fusion of two symmetry generators, the corresponding Dp -branes become coincident and are subject to non-Abelian dynamics involving the worldvolume gauge fields. In this work we demonstrate that the non-Abelian effects are

Symmetry Action	Symmetry Generator	Brane Configuration
$\mathbb{Z}_{2M}^{(0)}$	$\mathcal{N}_1(M^3)$	D5 wrapping $S^3 \subset T^{1,1}$
$\widehat{\mathbb{Z}}_{2M}^{(2)}$	$\mathcal{V}_1(\gamma^1)$	D1 wrapping Hopf fiber $S^1 \subset S^3$
$\mathbb{Z}_M^{(1)}$	$\mathcal{O}_1(W^2)$	D3-D1 bound state wrapping $S^2 \subset T^{1,1}$
$\widehat{\mathbb{Z}}_M^{(1)}$	$\mathcal{U}_1(\Sigma^2)$	D3 wrapping Hopf base $\widetilde{S}^2 \subset S^3$

Table 1. Symmetry generators/defects and their D-brane realizations.¹

excluded from the topological data of the symmetry defects; only the Abelian modes in the Dp -brane worldvolume theory participate in the fusion. It is kinematical information, rather than dynamical, captured in the fusion of the corresponding operators. Specifically, the center-of-mass mode for a given stack of branes yields the symmetry operator. For two stacks brought together, the resulting operator is associated to the *total* center-of-mass mode. The modes describing any *relative* motion between the two stacks decouple from the defects but survive as nontrivial fusion coefficients in the form of decoupled TQFTs.

Another interesting feature we point out, elaborated upon in the main text, is the non-invertibility of operators can be attributed to induced brane charges due to bulk-brane dynamics. For example, the operator $\mathcal{O}_1(W^2)$ presented in table 1 takes the explicit form,

$$\mathcal{O}_1(W^2) = \int \mathcal{D}\lambda_1 \mathcal{D}\varphi \exp \left(2\pi i \int_{W^2} \hat{c}_2 + \lambda_1 \mathcal{A}_1 + \varphi \mathcal{B}_2 - M\varphi d\lambda_1 \right), \quad (1.1)$$

where λ_1 and φ are modes localized on the D3-D1 bound state. The first term in this action encodes all the information contained in the D3- and D1-branes themselves. The second and third terms describe induced charges present on the D3-D1 worldvolume, while the final term is a standard BF-term describing a $\mathbb{Z}_M \times \mathbb{Z}_M$ gauge theory. Figure 3 provides a picture to keep in mind during this discussion. As will be seen later in the paper, the induced charges are associated with D1-branes and F1-strings in the worldvolume.

In addition to the above fusion, we study further the condensation defect present in many recent examples such as [8–10, 21–23]. We find that these defects are described naturally in string theory via tachyon condensation. When a Dp - and a \overline{Dp} -brane are brought together, these branes will not completely annihilate but leave behind a solitonic $D(p-2)$ -brane in the original worldvolume, determined by a choice of tachyon profile. By summing over all possible solitonic $D(p-2)$ -branes on the original worldvolume for a specific tachyon profile, we can reproduce exactly the condensation defect that arise when considering the operator fusion. There is a parallel discussion here as in the other cases of fusion; we can see the condensate being given by the center-of-mass mode for the original brane-anti-brane worldvolume. Including the appropriate condensates, we work out the full fusion algebra for the symmetry defects in the KT-KS background. The full list of the bulk fusion rules can be found in table 2. We also discuss the realization of these fusion rules in the boundary field theory in terms of allowed boundary conditions on the bulk supergravity fields.

This paper is organized as follows. In section 2, we construct the SymTFT for the KT-KS background, and use the associated Gauss Law constraints to derive the *gauge* symmetry

¹We assume a Euclidean spacetime throughout this paper, and use the terminology “generator”, “operator”, and “defect” interchangeably.

Brane Fusion	Defect Fusion
D5 \otimes D5	$\mathcal{N}_p(M^3) \otimes \mathcal{N}_{p'}(M^3) = \mathcal{A}^{M,pp'(p+p')} \mathcal{N}_{p+p'}(M^3)$
D5 \otimes $\overline{\text{D5}}$	$\mathcal{N}_p(M^3) \otimes \mathcal{N}_{-p}(M^3) = \mathcal{C}(M^3)$
D3 \otimes D3	$\mathcal{U}_q(\Sigma^2) \otimes \mathcal{U}_{q'}(\Sigma^2) = \mathcal{U}_{q+q'}(\Sigma^2)$
D3 \otimes D5	$\mathcal{U}_q(\Sigma^2) \otimes \mathcal{N}_p(M^3) = \mathcal{N}_p(M^3)$
D3 \otimes (\sum D3)	$\mathcal{U}_q(\Sigma^2) \otimes \mathcal{C}(M^3) = \mathcal{C}(M^3)$
D5 \otimes (\sum D3)	$\mathcal{N}_p(M^3) \otimes \mathcal{C}(M^3) = (\mathcal{Z}_M)_0 \mathcal{N}_p(M^3)$
(\sum D3) \otimes (\sum D3)	$\mathcal{C}(M^3) \otimes \mathcal{C}(M^3) = (\mathcal{Z}_M)_0 \mathcal{C}(M^3)$
(D3-D1) \otimes (D3-D1)	$\mathcal{O}_q(W^2) \otimes \mathcal{O}_{q'}(W^2) = \mathcal{X}^{M,qq'(q+q')^{-1}} \mathcal{O}_{q+q'}(W^2)$
(D3-D1) \otimes ($\overline{\text{D3-D1}}$)	$\mathcal{O}_q(W^2) \otimes \mathcal{O}_{-q}(W^2) = \tilde{\mathcal{C}}(W^2)$
D1 \otimes D1	$\mathcal{V}_p(\gamma^1) \otimes \mathcal{V}_{p'}(\gamma^1) = \mathcal{V}_{p+p'}(\gamma^1)$
D1 \otimes (D3-D1)	$\mathcal{V}_p(\gamma^1) \otimes \mathcal{O}_q(W^2) = \mathcal{O}_q(W^2)$
D3 \otimes (D3-D1)	$\mathcal{U}_q(\Sigma^2) \otimes \mathcal{O}_{q'}(W^2) = \mathcal{O}_{q'}(W^2)$
D1 \otimes (\sum (D1-F1))	$\mathcal{V}_p(\gamma^1) \otimes \tilde{\mathcal{C}}(W^2) = \tilde{\mathcal{C}}(W^2)$
D3 \otimes (\sum (D1-F1))	$\mathcal{U}_q(\Sigma^2) \otimes \tilde{\mathcal{C}}(W^2) = \tilde{\mathcal{C}}(W^2)$
(D3-D1) \otimes (\sum (D1-F1))	$\mathcal{O}_q(W^2) \otimes \tilde{\mathcal{C}}(W^2) = \mathcal{X}^{M,-1} \mathcal{X}^{M,-1} \mathcal{O}_q(W^2)$
(\sum (D1-F1)) \otimes (\sum (D1-F1))	$\tilde{\mathcal{C}}(W^2) \otimes \tilde{\mathcal{C}}(W^2) = \mathcal{X}^{M,-1} \mathcal{X}^{M,-1} \tilde{\mathcal{C}}(W^2)$

Table 2. Fusion rules for the symmetry defects. The left column indicates the schematic fusion in terms of the associated Dp -branes, while the right column shows the explicit fusion rules. The coefficients $\mathcal{A}^{M,p}$, $\mathcal{X}^{M,q}$, and $(\mathcal{Z}_M)_0$ denote decoupled TQFTs that are elaborated upon in the main text.

generators in this setup. In section 3, we switch gears and discuss how symmetry generators in a generic string theory background can also be realized using D-branes. In sections 4 and 5, we study in detail the fusion rules for the $(\mathcal{N}, \mathcal{U})$ and $(\mathcal{O}, \mathcal{V})$ defect pairs respectively. These two pairs of defects share a strikingly similar fusion algebra, e.g. the former undergoes non-invertible fusion to produce a condensation defect containing the latter. We offer an interpretation of these fusion rules in terms of the underlying brane kinematics and dynamics. In section 6, we derive the mixed fusion rules between all the symmetry defects, as well as their braiding relations. In section 7, we analyze the role of the choice of boundary conditions in realizing the bulk *gauge* symmetry generators as *global* symmetry generators in the boundary field theory. Last but not least, we follow up by a discussion in section 8.

2 Bulk supergravity gauge symmetries

2.1 Symmetry topological field theory

In holographic constructions, one can use anomaly inflow methods to obtain a Symmetry Topological Field Theory (SymTFT) that encodes the symmetry data of a lower-dimensional

QFT (e.g. [4–7, 24]). In the case of supergravity solutions in the low energy limit of Type IIB string theory, the SymTFT can be obtained by starting with the anomaly polynomial of the ten-dimensional supergravity theory and dimensionally reducing over the internal space. This anomaly polynomial was derived in [7]. The resulting topological theory, once augmented with kinetic terms for the relevant fields, is the SymTFT of the boundary field theory.

In the present work, we expand upon the SymTFT of the KT-KS solution as derived in [8], and particularly study the effect of additional terms that were previously neglected. A review of the KT-KS solution can be found in appendix A. We first start with the 11-form anomaly polynomial in Type IIB string theory,²

$$\mathcal{I}_{11} = \frac{1}{2} \mathcal{G}_5 \wedge d\mathcal{G}_5 + \mathcal{G}_5 \wedge H_3 \wedge G_3, \quad (2.1)$$

where G_3 and H_3 are the field strengths for the RR 2-form and NSNS 2-form potentials respectively, and \mathcal{G}_5 is defined such that $G_5 = (1 + \star_{10})\mathcal{G}_5$ is the self-dual RR 5-form. To dimensionally reduce the above polynomial, we perform an expansion of the supergravity field strengths onto the $U(1)_R$ -equivariant cohomology classes defined in (A.2) and (A.3), namely,

$$H_3 = h_3, \quad G_3 = M\omega_3 + g_1\omega_2 + g_3, \quad \mathcal{G}_5 = N\Omega_5 + g_2\omega_3. \quad (2.2)$$

The fluctuating fields h_3, g_1, g_2 , and g_3 are defined on the external space W^5 , while the coefficients M and N of ω_3 and Ω_5 reflect the nontrivial fluxes threading S^3 and $T^{1,1}$ respectively in the KT-KS setup.

Consistency of the IIB Bianchi identities for the RR field strengths, which in this background take the form $dH_3 = 0, dG_3 = 0$, and $dG_5 = -H_3 \wedge G_3$, impose constraints on the external field strengths written above. These extra constraints reduce to the 5d Bianchi identities

$$dh_3 = dg_3 = dF_2 = 0, \quad dg_1 = 2MF_2, \quad dg_2 = -Mh_3. \quad (2.3)$$

We may solve the leftmost equations via

$$h_3 = db_2 - \mathcal{H}_3, \quad g_3 = dc_2 + \mathcal{C}_3, \quad F_2 = dA_1 + \mathcal{F}_2, \quad (2.4)$$

where $\mathcal{H}_3, \mathcal{C}_3, \mathcal{F}_2 \in H^*(\mathcal{W}^5; \mathbb{Z})$ and b_2, c_2 , and A_1 are globally well-defined gauge potentials.³ In the later computations \mathcal{C}_3 will play no role, so without any loss of generality we will set it to zero. On the other hand, the nontrivial Bianchi identities for g_1 and g_2 indicate the presence of discrete gauge fields in the bulk theory. Specifically, these putative $U(1)$ gauge fields are respectively Higgsed down to \mathbb{Z}_{2M} and \mathbb{Z}_M gauge fields via the Stückelberg mechanism. As a result, we may solve the Bianchi identities as

$$g_1 = dc_0 + 2MA_1 + \mathcal{A}_1, \quad g_2 = d\beta_1 - Mb_2 + \mathcal{B}_2, \quad (2.5)$$

²Unless otherwise specified, we choose the normalization of all field strengths appearing in this work to be such that they have integral periods, and we also work in dimensionless units, i.e. $2\pi\alpha' = 1$ and $g_s = 1$, to avoid cluttering of notation. For the same reason, we will often suppress the wedge/cup products between differential forms/cochains.

³The fact that these are valued in integer cohomology follows from the flux quantization of the RR field strengths [25].

where $\mathcal{A}_1 \in H^1(\mathcal{W}^5; \mathbb{Z}_{2M})$ and $\mathcal{B}_2 \in H^2(\mathcal{W}^5; \mathbb{Z}_M)$, while c_0 and β_1 are globally well-defined gauge potentials.⁴ The quantities g_1 and g_2 , containing both free and torsional components, carry the same information as differential cochains. See [26–29] for a collection of reviews.

The coefficients of the cohomology groups in which \mathcal{A}_1 and \mathcal{B}_2 live can be understood as follows. The Bianchi identities in (2.3) put constraints on the basepoint fluxes \mathcal{F}_2 and \mathcal{H}_3 that we have written above, i.e. $2M\mathcal{F}_2$ and $M\mathcal{H}_3$ must vanish identically for the right-hand side of the relations to be an exact cocycle.⁵ This requirement implies that \mathcal{F}_2 and \mathcal{H}_3 are each contained in the image of a Bockstein homomorphism. Specifically, we may express

$$\mathcal{F}_2 = \tilde{\beta}(\mathcal{A}_1), \quad \mathcal{H}_3 = \tilde{\beta}'(\mathcal{B}_2). \quad (2.6)$$

In the above $\tilde{\beta}$ and $\tilde{\beta}'$ are respectively the Bockstein maps⁶

$$\tilde{\beta} : H^*(\mathcal{W}^5; \mathbb{Z}_{2M}) \rightarrow H^{*+1}(\mathcal{W}^5; \mathbb{Z}), \quad \tilde{\beta}' : H^*(\mathcal{W}^5; \mathbb{Z}_M) \rightarrow H^{*+1}(\mathcal{W}^5; \mathbb{Z}). \quad (2.7)$$

With this notation we can now compute the 5d SymTFT by reducing (2.1) over the conifold $T^{1,1}$. The topological action S_{top} takes the following form,

$$S_{\text{top}} = 2\pi \int_{\mathcal{W}^5} \left(Mb_2 d\hat{c}_2 - 2Mc_3 d\mathcal{A}_1 + b_2 \mathcal{A}_1 \mathcal{B}_2 - \mathcal{A}_1 \mathcal{B}_2^2 - \frac{\mathcal{K}}{M} \mathcal{A}_1 \beta(\mathcal{A}_1) \beta(\mathcal{A}_1) \right), \quad (2.8)$$

where \mathcal{K} is defined in (A.5), and we have defined the field $d\hat{c}_2 = kdc_2 + da_2$ in terms of a field a_2 that arises when computing the SymTFT.⁷ The details of this computation can be found in appendix A.

2.2 5d bulk gauge symmetries

The topological action S_{top} in (2.8) details the symmetry structure of the 5d bulk gauge theory, and it acts as the SymTFT for the 4d QFT living on the boundary. The first two terms of act as BF terms, so for instance, the fluctuating U(1) gauge fields b_2 and \hat{c}_2 in the former term are Higgsed down to \mathbb{Z}_M gauge fields at low energies via the Stückelberg mechanism [34]. As a consequence, the holonomies of \mathcal{A}_1 and b_2 in the IR turn into

$$\exp\left(2\pi i \int_{\gamma^1} \mathcal{A}_1\right) \xrightarrow{\text{IR}} \exp\left(\frac{2\pi i}{2M} \int_{\gamma^1} \mathcal{A}_1\right), \quad \exp\left(2\pi i \int_{\Sigma^2} b_2\right) \xrightarrow{\text{IR}} \exp\left(-\frac{2\pi i}{M} \int_{\Sigma^2} \mathcal{B}_2\right). \quad (2.9)$$

We shall remark that the U(1) gauge symmetry associated with \mathcal{A}_1 comes from the S^1 isometry circle inside $T^{1,1}$. This isometry, as was discussed in the introduction, corresponds to the R -symmetry in the 4d $\mathcal{N} = 1$ boundary field theory. The Higgsing of the gauge

⁴More precisely, \mathcal{A}_1 and \mathcal{B}_2 are cohomology class representatives. The difference between two representatives of the same class is exact, i.e. $\mathcal{A}_1 - \mathcal{A}'_1 = d\lambda_0$ for some globally well-defined 0-cochain λ_0 on \mathcal{W}^5 . We see that the choice of representative is immaterial since g_1 is invariant under the redefinitions $\mathcal{A}_1 \rightarrow \mathcal{A}_1 + d\lambda_0$ and $c_0 \rightarrow c_0 - \lambda_0$. Similarly, g_2 is invariant under $\mathcal{B}_2 \rightarrow \mathcal{B}_2 + d\lambda_1$ and $\beta_1 \rightarrow \beta_1 - \lambda_1$. See appendix B for a related discussion.

⁵In general one might demand $2M\mathcal{F}_2$ to be simply cohomologically trivial rather than to vanish identically, but such an exact piece can be removed via a field redefinition of \mathcal{A}_1 .

⁶A brief review of Bockstein homomorphisms can be found in appendix B.

⁷Technically, at the level of cochains, the term $\mathcal{B}_2 \cup \mathcal{B}_2$ is more appropriately written as $\mathcal{P}(\mathcal{B}_2)$, where \mathcal{P} denotes the Pontryagin square operation defined in, e.g. [30–33].

symmetry down to a \mathbb{Z}_{2M} subgroup is exactly the same as how the $U(1)$ global R -symmetry of the boundary theory is broken down to a \mathbb{Z}_{2M} subgroup by the ABJ anomaly. Reading off the form degrees of the gauge fields, the first term in S_{top} corresponds to $\mathbb{Z}_M^{(1)}$ and $\widehat{\mathbb{Z}}_M^{(1)}$ gauge symmetries in the 5d bulk theory, where the hat denotes the Pontryagin-dual group.⁸ Similarly, the latter term corresponds to $\mathbb{Z}_{2M}^{(0)}$ and $\widehat{\mathbb{Z}}_{2M}^{(2)}$ gauge symmetries.

The third and fourth terms in S_{top} are interaction terms describing an interplay between the $\mathbb{Z}_M^{(1)}$ and $\mathbb{Z}_{2M}^{(0)}$ gauge symmetries present in the bulk theory.⁹ The interplay, when considered from the perspective of the 4d boundary theory, is a mixed 't Hooft anomaly between the two symmetries. Similarly, the final term in S_{top} is a cubic interaction term in the bulk gauge theory, and becomes a 't Hooft anomaly purely for the $\mathbb{Z}_{2M}^{(0)}$ symmetry on the boundary.

2.3 Gauss Laws and symmetry generators

For each of the discrete gauge symmetries present in the SymTFT, we may construct a symmetry operator that generates the corresponding symmetry. This can be done by treating the radial direction of the external space \mathcal{W}^5 as a time coordinate and performing a Hamiltonian analysis of the 5d bulk action [20, 35, 36]. In this formalism, the time components of the bulk fields have vanishing conjugate momentum and thus impose (classical) Gauss Law constraints. Each of the fields $b_2, \hat{c}_2, A_1,$ and c_3 will impose a Gauss Law constraint, which were originally computed in [8] and we restate them here for convenience,

$$\begin{aligned} \mathcal{G}_{b_2} &= Md\hat{c}_2 + \mathcal{A}_1\mathcal{B}_2, & \mathcal{G}_{A_1} &= 2Mdc_3 + \mathcal{B}_2^2, \\ \mathcal{G}_{\hat{c}_2} &= Mdb_2, & \mathcal{G}_{c_3} &= 2MdA_1. \end{aligned} \tag{2.10}$$

The quantities \mathcal{G}_i may be thought of as dP_i where P_i is a ‘‘Page charge’’ generating the corresponding gauge transformation. For a *continuous* symmetry, e.g. $U(1)$, the operator $\exp(2\pi i \int_{M_i} P_i)$ acts as a symmetry generator supported on some submanifold $M_i \subset \mathcal{W}^5$ of the appropriate dimension. In contrast, if the symmetry is broken down to a *finite* group, e.g. \mathbb{Z}_k , then the proper generator to consider is the ‘‘ k -th’’ root of the aforementioned one, i.e. $\mathcal{D} \sim \exp((2\pi i/k) \int_{M_i} P_i)$, such that $\mathcal{D}^{\otimes k} \cong \mathbb{1}$ corresponds to the identity element of the finite group (up to condensation defects in the case of non-invertible symmetries).

To illustrate this, let us look at the Gauss Law constraint \mathcal{G}_{A_1} . Using the prescription above, one would naively construct an operator of the form,

$$\mathcal{N}_1(M^3, M^4) = \exp\left(2\pi i \int_{M^3} c_3 + \frac{2\pi i}{2M} \int_{M^4} \mathcal{B}_2^2\right). \tag{2.11}$$

This operator is well-defined when placed on a 3-manifold M^3 that is the boundary of some 4-manifold M^4 . To define the operator solely on M^3 , we may rewrite the second term as a

⁸Here the form degree of a symmetry (as labeled by the superscript) is defined with respect to a 4d ‘‘time’’ slice in the bulk resulting from Hamiltonian quantization. In this case, a p -form gauge field is associated with a $(3 - p)$ -form symmetry.

⁹The reason for b_2 being a gauge field for the $\mathbb{Z}_M^{(1)}$ symmetry rather than the $\widehat{\mathbb{Z}}_M^{(1)}$ symmetry will be made clear when we consider the boundary field theory later in section 7.

3d action by coupling \mathcal{B}_2 to a dynamical U(1) gauge field a_1 localized on M^3 [8, 22], i.e.

$$\mathcal{N}_1(M^3) = \int \mathcal{D}a_1 \exp \left(2\pi i \int_{M^3} c_3 + \frac{M}{2} a_1 da_1 + a_1 \mathcal{B}_2 \right). \quad (2.12)$$

As an aside, one may recognize the latter two terms as an effective action for a fractional quantum Hall state. We claim that such an operator generates a $\mathbb{Z}_{2M}^{(0)}$ symmetry, and as we will show in section 4, stacking $2M$ copies of (2.12) indeed produces a trivial operator up to a condensation defect.

It is instructive to point out that the defect $\mathcal{N}_1(M^3)$ can be schematically decomposed into two pieces, i.e.

$$\mathcal{N}_1(M^3) = \exp \left(2\pi i \int_{M^3} c_3 \right) \otimes \mathcal{A}^{M,1}[\mathcal{B}_2] := \widetilde{\mathcal{N}}_1(M^3) \otimes \mathcal{A}^{M,1}[\mathcal{B}_2], \quad (2.13)$$

where $\widetilde{\mathcal{N}}_1(M^3)$ generates a $\mathbb{Z}_{2M}^{(0)}$ R-symmetry rotation, and $\mathcal{A}^{M,1}[\mathcal{B}_2]$ is a 3d minimal TQFT associated with the $\widehat{\mathbb{Z}}_M^{(1)}$ symmetry [37]. Note that the simultaneous appearance of these two operators in $\mathcal{N}_1(M^3)$ results from the presence of the anomaly term $A_1 \mathcal{B}_2^2$ in (2.8). An analogous decomposition of the chiral symmetry defect can be found in [22] in the context of 4d massless QED (see also, e.g. [23, 38] for similar constructions). There, the defect is similarly composed of a naïve chiral rotation operator stacked with a minimal TQFT.

Applying the same philosophy, we can construct symmetry defects generating the remaining symmetries encoded by the SymTFT, namely, $\widehat{\mathbb{Z}}_{2M}^{(2)}$, $\mathbb{Z}_M^{(1)}$, and $\widehat{\mathbb{Z}}_M^{(1)}$, using the Gauss Law constraints \mathcal{G}_{c_3} , \mathcal{G}_{b_2} , and $\mathcal{G}_{\hat{c}_2}$. Explicitly, these defects take the respective forms,

$$\mathcal{V}_1(\gamma^1) = \exp \left(\frac{2\pi i}{2M} \int_{\gamma^1} \mathcal{A}_1 \right), \quad (2.14)$$

$$\mathcal{O}_1(W^2, W^3) = \exp \left(2\pi i \int_{W^2} (k c_2 + a_2) + \frac{2\pi i}{M} \int_{W^3} \mathcal{A}_1 \mathcal{B}_2 \right), \quad (2.15)$$

$$\mathcal{U}_1(\Sigma^2) = \exp \left(\frac{2\pi i}{M} \int_{\Sigma^2} \mathcal{B}_2 \right). \quad (2.16)$$

When constructing \mathcal{V}_1 and \mathcal{U}_1 , we have made use of the discussion surrounding (2.9).

Analogously to the previous case of \mathcal{N}_1 , the operator $\mathcal{O}_1(W^2, W^3)$ as written in (2.15) is well-defined when placed on a 2-manifold W^2 that is the boundary of some 3-manifold W^3 . To define the operator solely on W^2 , we may rewrite the last term in (2.15) as a 2d action by coupling \mathcal{A}_1 and \mathcal{B}_2 respectively to dynamical U(1) gauge fields λ_1 and φ localized on W^2 , i.e.

$$\mathcal{O}_1(W^2) = \int \mathcal{D}\lambda_1 \mathcal{D}\varphi \exp \left(2\pi i \int_{W^2} \hat{c}_2 + \lambda_1 \mathcal{A}_1 + \varphi \mathcal{B}_2 - M \varphi d\lambda_1 \right). \quad (2.17)$$

For convenience, let us define the shorthand notation,

$$\mathcal{X}^{M,q}[\mathcal{A}_1, \mathcal{B}_2] := \int \mathcal{D}\lambda_1 \mathcal{D}\varphi \exp \left(2\pi i q \int \lambda_1 \mathcal{A}_1 + \varphi \mathcal{B}_2 - M \lambda_1 d\varphi \right), \quad (2.18)$$

assuming $\gcd(M, q) = 1$. The anomaly label $q \sim q + M$ specifies the braiding statistics between λ_1 and φ . Note that in the absence of the coupling to \mathcal{A}_1 and \mathcal{B}_2 , the TQFT $\mathcal{X}^{M,1}$

is an untwisted 2d $\mathbb{Z}_M \times \mathbb{Z}_M$ BF theory.¹⁰ Similarly to the defect $\mathcal{N}_1(M^3)$, we observe that $\mathcal{O}_1(W^2)$ decomposes into

$$\mathcal{O}_1(W^2) = \tilde{\mathcal{O}}_1(W^2) \otimes \mathcal{X}^{M,1}[\mathcal{A}_1, \mathcal{B}_2], \tag{2.19}$$

where $\tilde{\mathcal{O}}_1(W^2) = \exp(2\pi i \int_{W^2} \hat{c}_2)$ is an invertible operator generating a $\mathbb{Z}_M^{(1)}$ transformation, and the appearance of the 2d TQFT $\mathcal{X}^{M,1}[\mathcal{A}_1, \mathcal{B}_2]$ is a consequence of the interaction term $b_2 \mathcal{A}_1 \mathcal{B}_2$ in (2.8).

3 Symmetries from branes

3.1 Bulk SymTFT vs. brane worldvolume action

We derived earlier the SymTFT for the bulk supergravity theory in the KT-KS background, from which the Gauss Law constraints give us a complete set of symmetry generators in the 5d bulk. In this and the subsequent sections, we show that each of these symmetry defects can be engineered as a certain probe D-brane configuration in the KT-KS geometry. As alluded to in section 1, recently in the literature there have been field-theoretic constructions of non-invertible symmetry defects with dimension $d \geq 2$. The hallmark of such defects is two-fold:

1. (Defect A) \otimes (Defect B) = (Decoupled TQFT Coefficient) \otimes (Defect C),
2. (Defect) \otimes (Oppositely Oriented Defect) = \sum (Lower-dimensional Defect).

The significance of the brane-defect correspondence in our work is that it allows us to interpret both of these properties purely in terms of brane kinematics and dynamics in the bulk. In fact, as we will discuss in section 7, there exist obstructions to realize certain bulk-symmetry-generating branes (or combinations of them) on the boundary field theory. The obstructions are determined by the anomaly terms in the SymTFT and choices of boundary conditions for the 5d bulk gauge fields. A comprehensive analysis of such branes therefore realizes a much richer structure of the topological data encoded by the SymTFT in the bulk, which would otherwise not be appreciated from the perspective of the boundary field theory.

In Type II string theory, the worldvolume action of a general Dp -brane consists of two parts: a Dirac-Born-Infeld (DBI) action encoding the dynamics of worldvolume gauge fields, and a Wess-Zumino (WZ) action describing the topological coupling of the Dp -brane to the bulk RR and NSNS fields. To construct a topological operator out of a probe Dp -brane, one can push it to the conformal boundary of the external spacetime. The dynamical degrees of freedom are frozen out since the brane tension scales as $T_p \sim r^p$, causing the DBI term to decouple. This leaves us with the WZ action (see, e.g. [40, 41]), given by

$$S_{WZ}^{Dp} = \mu_p \int_{\mathcal{M}^{p+1}} I_{p+1}^{Dp} := \mu_p \int_{\mathcal{M}^{p+1}} \sum_{q \leq p} C_{q+1} \wedge \text{ch}_B(\mathcal{E}) \wedge \sqrt{\frac{\hat{\mathcal{A}}(\mathcal{R}_T)}{\hat{\mathcal{A}}(\mathcal{R}_N)}} \Big|_{(p+1)\text{-form}}, \tag{3.1}$$

¹⁰By construction, the Wilson-like operators $e^{2\pi i \oint \lambda_1}$ and $e^{2\pi i \oint \varphi}$ generate the two \mathbb{Z}_M symmetries on W^2 . The cocycles \mathcal{A}_1 and \mathcal{B}_2 are respectively the BF-duals of $\lambda_1 \in H^1(W^2; \mathbb{Z}_M)$ and $\varphi \in H^0(W^2; \mathbb{Z}_M)$ [39]. This can be seen by rewriting $\mathcal{X}^{M,1}[\mathcal{A}_1, \mathcal{B}_2]$ in terms of continuum fields $\hat{\mathcal{A}}_1 = \mathcal{A}_1/M$ and $b_2 = \mathcal{B}_2/M$, and path-integrating over them instead of λ_1 and φ , which results in the equations of motion, $Md\lambda_1 = Md\varphi = 0$.

where μ_p is the charge of a single Dp -brane, and \mathcal{E} is the Chan-Paton bundle over the worldvolume \mathcal{M}^{p+1} . We sum over even q in Type IIA and odd q in Type IIB string theory. The quantities $\mathcal{R}_T = 2\pi R_T$ and $\mathcal{R}_N = 2\pi R_N$ denote respectively the (normalized) curvature 2-forms of the tangent and normal bundles of the worldvolume \mathcal{M}^{p+1} of the Dp -branes, while \hat{A} is the A -roof genus. In the analysis that follows, we will omit the gravitational couplings in the WZ action for simplicity.

For the purpose of our work, we would like to highlight two salient properties of the WZ action. Firstly, a Dp -brane is not only coupled to the $(p+1)$ -form RR field C_{p+1} , but also to the lower RR fields C_{p-1}, C_{p-3} , etc. via the (twisted) Chern character $\text{ch}_B(\mathcal{E}) = e^{B_2+F_2}$. Such couplings are physically interpreted as Dq -branes ($q < p$) being dissolved in \mathcal{M}^{p+1} [42]. The numbers of these Dq -branes are determined by the worldvolume Chan-Paton flux on the Dp -brane. As a simple example, suppose $\mathcal{M}^{p+1} = \tilde{\mathcal{M}}^2 \times \mathcal{M}^{p-1}$, and the Chan-Paton field strength decomposes as $F_2 = \tilde{f}_2 + f_2$ with $\tilde{f}_2 \in H^2(\tilde{\mathcal{M}}^2; \mathbb{Z})$ and $f_2 \in H^2(\mathcal{M}^{p-1}; \mathbb{Z})$. The Chern character factorizes as

$$e^{B_2+F_2} = e^{\tilde{f}_2} \wedge e^{B_2+f_2} = 1 + \tilde{f}_2 \wedge e^{B_2+f_2} + \dots, \tag{3.2}$$

and so the number, k , of $D(p-2)$ -branes within the Dp -brane is given by $k = \int_{\tilde{\mathcal{M}}^2} \tilde{f}_2$.

Let us now consider a single Dp -brane with zero net Dp charge, i.e. $\int_{\mathcal{M}^{p+1}} C_{p+1} = 0$, but with k units of worldvolume flux through $\tilde{\mathcal{M}}^2$. As far as the *center-of-mass mode* is concerned, such an object can also be thought of as a stack of k coincident $D(p-2)$ -branes. To see this, we use the fact that the Chan-Paton bundle \mathcal{E} on the brane stack has structure group $U(k)$. Suppose we parametrize the $u(k)$ -valued field strength $F_2 = \mathcal{F}_2 + f_2 \mathbf{1}_k$, with \mathcal{F}_2 being $\mathfrak{su}(k)$ -valued and $f_2 \in H^2(\mathcal{M}^{p+1}; \mathbb{Z})$, then the Chern character factorizes as

$$e^{B_2} \text{Tr}(e^{F_2}) = \text{Tr}(e^{\mathcal{F}_2}) \wedge e^{B_2+f_2} = k e^{B_2+f_2} + \dots. \tag{3.3}$$

We therefore recover the second term in (3.2) with $k = \int_{\tilde{\mathcal{M}}^2} \tilde{f}_2$. Note that the gauge field f_2 here is associated with the center-of-mass mode of the stack of $D(p-2)$ -branes, while \mathcal{F}_2 is associated with the remaining relative Chan-Paton modes. The relation between these two perspectives will prove useful later as we study the fusion between Dp -branes.

3.2 D-brane anomaly polynomial

As for any Chern-Simons action, we can naturally construct an anomaly polynomial associated with the Dp -brane via descent. It can be defined *locally* as¹¹

$$\mathcal{I}_{p+2}^{Dp} = dI_{p+1}^{Dp} = \sum_{q \leq p} G_{q+2} e^{B_2+F_2}. \tag{3.4}$$

The RR and NSNS field strengths locally take the forms,

$$G_{q+2} = dC_{q+1} + H_3 C_{q-1}, \quad H_3 = dB_2. \tag{3.5}$$

¹¹Formally, \mathcal{I}_{p+2}^{Dp} is defined on a manifold \mathcal{M}^{p+2} such that $\partial \mathcal{M}^{p+2} = \mathcal{M}^{p+1}$ is the worldvolume of the Dp -brane. We also assume the gauge fields originally defined on \mathcal{M}^{p+1} can be extended to \mathcal{M}^{p+2} .

In the presence of background D-branes, or when $[H_3] \in H^3(\mathcal{M}^{p+1}; \mathbb{Z})$ is nontrivial [43, 44], the RR field strengths G_{q+2} can be topologically nontrivial. Globally, they should be defined by the Bianchi identities,¹²

$$dG_{q+2} = -H_3 G_q, \tag{3.6}$$

in addition to the requirement that they are equivariant under all additional gauge symmetries present in the system (e.g. in our setup we gauged the R -symmetry circle in $T^{1,1}$). Throughout the rest of this work, we take the r.h.s. of (3.4) as the definition of the D-brane anomaly polynomial, subject to G_{q+2} satisfying the Bianchi identities (3.6). Specifically, we assume the ansatzes for the RR fields as defined in (2.2) to account for global topological data in the KS setup.

Bulk symmetry generators from wrapped D-branes. Recall that the 5d topological action (2.8) of the SymTFT is derived by reducing \mathcal{I}_{11} of Type IIB string theory on the conifold base $T^{1,1}$ of the KT-KS background. Similarly, one can reduce $\mathcal{I}_{p+2}^{\text{D}p}$ of a D p -brane on some n -manifold $X^n \subset T^{1,1}$, resulting in an effective action for a topological defect defined on a $(p+1-n)$ -submanifold M^{p+1-n} of the 5d bulk \mathcal{W}^5 .

We claim that each of the four symmetry generators, $\mathcal{N}_1(M^3)$, $\mathcal{V}_1(\gamma^1)$, $\mathcal{O}(W^2)$, and $\mathcal{U}_1(\Sigma^2)$, derived in section 2 with a Gauss Law constraint admits a D-brane origin. The first symmetry generator we are interested in is $\mathcal{N}_1 = \widetilde{\mathcal{N}}_1 \otimes \mathcal{A}^{M,1}[\mathcal{B}_2]$ which generates the $\mathbb{Z}_{2M}^{(0)}$ gauge symmetry. From the bulk string theory setup, this defect can be realized by reducing a D5-brane on $S^3 \subset T^{1,1}$ [8]. With this interpretation, the dynamical gauge field a_1 in $\mathcal{A}^{M,1}[\mathcal{B}_2]$ can be understood as the Chan-Paton field living on the D5-brane. These localized degrees of freedom thereby naturally encode the stacking of a minimal TQFT in order to construct the symmetry defect.

4 $\mathbb{Z}_{2M}^{(0)}$ and $\widehat{\mathbb{Z}}_M^{(1)}$ defect fusion

4.1 $\mathcal{N}_p \otimes \mathcal{N}_{p'}$ fusion and dielectric branes

Defect fusion. Let us study the parallel fusion between two copies of \mathcal{N}_1 defined on the same 3-manifold M^3 . Here we would like to neglect the D5-brane origin of \mathcal{N}_1 for a moment and treat them solely as topological defects acting on the Hilbert space of the theory. It is straightforward to deduce that $\widetilde{\mathcal{N}}_1$ fuse invertibly as $\widetilde{\mathcal{N}}_1 \otimes \widetilde{\mathcal{N}}_1 = \widetilde{\mathcal{N}}_2 = \exp(4\pi i \int_{M^3} c_3)$.¹³ On the other hand, the minimal TQFT $\mathcal{A}^{M,1}[\mathcal{B}_2]$ satisfies the fusion rule [22],

$$\mathcal{A}^{M,1}[\mathcal{B}_2] \otimes \mathcal{A}^{M,1}[\mathcal{B}_2] = \mathcal{A}^{M,2} \otimes \mathcal{A}^{M,2}[\mathcal{B}_2], \tag{4.1}$$

assuming $\text{gcd}(M, 2) = 1$, i.e. M is odd. Note that the first copy of $\mathcal{A}^{M,2}$ is a decoupled TQFT [47], while the second one couples to the \mathbb{Z}_M gauge field \mathcal{B}_2 . Combining the results, we conclude that the fusion of two R -symmetry defects is given by

$$\mathcal{N}_1 \otimes \mathcal{N}_1 = \widetilde{\mathcal{N}}_2 \otimes \mathcal{A}^{M,2} \otimes \mathcal{A}^{M,2}[\mathcal{B}_2] := \mathcal{A}^{M,2} \otimes \mathcal{N}_2. \tag{4.2}$$

¹²Equivalently, the Bianchi identities for the various RR fields G_{q+2} as differential forms can be collected and rewritten in a more suggestive way, $(d+H)G=0$, which is one of the supporting arguments for the claim that D-brane charges are formally classified by twisted K-theory [45, 46].

¹³To avoid notational clutter, we hereafter often omit the dependency of the operators on manifolds.

In general, one may construct an operator \mathcal{N}_p analogously to how we defined \mathcal{N}_2 above, i.e. we take the naïve operator $\widetilde{\mathcal{N}}_p$ and stack it with a minimal TQFT $\mathcal{A}^{M,p}[\mathcal{B}_2]$. The latter has an anomaly label $p \sim p + M$ dictating the braiding statistics of the anyons living on M^3 . This implies that $\mathcal{A}^{M,M} \cong \mathcal{A}^{M,0}$ is a trivial theory, so the operator $\mathcal{N}_M \cong \widetilde{\mathcal{N}}_M$ reduces to an invertible operator when we take $p = M$. The operator \mathcal{N}_M therefore generates the anomaly-free $\mathbb{Z}_2^{(0)}$ subgroup of the $\mathbb{Z}_{2M}^{(0)}$ R -symmetry that is associated with the $(-1)^F$ operator [19, 48]. It is important, however, to note that $\widetilde{\mathcal{N}}_M$ has a periodicity $p \sim p + 2M$, and the same applies to the full operator \mathcal{N}_p .

We show in appendix C that if $\gcd(M, p) = \gcd(M, p') = \gcd(M, p + p') = 1$,¹⁴ the fusion between general R -symmetry defects follows

$$\mathcal{N}_p \otimes \mathcal{N}_{p'} = \mathcal{A}^{M,pp'(p+p')} \otimes \mathcal{N}_{p+p'}, \quad (4.3)$$

which reduces to (4.2) when $p = p' = 1$. It should be emphasized again that the factor $\mathcal{A}^{M,pp'(p+p')}$ does not couple to \mathcal{B}_2 . Since $\mathcal{A}^{M,pp'(p+p')}$ is symmetric in the labels p and p' , the fusion rule (4.3) is manifestly commutative.

Dielectric branes. The above field-theoretic fusion for the \mathcal{N}_p operators can alternatively be understood geometrically through the dynamics of probe Dp -branes in the dual string theory. The $\mathcal{N}_1 \otimes \mathcal{N}_1$ fusion is modeled by two coincident D5-branes on the $M^3 \times S^3$ needed to generate the $\mathbb{Z}_{2M}^{(0)}$ R -symmetry. As the two branes are brought together, the worldvolume theory will undergo an enhancement of the gauge group from $U(1)$ to $U(2)$ as can be seen from the new stringy modes stretching between the two D5-branes. This enhancement will play a significant role in the behavior of the stack of D5-branes.

It is known in the string theory literature that, in the presence of a nontrivial G_{p+4} or B_2 background, k coincident Dp -branes will polarize and “puff up” into an S^2 transverse to the original stack [8, 49–51], known as the Myers Effect. The resulting configuration is equivalent to that of a $D(p+2)$ -brane with vanishing $D(p+2)$ -brane charge and k units of Dp -brane charge. In the case at hand, when two D5-branes wrapping $M^3 \times S^3$ are brought close to one another, the configuration becomes indistinguishable from a single D7-brane wrapping $M^3 \times T^{1,1}$ with two units of worldvolume flux through the $S^2 \subset T^{1,1}$.

There is a subtlety arising from the description of two D5-branes polarizing into a D7-brane. To illustrate the issue, let us reduce a D7-brane on $T^{1,1}$, with two units of worldvolume flux threading the S^2 orthogonal to the $S^3 \subset T^{1,1}$ on which the D5-branes are wrapped. The reduction leads to

$$\exp\left(2\pi i \int_{M^4 \times T^{1,1}} \mathcal{I}_9^{\text{D7}}\right) = \exp\left(4\pi i \int_{M^3} c_3 + \frac{M}{2} a_1 da_1 + a_1 \mathcal{B}_2\right) = \mathcal{N}_2(M^3), \quad (4.4)$$

where $\partial M^4 = M^3$. Note that here a_1 is the $U(1)$ Chan-Paton field living on the D7-brane. A quick comparison with (4.2) reveals the issue: the decoupled TQFT $\mathcal{A}^{M,2}$ seems to be absent from the naïve computation above.

To recover this missing information, we can simply model each D5-brane as a “neutral” D7-brane with one unit of worldvolume flux (see the discussion following (3.3)). Specifically,

¹⁴This condition can be satisfied for all possible values of $p, p' = 0, \dots, M - 1$ if $M > 2$ and is prime.

the anomaly polynomial of each of these neutral D7-branes can be expanded as

$$\mathcal{I}_{9,i}^{\text{D7}} = G_7(B_2 + da_1^i + f_2^i) + \frac{1}{2} G_5(B_2 + da_1^i + f_2^i)^2 + \frac{1}{6} G_3(B_2 + da_1^i + f_2^i)^3, \quad (4.5)$$

where a_1^i denotes the Chan-Paton gauge field along S^3 , and f_2^i represents the worldvolume flux such that $\int_{S^2} f_2^i = 1$. We can then perform the reduction over $T^{1,1}$ using (2.2) to find

$$\begin{aligned} \int_{M^4 \times T^{1,1}} \mathcal{I}_{9,1}^{\text{D7}} + \mathcal{I}_{9,2}^{\text{D7}} &= \int_{M^4} 2dc_3 + \frac{M}{2} (da_1^1 da_1^1 + da_1^2 da_1^2) + d(a_1^1 + a_1^2) \mathcal{B}_2 \\ &= \int_{M^3} \frac{2^{-1}M}{2} \alpha'_1 d\alpha'_1 + 2 \left(c_3 + \frac{M}{2} \alpha_1 d\alpha_1 + \alpha_1 \mathcal{B}_2 \right), \end{aligned} \quad (4.6)$$

where we have made the field redefinitions $2\alpha_1 = a_1^1 + a_1^2$ and $\alpha'_1 = -a_1^1 + a_1^2$. Not only have we recovered the action for $\mathcal{N}_2(M^3)$, but we have also gained an additional term. This term encodes the information of the minimal TQFT $\mathcal{A}^{M,2^{-1}}$.¹⁵ Since the anomaly p of the minimal TQFT is only defined modulo M , the quantity 2^{-1} denotes the unique multiplicative inverse of 2 modulo M assuming $\gcd(M, 2) = 1$. Importantly, it is an integer so that (4.6) is properly quantized. We now further utilize the equivalence of minimal TQFTs [37],

$$\mathcal{A}^{M,p} \cong \mathcal{A}^{M,pr^2}, \quad (4.7)$$

by rescaling the generating line $a = e^{2\pi i} \oint \alpha'_1$ of a minimal TQFT as $a \rightarrow a^r$, for any r such that $\gcd(M, r) = 1$. Setting $r = 2$ gives $\mathcal{A}^{M,2^{-1}} \cong \mathcal{A}^{M,2}$, and so we have fully reproduced the r.h.s. of (4.2).

The question then arises of how to address the more general fusion $\mathcal{N}_p \otimes \mathcal{N}_{p'}$. We can repeat the above computation for stacks of p and p' coincident D5-branes, by viewing each stack as a single D7-brane with p and p' units of D5-brane charge respectively, i.e. $\int_{S^2} f_2^1 = p$ and $\int_{S^2} f_2^2 = p'$. The reduction over $T^{1,1}$ now becomes

$$\begin{aligned} \int_{M^4 \times T^{1,1}} \mathcal{I}_{9,1}^{\text{D7}} + \mathcal{I}_{9,2}^{\text{D7}} &= \int_{M^4} (p + p') dc_3 + \frac{M}{2} (p da_1^1 da_1^1 + p' da_1^2 da_1^2) + d(pa_1^1 + p' a_1^2) \mathcal{B}_2 \\ &= \int_{M^3} \frac{pp'(p + p')^{-1}M}{2} \alpha'_1 d\alpha'_1 + (p + p') \left(c_3 + \frac{M}{2} \alpha_1 d\alpha_1 + \alpha_1 \mathcal{B}_2 \right), \end{aligned} \quad (4.8)$$

where the required field redefinition here is $(p + p')\alpha_1 = pa_1^1 + p'a_1^2$ and $\alpha'_1 = -a_1^1 + a_1^2$. This exactly reproduces the general fusion in (4.3), with the first term describing $\mathcal{A}^{M,pp'(p+p')^{-1}} \cong \mathcal{A}^{M,pp'(p+p')}$, and the terms in parentheses describing $\mathcal{N}_{p+p'}$. Note that the latter can equivalently be described as a single D7-brane with $p + p'$ units of worldvolume flux.

¹⁵We should be careful as there is an ambiguity in this description. As was discussed in [37], $U(1)_{mn} \cong \mathcal{A}^{mn,1} \cong \mathcal{A}^{m,n} \otimes \mathcal{A}^{n,m}$ are isomorphic as topological field theories. Furthermore, the effective Lagrangian descriptions for $\mathcal{A}^{m,n}$, $\mathcal{A}^{n,m}$, and $U(1)_{mn}$ are essentially identical. Technically, they can only be distinguished by coupling to appropriate background fields. Our choice of the identification is motivated by the field-theoretic results detailed in appendix C.

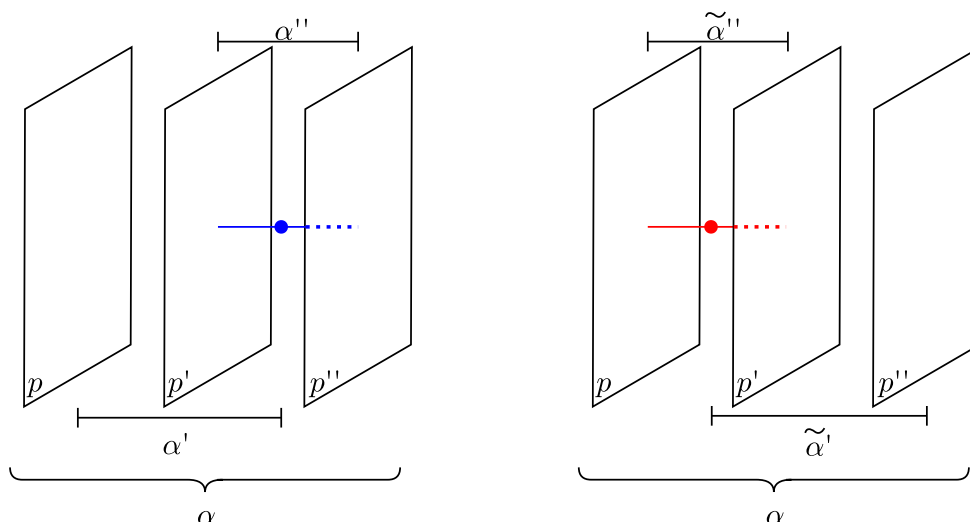


Figure 1. Fusion of three stacks of D5-branes. The left figure depicts the fusion $\mathcal{N}_p \otimes (\mathcal{N}_{p'} \otimes \mathcal{N}_{p''})$, while the right depicts the fusion $(\mathcal{N}_p \otimes \mathcal{N}_{p'}) \otimes \mathcal{N}_{p''}$. In the former case, there is a center-of-mass mode α for the overall $p + p' + p''$ stack, and two other relative modes α' and α'' respectively for the $p - (p' + p'')$ and $p' - p''$ stacks. The same applies to the figure on the right.

Collective degrees of freedom. The conclusions above can be recasted from the perspective of the original stacks of p and p' D5-branes. We see that the gauge field $\alpha_1 \sim pa_1^1 + p'a_1^2$ defined on $\mathcal{N}_{p+p'}$ is the overall *center-of-mass* mode for the combined stack of $p + p'$ D5-branes. Likewise, a_1^1 and a_1^2 are the center-of-mass modes of the individual stacks.

The other gauge field, α'_1 , should be interpreted as a mode describing the *relative* motion of the two stacks of p and p' D5-branes. This can be understood in the following way. The Chan-Paton bundle for the stack of p D5-branes has a gauge group $U(p)$. It is known that the center-of-mass mode and the non-Abelian modes are decoupled from each other, so that the gauge group can be schematically considered as $SU(p) \times U(1)_{\text{CM}}$. The same is true of course for the stack of p' D5-branes as well as the final stack of $p + p'$ D5-branes. However, this final stack has another Abelian mode contained within it. Within $SU(p + p')$ there is a maximal subgroup given by

$$SU(p + p') \supset SU(p) \times SU(p') \times U(1)_{\text{rel}}, \tag{4.9}$$

where the $U(1)_{\text{rel}}$ is generated by the Cartan element $\text{diag}(p'\mathbb{1}_p, -p\mathbb{1}_{p'})$. It is this $U(1)_{\text{rel}}$ that is describing the relative motion of the D5-brane stacks, which manifests itself as the decoupled minimal TQFT $\mathcal{A}^{M,pp'(p+p')}$ at the level of the fusion algebra (4.3). Note that if we were to start with stacks of, say, $p + k$ and $p' - k$ D5-branes, then we would end up with a different $U(1)_{\text{rel}}$, as is reflected by a different decoupled minimal TQFT $\mathcal{A}^{M,(p+k)(p'-k)(p+p')}$ in the fusion.

We can carry out the same argument for more than two stacks of D5-branes. Consider, for instance, three stacks of p , p' , and p'' D5-branes as in figure 1. Each stack of D5-branes will itself have a center-of-mass mode describing the collective motion, i.e. a_1^1 , a_1^2 , and a_1^3 , but we can repackage these degrees of freedom in different combinations. We may first consider

an overall center-of-mass mode, α_1 , describing the collective motion of the three stacks. We can then consider a mode describing the relative motion of the leftmost D5-brane stack compared to the collective motion of the rightmost two D5-brane stacks given by α'_1 , as well as a mode describing the relative motion of the two rightmost D5-brane stacks given by α''_1 . The explicit expressions for $\alpha_1, \alpha'_1, \alpha''_1$ can be written as

$$\begin{aligned}\alpha_1 &= \frac{1}{p+p'+p''} \left(pa_1^1 + p'a_1^2 + p''a_1^3 \right), \\ \alpha'_1 &= \frac{1}{p(p'+p'')} \left((p'+p'')(pa_1^1) - p(p'a_1^2) - p(p''a_1^3) \right), \\ \alpha''_1 &= \frac{1}{p'p''} \left(p''(p'a_1^2) - p'(p''a_1^3) \right).\end{aligned}\tag{4.10}$$

By regrouping our degrees of freedom in this way, we are effectively performing the fusion

$$\mathcal{N}_p \otimes (\mathcal{N}_{p'} \otimes \mathcal{N}_{p''}) \cong \mathcal{A}^{M,p'p''(p'+p'')} \otimes \mathcal{A}^{M,p(p'+p'')(p+p'+p'')} \otimes \mathcal{N}_{p+p'+p''}.\tag{4.11}$$

The above describes the left side of figure 1. If we grouped the stacks of D5-branes in the other way as in the right side, we could define similar modes and would be performing the fusion

$$(\mathcal{N}_p \otimes \mathcal{N}_{p'}) \otimes \mathcal{N}_{p''} \cong \mathcal{A}^{M,pp'(p+p')} \otimes \mathcal{A}^{M,p''(p+p')(p+p'+p'')} \otimes \mathcal{N}_{p+p'+p''}.\tag{4.12}$$

Since the only difference between the two processes is the order in which we fuse the stacks of D5-branes, it is natural to argue that locality of the theory demands the fusion to be associative, i.e. (4.11) and (4.12) should be physically equivalent. In appendix C, we show that these two triple fusion rules are indeed isomorphic.

It is worth taking a moment to reassess the previous results. We are able to model the symmetry defects \mathcal{N}_p as stacks of p D5-branes, which themselves can equivalently be described as a single D7-brane with p units of worldvolume flux. The fusion (4.3) of the defects themselves does not seem to favor a particular approach, as it extracts only the Abelian information from the Dp -branes. All the brane information captured by the fusion rules comes from a single center-of-mass mode as well as a single mode describing relative motion, or multiple relative modes in the case of more instances of fusion. In this sense, the dynamics of the non-Abelian modes in the worldvolume theory are “washed out” in the symmetry defects.

4.2 $\mathcal{N}_p \otimes \mathcal{N}_{-p}$ fusion and Tachyon Condensation

Condensation defect. The case of $p' = -p \pmod M$ requires special treatment. As discussed above, the $\widetilde{\mathcal{N}}_p(M^3)$ component of the defect is invertible and thus becomes the identity for the $\mathcal{N}_p \otimes \mathcal{N}_{-p}$ fusion. Regarding the fusion between the minimal TQFTs, we have, in terms of explicit worldvolume actions,

$$\begin{aligned}\mathcal{A}^{M,p}[\mathcal{B}_2] \otimes \mathcal{A}^{M,-p}[\mathcal{B}_2] &= \int \mathcal{D}a_1 \mathcal{D}a'_1 \exp \left(2\pi i \int_{M^3} \frac{pM}{2} a_1 da_1 - \frac{pM}{2} a'_1 da'_1 + p(a_1 - a'_1) \mathcal{B}_2 \right) \\ &= \int \mathcal{D}\bar{a}_1 \mathcal{D}a'_1 \exp \left(2\pi i \int_{M^3} M \bar{a}_1 da'_1 + \frac{p^{-1}M}{2} \bar{a}_1 d\bar{a}_1 + \bar{a}_1 \mathcal{B}_2 \right) \\ &= (\mathcal{Z}_M)_0[\mathcal{B}_2],\end{aligned}\tag{4.13}$$

where a_1 and a'_1 respectively denote the auxiliary fields living on $\mathcal{A}^{M,p}$ and $\mathcal{A}^{M,-p}$, while $\bar{a}_1 = p(a_1 - a'_1)$. In the third line, we used the fact that if M is odd, the Dijkgraaf-Witten (DW) term in $(\mathcal{Z}_M)_K$ is trivial when K is an integer multiple of M , i.e. $K \sim K + M$ (assuming M^3 is a spin manifold) [52].¹⁶ Integrating out a'_1 enforces $M\bar{a}_1 \in H^1(M^3; \mathbb{Z})$ to be an integral lift of a 1-cocycle in $H^1(M^3; \mathbb{Z}_M)$. We then rewrite (4.13) as a discrete sum of the $\widehat{\mathbb{Z}}_M^{(1)}$ symmetry defects $\mathcal{U}_1(\Sigma^2)$ in (2.16),¹⁷

$$\mathcal{N}_p \otimes \mathcal{N}_{-p} = \frac{1}{|H^0(M^3; \mathbb{Z}_M)|} \sum_{\Sigma^2 \in H_2(M^3; \mathbb{Z}_M)} \mathcal{U}_1(\Sigma^2) := \mathcal{C}, \tag{4.14}$$

where Σ^2 is the Poincaré dual of the torsional reduction of \bar{a}_1 with respect to M^3 . We will refer to \mathcal{C} as the “condensation defect” as is common in the literature. This nomenclature stems from the interpretation that the sum of operators signifies a “higher gauging” of the $\widehat{\mathbb{Z}}_M^{(1)}$ symmetry [47]. The symmetry operators condense on the submanifold M^3 of spacetime, hence the name. Note that the factor of $|H^0(M^3; \mathbb{Z}_M)|$ is a standard gauge-theoretic normalization coming from the normalization of the path integral measure [1, 23].

Tachyon condensation. The presence of the condensation defect can be understood completely in terms of the dynamical process of tachyon condensation, suggesting an interesting interpretation of non-invertible fusion in holographic setups. Here we review some relevant generic features of tachyon condensation [53–55].

Our starting point is a Dp - \overline{Dp} -brane pair wrapped on a submanifold \mathcal{M}^{p+1} of the total spacetime, where an anti- Dp -brane is the orientation reversal of a Dp -brane. Over the Dp -brane we denote the Chan-Paton bundle by \mathcal{E}^+ , while over the anti- Dp -brane we denote the bundle \mathcal{E}^- . Without loss of generality, we may take $\mathcal{E}^- = \mathcal{K}$ and $\mathcal{E}^+ = \mathcal{L} \otimes \mathcal{K}$ for \mathcal{L} and \mathcal{K} complex line bundles.¹⁸ It is known that the worldvolume theory for a brane-antibrane pair suffers from a tachyonic mode. The tachyon in this case should be viewed as a section of $\mathcal{E}^+ \otimes (\mathcal{E}^-)^* \cong \mathcal{L}$, where $(\mathcal{E}^-)^*$ denotes the dual bundle, reflecting that it is charged under the $U(1) \times U(1)$ gauge symmetry of the worldvolume theory. We may consider vortex-like solutions corresponding to a choice of tachyon profile that vanishes in a specified region of spacetime. To be precise, let us take the tachyon profile to be

$$T = c \cdot s, \tag{4.15}$$

where s is a section of \mathcal{L} which vanishes over a codimension-2 submanifold $\mathcal{M}^{p-1} \subset \mathcal{M}^{p+1}$, and c is a constant chosen such that the tachyon takes its vev away from \mathcal{M}^{p-1} . By construction, this localizes a codimension-2 solution to exist only along \mathcal{M}^{p-1} , and far from \mathcal{M}^{p-1} the system is indistinguishable from the vacuum. In the above construction, we are only

¹⁶The continuum action for a 3d DW theory $(\mathcal{Z}_M)_K$ is given by $\int M a_1 da_1 + (k/2) a_1 dc_1$.

¹⁷For general M , there is an additional factor in the condensation defect coming from a choice of *discrete torsion*, or the choice to stack with an SPT phase when gauging. For M odd, which we are assuming throughout this work, this discrete torsion factor will be trivial and thus not present in the condensation defect.

¹⁸The proper classification of D-branes is well-known to be formalized not by ordinary cohomology but by K-theory [45, 46, 56, 57]. In this formalism, the relevant quantity which classifies the D-branes is the K-theoretic “difference” between the bundles \mathcal{E}^\pm . Hence, the choice of parametrization in the main text is for convenience.

considering a single Dp -brane and \overline{Dp} -brane producing a single $D(p-2)$ -brane, but in general one may consider 2^{k-1} each of branes and antibranes to produce a $D(p-2k)$ -brane [46].

More explicitly, note that the worldvolume bundle \mathcal{E} has a natural \mathbb{Z}_2 grading induced by the splitting between the brane and antibrane degrees of freedom, thereby giving the structure of a *superbundle*. Taking advantage of the superbundle structure, we may collect the worldvolume degrees of freedom into a single superconnection [58],

$$\mathbb{A} = \begin{pmatrix} a_1^+ & iT \\ i\overline{T} & a_1^- \end{pmatrix}, \tag{4.16}$$

where a_1^\pm denote the Chan-Paton gauge fields over \mathcal{E}^\pm , and T is the tachyon field as a section. Using this superconnection, we may write an action that allows us to go directly from a Dp - \overline{Dp} configuration to a stable $D(p-2)$ -brane. This action was proposed in [59], later elaborated on in [60–63], and can be written in the form

$$S_{\text{WZ}}^{\text{D}p\text{-}\overline{\text{D}}p} = \mu_p \int_{\mathcal{M}^{p+1}} C \wedge e^B \wedge \text{Tr}_s e^{\mathbb{F}} \wedge \sqrt{\frac{\hat{\mathcal{A}}(\mathcal{R}_T)}{\hat{\mathcal{A}}(\mathcal{R}_N)}} \Big|_{(p+1)\text{-form}}, \tag{4.17}$$

where $\mathbb{F} = d\mathbb{A} + \mathbb{A}^2$ is the supercurvature given by

$$\mathbb{F} = \begin{pmatrix} da_1^+ - T\overline{T} & iDT \\ i\overline{DT} & da_1^- - T\overline{T} \end{pmatrix}. \tag{4.18}$$

In the above, Tr_s is the \mathbb{Z}_2 -graded supertrace and the covariant derivatives DT and \overline{DT} are defined as

$$DT := dT + (a_1^+ - a_1^-)T, \quad \overline{DT} := d\overline{T} - (a_1^+ - a_1^-)\overline{T}. \tag{4.19}$$

By writing $a_1^+ = a_{\mathcal{L}} + a_{\mathcal{K}}$ and $a_1^- = a_{\mathcal{K}}$ to reflect the structure of the line bundles, we can see that the tachyon field T couples only to $a_{\mathcal{L}}$ and thus is tied only to the bundle \mathcal{L} . The action (4.17) may be rewritten as a WZ action for a $D(p-2)$ -brane living inside the Dp - \overline{Dp} worldvolume, i.e.

$$S_{\text{WZ}}^{\text{D}p\text{-}\overline{\text{D}}p} = \mu_p \int_{\mathcal{M}^{p+1}} C e^B \text{ch}(\mathcal{K}) (\text{ch}(\mathcal{L}) - 1) = \mu_{p-2} \int_{\mathcal{M}^{p-1}} C e^{B+F_{\mathcal{K}}} = S_{\text{WZ}}^{\text{D}(p-2)}, \tag{4.20}$$

where in the final equality we integrated out the two directions transverse to the $D(p-2)$ -brane's worldvolume \mathcal{M}^{p-1} . Due to our choice of bundle structure, this is equivalent to integrating out the dependence on the line bundle \mathcal{L} , and thus the tachyon.¹⁹

¹⁹It is worth pointing out that the factor involving $\text{ch}(\mathcal{L})$ is encoding the K-theoretic difference alluded to in the previous footnote. Had we not made the previous simplification and instead chosen $\mathcal{E}^+ = \mathcal{L}_1 \otimes \mathcal{K}$ and $\mathcal{E}^- = \mathcal{L}_2 \otimes \mathcal{K}$ (we can always consider a common line bundle \mathcal{K}), this term would have taken the form $\text{ch}(\mathcal{L}_1) - \text{ch}(\mathcal{L}_2)$. For the purpose of illustration, we assume in (4.20) that the integral of $\text{ch}(\mathcal{L}) - 1$ is equal to 2π . In general, the ratio of Dp -brane charges is given by $\mu_p/\mu_{p'} = (2\pi\sqrt{\alpha'})^{p'-p}$ [64], but we also adopt the convention that $2\pi\alpha' = 1$, otherwise $B_2 + 2\pi\alpha'F_2$ should be the gauge-invariant combination instead.

Condensation defect from tachyon condensation. Let us apply the general discussion above to our case where a D5-brane and an anti-D5-brane wrap the submanifold $M^3 \times S^3$ in the KT-KS geometry $\mathcal{W}^5 \times T^{1,1}$. When the branes are brought together there will be a solitonic D3-brane localized on a worldvolume $\mathcal{M}^4 \subset M^3 \times S^3$, with a WZ action described by (4.20).

To reconstruct the condensation defect $\mathcal{C} \sim \sum \mathcal{U}_1$ in (4.14), we must first specify the submanifold \mathcal{M}^4 that the solitonic D3-brane is wrapping. Equivalently, this amounts to specifying the bundle \mathcal{L} defined along the two transverse directions of \mathcal{M}^4 with respect to $M^3 \times S^3$. The integration in (4.20) can be viewed as a two-step process: a fiber integration from the S^3 down to the Hopf base \widetilde{S}^2 , and an application of Poincaré duality to land on $\Sigma^2 \in H_2(M^3; \mathbb{Z}_M)$. Both of these are encoded by the Chan-Paton flux $F_{\mathcal{L}}$. Specifically, we can construct a $U(1)_R$ -equivariant ansatz for the Chan-Paton flux as

$$F_{\mathcal{L}} = f_2 + M\eta_1 \frac{D\psi}{2\pi} + a_0^i V_i, \tag{4.21}$$

where f_2, η_1 , and a_0^i are fields defined over the external spacetime, and V_i are the volume forms of the spheres present in the $T^{1,1}$. Enforcing the Bianchi identity $dF_{\mathcal{L}} = 0$ leads us to identify $a_0^1 = a_0^2 = a_0$ and $M\eta_1 = da_0$, so that

$$F_{\mathcal{L}} = M\eta_1 \frac{D\psi}{2\pi} - a_0 (V_1 + V_2 - F_2) + \Lambda_2 = da_{\mathcal{L}} + \Lambda_2, \tag{4.22}$$

where $a_{\mathcal{L}} = a_0(D\psi/2\pi)$ is a locally defined gauge field associated with \mathcal{L} . The 2-form Λ_2 is closed, and it plays the role of a basepoint flux for $F_{\mathcal{L}}$, much like f_2^i in the discussion surrounding (4.5). Performing the integration in (4.20) using this field strength yields the desired WZ action for a solitonic D3-brane localized on $\mathcal{M}^4 = \Sigma^2 \times \widetilde{S}^2$.

Note that the expression for $F_{\mathcal{L}}$ give two possible sources for a D3-brane in the D5- $\overline{\text{D5}}$ worldvolume. The first is as a soliton arising from tachyon condensation as discussed above. The second is from any worldvolume flux present on the original D5-brane and anti-D5-brane. The D5-branes used to construct $\mathcal{N}_1(M^3)$ did not carry any worldvolume flux, so we may take the corresponding term to be zero if desired.

At this point, one may try to reduce the anomaly polynomial $\mathcal{I}_5^{\text{D3}}$ for the D3-brane on the Hopf base \widetilde{S}^2 , but end up finding an apparently vanishing result, i.e.

$$\int_{\widetilde{S}^2 \subset S^3} \mathcal{I}_5^{\text{D3}} = \mathcal{I}_3 \text{ “=” } 0. \tag{4.23}$$

However, this is too quick. The operator $\mathcal{U}_1(\Sigma^2)$ we are ultimately concerned with finding is a holonomy of a discrete gauge field $\mathcal{B}_2 \in H^2(M^3; \mathbb{Z}_M)$. The standard descent procedure falls short as the reduction above merely indicates that there is no nontrivial holonomy for *continuous* gauge fields; we may still have *discrete* holonomies. Roughly speaking, we can again think of \mathcal{I}_3 as a “field strength” for torsional flux and thus would vanish when evaluated on integer-valued cycles. If not integer-valued, then what cycles *should* we evaluate this on? Since we aim to recover a \mathbb{Z}_M -valued operator, we shall follow our nose and consider reducing on \mathbb{Z}_M -valued *torsional* cycles.

The correct topological operator is then not the standard holonomy of \mathcal{I}_3 on an integer-valued cycle, but a holonomy along a \mathbb{Z}_M -valued cycle. Namely, the proper operator to

consider is²⁰

$$\mathcal{U}_1(\Sigma^2) = \exp\left(\frac{2\pi i}{M} \int_{\tilde{\Sigma}^3} \mathcal{I}_3\right) = \exp\left(\frac{2\pi i}{M} \langle \tilde{\Sigma}^3, \mathcal{I}_3 \rangle\right), \quad (4.24)$$

where $\tilde{\Sigma}^3$ is a formal extension of Σ^2 such that $\partial\tilde{\Sigma}^3 = M\Sigma^2$. In the final expression we have rewritten the integral as a pairing between \mathbb{Z}_M -valued cycles and cocycles, and should now consider everything to be valued in (co)homology with \mathbb{Z}_M coefficients. As we mentioned, \mathcal{I}_3 should be viewed as a “field strength” for a discrete field, so we may express $\mathcal{I}_3 = \beta'(\mathcal{B}_2) \in H^3(M^3; \mathbb{Z}_M)$. This justifies our decision to consider the reduction on \mathbb{Z}_M -valued cycles, as opposed to cycles valued in any other coefficients, as the only discrete 2-form gauge field present in the supergravity setup is $\mathcal{B}_2 \in H^2(M^3; \mathbb{Z}_M)$.²¹

With this choice, we can evaluate the holonomy of \mathcal{I}_3 as

$$\exp\left(\frac{2\pi i}{M} \langle \tilde{\Sigma}^3, \beta'(\mathcal{B}_2) \rangle\right) = \exp\left(\frac{2\pi i}{M} \langle \beta'(\tilde{\Sigma}^3), \mathcal{B}_2 \rangle\right) = \exp\left(\frac{2\pi i}{M} \langle \Sigma^2, \mathcal{B}_2 \rangle\right). \quad (4.25)$$

In the first equality, we switched the Bockstein homomorphism in cohomology to its dual in homology, and in the second equality, we used the property that $\beta' = \partial/M$ for the Bockstein homomorphism in homology (see appendix B for details). This holonomy is nothing but the $\mathcal{U}_1(\Sigma^2)$ operator we saw in the condensation defect (4.14).

The arguments above regarding the D3-brane worldvolume action can be rephrased in the language of differential cohomology [26–29, 65]. The standard characteristic class construction via the descent procedure constructs the curvature form for a (higher) anomaly bundle over the base spacetime. Naïvely, this curvature form only captures the continuous data of a Cheeger-Simons differential cocycle; the torsional data is overlooked. What we see above is that the torsional data is not completely lost, but can be reconstructed through the presence of torsional flux.

Now that we have constructed an operator for the D3-brane wrapping $\Sigma^2 \times \tilde{S}^2$, there is one last thing to consider about the D3-brane soliton arising from the D5- $\overline{\text{D5}}$ worldvolume. The Chan-Paton field on the D5- $\overline{\text{D5}}$ brane pair is dynamical and therefore should be summed over in a quantum theory. This includes a summing over the gauge field not only from the solitonic D3-brane itself, but also from $a_{\mathcal{L}}$ along the other two transverse directions in the original D5- $\overline{\text{D5}}$ worldvolume. A sum over $a_{\mathcal{L}}$ configurations is equivalent to a sum over the configurations of a_0 defined in (4.22). This in turn, as a_0 dictates the torsional 2-cycle Σ^2 that the D3-brane is wrapping (via Poincaré duality), yields a sum over all possible 2-cycles $\Sigma^2 \in H_2(M^3; \mathbb{Z}_M)$. Hence, taking into account the proper gauge theoretic normalization, our final defect is given by

$$\mathcal{C} = \frac{1}{|H^0(M^3; \mathbb{Z}_M)|} \sum_{\Sigma^2 \in H_2(M^3; \mathbb{Z}_M)} \exp\left(\frac{2\pi i}{M} \int_{\Sigma^2} \mathcal{B}_2\right), \quad (4.26)$$

²⁰We can alternatively phrase this in terms of the RR field strengths, by originally having a term in G_5 of the form $M\eta_3 \wedge (V_1 + V_2)$ with η_3 a \mathbb{Z}_M -valued field strength. Due to the torsion, this term would formally be zero and naïvely wouldn't be present in the expansion.

²¹By exactness of the short exact sequence $0 \rightarrow \mathbb{Z}_M \rightarrow \mathbb{Z}_{M^2} \rightarrow \mathbb{Z}_M \rightarrow 0$, the quantity $M\mathcal{B}_2$ is trivial, and so is $\beta'(M\mathcal{B}_2)$. This means that $\mathcal{U}_M(\Sigma^2) = \mathcal{U}_1(M\Sigma^2) = \exp\left(2\pi i \int_{\tilde{\Sigma}^3} \mathcal{I}_3\right) = 0$ as we saw in (4.23).

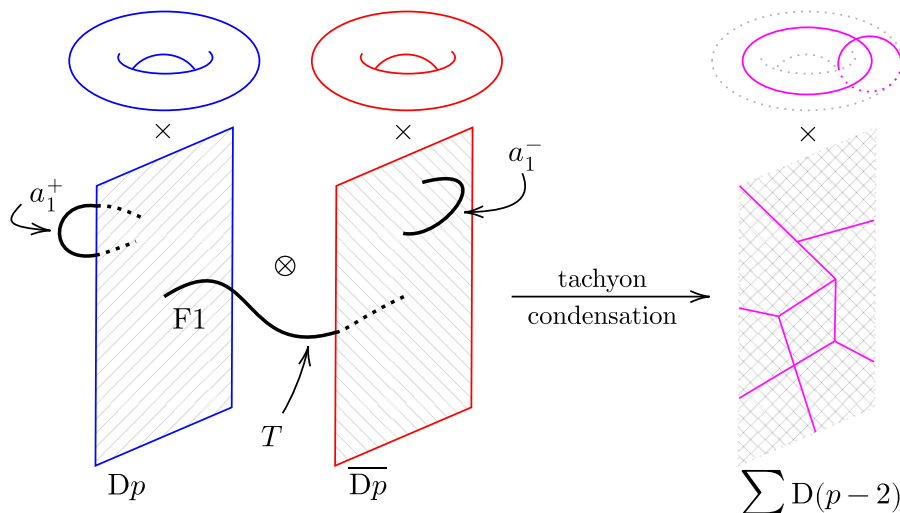


Figure 2. Condensation defect arising from tachyon condensation on a $Dp\text{-}\overline{Dp}$ system wrapping some internal manifold. We schematically decomposed the worldvolume of the branes as a direct product of internal and external manifolds. Chan-Paton modes exist both for open strings ending on the same brane or on different branes. The latter includes tachyonic modes, which, upon integrating out, lead to a condensate of lower branes within the (both internal and external) worldvolume of the original $Dp\text{-}\overline{Dp}$ system.

which is in exact agreement with (4.14) for $p = 1$. We have understood the condensation defect arising in the fusion $\mathcal{N}_1 \otimes \mathcal{N}_{-1}$ as being a sum of all possible D3-brane configurations in the original D5- and $\overline{D5}$ -brane system. It is worth noting that the $a_{\mathcal{L}}$ that the condensation defect couples to is the center-of-mass mode for the brane-antibrane configuration. Even though it arises as a difference of gauge fields, this “difference” is really a sum with an orientation reversal coming from the anti-D5-brane. The condensation defect aligns with the earlier observation (in the case of $\mathcal{N}_p \otimes \mathcal{N}_{p'}$) that the symmetry defects are those coupling to the center-of-mass modes for the stacks of D p -branes.

We can extrapolate the above argument to the more general fusion of $\mathcal{N}_p \otimes \mathcal{N}_{-p}$. In this case, tachyon condensation will produce p D3-branes defined on the $\Sigma^2 \times \widetilde{S}^2$ submanifold. As we will elaborate upon later, this will cause each term in the sum over 2-cycles to have charge p rather than charge one. As the sum is over *all* 2-cycles in M^3 , this rescaling will simply reshuffle the terms in the sum and has no effect otherwise. We can then immediately see the brane realization of the more general fusion $\mathcal{N}_p \otimes \mathcal{N}_{-p} = \mathcal{C}$ as derived in (4.14).

There is one last comment to make regarding the computation of the condensation defect. When constructing the solitonic D3-brane on the D5- $\overline{D5}$ worldvolume, the base \widetilde{S}^2 of the Hopf fibration was really one of the many possible choices on which to wrap the D3-brane. We could have chosen any possible linear combination of the two spheres in the base of the $T^{1,1}$ fibration and would have produced the same defect $\mathcal{U}_q(\Sigma^2)$. From the point of view of the external spacetime, the defects are identical. In fact, one might argue that a more natural choice for the worldvolume of the D3-brane is one wrapping the equatorial S^2 of the S^3 cycle. For this choice the solitonic nature is more apparent as follows. When implementing tachyon condensation for a D5- $\overline{D5}$ system, the tachyon profile is crucial in determining where the D3-brane is located. We could have picked the tachyon profile such that it approaches T_0 , its

minimum, at one pole of the S^3 , and $-T_0$ at the other pole.²² The tachyon will then have a profile such that it vanishes exactly along the equatorial S^2 specified above. The fact that this choice of internal directions for the D3-brane has no effect on the produced operator is an interesting phenomenon that we will encounter again later in this paper.

4.3 Nature of the $\widehat{\mathbb{Z}}_M^{(1)}$ Defect

We have so far seen that the existence of the $\mathbb{Z}_{2M}^{(0)}$ symmetry defect $\mathcal{N}_1(M^3)$ in the 5d bulk \mathcal{W}^5 can be attributed to a D5-brane wrapping $M^3 \times S^3$. Furthermore, the fusion of such a D5-brane with its orientation reversal produces a condensate of the $\widehat{\mathbb{Z}}_M^{(1)}$ defects $\mathcal{U}_1(\Sigma^2)$ as solitonic D3-branes localized on M^3 . A natural question to ask then is whether a $\mathcal{U}_1(\Sigma^2)$ defect defined on a generic $\Sigma^2 \not\subset M^3$ can be realized as a bona fide D3-brane existing on its own right.

The answer is actually baked in already in the presentation of (4.20). Instead of considering solitonic D3-brane excitations on a D5- $\overline{\text{D5}}$ worldvolume, we can consider a D3-brane on its own wrapping the Hopf base $\widehat{S^2} \subset S^3$ inside the internal $T^{1,1}$, and an arbitrary external \mathbb{Z}_M -valued cycle $\Sigma^2 \in H_2(\mathcal{W}^5; \mathbb{Z}_M)$. Following the identical arguments given previously, the D3-brane reproduces the $\widehat{\mathbb{Z}}_M^{(1)}$ magnetic symmetry defect,

$$\mathcal{U}_1(\Sigma^2) = \exp\left(\frac{2\pi i}{M} \int_{\Sigma^2} \mathcal{B}_2\right). \tag{4.27}$$

It is interesting to note that this symmetry defect can also be constructed by a closed fundamental (F1) string sitting in the external spacetime \mathcal{W}^5 . As F1-strings do not couple to the Ramond-Ramond sector, the anomaly polynomial for this F1-string is simply given by

$$\mathcal{I}_3^{\text{F1}} = h_3 = db_2 - \widetilde{\beta}'(\mathcal{B}_2), \tag{4.28}$$

where we used (2.5) and (2.6). The fluctuating field b_2 is suppressed as we push the F1-string towards the conformal boundary, so what is left behind is a holonomy $\sim \exp(2\pi i \int_{\Sigma^2} \mathcal{B}_2/M)$ (up to a sign convention). We have thus showed that these F1-strings can also give rise to the symmetry defect $\mathcal{U}_1(\Sigma^2)$.

An alternative understanding of this comes from a modification of the ‘‘baryon vertex’’ introduced in [66]. This utilizes the fact that F1-strings act as Wilson operators in the boundary field theory, which can be screened by operators carrying the appropriate charge. In the background we are concerned with, there are two possible causes for the screening of F1-strings: a D5-brane wrapping the entirety of the $T^{1,1}$, or a D3-brane wrapping only the $S^3 \subset T^{1,1}$ [67]. From the Bianchi identities,

$$\int_{T^{1,1}} dG_7 = - \int_{T^{1,1}} H_3 G_5 = -N db_2, \quad \int_{S^3} dG_5 = - \int_{S^3} H_3 G_3 = -M db_2, \tag{4.29}$$

we can see that the former will screen N F1-strings while the latter will screen M F1-strings. The combined effect will be to screen $\text{gcd}(N, M) = M$ F1-strings. This implies that the defect produced by an F1-string will generate a \mathbb{Z}_M (rather than \mathbb{Z}_N) symmetry.

Due to the baryon vertex, a collection of k F1-strings also generates the full $\widehat{\mathbb{Z}}_M^{(1)}$ symmetry as long as $\text{gcd}(M, k) \neq 1$ (otherwise it will only generate a $\mathbb{Z}_{M/k}$ subgroup). With an eye

²²We would need to pick similar configurations in the external spacetime within M^3 .

towards the construction of the dual symmetry defect in section 5, it is convenient for us to take $k = N/M$. One may then interpret the defect $\mathcal{U}_1(\Sigma^2)$ to be either one D3-brane or k coincident F1-strings. However, these descriptions are equivalent when considering the Myers Effect discussed earlier.²³ To see this, it is best to first consider the construction of [68] which describes an S-dual description to the KS/KT background. In that analysis, there exists nontrivial H_3 flux threading the S^3 of the internal geometry. If one imagines k coincident D1-branes placed in the external spacetime of this background, the D1-branes will “puff up” into a single D3-brane with k units of worldvolume flux through an $S^2 \subset S^3$ via the standard Myers Effect. Transforming to the KS/KT background,²⁴ this tells us that a stack of k F1-strings in the presence of a nontrivial G_3 background is equivalent to a D3-brane with k units of worldvolume flux threading a transverse $S^2 \subset S^3$, which we identify as the Hopf base \widetilde{S}^2 above. We thus see that the two descriptions presented above for $\mathcal{U}_1(\Sigma^2)$ converge. We will use the D3-brane description, as opposed to the one involving F1-strings, in the discussion that follows.

Recall that earlier we constructed the $\mathbb{Z}_{2M}^{(0)}$ defects $\mathcal{N}_p(M^3)$ by stacking p D5-branes on $M^3 \times S^3$, thanks to the fact that only the (Abelian) center-of-mass mode, but not the non-Abelian modes, of the stack contributes to this topological operator. Equivalently, we may take a D7-brane with p units of worldvolume flux threading $S^2 \subset T^{1,1}$. It is straightforward to apply the same argument here to construct the general $\widehat{\mathbb{Z}}_M^{(1)}$ defects,

$$\mathcal{U}_q(\Sigma^2) = \exp\left(\frac{2\pi i q}{M} \int_{\Sigma^2} \mathcal{B}_2\right), \tag{4.30}$$

from a stack of D3-branes wrapping $\Sigma^2 \times \widetilde{S}^2$. Again equivalently, this can also be realized by taking a D5-brane with q units of worldvolume flux threading $S^2 \subset T^{1,1}$.

Defect fusion. Unlike the D5-brane operator $\mathcal{N}_p(M^3)$ on which the Chan-Paton fields form a nontrivial TQFT, the D3-brane operator $\mathcal{U}_q(\Sigma^2)$ has a relatively simple structure as seen in (4.30). For two such defects wrapping the same external 2-cycle (mod M) $\Sigma^2 \subset \mathcal{W}^5$, their fusion obeys

$$\mathcal{U}_q(\Sigma^2) \otimes \mathcal{U}_{q'}(\Sigma^2) = \mathcal{U}_{q+q'}(\Sigma^2) = \exp\left(\frac{2\pi i(q+q')}{M} \int_{\Sigma^2} \mathcal{B}_2\right), \tag{4.31}$$

where the label q of \mathcal{U}_q is only defined modulo M . In other words, the \mathcal{U}_q operators realize a group-like fusion algebra corresponding to a \mathbb{Z}_M group.

5 $\mathbb{Z}_M^{(1)}$ and $\widehat{\mathbb{Z}}_{2M}^{(2)}$ defect fusion

Let us proceed to discuss the remaining two symmetry defects resulting from the Gauss Law constraints presented in section 2, namely, \mathcal{O}_q and \mathcal{V}_p , which respectively generate the $\mathbb{Z}_M^{(1)}$ and $\widehat{\mathbb{Z}}_{2M}^{(2)}$ symmetries. As we will show, these operators can also be realized by certain D-brane configurations in the KT-KS background. Remarkably, their fusion rules share some

²³We thank Oren Bergman for pointing this out to us.

²⁴This was described in [69], and a similar effect was discussed in [70].

qualitative features with those we studied in the previous section, thus shedding light on the underlying brane kinematics and dynamics.

The $\mathbb{Z}_M^{(1)}$ symmetry defect $\mathcal{O}_q(W^2)$, as written in (2.15), is composed of multiple 5d bulk gauge fields of different origins. Therefore, it is natural to expect that the string-theoretic construction is not given by a single type of Dp -brane. In fact, we can construct the desired operator through a bound state of a single D3-brane and k D1-branes.²⁵ Let us first consider a D3-brane wrapping the (homologically) nontrivial $S^2 \subset T^{1,1}$. To reduce its WZ action, we have to carefully isolate the components of G_5 along the directions of the S^2 . With (2.2), the full expression for $G_5 = (1 + \star_{10})\mathcal{G}_5$ is given by

$$G_5 = N\Omega_5 + g_2\omega_3 + Ng_5 - \frac{N}{2}(V_1 + V_2) \star_5 F_2 + \star_5 g_2\omega_2, \quad (5.1)$$

where g_5 is an external gauge field. By imposing the IIB Bianchi identities, we find that $\star_5 g_2$ can be solved as

$$\star g_2 = da_2 + \frac{1}{M} g_1 g_2, \quad (5.2)$$

for some globally well-defined a_2 . Now that we have this expansion, the reduction of the D3-brane characteristic class over S^2 becomes

$$\int_{S^2} \mathcal{I}_5^{\text{D3}} = \int_{S^2} G_5 + G_3(B_2 + da_1) = da_2 + \frac{1}{M} g_1 \mathcal{B}_2 + g_1 \left(da_1 + \frac{d\beta_1}{M} \right), \quad (5.3)$$

where we have made use of the expression for g_2 in (2.5). We also hereafter absorb β_1 into the worldvolume Chan-Paton gauge field a_1 via a simple field redefinition.

As argued previously in section 4, the Chan-Paton field is dynamical and hence should be summed over in the worldvolume gauge theory. For the D3-brane operator at hand, performing the path integral enforces the condition $dg_1 = 0$, so we retain only the cohomological class (representative) \mathcal{A}_1 of g_1 as in (2.5).²⁶ The topological operator associated to the D3-brane wrapping $S^2 \subset T^{1,1}$ is thus given by

$$\exp \left(2\pi i \int_{W^2} a_2 + \frac{2\pi i}{M} \int_{W^3} \mathcal{A}_1 \mathcal{B}_2 \right), \quad (5.4)$$

where $\partial W^3 = W^2 \subset \mathcal{W}^5$.

In addition to the D3-brane configuration described above, we also consider a D1-brane in the external \mathcal{W}^5 , whose characteristic class is simply

$$\mathcal{I}_3^{\text{D1}} = G_3|_{\mathcal{W}^5} = dc_2. \quad (5.5)$$

To reproduce the $\mathbb{Z}_M^{(1)}$ symmetry generator, we need k such D1-branes wrapping the W^2 of the D3-brane worldvolume, combining to give

$$\mathcal{O}_1(W^2, W^3) = \exp \left(2\pi i \int_{W^2} (kc_2 + a_2) + \frac{2\pi i}{M} \int_{W^3} \mathcal{A}_1 \mathcal{B}_2 \right). \quad (5.6)$$

²⁵We previously defined k via $N = kM$. As long as $M^2 \gg N \gg M$, there is no significant backreaction from the stack of k probe D1-branes. We assume this limit for the above discussion.

²⁶The potential c_0 may be turned off via a gauge transformation.

Alternatively, we can derive the above expression for $\mathcal{O}_1(W^2, W^3)$ by considering a D3-brane with k units of worldvolume flux threading the $S^2 \subset T^{1,1}$.

This combination of branes pairs nicely with the discussion of the $\mathcal{U}_1(\Sigma^2)$ operator surrounding (4.29). There, a bound state of a single D3-brane and k F1-strings gave rise to a $\widehat{\mathbb{Z}}_M^{(1)}$ symmetry generator. It is natural to expect that the symmetry generator for the Pontryagin dual symmetry $\mathbb{Z}_M^{(1)}$ should be generated by the S-dual brane configuration, i.e. a bound state of a single D3-brane and k D1-branes, which is precisely what was used to construct $\mathcal{O}_1(W^2, W^3)$.

As mentioned in section 2, this operator is only well-defined when $W^2 = \partial W^3$. In order to find an operator that is well-defined even when W^2 is closed but is *not* a boundary, we may rewrite it in terms of auxiliary fields which we denote as λ_1 and φ ,

$$\mathcal{O}_1(W^2) = \int \mathcal{D}\lambda_1 \mathcal{D}\varphi \exp \left(2\pi i \int_{W^2} \hat{c}_2 + \lambda_1 \mathcal{A}_1 + \varphi \mathcal{B}_2 - M\varphi d\lambda_1 \right). \quad (5.7)$$

We emphasize that, similarly to the worldvolume Chan-Paton field a_1 in $\mathcal{A}^{M,p}[\mathcal{B}_2]$ (see (2.12)), the presence of the worldvolume gauge fields λ_1 and φ can be understood in terms of anomaly inflow. The $\mathcal{A}_1 \mathcal{B}_2 / M$ term in (5.6) should be interpreted as an anomaly defined in one dimension higher than the D3-D1 worldvolume. Specifically, it describes a mixed anomaly between the $\mathbb{Z}_M^{(0)} \subset \mathbb{Z}_{2M}^{(0)}$ and $\mathbb{Z}_M^{(1)}$ global symmetries on the D3-D1 bound state. It is known in general that a theory with a *global* symmetry $G^{(k)}$ can be equivalently viewed as a *gauge* theory for the Pontryagin-dual symmetry $\widehat{G}^{(d-k-2)}$ [1]. In our case, this corresponds to a $\widehat{\mathbb{Z}}_M^{(0)} \times \widehat{\mathbb{Z}}_M^{(-1)}$ gauge theory on the D3-D1 worldvolume, which is exactly the TQFT $\mathcal{X}^{M,1}$ within $\mathcal{O}_1(W^2)$.

Before proceeding, let us pause to examine the field content of the $\mathcal{O}_1(W^2)$ operator. The first term with \hat{c}_2 describes the D3- and D1-brane charges present in the bound state, while the final term is simply a BF term that Higgses the two gauge fields from U(1) down to \mathbb{Z}_M . On the other hand, the second and third terms can be interpreted as *induced* charges from F1-strings and D1-branes. To see this explicitly, note that $\varphi \in H^0(W^2; \mathbb{Z}_M)$ and $\lambda_1 \in H^1(W^2; \mathbb{Z}_M)$ by virtue of the discussion in footnote 10. The coupling to \mathcal{B}_2 in (5.7) then implies that the D3-D1 worldvolume has a φ number of induced F1-string charge, where φ is summed from 0 to $M - 1$ in order to create a gauge-invariant operator. Similarly, the coupling to \mathcal{A}_1 in (5.7) describes induced D1-brane charge localized on 1-cycles γ^1 which is Poincaré-dual to λ_1 (with respect to W^2). We will soon explain why \mathcal{A}_1 is associated with D1-branes when we discuss the $\mathcal{V}_1(\gamma^1)$ operator. See figure 3 for a schematic picture of these induced charges on $\mathcal{O}_1(W^2)$.

5.1 $\mathcal{O}_q \otimes \mathcal{O}_{q'}$ fusion

We are now interested in the fusion of these operators, which can be computed using methods similar to those utilized in section 4. First we can consider the self-fusion of $\mathcal{O}_1(W^2, W^3)$ where W^2 is a boundary such that $W^2 = \partial W^3$. In this case, we may compute the fusion by taking the D3-D1 configuration to be a neutral D5-brane with the appropriate units of worldvolume flux. Specifically, we take two D5-branes each with worldvolume $W^2 \times S^2 \times \widetilde{S}^2$ and one unit of worldvolume flux threading the \widetilde{S}^2 . Each D5-brane will have a characteristic

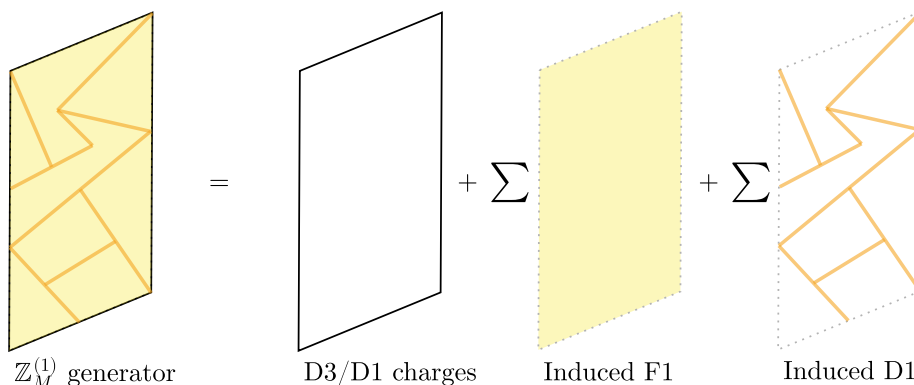


Figure 3. A schematic picture for the induced charges on the $\mathbb{Z}_M^{(1)}$ generator $\mathcal{O}_q(W^2)$. In addition to the D3- and D1-brane charges from the constituent branes, there are additional induced charges describing F1-strings and D1-branes. The F1-charge is spread throughout the manifold W^2 , while the D1-charge is localized to codimension-1 submanifolds.

class $\mathcal{I}_{7,i}$ similar to what we used before in (4.5), that can be written as

$$\mathcal{I}_{7,i}^{\text{D5}} = G_5(B_2 + da_1^i + f_2^i) + \frac{1}{2} G_3(B_2 + da_1^i + f_2^i)^2, \tag{5.8}$$

where $\int_{\tilde{S}^2} f_2^i = 1$ and $\int_{S^2} f_2^i = k$ to account for the k D1-branes in the bound state. Summing the two characteristic classes and performing the reduction over the internal geometry yields

$$\int_{S^2 \times \tilde{S}^2} \mathcal{I}_{7,1}^{\text{D5}} + \mathcal{I}_{7,2}^{\text{D5}} = 2 \int_{S^2} G_5 + G_3(B_2 + da_1 + f_2^1 + f_2^2), \tag{5.9}$$

which is twice that of the reduction for a single D3-D1 bound state. We therefore find that the operators obey a group-like fusion,

$$\mathcal{O}_1(W^2, W^3) \otimes \mathcal{O}_1(W^2, W^3) = \mathcal{O}_2(W^2, W^3). \tag{5.10}$$

The same can be shown to hold for general $\mathcal{O}_q(W^2, W^3)$ and $\mathcal{O}_{q'}(W^2, W^3)$. We emphasize that this fusion, and by extension this method, will only be valid in the case $W^2 = \partial W^3$. If W^2 is a non-contractible closed manifold, we will need to take more care in handling the discrete worldvolume fields φ and λ_1 discussed previously.

To account for these additional worldvolume fields, we can compute the fusion directly at the level of $\mathcal{O}_1(W^2)$ or, in general, $\mathcal{O}_q(W^2)$. We find, for $q' \neq -q$,

$$\mathcal{O}_q(W^2) \otimes \mathcal{O}_{q'}(W^2) = \mathcal{X}^{M, qq'(q+q')^{-1}}(W^2) \otimes \mathcal{O}_{q+q'}(W^2), \tag{5.11}$$

where the coefficient is a decoupled 2d TQFT using the notation introduced in (2.19). The full details of this fusion can be found in appendix C, but we highlight some important features here. It is first worth noting that the gauge fields φ and λ_1 are localized worldvolume degrees of freedom coupled respectively to the 5d bulk fields \mathcal{B}_2 and \mathcal{A}_1 . To arrive at the fusion rule (5.11), we make a field redefinition analogously to what we did in (4.8), i.e.

$$\begin{aligned} (q + q')\tilde{\varphi} &= q\varphi + q'\varphi', & (q + q')\tilde{\lambda}_1 &= q\lambda_1 + q'\lambda'_1, \\ \tilde{\varphi}' &= -\varphi + \varphi', & \tilde{\lambda}'_1 &= -\lambda_1 + \lambda'_1. \end{aligned} \tag{5.12}$$

Ultimately, the $\tilde{\varphi}$ and $\tilde{\lambda}_1$ fields are associated to the $\mathcal{O}_{q+q'}$ operator while the $\tilde{\varphi}'$ and $\tilde{\lambda}'_1$ fields comprise the decoupled TQFT coefficient. This lends itself to an interpretation reminiscent of that given for the fusion of the \mathcal{N}_p operators; the symmetry defects themselves couple to the center-of-mass modes of their corresponding brane stacks, whereas the topological sector that decouples from the system is encoded by the relative modes. In this case, the relative modes yield the 2d TQFT $\mathcal{X}^{M,qq'(q+q')^{-1}}$ arising as the fusion coefficient.

Despite the similarity, the mechanisms from which the 3d TQFT $\mathcal{A}^{M,pp'(p+p')}$ and the 2d TQFT $\mathcal{X}^{M,qq'(q+q')^{-1}}$ arise are slightly different. In the former case, since the Chan-Paton field a_1 is what couples to \mathcal{B}_2 in the 3d worldvolume action of \mathcal{N}_p , it captures both the center-of-mass and relative modes of the brane stacks. However, in the latter case, these modes are instead captured by the pair of gauge fields φ and λ_1 , which respectively couple to \mathcal{B}_2 and \mathcal{A}_1 in the 2d worldvolume action of \mathcal{O}_q .

5.2 $\mathcal{O}_q \otimes \mathcal{O}_{-q}$ fusion

Due to the existence of the nontrivial 2d TQFT $\mathcal{X}^{M,q}[\mathcal{A}_1, \mathcal{B}_2]$ within the $\mathcal{O}_q(W^2)$ operator, one naturally expects it to be non-invertible. The fusion of \mathcal{O}_q with its orientation reversal produces another condensation defect $\tilde{\mathcal{C}}$ through

$$\mathcal{O}_q(W^2) \otimes \mathcal{O}_{-q}(W^2) = \frac{1}{|H^0(W^2; \mathbb{Z}_M)|} \bigoplus_{\substack{\gamma^1 \in H_1(W^2; \mathbb{Z}_{2M}) \\ \Sigma^2 \in H_2(W^2; \mathbb{Z}_M)}} \mathcal{V}_2(\gamma^1) \otimes \mathcal{U}_1(\Sigma^2) := \tilde{\mathcal{C}}(W^2). \quad (5.13)$$

More technical details are provided in appendix C. The presence of this sum over operators can be understood as a consequence of the discussion following (5.7), as it is simply enumerating the additional induced charges on the worldvolume W^2 of the D3-D1 operator.

We can interpret this condensation defect analogously to what we did to that in section 4: through tachyon condensation between the D3-D1 bound state and its orientation reversal. In this case, the overall D3-brane charge and the overall D1-brane charge (from the k D1-branes in \mathcal{W}^5) will annihilate completely, but the *induced* charges will not. This logic is the same as that of the non-invertible fusion of [9]. As the initial configuration is a product of *sums* of the induced states, the final result will be a sum over charges that have not been completely annihilated. The charges remaining in the (D3-D1)-($\overline{\text{D3-D1}}$) configuration are those of the F1-strings induced in the worldvolume, and some remaining D1-brane charge that can be seen coming from the coupling to λ_1 in the definition of $\mathcal{O}_q(W^2)$. The reason why the sum includes terms of the form $\mathcal{V}_2 \otimes \mathcal{U}_1$ can be understood from the BF term in (5.7). It acts as a constraint that the induced charges are equal for the F1-strings and D1-branes, so that we do not sum over *all* insertions of the $\widehat{\mathbb{Z}}_{2M}^{(2)}$ symmetry generator but only those which satisfy the imposed charge constraint.

It is clear from the discussion following (4.27) that the F1-strings in the external spacetime represent the $\mathcal{U}_1(\Sigma^2)$ factors in the condensation defect $\tilde{\mathcal{C}}(W^2)$. The question remains of where the remaining D1-brane charge is localized. As stated previously, this D1-brane charge is induced on the original D3-D1 worldvolume and should not be thought of as a stand-alone D1-brane. We should interpret the induced charge as a D1-brane soliton on the worldvolume of the bound state, which is picked out by the tachyon condensation process. Specifically, we see that the net D1-brane charge is localized to a torsional 1-cycle $\gamma^1 \in H_1(W^2; \mathbb{Z}_M)$

in the external spacetime. In order for the soliton D1-brane to be stable, there must be a locus in the internal geometry where the tachyon profile vanishes. Analogously to the discussion in section 4 for the D5- $\overline{\text{D5}}$ condensation, we can pick the tachyon profile such that it reaches its minimum T_0 at one pole of the internal S^2 , and takes the value $-T_0$ at the other. This leaves a codimension-1 locus where the tachyon vanishes, and thus where a solitonic D1-brane can live. For simplicity, we take this to be the $\widetilde{S^1}$ equator of the S^2 cycle wrapped by the original D3-D1 bound states. The end result is then a collection of F1-string operators $\mathcal{U}_1(\Sigma^2)$, and a solitonic D1-brane operator $\mathcal{V}_2(\gamma^1)$.

We shall emphasize that the D1-brane operator $\mathcal{V}_2(\gamma^1)$ resulting from tachyon condensation is stable only because of its solitonic nature. To construct a generic $\widehat{\mathbb{Z}}_{2M}^{(2)}$ defect from a stable D1-brane configuration, as we will do in the next subsection, we have to consider a different S^1 to wrap in the internal geometry of the KT-KS background. This is the same phenomenon we witnessed in constructing the condensation defect \mathcal{C} in section 4, namely, when constructing symmetry operators from Dp -branes, there is an ambiguity arising from which internal manifolds the branes wrap. In fact, multiple distinct configurations may lead to the same 5d bulk operator that cannot otherwise be distinguished. For the setup of interest, it is plausible that such a degeneracy can be lifted if we also consider the charges of the operators under the $SU(2) \times SU(2)$ isometries, corresponding to the $S^2 \times S^2$ base of $T^{1,1}$ in the KT-KS background.

5.3 Nature of the $\widehat{\mathbb{Z}}_{2M}^{(2)}$ defect

As alluded to earlier, one can construct a stand-alone $\widehat{\mathbb{Z}}_{2M}^{(2)}$ symmetry operator,

$$\mathcal{V}_1(\gamma^1) = \exp\left(\frac{2\pi i}{2M} \int_{\gamma^1} \mathcal{A}_1\right), \tag{5.14}$$

with $\gamma^1 \in H_1(\mathcal{W}^5; \mathbb{Z}_{2M})$, by wrapping a D1-brane on the S^1 Hopf fiber of the $T^{1,1}$. The stability of this D1-brane can be attributed to the fact that the S^1 fiber is non-vanishing over the base $\widetilde{S^2}$, such that the brane is protected from shrinking to zero.

The explicit construction of $\mathcal{V}_1(\gamma^1)$ follows closely to that of $\mathcal{U}_q(\Sigma^2)$ in section 4, i.e. if we naïvely reduce the characteristic class over the Hopf fiber, we find

$$\int_{S^1 \subset S^3} \mathcal{I}_3^{\text{D1}} = \mathcal{I}_2 \text{ “=” } 0 \tag{5.15}$$

Much akin to the construction of the $\widehat{\mathbb{Z}}_M^{(1)}$ operator, this is simply a side effect of the descent formalism overlooking information regarding discrete holonomies in our theory. The appropriate torsional flux that should appear this time can be anticipated, using the simple fact that the S^1 Hopf fiber is associated with the $U(1)$, or more precisely, \mathbb{Z}_{2M} R -symmetry. Following the same arguments as presented around (4.24), we may express $\mathcal{I}_2 = \widetilde{\beta}(\mathcal{A}_1) \in H^2(\mathcal{W}^5; \mathbb{Z}_{2M})$, and by wrapping the D1-brane on a cycle γ^1 such that $\partial\widetilde{\gamma}^2 = 2M\gamma^1$, one can reproduce (5.14). More generally, the operator

$$\mathcal{V}_p(\gamma^1) = \exp\left(\frac{2\pi ip}{2M} \int_{\gamma^1} \mathcal{A}_1\right), \tag{5.16}$$

where $p \sim p + 2M$, can be constructed from a stack of p D1-branes.

It is straightforward to show that these operators obey a group-like fusion rule, i.e.

$$\mathcal{V}_p(\gamma^1) \otimes \mathcal{V}_{p'}(\gamma^1) = \mathcal{V}_{p+p'}(\gamma^1). \tag{5.17}$$

Similarly to before, the fusion rule can be understood by considering the D1-brane used to construct $\mathcal{V}_1(\gamma^1)$ as being instead a neutral D3-brane wrapping $\gamma^1 \times S^1 \times S^2$ with one unit of worldvolume flux through the sphere S^2 . Bringing two such D3-branes together is equivalent to a single D3-brane with two units of worldvolume flux, which we can then extrapolate to arbitrary $p = 1, 2, \dots, 2M - 1 \in \mathbb{Z}_{2M}$.

6 Cross-symmetry defect fusion and braiding

6.1 Dissolved branes

We discussed in sections 4 and 5 the brane kinematics and dynamics underlying the parallel fusion rules for $\mathcal{N}_p \otimes \mathcal{N}_{p'}$, $\mathcal{O}_q \otimes \mathcal{O}_{q'}$, $\mathcal{U}_q \otimes \mathcal{U}_{q'}$, and $\mathcal{V}_p \otimes \mathcal{V}_{p'}$. These fusion rules alone, however, only form part of the full-fledged story involving the interplay between defects associated with different bulk gauge symmetries, as summarized in table 2 and derived in appendix C. In the following, we will discuss the brane origin of the rest of these fusion rules.

Let us begin with the fusion rule,

$$\mathcal{U}_q(\Sigma^2) \otimes \mathcal{N}_p(M^3) = \mathcal{N}_p(M^3). \tag{6.1}$$

The D-branes associated with $\mathcal{U}_q(\Sigma^2)$ and $\mathcal{N}_p(M^3)$ are respectively q D3-branes wrapping $\Sigma^2 \times \widetilde{S}^2$ and p D5-branes wrapping $M^3 \times S^3$, where $\Sigma^2 \subset M^3$ and $\widetilde{S}^2 \subset S^3$. It now becomes clear why (6.1) should be the way it is. Since the entire worldvolume of the D3-branes is a submanifold of the D5-branes, the former can be absorbed by the latter without a cost. This can be understood by the fact that D3-branes are dissolved in the D5-branes through the WZ coupling $da_1 C_4$, where we integrate over all configurations of the (dynamical) Chan-Paton field a_1 . The effect of embedding extra D3-branes into the D5-branes then is merely to reshuffle the summands in the said integration.²⁷

Moreover, adding D3-branes to the condensation defect $\mathcal{C}(M^3)$, which is itself a sum of all D3-branes localized on $M^3(\times S^3)$, has the identical effect of reshuffling the summands in the condensation defect. This naturally explains the fusion rule,

$$\mathcal{U}_q(\Sigma^2) \otimes \mathcal{C}(M^3) = \mathcal{C}(M^3). \tag{6.2}$$

Alternatively, locality of the fusion allows us to interpret (6.2) from a different but equivalent perspective, i.e.

$$\mathcal{U}_q \otimes \mathcal{C} = \mathcal{U}_q \otimes (\mathcal{N}_p \otimes \mathcal{N}_{-p}) = (\mathcal{U}_q \otimes \mathcal{N}_p) \otimes \mathcal{N}_{-p} = \mathcal{N}_p \otimes \mathcal{N}_{-p} = \mathcal{C}. \tag{6.3}$$

The expression above tells us that $\mathcal{U}_q \otimes \mathcal{C}$ can equivalently be thought of as a two-step process, namely, absorption of D3-branes by D5-branes, followed by tachyon condensation

²⁷If we instead interpret the defect $\mathcal{U}_q(\Sigma^2)$ to be coming from F1-strings (or more generally, a D3-F1 bound state), we can understand the fusion rule as the simple statement that a D5-brane freely absorbs F1-strings.

on the D5- $\overline{\text{D5}}$ system. Analogous arguments can be used to provide a brane interpretation of the following fusion rules,

$$\begin{aligned} \mathcal{V}_p(\gamma^1) \otimes \mathcal{O}_q(W^2) &= \mathcal{O}_q(W^2), & \mathcal{V}_p(\gamma^1) \otimes \tilde{\mathcal{C}}(W^2) &= \tilde{\mathcal{C}}(W^2), \\ \mathcal{U}_q(\Sigma^2) \otimes \mathcal{O}_{q'}(W^2) &= \mathcal{O}_{q'}(W^2), & \mathcal{U}_q(\Sigma^2) \otimes \tilde{\mathcal{C}}(W^2) &= \tilde{\mathcal{C}}(W^2). \end{aligned} \tag{6.4}$$

We now turn our focus to the fusion between the $\mathbb{Z}_{2M}^{(0)}$ defect $\mathcal{N}_p(M^3)$ with its associated condensation defect $\mathcal{C}(M^3)$. To motivate the subsequent explanation, we utilize the associativity of the fusion algebra again to rewrite the fusion in the following manner,

$$\begin{aligned} \mathcal{N}_p \otimes \mathcal{C} &= \mathcal{N}_p \otimes (\mathcal{N}_{p'} \otimes \mathcal{N}_{-p'}) \\ &= (\mathcal{N}_p \otimes \mathcal{N}_{p'}) \otimes \mathcal{N}_{-p'} \\ &= \mathcal{A}^{M,pp'(p+p')} \otimes \mathcal{N}_{p+p'} \otimes \mathcal{N}_{-p'} \\ &= \mathcal{A}^{M,pp'(p+p')} \otimes \mathcal{A}^{M,-pp'(p+p')} \otimes \mathcal{N}_p, \end{aligned} \tag{6.5}$$

assuming $\text{gcd}(M, p + p') = 1$. As per the discussion in section 4, there is a center-of-mass mode associated with each stack of D5-branes (or $\overline{\text{D5}}$ -branes). Suppose we first fuse p and p' D5-branes. This results in a stack of $p + p'$ D5-branes, in addition to the relative mode $\mathcal{A}^{M,pp'(p+p')}$ between these two stacks. Next, we fuse this new stack of $p + p'$ D5-branes with p' $\overline{\text{D5}}$ -branes, which results in a stack of p D5-branes, in addition to the relative mode $\mathcal{A}^{M,-pp'(p+p')}$ between the two original stacks. It turns out that these two minimal TQFTs, each of which is associated with an independent \mathbb{Z}_M symmetry, combines to form a $\mathbb{Z}_M \times \mathbb{Z}_M$ 3d DW theory $(\mathcal{Z}_M)_0$ [37] (see appendix C for details).

This result suggests that conceptually, rather than thinking of the condensation defect as a collection of D3-branes, it is more appropriate to think of the defect as a “zero-charge” D5-brane with solitonic D3-branes dissolved within, with an emphasis on “D5-brane” such that we can associate it with an appropriate notion of a center-of-mass mode. Accordingly, the decoupled DW theory in the fusion $\mathcal{N}_p \otimes \mathcal{C} = (\mathcal{Z}_M)_0 \otimes \mathcal{N}_p$ can be interpreted as the relative mode between the p D5-branes and such a neutral D5-brane. Similar remarks apply to the remaining nontrivial fusion rules in table 2, i.e.

$$\mathcal{C} \otimes \mathcal{C} = (\mathcal{Z}_M)_0 \otimes \mathcal{C}, \quad \mathcal{O}_q \otimes \tilde{\mathcal{C}} = (\mathcal{X}^{M,-1})^{\otimes 2} \otimes \mathcal{O}_q, \quad \tilde{\mathcal{C}} \otimes \tilde{\mathcal{C}} = (\mathcal{X}^{M,-1})^{\otimes 2} \otimes \tilde{\mathcal{C}}. \tag{6.6}$$

6.2 Commutation relations

What we have worked out earlier in this paper is the full set of *parallel* fusion rules for the symmetry operators we constructed from D-branes. While the fusion algebra plays a crucial role in the interplay between the symmetry defects in the bulk, an equally important piece of information is the braiding rules of these defects. Here we study the case where there is a nontrivial linking between the manifolds on which the defects are defined. Depending on the field content of these defects, there may be nontrivial commutation relations, meaning that they are charged under each other.

To determine which pairs of operators are mutually charged, we first recall from the SymTFT (2.8) that the conjugate momenta of the various 5d gauge fields are given by

$$\begin{aligned} \Pi_{b_2} &= -2\pi M \hat{c}_2, & \Pi_{\hat{c}_2} &= 2\pi M b_2 \rightarrow -2\pi \mathcal{B}_2, \\ \Pi_{A_1} &= 4\pi M c_3, & \Pi_{c_3} &= -4\pi M A_1 \rightarrow -2\pi \mathcal{A}_1. \end{aligned} \tag{6.7}$$

Importantly, we see that (b_2, \hat{c}_2) and (A_1, c_3) are conjugate pairs. Note that at low energies, the fluctuating fields Mb_2 and $2MA_1$ are respectively Higgsed to the discrete gauge fields $-\mathcal{B}_2 \in H^2(\mathcal{W}^5; \mathbb{Z}_M)$ and $\mathcal{A}_1 \in H^1(\mathcal{W}^5; \mathbb{Z}_{2M})$ via the Stückelberg mechanism.

It follows that the commutation relation between the $\widehat{\mathbb{Z}}_M^{(1)}$ defect $\mathcal{U}_q(\Sigma^2)$ and the $\mathbb{Z}_M^{(1)}$ defect $\mathcal{O}_{q'}(W^2)$ is

$$\mathcal{U}_q(\Sigma^2) \mathcal{O}_{q'}(W^2) = \exp\left(\frac{2\pi i q q'}{M} \ell(\Sigma^2, W^2)\right) \mathcal{O}_{q'}(W^2) \mathcal{U}_q(\Sigma^2), \quad (6.8)$$

where $\ell(\Sigma^2, W^2)$ denotes the linking number of Σ^2 and W^2 with respect to \mathcal{W}^5 .²⁸ As discussed in [12, 27, 72, 73], the nontriviality of the commutation relation is very natural when considering the Dp-branes associated with these operators. In Type II string theory, Dp-branes are (electromagnetically) dual to D(6-p)-branes, and particularly, D3-branes are self-dual.²⁹ Therefore one does expect the commutation relation between $\mathcal{U}_q(\Sigma^2)$ and $\mathcal{O}_{q'}(W^2)$, both of which are constructed from D3-branes, to be nontrivial. By the same token, the $\widehat{\mathbb{Z}}_{2M}^{(2)}$ defect $\mathcal{V}_p(\gamma^1)$ (from D1-branes) and the $\mathbb{Z}_{2M}^{(0)}$ defect $\mathcal{N}_{p'}(M^3)$ (from D5-branes) do not commute. More explicitly, we find

$$\mathcal{V}_p(\gamma^1) \mathcal{N}_{p'}(M^3) = \exp\left(-\frac{2\pi i p p'}{2M} \ell(\gamma^1, M^3)\right) \mathcal{N}_{p'}(M^3) \mathcal{V}_p(\gamma^1), \quad (6.9)$$

where $\ell(\gamma^1, M^3)$ again denotes the linking number of γ^1 and M^3 with respect to \mathcal{W}^5 .

For completeness, let us provide a broad-brush argument for the commutation relation between $\mathcal{N}_p(M^3)$ and $\mathcal{O}_q(W^2)$, as was also studied by [8]. The corresponding manifolds typically do not link in \mathcal{W}^5 , in which case these two operators commute. However, if we consider a refined case where $W^2 \cong S^1 \times L^1$, with $S^1 \subset M^3$, then heuristically the commutation relation takes the following form,

$$\mathcal{N}_p(M^3) \mathcal{O}_q(S^1 \times L^1) \sim \exp\left(-2\pi i p q \ell(L^1, M^3) \int_{S^1} a_1\right) \mathcal{O}_q(S^1 \times L^1) \mathcal{N}_p(M^3). \quad (6.10)$$

The ‘‘charge’’ here is not a c-number phase, but rather a holonomy of the Chan-Paton field of (multiple) open F1-strings, whose endpoints trace out the two copies of S^1 respectively embedded in M^3 and W^2 . Via the open-closed string duality, one may equivalently interpret this process as the creation of a 2d worldsheet of closed F1-strings when the D3-brane operator $\mathcal{O}_q(S^1 \times L^1)$ is dragged through the D5-brane operator $\mathcal{N}_p(M^3)$. Note that the Chan-Paton field a_1 is a trivialization of the NSNS field b_2 on the boundary of this worldsheet (with $b_2 \rightarrow -\mathcal{B}_2/M$ at low energies), so we may schematically express the action of $\mathcal{N}_p(M^3)$ on $\mathcal{O}_q(S^1 \times L^1)$ as

$$\mathcal{N}_p(M^3) : \mathcal{O}_q(S^1 \times L^1) \rightarrow \exp\left(\frac{2\pi i p q}{M} \ell(L^1, M^3) \int_{S^1 \times I} \mathcal{B}_2\right) \mathcal{O}_q(S^1 \times L^1), \quad (6.11)$$

where I is the interval along the transverse direction between $\mathcal{N}_p(M^3)$ and $\mathcal{O}_q(S^1 \times L^1)$. Such a phenomenon is akin to the Hanany-Witten transition [74], where a D3-brane is created when

²⁸The linking number we are using in this context is defined in the sense of [71]. By construction, $\ell(\Sigma^2, W^2) = 0$ if $\Sigma^2 \subset W^2$, such that it is compatible with the parallel fusion rule $\mathcal{U}_q(\Sigma^2) \otimes \mathcal{O}_{q'}(W^2) = \mathcal{O}_{q'}(W^2)$.

²⁹This duality is not to be confused with the $SL(2, \mathbb{Z})$ duality of Type IIB string theory, under which the pair (B_2, C_2) transforms as a doublet while C_4 transforms as a singlet.

a D5-brane crosses an NS5-brane (each wrapping manifolds along appropriate directions). By successively applying T and S dualities on this brane system, i.e.

$$(\text{NS5-D3-D5}) \xrightarrow{\text{T}^2} (\text{NS5-D1-D3}) \xrightarrow{\text{S}} (\text{D5-F1-D3}),$$

we arrive at the desired combination of branes in our construction.

7 Boundary conditions and field theory

In the previous sections, we have been primarily concerned with the presence and action of the symmetry defects in the 5d bulk gauge theory, where all the discussed operators are realized and interact in nontrivial ways. While the bulk alone has an involved structure, there is another half of the story, namely, the symmetry structure of the 4d boundary field theory is determined by choices of (Neumann or Dirichlet) boundary conditions for the bulk gauge fields [1, 2, 6, 20, 75]. Importantly, these choices also prescribe the spectrum of topological operators existing on the boundary.

7.1 Allowed choices from the SymTFT

Owing to the canonical quantization imposed on conjugate gauge fields, one cannot simultaneously pick Neumann (or Dirichlet) boundary conditions on the pair of fields. In addition, there are potential obstructions from the anomaly terms in the SymTFT that forbids us from imposing Neumann boundary condition on a given gauge field. In the following, we are going to systematically study all the allowed choices of boundary conditions for the 5d bulk gauge fields in the KT-KS setup.

Recall that the SymTFT is given by

$$S_{\text{top}} = 2\pi \int_{\mathcal{W}^5} \left(Mb_2 d\hat{c}_2 - 2Mc_3 d\mathcal{A}_1 + b_2 \mathcal{A}_1 \mathcal{B}_2 - \mathcal{A}_1 \mathcal{B}_2^2 - \frac{\mathcal{K}}{M} \mathcal{A}_1 \beta(\mathcal{A}_1) \beta(\mathcal{A}_1) \right), \quad (7.1)$$

where the first two terms are BF couplings respectively pairing b_2 and \hat{c}_2 , and c_3 and \mathcal{A}_1 . For concreteness, let us consider the former pair; the story for the latter pair is completely analogous. Varying the topological action, we find

$$\delta S_{\text{BF}} = 2\pi \int_{\partial\mathcal{W}^5} Mb_2 \delta \hat{c}_2, \quad (7.2)$$

where we have made a choice in having the variation on \hat{c}_2 . By adding a boundary term we may equivalently view the variation as $\delta S'_{\text{BF}} = 2\pi \int_{\partial\mathcal{W}^5} M \hat{c}_2 \delta b_2$. Because the variation is first-order in derivatives, we must impose Dirichlet boundary condition on one of the gauge fields, and Neumann boundary condition on the other to have a well-defined variational problem.³⁰ The difference between these two choices corresponds to different global structures for the gauge group of the 4d $\mathcal{N} = 1$ SYM field theory living on the boundary $\mathcal{W}^4 = \partial\mathcal{W}^5$.

³⁰These boundary conditions are *topological* boundary conditions as they do not require a metric on the boundary manifold, and are indeed invariant under diffeomorphisms of the boundary manifold. One can also discuss topological boundary conditions in theories other than BF theories. One such example is 3d Abelian Chern-Simons theory, where the existence of topological boundary conditions depends on the choice of the coefficient matrix for the theory [76].

Boundary Conditions	(D,N,N,D)	(N,D,N,D)
Gauge Group	$SU(M)$	$PSU(M)$
Global Symmetries	$\mathbb{Z}_{2M}^{(0)}, \mathbb{Z}_M^{(1)}$	$\mathbb{Z}_{2M}^{(0)}, \widehat{\mathbb{Z}}_M^{(1)}$
Gauged Symmetries	$\widehat{\mathbb{Z}}_{2M}^{(2)}, \widehat{\mathbb{Z}}_M^{(1)}$	$\widehat{\mathbb{Z}}_{2M}^{(2)}, \mathbb{Z}_M^{(1)}$
Symmetry Generators	$\widetilde{\mathcal{N}}_p(M^3), \widetilde{\mathcal{O}}_q(W^2)$	$\mathcal{N}_p(M^3), \mathcal{U}_q(\Sigma^2)$
Charged Operators	$\overline{\mathcal{V}}(\overline{x}), \overline{\mathcal{U}}(\overline{\gamma}^1)$	$\overline{\mathcal{V}}(\overline{x}), \overline{\mathcal{O}}(\overline{W}^1)$

Table 3. The allowed boundary conditions for the four bulk gauge fields $(b_2, \hat{c}_2, c_3, A_1)$, and their effect on the boundary field theory. Here and in the main text “N” (“D”) denotes Neumann (Dirichlet) boundary condition imposed on a given gauge field.

Specifically, an $SU(M)$ gauge theory has a $\mathbb{Z}_M^{(1)}$ (electric) global symmetry, whereas a $PSU(M)$ gauge theory has a $\widehat{\mathbb{Z}}_M^{(1)}$ (magnetic) global symmetry. In general, one may consider a larger family of boundary conditions when $M = kk'$ is not prime. This realizes a $\mathbb{Z}_k \times \mathbb{Z}_{k'}$ global symmetry on the boundary 4d $\mathcal{N} = 1$ $SU(M)/\mathbb{Z}_k$ SYM theory. We leave the study of such cases to future work.

The remaining cubic terms in (7.1) place additional constraints on the allowed choices of boundary conditions for the bulk gauge fields. From the point of view of the boundary field theory, these interaction terms describe ’t Hooft anomalies between different global symmetries. For example, the $A_1 \mathcal{B}_2^2$ term is a mixed ’t Hooft anomaly between the $\mathbb{Z}_{2M}^{(0)}$ and $\mathbb{Z}_M^{(1)}$ symmetries. Imposing Neumann boundary conditions on b_2 (and \mathcal{B}_2 by extension) amounts to gauging the $\mathbb{Z}_M^{(1)}$ symmetry on the boundary, which causes the $\mathbb{Z}_{2M}^{(0)}$ symmetry to become non-invertible [8, 23, 67]. Meanwhile, the self-anomaly term $\mathcal{A}_1 \beta(\mathcal{A}_1) \beta(\mathcal{A}_1)$ indicates A_1 (and \mathcal{A}_1) *cannot* be given Neumann boundary conditions, reflecting the fact that one cannot gauge the anomalous $\mathbb{Z}_{2M}^{(0)}$ R-symmetry in the 4d $\mathcal{N} = 1$ SYM field theory.

7.2 Boundary conditions and branes

The choice of boundary conditions for the bulk supergravity fields determines the set of allowed configurations of the probe Dp -branes in \mathcal{W}^5 . *A priori*, there are $2^4 = 16$ choices of boundary conditions on the set of fields $(b_2, \hat{c}_2, c_3, A_1)$. However, A_1 only admits Dirichlet boundary condition due to its self-anomaly, so its conjugate, c_3 , only admits Neumann boundary condition. This narrows the number of combinations down to 2. Table 3 summarizes the consequences of each of these two choices in the corresponding boundary field theories.

(D,N,N,D) boundary conditions. To see the effect of boundary conditions concretely, let us again focus on the (b_2, \hat{c}_2) BF pair discussed in the previous subsection, keeping in mind that c_3 and A_1 must be given Neumann and Dirichlet boundary conditions respectively. We first consider the case where we impose Neumann boundary condition on \hat{c}_2 . By the variational principle,

$$Mb_2 \delta \hat{c}_2|_{\mathcal{W}^4} = 0, \tag{7.3}$$

we must require that Mb_2 , and by extension \mathcal{B}_2 , vanish on the conformal boundary \mathcal{W}^4 . In terms of the symmetry structure of the boundary field theory, this is equivalent to choosing a gauge group of $SU(M)$ rather than $PSU(M)$.

With this choice of boundary conditions it is straightforward to see from (4.27) that $\mathcal{U}_q(\Sigma^2)$ becomes a trivial operator when pushed to the boundary. This is expected for a theory with gauge group $SU(M)$, as the defects \mathcal{U}_q would generate the magnetic 1-form symmetry not present in this theory. Instead of having the D3-branes being pushed *parallel* to \mathcal{W}^4 , we can consider *transverse* D3-branes such that one of their external dimensions is aligned with the radial coordinate of \mathcal{W}^5 . Using (7.3), we can write b_2 in terms of a \mathbb{Z}_M -valued flat connection \mathcal{B}_1 defined only on the conformal boundary, i.e.

$$Mb_2|_{\mathcal{W}^4} = d\mathcal{B}_1 = 0, \tag{7.4}$$

in which case the resultant operator becomes

$$\bar{\mathcal{U}}(\bar{\gamma}^1 \subset \mathcal{W}^4) = \exp\left(\frac{2\pi i}{M} \int_{\bar{\gamma}^1} \mathcal{B}_1\right), \tag{7.5}$$

with γ^1 closed. The commutation relation (6.8), when localized to \mathcal{W}^4 , tells us that such a line operator living in the boundary field theory is charged under the $\mathbb{Z}_M^{(1)}$ electric symmetry generated by the \mathcal{O}_q defects. Following the discussion below (4.27), this could have been equivalently argued from an alternative construction of the \mathcal{U}_q defects using F1-strings. It is known in the literature that F1-strings ending on the conformal boundary trace out Wilson lines [77]. The line operators $\bar{\mathcal{U}}(\bar{\gamma}^1)$ should therefore be identified with these Wilson lines in the $SU(M)$ gauge theory.³¹

Similarly to before, imposing Neumann boundary condition on c_3 leads to

$$2MA_1|_{\mathcal{W}^4} = d\mathcal{A}_0 = 0, \tag{7.6}$$

where $\mathcal{A}_0 \in \{0, \dots, 2M - 1\}$ is a flat \mathbb{Z}_{2M} -valued 0-form field. This corresponds to a local operator $\bar{\mathcal{V}}(\bar{x} \in \mathcal{W}^4)$, where $\bar{x} = \partial\gamma^1$, that is charged under the $\mathbb{Z}_{2M}^{(0)}$ symmetry generated by the \mathcal{N}_p defects according to (6.9).

The (D,N,N,D) boundary conditions also affect the non-invertibility of the $\mathbb{Z}_M^{(1)}$ symmetry generator. From either (5.6) or (5.7), we see that \mathcal{O}_q simplifies dramatically when \mathcal{A}_1 and \mathcal{B}_2 have Dirichlet boundary conditions, i.e.

$$\tilde{\mathcal{O}}_q(W^2 \subset \mathcal{W}^4) = \exp\left(2\pi i q \int_{W^2} \hat{c}_2\right). \tag{7.7}$$

Note that the $b_2\mathcal{A}_1\mathcal{B}_2$ anomaly trivializes as both fields \mathcal{A}_1 and \mathcal{B}_2 are turned off on \mathcal{W}^4 , so the D3-D1 operator no longer couples to the TQFT $\mathcal{X}^{M,q}$ on which the auxiliary fields λ_1 and φ live. Importantly, $\tilde{\mathcal{O}}_q$ is an invertible operator, which agrees with our expectation that this defect generates the (invertible) $\mathbb{Z}_M^{(1)}$ center symmetry in the $SU(M)$ gauge theory. By the same token, the $\mathcal{A}_1\mathcal{B}_2^2$ anomaly trivializes on \mathcal{W}^4 , so the $\mathbb{Z}_{2M}^{(0)}$ defect also becomes an invertible operator [8], i.e.

$$\tilde{\mathcal{N}}_p(M^3 \subset \mathcal{W}^4) = \exp\left(2\pi i p \int_{M^3} c_3\right). \tag{7.8}$$

This conforms with the fact that the $\mathbb{Z}_{2M}^{(0)}$ R -symmetry is invertible in the $SU(M)$ gauge theory.

³¹More accurately, the $\bar{\mathcal{U}}$ operators describe the *topological data* of a Wilson line in the picture of [2, 3].

(N,D,N,D) boundary conditions. Let us now consider the case where \hat{c}_2 is chosen to have Dirichlet boundary conditions instead of b_2 . This is equivalent to having a $\text{PSU}(M)$ gauge theory on the boundary. As has been discussed previously, this causes the \mathcal{U}_q operators to generate an invertible $\widehat{\mathbb{Z}}_M^{(1)}$ symmetry and the \mathcal{N}_p operators to generate a non-invertible $\mathbb{Z}_{2M}^{(0)}$ symmetry on the boundary. Furthermore, the bulk line operator \mathcal{V}_p ends on the boundary to become the local operator $\overline{\mathcal{V}}(\overline{x} \in \mathcal{W}^4)$. What remains to be discussed is the fate of the \mathcal{O}_q operators. When \hat{c}_2 is chosen to vanish on \mathcal{W}^4 , the defect \mathcal{O}_q takes the form,

$$\mathcal{O}_q(W^2 \subset \mathcal{W}^4) = \int \mathcal{D}\varphi \mathcal{D}\lambda_1 \exp\left(2\pi i \int_{W^2} \varphi \mathcal{B}_2 - M\varphi d\lambda_1\right), \tag{7.9}$$

where we have kept in mind that $\mathcal{A}_1|_{\mathcal{W}^4} = 0$ due to the self-anomaly term. In the above, λ_1 acts as a Lagrange multiplier causing φ to take on integer values, such that the remaining operator turns into a (normalized) sum over trivial operators.³² This implies that the D3-D1 operator trivializes when placed *parallel* to the conformal boundary.

Nonetheless, we may consider D3-D1 bound states that instead end transversely to \mathcal{W}^4 , i.e. we take one of the directions in W^2 to align with the radial coordinate of \mathcal{W}^5 . From the variation (7.3), we can write on the boundary,

$$M\hat{c}_2|_{\mathcal{W}^4} = d\hat{\mathcal{C}}_1 = 0, \tag{7.10}$$

where $\hat{\mathcal{C}}_1$ is a \mathbb{Z}_M -valued flat connection. The associated operator on \mathcal{W}^4 is then given by

$$\overline{\mathcal{O}}(\overline{W}^1 \subset \mathcal{W}^4) = \exp\left(\frac{2\pi i}{M} \int_{\overline{W}^1} \hat{\mathcal{C}}_1\right), \tag{7.11}$$

which is charged under the $\widehat{\mathbb{Z}}_M^{(1)}$ magnetic symmetry generated by the \mathcal{U}_q defects. Hence, the line operators $\overline{\mathcal{O}}(\overline{W}^1)$ should be identified with 't Hooft lines in the $\text{PSU}(M)$ gauge theory.

8 Discussion and outlook

In this paper, we have studied in detail the complete fusion algebra of *gauge* symmetry generators in the 5d bulk \mathcal{W}^5 of the KT-KS background (see table 2 for a summary). More precisely, the gauge symmetries of the bulk are encoded by the SymTFT (2.8), whose kinetic terms are suppressed near the conformal boundary $\mathcal{W}^4 \subset \mathcal{W}^5$. Via the Gauss Law constraints, the SymTFT indicates the existence of topological extended operators (near \mathcal{W}^4) which generate the gauge symmetries.

Intuitively, the candidates for these operators are nothing but the solitonic objects in Type II string theory, i.e. D-branes, wrapping suitable internal submanifolds. We find that this is indeed the case, and particularly, the WZ coupling of D-branes to lower-degree RR fields naturally reproduces the mixed anomaly terms in the SymTFT. The agreement between the symmetry generators and the D-branes therefore serves as a nontrivial check for the consistency between the Type IIB supergravity action and the WZ action of D-branes. An

³²Alternatively, we can see that the $\mathcal{A}_1\mathcal{B}_2$ anomaly again vanishes on the conformal boundary due to the Dirichlet boundary condition on \mathcal{A}_1 . The associated TQFT involving φ and λ_1 will then vanish and the operator \mathcal{O}_q will still be trivial.

even more stringent test can be carried out by further including the gravitational terms respectively in the 11-form Type IIB anomaly polynomial \mathcal{I}_{11} and the WZ action of the D-branes.³³ We leave a careful analysis of this to future work.

The consistency between the supergravity bulk and the D-branes teaches us three important lessons. Firstly, the construction of, say, non-invertible *global* symmetry generators in the dual field theory typically requires a manual stacking of individually non-topological operators (e.g. [22]), whereas we can obtain the full operator at once from a straightforward dimensional reduction of the D-brane WZ action. For example, the 3d TQFT $\mathcal{A}^{M,1}$ within the R -symmetry defect \mathcal{N}_1 comes for free when we reduce a D5-brane over S^3 . The fact that the WZ action is compatible with the bulk supergravity action enables the former to capture the necessary anomaly data from the latter.

Secondly, the explicit fusion algebra of symmetry generators in the boundary field theory can be understood holographically as brane interactions in the bulk. For concreteness, in the case of the 4d $\mathcal{N} = 1$ PSU(M) SYM theory, charge- p and charge- p' R -symmetry defects fuse not only into a charge- $(p + p')$ defect, but also an extra decoupled TQFT. We argued in section 4 that this fusion rule is fully characterized by brane *kinematics*, i.e. the former comes from the overall center-of-mass mode of the combined stack of branes, and the latter comes from the relative contributions from the center-of-mass modes of the individual stacks.

On the contrary, a condensation defect arises from the fusion between a charge- p defect with its orientation reversal. We suggest that this can instead be interpreted as a *dynamical* process known as tachyon condensation. In fact, the dressing of an otherwise invertible operator with a nontrivial TQFT (as in (2.13)) is indicative of lower-dimensional D3-branes being dissolved within the worldvolume of the D5-brane. As a proof of concept, we showed that by engineering a certain tachyon profile, the D5- $\overline{\text{D5}}$ fusion indeed results in a sum of (codimension-2) solitonic D3-brane operators that matches the field-theoretic fusion rule.

That being said, it is not entirely clear to us how unique the specific tachyon profile is. To illustrate this, recall that the worldvolume of the D5-branes is $M^3 \times S^3$, so there are at least three naïve ways to construct a codimension-2 soliton, namely,

$$\gamma^1 \times S^3, \quad \Sigma^2 \times \widetilde{S}^2, \quad M^3 \times S^1,$$

where $\gamma^1, \Sigma^2 \subset M^3$, and \widetilde{S}^2, S^1 are respectively the Hopf base and fiber of S^3 . The first option amounts to integrating out a 2-cycle in M^3 , and the second option amounts to integrating out a one-cycle in M^3 plus a fiber integration in S^3 , whereas the third option does not seem to be viable because we cannot perform a “base integration” in S^3 . Ruling out the first option requires some technical details, but essentially one can show that the resultant operator is incompatible with the fluxes present in the KT-KS background. This then singles out the second option, as far as the condensation defect is concerned. Nevertheless, there is still a degeneracy in the choice of the internal manifold on which the tachyon profile is defined. We may for example replace the Hopf base \widetilde{S}^2 with the equatorial S^2 of S^3 , and one cannot tell the distinction between these two D3-brane configurations from the perspective of the 5d bulk operator.

³³We expect in this case that the mixed chiral-gravitational anomaly of 4d $\mathcal{N} = 1$ SU(M) SYM theory disguises as some nontrivial TQFT attached to the appropriate D-brane operator. The resultant defect should be similar to that constructed by [78].

This degeneracy *may* be lifted by considering the isometries of the $S^2 \times S^2$ base in the $T^{1,1}$ fibration. Including these isometries would introduce an additional $SU(2)_L \times SU(2)_R$ gauge symmetry in the bulk; in the dual field theory there are two sets of chiral superfields A_r, B_u in bifundamental representations of the UV gauge group $SU(N + M) \times SU(N)$. Each set forms a doublet under an $SU(2)$ global symmetry, which gives a combined $SU(2)_L \times SU(2)_R$ global symmetry in the boundary field theory. In the supergravity theory, the effect of this symmetry can be seen in the following way. A D3-brane wrapping a submanifold of the S^3 cycle will act as a point particle on S^2 in the presence of a nonzero “magnetic field” arising from the G_5 flux threading $T^{1,1}$. This will cause the corresponding operator to have a Landau-level label as discussed in [79, 80]. It is possible that this additional label will remove the degeneracy in the choice of internal manifold for the tachyon condensation.

Moreover, as we saw in section 4, the D3-brane operators \mathcal{U}_q obey a group-like fusion rule, and notably, it annihilates with its orientation reversal. In other words, they do not undergo further tachyon condensation to form D1-branes. One may heuristically argue that there is a qualitative difference between the $\mathcal{N}_p \otimes \mathcal{N}_{-p}$ and $\mathcal{U}_q \otimes \mathcal{U}_{-q}$ fusions. The operator \mathcal{N}_p consists of a 3d TQFT $\mathcal{A}^{M,p}$ on which Chan-Paton fields reside, whereas all the Chan-Paton modes are already integrated out in \mathcal{U}_q . Since the presence of worldvolume Chan-Paton modes are necessary (but not sufficient) for tachyon condensation to happen, as effective 5d bulk operators the former is non-invertible while the latter is not. It is likely that we can shed some light on these open questions by examining more constructions of condensation defects in other holographic setups.

The third lesson that we can learn is the bulk SymTFT contains a rich symmetry structure that encodes the data of multiple possible boundary field theories. As explained in section 7, the global symmetries present in the boundary field theory is not only specified by the SymTFT itself, but also by a choice of boundary conditions for the various supergravity gauge fields. Consequently, this determines the field content of a given D-brane operator when placed at the conformal boundary (transverse to the radial direction of \mathcal{W}^5). For instance, the electric $\mathbb{Z}_M^{(1)}$ symmetry defect $\mathcal{O}_q(W^2)$ can only be invertible if $W^2 \subset \mathcal{W}^4$, but once we move slightly into the bulk (but still near \mathcal{W}^4), the 2d TQFT $\mathcal{X}^{M,q}$ attached to $\mathcal{O}_q(W^2 \subset \mathcal{W}^5)$ renders the defect non-invertible. This illustrates a crucial point: a non-invertible *gauge* symmetry can be trivialized to an invertible *global* symmetry depending on the choice of boundary conditions.

There is a further subtlety that one should consider when using the Gauss Laws to derive the symmetry structure from the SymTFT. The analysis performed in this paper was only concerned with the *classical* Gauss Laws of the associated symmetries. In general there may be nontrivial and important effects from the *quantum* Gauss Laws as considered in [20]. These effects may contribute extra torsional phases to the symmetry generators. It would be interesting to study the effect of these phases as well as their origin in holographic constructions.

Recently in the literature, the study of symmetry-generating D-brane operators has provided us with fruitful insights into the nature of non-invertible and/or higher-form global symmetries in quantum field theories. Multiple examples of non-invertible symmetries have indeed been uncovered in different string theory solutions using D-branes. On top of that, we have shown in this paper brane interactions in the bulk play a significant role in characterizing the fusion algebra of the boundary field theory. It is widely believed a fusion algebra involving

defects of various dimensions furnishes part of the data of some higher fusion category (e.g. [81–86]). The data contained in the bulk theory should then be interpreted as describing the Drinfeld center of this higher fusion category [87–91]. We therefore hope our work will offer a complementary understanding of the intricacies of such higher categorical structures.

Acknowledgments

We are grateful to Oren Bergman, Federico Bonetti, T. Daniel Brennan, Diego García-Sepúlveda, Pierre Heidmann, Patrick Jefferson, Ho Tat Lam, Ruben Minasian, Salvatore Pace, Sebastian Schulz, Hannah Tillim, and Peter Weck for interesting conversations and correspondence. We also thank the authors of [13] for communicating about a related work and coordinating submission. IB, EL, and TW are supported by NSF grant PHY-2112699, and also in part by the Simons Collaboration on Global Categorical Symmetries.

A SymTFT computation

We start with a brief review of the Klebanov-Tseylin (KT) [18] and Klebanov-Strassler (KS) setups [19] dual to the cascading field theory. There are two relevant features. The first is a stack of N D3-branes extending along flat space $\mathbb{R}^{1,3}$ and placed at the tip of the conifold background as was studied in [92]. The conifold is a non-compact Calabi-Yau cone over a Sasaki-Einstein 5-manifold denoted $T^{1,1}$. The space $T^{1,1}$ is constructed as a Hopf fibration over $S^2 \times S^2$ with Einstein metric given by [8, 93, 94]

$$ds^2(T^{1,1}) = \frac{4}{9} \left(d\psi + \frac{1}{2} \sum_{i=1}^2 \cos \theta_i d\phi_i \right)^2 + \frac{1}{6} \sum_{i=1}^2 \left(d\theta_i^2 + \sin^2 \theta_i d\phi_i^2 \right), \quad (\text{A.1})$$

where in the above $\{\theta_i, \phi_i\}$ denote the coordinates of the two S^2 geometries each with periodicity 2π , and ψ is the coordinate of the S^1 fiber with 4π periodic. The $T^{1,1}$ space admits the topology of $S^2 \times S^3$ and cohomologically nontrivial one 2-cycle, one 3-cycle, and 5-cycle that we denote by ω_2 , ω_3 , and Ω_5 respectively. The near-horizon geometry of the stack of D3-branes in Type IIB string theory is correspond to $AdS_5 \times T^{1,1}$, with N units of G_5 flux.

We will be interested in gauging the U(1) isometry of the fiber coordinate ψ through the introduction of a connection $d\psi \rightarrow d\psi + 2\pi A_1$, where the normalization is chosen such that the flux of A_1 has integral periods. In general one may consider gauging the $SU(2) \times SU(2)$ isometries of the $S^2 \times S^2$ base of $T^{1,1}$, in which case one would then have to incorporate global angular forms to account for the equivariant completion [24, 95–97]. The effect of these forms do not play a role in this paper, so we refrain from gauging these isometries. We can construct a U(1)-equivariant completion of the volume forms mentioned earlier as

$$\omega_2 = -\frac{1}{2}(V_1 - V_2), \quad \omega_3 = (V_1 - V_2) \frac{D\psi}{2\pi}, \quad (\text{A.2})$$

where V_i denote the normalized volume forms of the 2-spheres in the base of the $T^{1,1}$ fibration. The normalization of these forms comes from demanding that $\int_{T^{1,1}} \omega_2 \omega_3 = 1$. The U(1)-equivariant completion of Ω_5 is written

$$\Omega_5 = \omega_2 \omega_3 + \frac{1}{2} F_2 (V_1 + V_2) \frac{D\psi}{2\pi}, \quad (\text{A.3})$$

where F_2 is the field strength of A_1 .

The second main ingredient of the KT and KS setup is a stack of M D5-branes wrapping the topological S^2 of the conifold and extending along the flat space $\mathbb{R}^{1,3}$. The backreaction of this stack of branes has two effects on the supergravity setup, the first of which being that the conifold metric is replaced with that of the deformed conifold³⁴ where the S^3 cycle is blown up at the tip of the cone. The change in metric is felt very close to the original stack of N D3-branes and similarly in the deep IR of the dual field theory, but far away from the origin the space is indistinguishable from the original conifold. In the near horizon limit, the effect of the stack of M D5-branes is that the G_5 flux runs along the AdS_5 radius. The asymptotic region of the KS background corresponds to the KT [18]. This latter geometry is enough to determine the symmetry structure of interest in this paper.

The dual field theory of the setup is $\mathcal{N} = 1$ Super Yang-Mills with gauge group $SU(N + M) \times SU(N)$ coupled to bifundamental matter [18]. This is a non-conformal background where the gauge couplings run. As this theory flows to the IR a sequence of Seiberg duality transformations occurs, often referred to as the “duality cascade”. For N a multiple of M , this cascade ends with pure $\mathcal{N} = 1$ Super Yang-Mills theory with gauge group $SU(M)$. In this paper we will study the symmetry structure of the $\mathcal{N} = 1$ $SU(M)$ theory from the SymTFT derived from the dual gravitational background. In particular we will use various brane probes of the supergravity to construct the possible symmetry generators and their fusion rules.

To construct the SymTFT in this background, we first write down the 5d topological action using the nontrivial fluxes for h_3 and F_2 as discussed in section 2,

$$2\pi \int_{W^5} \left[Nb_2g_3 - \mathcal{A}_1(\mathcal{B}_2)^2 + \mathcal{A}_1b_2\mathcal{B}_2 - \frac{1}{2M} \mathcal{A}_1\tilde{g}_2^2 + \frac{\mathcal{K}}{2M} \mathcal{A}_1\beta(\mathcal{A}_1)\beta(\mathcal{A}_1) \right. \\ \left. - \frac{1}{2M}\tilde{g}_1 \left(\tilde{g}_2^2 + 2\tilde{g}_2\mathcal{B}_2 - \mathcal{K} \left((d\mathcal{A}_1)^2 + 3d\mathcal{A}_1\beta(\mathcal{A}_1) + 3\beta(\mathcal{A}_1)^2 \right) \right) \right], \quad (\text{A.4})$$

where the \mathcal{A}_1 cubic coefficient is given by

$$\mathcal{K} = \frac{N(N + M)}{4} + \frac{M^2}{6} - \frac{1}{3}. \quad (\text{A.5})$$

The subleading corrections in N may be computed directly from the UV quiver of the KT-KS setup. The full SymTFT is given by the above action augmented with standard kinetic terms for g_1, g_2, g_3, h_3 , and F_2 with associated couplings $\kappa_1, \kappa_2, \kappa_3, \kappa_h$, and κ_F .

To elucidate the symmetry structure of the boundary field theory, we may dualize some of the fields in the above SymTFT. We first dualize g_1 by introducing a 3-form c_3 as a Lagrange multiplier to enforce the Bianchi identity.

$$S_{\text{mult}} = 2\pi \int_{W^5} c_3 \wedge (2Md\mathcal{A}_1 - dg_1). \quad (\text{A.6})$$

The equation of motion for g_1 yields

$$\star_5 g_1 = \kappa_1 \left(dc_3 - \frac{1}{2M} \left(\tilde{g}_2^2 + 2\tilde{g}_2\mathcal{B}_2 - \mathcal{K} \left((d\mathcal{A}_1)^2 + 3d\mathcal{A}_1\beta(\mathcal{A}_1) + 3\beta(\mathcal{A}_1)^2 \right) \right) \right) := g_4. \quad (\text{A.7})$$

³⁴There is another construction which considers the resolved conifold rather than the deformed conifold [98]. The difference is in which cycle is blown up, but the physical theory is still that of 4d $\mathcal{N} = 1$ Super Yang-Mills. It would be interesting to explore the connections between the two theories with regard to the operators constructed out of branes.

After integrating out g_1 by replacing it with its equation of motion, we may also dualize g_2 through an additional field a_2 acting as a Lagrange multiplier

$$S'_{\text{mult}} = -2\pi \int_{W^5} a_2 \wedge (dg_2 + Mdb_2). \tag{A.8}$$

The equation of motion for g_2 can then be found as

$$\star_5 g_2 = -\kappa_2 \left(da_2 + \frac{1}{M} \mathcal{A}_1 \mathcal{B}_2 \right). \tag{A.9}$$

After performing both dualizations, we may write our final action S_{top} as

$$2\pi \int_{W^5} \left(Mb_2 d\hat{c}_2 - 2Mc_3 d\mathcal{A}_1 - \mathcal{A}_1 \mathcal{B}_2^2 + \mathcal{A}_1 b_2 \mathcal{B}_2 - \frac{\mathcal{K}}{M} \mathcal{A}_1 \beta(\mathcal{A}_1) \beta(\mathcal{A}_1) \right), \tag{A.10}$$

thus producing (2.8). In the low energy limit, the kinetic terms present in the full SymTFT are suppressed, so we often refer to S_{top} as “the SymTFT” unless otherwise specified.

B Brief review of Bockstein homomorphisms

For a detailed introduction to Bockstein homomorphisms, we refer the reader to standard references in the literature, e.g. [99, 100]. Here we briefly review and explain the notation adopted in the main text. Let us first consider the following short exact sequence with k prime,³⁵

$$0 \rightarrow \mathbb{Z}_k \xrightarrow{\times k} \mathbb{Z}_{k^2} \xrightarrow{\text{mod } k} \mathbb{Z}_k \rightarrow 0, \tag{B.1}$$

where the “mod k ” map means reduction modulo k . Associated with it is a long exact sequence in homology,

$$\dots \rightarrow H_n(M; \mathbb{Z}_{k^2}) \xrightarrow{\text{mod } k} H_n(M; \mathbb{Z}_k) \xrightarrow{\beta} H_{n-1}(M; \mathbb{Z}_k) \xrightarrow{\times k} H_{n-1}(M; \mathbb{Z}_{k^2}) \rightarrow \dots, \tag{B.2}$$

where the Bockstein homomorphism is defined to be a map

$$\beta : H_n(M; \mathbb{Z}_k) \rightarrow H_{n-1}(M; \mathbb{Z}_k). \tag{B.3}$$

Explicitly, it acts on cycles (mod k), $\mathcal{C}^n \in H_n(M; \mathbb{Z}_k)$, as

$$\beta(\mathcal{C}^n) = \frac{1}{k} \partial \mathcal{C}^n. \tag{B.4}$$

A closely related construction starts with the short exact sequence

$$0 \rightarrow \mathbb{Z} \xrightarrow{\times k} \mathbb{Z} \xrightarrow{\text{mod } k} \mathbb{Z}_k \rightarrow 0, \tag{B.5}$$

which leads to the long exact sequence

$$\dots \rightarrow H_n(M; \mathbb{Z}) \xrightarrow{\text{mod } k} H_n(M; \mathbb{Z}_k) \xrightarrow{\tilde{\beta}} H_{n-1}(M; \mathbb{Z}) \xrightarrow{\times k} H_{n-1}(M; \mathbb{Z}) \rightarrow \dots \tag{B.6}$$

³⁵If k is not prime, then there are possibly additional torsion subgroups that we have to worry about, but we refrain from considering that in this appendix.

The codomain of this Bockstein homomorphism is different from that in the previous case, namely,

$$\tilde{\beta} : H_n(M; \mathbb{Z}_k) \rightarrow H_{n-1}(M; \mathbb{Z}). \quad (\text{B.7})$$

On a cycle $\mathcal{C}^n \in H_n(M; \mathbb{Z}_k)$, the map similarly acts as

$$\tilde{\beta}(\mathcal{C}^n) = \frac{1}{k} \partial \mathcal{C}^n, \quad (\text{B.8})$$

where the two connecting homomorphisms are related via a reduction map,

$$\beta(\mathcal{C}^n) = (\text{mod } k) \circ \tilde{\beta}(\mathcal{C}^n). \quad (\text{B.9})$$

In summary, the relations between the various maps are depicted by the following commutative diagram.

$$\begin{array}{ccccccc} H_n(M; \mathbb{Z}) & \xrightarrow{\text{mod } k} & H_n(M; \mathbb{Z}_k) & \xrightarrow{\tilde{\beta}} & H_{n-1}(M; \mathbb{Z}) & \xrightarrow{\times k} & H_{n-1}(M; \mathbb{Z}) \\ \downarrow \text{mod } k^2 & & \downarrow \text{id} & & \downarrow \text{mod } k & & \downarrow \text{mod } k^2 \\ H_n(M; \mathbb{Z}_{k^2}) & \xrightarrow{\text{mod } k} & H_n(M; \mathbb{Z}_k) & \xrightarrow{\beta} & H_{n-1}(M; \mathbb{Z}_k) & \xrightarrow{\times k} & H_{n-1}(M; \mathbb{Z}_{k^2}) \end{array} \quad (\text{B.10})$$

We see that if $\partial \mathcal{C}^n = k \mathcal{C}^{n-1}$ for some $\mathcal{C}^{n-1} \in H_{n-1}(M; \mathbb{Z})$, i.e. \mathcal{C}^{n-1} is a torsion n -cycle of order k , then $\tilde{\beta}(\mathcal{C}^n) = \mathcal{C}^{n-1}$. In addition, since the kernel of $\tilde{\beta}$ is the image of the reduction map by exactness, we have $\tilde{\beta}(k \mathcal{C}^n) = 0 \in H_{n-1}(M; \mathbb{Z})$ for any $\mathcal{C}^n \in H_n(M; \mathbb{Z}_k)$. Similar remarks apply to the β map. Note that by construction, the Bockstein homomorphisms are nilpotent, e.g.

$$\beta \circ \beta = 0, \quad (\text{B.11})$$

and they satisfy the Leibniz rule,

$$\beta(a_m \cup b_n) = \beta(a_m) \cup b_n + (-1)^m a_m \cup \beta(b_n). \quad (\text{B.12})$$

One can obtain dual Bockstein homomorphisms by constructing long exact sequences in cohomology instead, also denoted as β and $\tilde{\beta}$ with an abuse of notation. Again, the relevant maps are described by the following commutative diagram.

$$\begin{array}{ccccccc} H^n(M; \mathbb{Z}) & \xrightarrow{\text{mod } k} & H^n(M; \mathbb{Z}_k) & \xrightarrow{\tilde{\beta}} & H^{n+1}(M; \mathbb{Z}) & \xrightarrow{\times k} & H^{n+1}(M; \mathbb{Z}) \\ \downarrow \text{mod } k^2 & & \downarrow \text{id} & & \downarrow \text{mod } k & & \downarrow \text{mod } k^2 \\ H^n(M; \mathbb{Z}_{k^2}) & \xrightarrow{\text{mod } k} & H^n(M; \mathbb{Z}_k) & \xrightarrow{\beta} & H^{n+1}(M; \mathbb{Z}_k) & \xrightarrow{\times k} & H^{n+1}(M; \mathbb{Z}_{k^2}) \end{array} \quad (\text{B.13})$$

Just like how the Bockstein homomorphisms act on cycles as “boundary operators,” they act on cocycles as “coboundary operators.” More explicitly,

$$\beta(c_n) = \frac{1}{k} \delta c_n \quad (\text{B.14})$$

for any $c_n \in H^n(M; \mathbb{Z}_k)$.

The Bockstein homomorphisms have important applications in physics, especially when torsional manifolds or fluxes are involved. One of them being the definition of the integral Stiefel-Whitney classes. For a degree- n (generalized) Stiefel-Whitney class $w_n \in H^n(M; \mathbb{Z}_k)$, the corresponding integral Stiefel-Whitney class is defined as

$$W_{n+1} := \tilde{\beta}(w_n) \in H^{n+1}(M; \mathbb{Z}). \tag{B.15}$$

If $n = k = 2$, the degree-3 integral class $W_3 = \beta(w_2)$ measures the obstruction to having a spin^c structure on the manifold M . Another example, used in our construction, is the flat but topologically nontrivial part of the NSNS field $\mathcal{B}_2 \in H^2(\mathcal{W}^5; \mathbb{Z}_M)$, which corresponds to the cohomological class of the 't Hooft magnetic flux via $\mathcal{H}_3 = \tilde{\beta}'(\mathcal{B}_2)$ as in (2.6). In fact, these two examples are closely related through the Freed-Witten anomaly [43, 44].

Moreover, the Bockstein homomorphism for $k = 2$ coincides with the first Steenrod square (see, e.g. [101]),

$$\beta(c_n) = \text{Sq}^1(c_n) := w_1(M) \cup c_n \in H^2(M; \mathbb{Z}_2), \tag{B.16}$$

where $w_1(M) \in H^1(M; \mathbb{Z}_2)$ is the first Stiefel-Whitney class of M (or more precisely, its normal bundle).

As an aside, we clarify that the Bockstein homomorphisms invoked in this paper shall be distinguished from that commonly used in ordinary differential cohomology, i.e.

$$\hat{\beta} : H^n(M; \mathbb{R}/\mathbb{Z}) \rightarrow H^{n+1}(M; \mathbb{Z}), \tag{B.17}$$

which is associated with the short exact sequence, $0 \rightarrow \mathbb{Z} \rightarrow \mathbb{R} \rightarrow \mathbb{R}/\mathbb{Z} \rightarrow 0$. These two distinct Bockstein homomorphisms are in fact related to each other, as can be inferred from the commutative diagram [102],

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\times k} & \mathbb{Z} & \xrightarrow{\text{mod } k} & \mathbb{Z}_k & \longrightarrow & 0 \\ & & \downarrow & & \downarrow \text{id} & & \downarrow \times 1/k & & \downarrow \\ 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{i} & \mathbb{R} & \longrightarrow & \mathbb{R}/\mathbb{Z} & \longrightarrow & 0 \end{array} \tag{B.18}$$

The map ϕ induces a homomorphism $\phi_* : H^n(M; \mathbb{Z}_k) \rightarrow H^n(M; \mathbb{R}/\mathbb{Z})$, so it follows that $\tilde{\beta} = \hat{\beta} \circ \phi_*$ by exactness.

C Derivation of fusion rules

In this appendix, we are going to derive the nontrivial fusion rules between the symmetry defects. The reader may find similar derivations in [103]. We assume all manifolds are spin throughout the appendix.

$\mathcal{N}_p \otimes \mathcal{N}_{p'}$ fusion. We already derived in section 4 the fusion rule for the case when $p' = -p \pmod M$. Let us turn to the case when $p' \neq -p \pmod M$ and assume $\text{gcd}(M, p + p') = 1$ (in addition to $\text{gcd}(M, p) = \text{gcd}(M, p') = 1$). Recall that the defect $\mathcal{N}_p(M^3)$ can be decomposed as

$$\mathcal{N}_p = \tilde{\mathcal{N}}_p \otimes \mathcal{A}^{M,p}[\mathcal{B}_2], \tag{C.1}$$

where $\widetilde{\mathcal{N}}_p = \exp(2\pi i \int_{M^3} c_3)$ obeys a group-like fusion rule,

$$\widetilde{\mathcal{N}}_p \otimes \widetilde{\mathcal{N}}_{p'} = \widetilde{\mathcal{N}}_{p+p'}, \tag{C.2}$$

thus the only nontrivial part of the fusion $\mathcal{N}_p \otimes \mathcal{N}_{p'}$ is that between the minimal TQFTs, i.e. $\mathcal{A}^{M,p}[\mathcal{B}_2] \otimes \mathcal{A}^{M,p'}[\mathcal{B}_2]$. An efficient way to compute the fusion rule is as follows.³⁶

The minimal TQFT $\mathcal{A}^{M,p}$ is a 3d theory with a \mathbb{Z}_M 1-form symmetry generated by Wilson lines denoted by a^s where $s = 0, \dots, M-1$ [37], and their topological spins are labeled by

$$h[a^s] = \frac{ps^2}{2M}. \tag{C.3}$$

The theory is modular given that its S -matrix,

$$S(a^s, a^{s'}) = \frac{1}{\sqrt{M}} \exp\left(2\pi i \{h[a^s] + h[a^{s'}] - h[a^{s+s'}]\}\right) = \frac{1}{\sqrt{M}} \exp\left(-\frac{2\pi i p s s'}{M}\right), \tag{C.4}$$

is non-degenerate, i.e. only the transparent line $a^0 = \mathbf{1}$ braids trivially with all other lines.³⁷ Similarly, $\mathcal{A}^{M,p'}$ has lines a'^r with spins $h[a'^r] = p'r^2/2M$. We claim that for any (p, p') , there always exists an equivalent factorization of the minimal TQFTs of the form,

$$\mathcal{A}^{M,p} \otimes \mathcal{A}^{M,p'} = \mathcal{A}^{M,n} \otimes \mathcal{A}^{M,p+p'}, \tag{C.5}$$

for some integer n yet to be determined. Suppose we denote the sets of lines in $\mathcal{A}^{M,n}$ and $\mathcal{A}^{M,p+p'}$ respectively as

$$b^s = (a^x a'^y)^s = a^{xs} a'^{ys}, \quad b'^r = (a a')^r = a^r a'^r, \tag{C.6}$$

then their spins are

$$\begin{aligned} h[b^s] &= \frac{p(xs)^2}{2M} + \frac{p'(ys)^2}{2M} = \frac{(px^2 + p'y^2)s^2}{2M}, \\ h[b'^r] &= \frac{pr^2}{2M} + \frac{p'r^2}{2M} = \frac{(p+p')r^2}{2M}. \end{aligned} \tag{C.7}$$

One may see that $p+p'$ corresponds to the anomaly of $\mathcal{A}^{M,p+p'}$ as desired.

To impose the condition that $\mathcal{A}^{M,n}$ and $\mathcal{A}^{M,p+p'}$ are decoupled from each other, we demand the line b^s to braid trivially with b'^r for any (s, r) , i.e.

$$h[b^s] + h[b'^r] - h[b^s b'^r] = -\frac{sr(px + p'y)}{M} = 0. \tag{C.8}$$

The simplest nontrivial integer solution for general (p, p') is $(x, y) = (p', -p)$, so the spin of b^s becomes

$$h[b^s] = \frac{pp'(p+p')s^2}{2M}, \tag{C.9}$$

³⁶We thank Ho Tat Lam for discussions about this point.

³⁷In a braided fusion category with fusion rules $a \otimes b = \sum_c N_{ab}^c c$, the S -matrix is given by $S_{ab} = \sum_c N_{ab}^c (\theta_c / \theta_a \theta_b) (d_c / \mathcal{D})$, where $\theta_a = \exp(-2\pi i h)$ is the topological twist of the simple object (or anyon) a . If we represent a finite group G with a fusion category, then the quantum dimension of any anyon is $d_a = 1$, and the total quantum dimension of the category is $\mathcal{D} = \sqrt{\sum_a d_a^2} = \sqrt{|G|}$.

from which we can read off the anomaly label to be $(p+p')pp'$. We have therefore found that³⁸

$$\mathcal{A}^{M,p} \otimes \mathcal{A}^{M,p'} = \mathcal{A}^{M,pp'(p+p')} \otimes \mathcal{A}^{M,p+p'}. \quad (\text{C.10})$$

Note that $\gcd(M, p+p') = \gcd(M, p) = \gcd(M, p') = 1$ guarantees $\gcd(M, pp'(p+p')) = 1$. Furthermore, since the lines $b'^r = a^r a'^r$ are those that are coupled to \mathcal{B}_2 in the product $\mathcal{A}^{M,p} \otimes \mathcal{A}^{M,p'}$ (whereas b^s are decoupled), we conclude that the fusion rule between \mathcal{N}_p and $\mathcal{N}_{p'}$ is given by

$$\mathcal{N}_p \otimes \mathcal{N}_{p'} = \widetilde{\mathcal{N}}_{p+p'} \otimes \mathcal{A}^{M,pp'(p+p')} \otimes \mathcal{A}^{M,p+p'}[\mathcal{B}_2] = \mathcal{A}^{M,pp'(p+p')} \otimes \mathcal{N}_{p+p'}. \quad (\text{C.11})$$

Alternatively, one may show this result using an effective Lagrangian description of $\mathcal{A}^{M,p}[\mathcal{B}_2] \otimes \mathcal{A}^{M,p'}[\mathcal{B}_2]$, in which case we have

$$\int_{M^3} \left(\frac{pM}{2} a_1 da_1 + \frac{p'M}{2} a'_1 da'_1 + (pa_1 + p'a'_1) \mathcal{B}_2 \right). \quad (\text{C.12})$$

Motivated by the ‘‘correct’’ answer from the previous approach, we define $pp'(p+p')\tilde{a}_1 = p(p'a_1) + p'(-pa'_1)$ and $(p+p')\tilde{a}'_1 = pa_1 + p'a'_1$. The Lagrangian then turns into

$$pp'(p+p') \int_{M^3} \frac{M}{2} \tilde{a}_1 d\tilde{a}_1 + (p+p') \int_{M^3} \left(\frac{M}{2} \tilde{a}'_1 d\tilde{a}'_1 + \tilde{a}'_1 \mathcal{B}_2 \right), \quad (\text{C.13})$$

which can be identified as an effective Lagrangian for $\mathcal{A}^{M,(p+p')pp'} \otimes \mathcal{A}^{M,p+p'}[\mathcal{B}_2]$. Note that the former minimal TQFT is decoupled from \mathcal{B}_2 , while the latter is associated with a stack of $p+p'$ D5-branes (reduced on S^3).

The fusion rule (C.11) is manifestly commutative, i.e. $\mathcal{N}_p \otimes \mathcal{N}_{p'} = \mathcal{N}_{p'} \otimes \mathcal{N}_p$. We also claim that it is associative, and in fact any triple fusion admits the following factorization (assuming $\gcd(M, p+p'+p'') = 1$),

$$\mathcal{N}_p \otimes \mathcal{N}_{p'} \otimes \mathcal{N}_{p''} = \mathcal{A}^{M,pp'p''P(p+p'+p'')} \otimes \mathcal{A}^{M,P} \otimes \mathcal{N}_{p+p'+p''}, \quad (\text{C.14})$$

where $P = (p+p')(p+p'')(p'+p'')$. To examine such a claim, we first apply (C.11) twice to write down

$$(\mathcal{N}_p \otimes \mathcal{N}_{p'}) \otimes \mathcal{N}_{p''} = \mathcal{A}^{M,(p+p')pp'} \otimes \mathcal{A}^{M,(p+p'+p'')(p+p'')p''} \otimes \mathcal{N}_{p+p'+p''}. \quad (\text{C.15})$$

Denoting the topological Wilson lines in $\mathcal{A}^{M,pp'(p+p')}$ and $\mathcal{A}^{M,(p+p')p''(p+p'+p'')}$ respectively as a^s and a'^r , we define two sets of new lines,

$$b^s = (a^{p''(p+p'+p'')}) a'^{-pp'}{}^s, \quad b'^r = (aa')^r, \quad (\text{C.16})$$

whose spins are given by

$$\begin{aligned} h[b^s] &= \frac{pp'(p+p')[p''(p+p'+p'')s]^2}{2M} + \frac{(p+p')p''(p+p'+p'')[-pp's]^2}{2M} \\ &= \frac{pp'p''(p+p')(p+p'')(p'+p'')(p+p'+p'')s^2}{2M}, \end{aligned} \quad (\text{C.17})$$

$$\begin{aligned} h[b'^r] &= \frac{pp'(p+p')r^2}{2M} + \frac{(p+p')p''(p+p'+p'')r^2}{2M} \\ &= \frac{(p+p')(p+p'')(p'+p'')r^2}{2M}. \end{aligned} \quad (\text{C.18})$$

³⁸This is consistent with the result of [103] where they chose the solution $(x, y) = (p^{-1}, -p'^{-1})$, thus producing $\mathcal{A}^{M,p^{-1}+p'^{-1}} \cong \mathcal{A}^{M,(p+p')pp'}$. Here we have used the fact that $\mathcal{A}^{M,pr^2} \cong \mathcal{A}^{M,p}$ if $\gcd(M, r) = 1$.

These two sets of lines braid trivially with each other, i.e.

$$h[b^s] + h[b'^r] - h[b^s b'^r] = 0, \quad (\text{C.19})$$

so we have successfully reproduced the triple fusion rule (C.14). By the same token, we may write down

$$\mathcal{N}_p \otimes (\mathcal{N}_{p'} \otimes \mathcal{N}_{p''}) = \mathcal{A}^{M,p'p''(p'+p'')} \otimes \mathcal{A}^{M,p(p'+p'')(p+p'+p'')} \otimes \mathcal{N}_{p+p'+p''}. \quad (\text{C.20})$$

In this case, one can check that the following redefinition of lines,

$$b^s = (a^{p'p''} a'^{-p(p'+p'')})^s, \quad b'^r = (aa')^r, \quad (\text{C.21})$$

gives rise to (C.14) as well. Since the fusion is also commutative, this triple fusion rule indeed holds for arbitrary permutations of (p, p', p'') with an appropriate redefinition of lines. Holographically, this implies that we can freely reshuffle the three stacks of D5-branes without altering the end result, although the decoupled modes are sensitive to the number of branes in each individual stack.

$\mathcal{N}_p \otimes \mathcal{C}$ fusion. The invertible operator $\widetilde{\mathcal{N}}_p$ essentially acts as a spectator, so it suffices to work out the fusion rule for $\mathcal{A}^{M,p}[\mathcal{B}_2] \otimes (\mathcal{Z}_M)_0[\mathcal{B}_2]$. In terms of the effective Lagrangian description, we have

$$\int_{M^3} \left(\frac{pM}{2} a_1 da_1 + pa_1 \mathcal{B}_2 \right) + \int_{M^3} \left(M\Lambda_1 da'_1 + \Lambda_1 \mathcal{B}_2 \right). \quad (\text{C.22})$$

Defining $\hat{a}_1 = a_1 + p^{-1}\Lambda_1$ and $\hat{a}'_1 = a'_1 - \hat{a}_1$, we can rewrite the Lagrangian as

$$\int_{M^3} \left(M\Lambda_1 d\hat{a}'_1 + \frac{p^{-1}M}{2} \Lambda_1 d\Lambda_1 \right) + \int_{M^3} \left(\frac{pM}{2} \hat{a}_1 d\hat{a}_1 + p\hat{a}_1 \mathcal{B}_2 \right), \quad (\text{C.23})$$

which may be identified as $(\mathcal{Z}_M)_{p^{-1}M} \otimes \mathcal{A}^{M,p}[\mathcal{B}_2]$. Assuming M is odd, the DW twist in $(\mathcal{Z}_M)_K$ has a periodicity $K \sim K + M$, so $(\mathcal{Z}_M)_{p^{-1}M} \cong (\mathcal{Z}_M)_0$ given $p^{-1} \in \mathbb{Z}_M$ is an integer. We thus conclude that

$$\mathcal{N}_p \otimes \mathcal{C} = \mathcal{C} \otimes \mathcal{N}_p = (\mathcal{Z}_M)_0 \otimes \mathcal{N}_p, \quad (\text{C.24})$$

where $(\mathcal{Z}_M)_0$ is decoupled from \mathcal{B}_2 and acts as a coefficient.

We can immediately verify that associativity (and commutativity) holds for the following triple fusion,

$$\mathcal{N}_p \otimes \mathcal{N}_{p'} \otimes \mathcal{C}. \quad (\text{C.25})$$

More explicitly, performing the fusion in different orders yield the same answer,

$$(\mathcal{N}_p \otimes \mathcal{N}_{p'}) \otimes \mathcal{C} = \mathcal{A}^{M,pp'(p+p')} \otimes (\mathcal{Z}_M)_0 \otimes \mathcal{N}_{p+p'} = \mathcal{N}_p \otimes (\mathcal{N}_{p'} \otimes \mathcal{C}). \quad (\text{C.26})$$

$\mathcal{C} \otimes \mathcal{C}$ fusion. The most efficient way to compute this fusion rule is to observe that

$$\mathcal{C} \otimes \mathcal{C} = (\mathcal{N}_p \otimes \mathcal{N}_{-p}) \otimes \mathcal{C} = \mathcal{N}_p \otimes (\mathcal{N}_{-p} \otimes \mathcal{C}) = \mathcal{N}_p \otimes (\mathcal{Z}_M)_0 \otimes \mathcal{N}_{-p} = (\mathcal{Z}_M)_0 \otimes \mathcal{C}. \quad (\text{C.27})$$

Alternatively, one may derive this using a Lagrangian description, i.e.

$$\int_{M^3} (M\Lambda_1 da_1 + \Lambda_1 \mathcal{B}_2) + \int_{M^3} (M\Lambda'_1 da'_1 + \Lambda'_1 \mathcal{B}_2). \quad (\text{C.28})$$

If we define $\hat{\Lambda}_1 = \Lambda_1 + \Lambda'_1$ and $\hat{\Lambda}'_1 = a'_1 - a_1$, then we will obtain

$$\int_{M^3} M\hat{\Lambda}'_1 d\Lambda'_1 + \int_{M^3} (M\hat{\Lambda}_1 da_1 + \hat{\Lambda}_1 \mathcal{B}_2), \quad (\text{C.29})$$

which can be identified as $(\mathcal{Z}_M)_0 \otimes \mathcal{C}$. It can be readily checked that $\mathcal{N}_p \otimes \mathcal{C} \otimes \mathcal{C}$ and $\mathcal{C} \otimes \mathcal{C} \otimes \mathcal{C}$ are both commutative and associative.

$\mathcal{U}_q(\Sigma^2) \otimes \mathcal{N}_p$ fusion. Suppose we define \mathcal{U}_q on a 2-cycle $\Sigma^2 \in H_2(M^3; \mathbb{Z}_M)$, then we may use Poincaré duality to express

$$\mathcal{U}_q(\Sigma^2) = \exp\left(\frac{2\pi i q}{M} \int_{\Sigma^2} \mathcal{B}_2\right) = \exp\left(\frac{2\pi i q}{M} \int_{M^3} \bar{\Lambda}_1 \mathcal{B}_2\right) = \exp\left(2\pi i q \int_{M^3} \Lambda_1 \mathcal{B}_2\right), \quad (\text{C.30})$$

where $\bar{\Lambda}_1 = \text{PD}(\Sigma^2) \in H^1(M^3; \mathbb{Z}_M)$ and $M\Lambda_1 = \bar{\Lambda}_1$. Hence, the worldvolume Lagrangian of $\mathcal{U}_q(\Sigma^2) \otimes \mathcal{N}_p$ can be written as

$$\begin{aligned} \int_{M^3} \left(q\Lambda_1 \mathcal{B}_2 + \frac{pM}{2} a_1 da_1 + pa_1 \mathcal{B}_2 \right) &= \int_{M^3} \left(\frac{q^2 p^{-1} M}{2} \Lambda_1 d\Lambda_1 + \frac{pM}{2} a'_1 da'_1 + pa'_1 \mathcal{B}'_2 \right) \\ &= \int_{M^3} \left(\frac{q^2 p^{-1}}{2} \bar{\Lambda}_1 \beta(\bar{\Lambda}_1) + \frac{pM}{2} a'_1 da'_1 + pa'_1 \mathcal{B}'_2 \right), \end{aligned} \quad (\text{C.31})$$

where $a'_1 = a_1 + qp^{-1}\Lambda$ and $\mathcal{B}'_2 = \mathcal{B}_2 - qp^{-1}M d\Lambda_1 = \mathcal{B}_2$. Note that the last two terms can be identified as $\mathcal{A}^{M,p}[\mathcal{B}_2]$. The Bockstein homomorphism is associated with the short exact sequence $0 \rightarrow \mathbb{Z}_M \rightarrow \mathbb{Z}_{M^2} \rightarrow \mathbb{Z}_M \rightarrow 0$, and acts explicitly as $\beta(\bar{\Lambda}_1) = d\bar{\Lambda}_1/M$. Suppose we define the shorthand notation,

$$Q(\Sigma^2) = \int_{M^3} \bar{\Lambda}_1 \beta(\bar{\Lambda}_1), \quad (\text{C.32})$$

then we may express

$$\mathcal{U}_q(\Sigma^2) \otimes \mathcal{N}_p = \mathcal{N}_p \otimes \mathcal{U}_q(\Sigma^2) = (-1)^{q^2 p^{-1} Q(\Sigma^2)} \otimes \mathcal{N}_p. \quad (\text{C.33})$$

Similarly to before, the fusions $\mathcal{U}_q(\Sigma^2) \otimes \mathcal{U}_{q'}(\Sigma^2) \otimes \mathcal{N}_p$ and $\mathcal{U}_q(\Sigma^2) \otimes \mathcal{N}_p \otimes \mathcal{N}_{p'}$ are both commutative and associative. If M is odd, the phase $(-1)^{Q(\Sigma^2)}$ is always trivial [103], so we recover the fusion rule presented in section 4.

Since the condensation defect \mathcal{C} is essentially a sum of magnetic symmetry defects, i.e.

$$\mathcal{C} = \frac{1}{|H^0(M^3; \mathbb{Z}_M)|} \bigoplus_{\Sigma^2 \in H_2(M^3; \mathbb{Z}_M)} \mathcal{U}_1(\Sigma^2), \quad (\text{C.34})$$

we find that

$$\begin{aligned} \frac{1}{|H^0(M^3; \mathbb{Z}_M)|} \bigoplus_{\Sigma^2 \in H_2(M^3; \mathbb{Z}_M)} \mathcal{U}_1(\Sigma^2) \otimes \mathcal{N}_p &= \frac{1}{|H^0(M^3; \mathbb{Z}_M)|} \bigoplus_{\Sigma^2 \in H_2(M^3; \mathbb{Z}_M)} (-1)^{p-1Q(\Sigma^2)} \otimes \mathcal{N}_p \\ &= (\mathcal{Z}_M)_{p-1M} \otimes \mathcal{N}_p \\ &= (\mathcal{Z}_M)_0 \otimes \mathcal{N}_p. \end{aligned} \tag{C.35}$$

This result is consistent with the previously derived fusion rule for $\mathcal{C} \otimes \mathcal{N}_p$.

$\mathcal{U}_q(\Sigma^2) \otimes \mathcal{C}$ fusion. We can again use (C.34) to deduce that

$$\begin{aligned} \mathcal{U}_q(\Sigma^2) \otimes \mathcal{C} &= \frac{1}{|H^0(M^3; \mathbb{Z}_M)|} \bigoplus_{\widehat{\Sigma}^2 \in H_2(M^3; \mathbb{Z}_M)} \mathcal{U}_q(\Sigma^2) \otimes \mathcal{U}_1(\widehat{\Sigma}^2) \\ &= \frac{1}{|H^0(M^3; \mathbb{Z}_M)|} \bigoplus_{\widetilde{\Sigma}^2 \in H_2(M^3; \mathbb{Z}_M)} \mathcal{U}_1(\widetilde{\Sigma}^2) = \mathcal{C}, \end{aligned} \tag{C.36}$$

where $\widetilde{\Sigma}^2 = \widehat{\Sigma}^2 + q\Sigma^2 \in H_2(M^3; \mathbb{Z}_M)$. In other words, the effect of such a fusion is merely to relabel the 2-cycles (mod M) that are being summed over in the condensation defect. We can again easily verify that this fusion rule is compatible with the previous ones that we have derived.

$\mathcal{O}_q \otimes \mathcal{O}_{q'}$ fusion. Let us consider first the case where $q' \neq -q \pmod{M}$. The worldvolume Lagrangian of the $\mathcal{O}_q \otimes \mathcal{O}_{q'}$ fusion reads

$$\int_{W^2} \left((q+q')\hat{c}_2 + (q\lambda_1 + q'\lambda'_1)\mathcal{A}_1 + (q\varphi + q'\varphi')\mathcal{B}_2 - qM\lambda_1 d\varphi - q'M\lambda'_1 d\varphi' \right). \tag{C.37}$$

Rewriting the Lagrangian above in terms of the redefined fields, $\widetilde{\lambda}_1 = -\lambda_1 + \lambda'_1$, $\widetilde{\varphi} = -\varphi + \varphi'$, $(q+q')\widetilde{\lambda}'_1 = q\lambda_1 + q'\lambda'_1$, $(q+q')\widetilde{\varphi}' = q\varphi + q'\varphi'$,³⁹ we obtain

$$-qq'(q+q')^{-1} \int_{W^2} M\widetilde{\lambda}_1 d\widetilde{\varphi} + (q+q') \int_{W^2} \left(\hat{c}_2 + \widetilde{\lambda}'_1 \mathcal{A}_1 + \widetilde{\varphi}' \mathcal{B}_2 - M\widetilde{\lambda}'_1 d\widetilde{\varphi}' \right). \tag{C.38}$$

Consequently, we may express

$$\mathcal{O}_q \otimes \mathcal{O}_{q'} = \mathcal{X}^{M, qq'(q+q')^{-1}} \otimes \mathcal{O}_{q+q'}. \tag{C.39}$$

On the other hand, if $q' = -q \pmod{M}$, then the Lagrangian simply reduces to

$$q \int_{W^2} \left(\widetilde{\lambda}_1 \mathcal{A}_1 + \widetilde{\varphi} \mathcal{B}_2 - M\lambda'_1 d\widetilde{\varphi} - M\widetilde{\lambda}_1 d\varphi \right). \tag{C.40}$$

Integrating out λ'_1 and φ respectively enforces $M\lambda'_1 \in H^1(W^2; \mathbb{Z})$ and $M\varphi \in H^0(W^2; \mathbb{Z})$. Applying similar arguments in the discussion following (4.13), we conclude that

$$\begin{aligned} \mathcal{O}_q \otimes \mathcal{O}_{-q} &= \frac{1}{|H^0(W^2; \mathbb{Z}_M)|} \sum_{\substack{\gamma^1 \in H_1(W^2; \mathbb{Z}_M) \\ \Sigma^2 \in H_2(W^2; \mathbb{Z}_M)}} \exp\left(\frac{2\pi i}{M} \int_{\gamma^1} \mathcal{A}_1\right) \exp\left(\frac{2\pi i}{M} \int_{\Sigma^2} \mathcal{B}_2\right) \\ &= \frac{1}{|H^0(W^2; \mathbb{Z}_M)|} \bigoplus_{\substack{\gamma^1 \in H_1(W^2; \mathbb{Z}_{2M}) \\ \Sigma^2 \in H_2(W^2; \mathbb{Z}_M)}} \mathcal{V}_2(\gamma^1) \otimes \mathcal{U}_1(\Sigma^2) := \widetilde{\mathcal{C}}. \end{aligned} \tag{C.41}$$

³⁹Note the resemblance of these field redefinitions to those in the discussion preceding (C.13).

Note that if W^2 is connected, then $\Sigma^2 = sW^2$ for $s = 1, \dots, M$. The condensation defect $\tilde{\mathcal{C}}$ is essentially a sum of all the $\hat{\mathbb{Z}}_{2M}^{(2)}$ and $\hat{\mathbb{Z}}_M^{(1)}$ defects localized on W^2 .⁴⁰

$\mathcal{O}_q \otimes \tilde{\mathcal{C}}$ and $\tilde{\mathcal{C}} \otimes \tilde{\mathcal{C}}$ fusions. The worldvolume Lagrangian corresponding to this fusion is

$$\int_{W^2} \left(q\hat{c}_2 + q\lambda_1\mathcal{A}_1 + q\varphi\mathcal{B}_2 - qM\lambda_1d\varphi + \tilde{\lambda}_1\mathcal{A}_1 + \tilde{\varphi}\mathcal{B}_2 - M\lambda'_1d\tilde{\varphi} - M\tilde{a}d\varphi' \right). \quad (\text{C.42})$$

Defining $\hat{\lambda}_1 = \lambda_1 + q^{-1}\tilde{\lambda}_1$, $\hat{\varphi} = \varphi + q^{-1}\tilde{\varphi}$, $\hat{\lambda}'_1 = \lambda_1 - \lambda'_1 + q^{-1}\tilde{\lambda}_1$, $\hat{\varphi}' = \varphi - \varphi'$, we can rewrite the Lagrangian above as

$$\int_{W^2} \left(M\tilde{\lambda}_1d\tilde{\varphi}' + M\hat{\lambda}'_1d\hat{\varphi} \right) + q \int_{W^2} \left(\hat{c}_2 + \hat{\lambda}_1\mathcal{A}_1 + \hat{\varphi}\mathcal{B}_2 - M\hat{\lambda}_1d\hat{\varphi} \right). \quad (\text{C.43})$$

Therefore, we find that

$$\mathcal{O}_q \otimes \tilde{\mathcal{C}} = \mathcal{X}^{M,-1} \otimes \mathcal{X}^{M,-1} \otimes \mathcal{O}_q. \quad (\text{C.44})$$

It also follows immediately from (C.41) and (C.44) that

$$\tilde{\mathcal{C}} \otimes \tilde{\mathcal{C}} = \mathcal{X}^{M,-1} \otimes \mathcal{X}^{M,-1} \otimes \tilde{\mathcal{C}}_q. \quad (\text{C.45})$$

$\mathcal{V}_p(\gamma^1) \otimes \mathcal{O}_q$, $\mathcal{U}_q(\Sigma^2) \otimes \mathcal{O}_{q'}$, $\mathcal{V}_p(\gamma^1) \otimes \tilde{\mathcal{C}}$, and $\mathcal{U}_q(\Sigma^2) \otimes \tilde{\mathcal{C}}$ fusions. Analogously to the earlier derivation for $\mathcal{U}_q(\Sigma^2) \otimes \mathcal{N}_p$, one can show that

$$\mathcal{V}_p(\gamma^1) \otimes \mathcal{O}_q = \mathcal{O}_q, \quad \mathcal{U}_q(\Sigma^2) \otimes \mathcal{O}_{q'} = \mathcal{O}_{q'}, \quad (\text{C.46})$$

assuming $\gamma^1 \in H_1(W^2; \mathbb{Z}_{2M})$ and $\Sigma^2 \in H_2(W^2; \mathbb{Z}_M)$. We stress again that here we are only considering parallel fusion, in which case Σ^2 and W^2 do not link, thus there are no nontrivial commutation relations involved. Using (C.41), we also have

$$\mathcal{V}_p(\gamma^1) \otimes \tilde{\mathcal{C}} = \tilde{\mathcal{C}}, \quad \mathcal{U}_q(\Sigma^2) \otimes \tilde{\mathcal{C}} = \tilde{\mathcal{C}}. \quad (\text{C.47})$$

Open Access. This article is distributed under the terms of the Creative Commons Attribution License ([CC-BY4.0](https://creativecommons.org/licenses/by/4.0/)), which permits any use, distribution and reproduction in any medium, provided the original author(s) and source are credited.

References

- [1] D. Gaiotto, A. Kapustin, N. Seiberg and B. Willett, *Generalized Global Symmetries*, *JHEP* **02** (2015) 172 [[arXiv:1412.5148](https://arxiv.org/abs/1412.5148)] [[INSPIRE](https://inspirehep.net/literature/127713)].
- [2] D. Gaiotto and J. Kulp, *Orbifold groupoids*, *JHEP* **02** (2021) 132 [[arXiv:2008.05960](https://arxiv.org/abs/2008.05960)] [[INSPIRE](https://inspirehep.net/literature/186500)].
- [3] D.S. Freed, G.W. Moore and C. Teleman, *Topological symmetry in quantum field theory*, [arXiv:2209.07471](https://arxiv.org/abs/2209.07471) [[INSPIRE](https://inspirehep.net/literature/210000)].

⁴⁰Technically, \mathcal{V}_p is defined on 1-cycles $\gamma^1 \in H_1(W^2; \mathbb{Z}_{2M})$. We assume that there are no extra torsion elements in $H_1(W^2; \mathbb{Z}_{2M})$ compared to $H_1(W^2; \mathbb{Z}_M)$, so that the sum over $\gamma^1 \in H_1(W^2; \mathbb{Z}_M)$ in (C.41) makes sense.

- [4] F. Apruzzi et al., *Symmetry TFTs from String Theory*, *Commun. Math. Phys.* **402** (2023) 895 [[arXiv:2112.02092](#)] [[INSPIRE](#)].
- [5] I. Bah, F. Bonetti, R. Minasian and E. Nardoni, *Anomalies of QFTs from M-theory and Holography*, *JHEP* **01** (2020) 125 [[arXiv:1910.04166](#)] [[INSPIRE](#)].
- [6] I. Bah, F. Bonetti and R. Minasian, *Discrete and higher-form symmetries in SCFTs from wrapped M5-branes*, *JHEP* **03** (2021) 196 [[arXiv:2007.15003](#)] [[INSPIRE](#)].
- [7] I. Bah, F. Bonetti, R. Minasian and P. Weck, *Anomaly Inflow Methods for SCFT Constructions in Type IIB*, *JHEP* **02** (2021) 116 [[arXiv:2002.10466](#)] [[INSPIRE](#)].
- [8] F. Apruzzi, I. Bah, F. Bonetti and S. Schäfer-Nameki, *Noninvertible Symmetries from Holography and Branes*, *Phys. Rev. Lett.* **130** (2023) 121601 [[arXiv:2208.07373](#)] [[INSPIRE](#)].
- [9] I. García Etzebarria, *Branes and Non-Invertible Symmetries*, *Fortsch. Phys.* **70** (2022) 2200154 [[arXiv:2208.07508](#)] [[INSPIRE](#)].
- [10] J.J. Heckman, M. Hübner, E. Torres and H.Y. Zhang, *The Branes Behind Generalized Symmetry Operators*, *Fortsch. Phys.* **71** (2023) 2200180 [[arXiv:2209.03343](#)] [[INSPIRE](#)].
- [11] J.J. Heckman et al., *Top down approach to topological duality defects*, *Phys. Rev. D* **108** (2023) 046015 [[arXiv:2212.09743](#)] [[INSPIRE](#)].
- [12] M. Etheredge, I. García Etzebarria, B. Heidenreich and S. Rauch, *Branes and symmetries for $\mathcal{N} = 3$ S-folds*, *JHEP* **09** (2023) 005 [[arXiv:2302.14068](#)] [[INSPIRE](#)].
- [13] F. Apruzzi, F. Bonetti, D.S.W. Gould and S. Schäfer-Nameki, *Aspects of Categorical Symmetries from Branes: SymTFTs and Generalized Charges*, [arXiv:2306.16405](#) [[INSPIRE](#)].
- [14] M. Cvetič, J.J. Heckman, M. Hübner and E. Torres, *Fluxbranes, Generalized Symmetries, and Verlinde's Metastable Monopole*, [arXiv:2305.09665](#) [[INSPIRE](#)].
- [15] M. Dierigl, J.J. Heckman, M. Montero and E. Torres, *R7-Branes as Charge Conjugation Operators*, [arXiv:2305.05689](#) [[INSPIRE](#)].
- [16] V. Bashmakov, M. Del Zotto, A. Hasan and J. Kaidi, *Non-invertible symmetries of class S theories*, *JHEP* **05** (2023) 225 [[arXiv:2211.05138](#)] [[INSPIRE](#)].
- [17] V. Bashmakov, M. Del Zotto and A. Hasan, *On the 6d origin of non-invertible symmetries in 4d*, *JHEP* **09** (2023) 161 [[arXiv:2206.07073](#)] [[INSPIRE](#)].
- [18] I.R. Klebanov and A.A. Tseytlin, *Gravity duals of supersymmetric $SU(N) \times SU(N + M)$ gauge theories*, *Nucl. Phys. B* **578** (2000) 123 [[hep-th/0002159](#)] [[INSPIRE](#)].
- [19] I.R. Klebanov and M.J. Strassler, *Supergravity and a confining gauge theory: Duality cascades and chi SB resolution of naked singularities*, *JHEP* **08** (2000) 052 [[hep-th/0007191](#)] [[INSPIRE](#)].
- [20] D. Belov and G.W. Moore, *Conformal blocks for AdS_5 singletons*, [hep-th/0412167](#) [[INSPIRE](#)].
- [21] Y. Choi, H.T. Lam and S.-H. Shao, *Non-invertible Gauss law and axions*, *JHEP* **09** (2023) 067 [[arXiv:2212.04499](#)] [[INSPIRE](#)].
- [22] Y. Choi, H.T. Lam and S.-H. Shao, *Noninvertible Global Symmetries in the Standard Model*, *Phys. Rev. Lett.* **129** (2022) 161601 [[arXiv:2205.05086](#)] [[INSPIRE](#)].
- [23] J. Kaidi, K. Ohmori and Y. Zheng, *Kramers-Wannier-like Duality Defects in $(3 + 1)D$ Gauge Theories*, *Phys. Rev. Lett.* **128** (2022) 111601 [[arXiv:2111.01141](#)] [[INSPIRE](#)].
- [24] I. Bah and E. Nardoni, *Structure of Anomalies of 4d SCFTs from M5-branes, and Anomaly Inflow*, *JHEP* **03** (2019) 024 [[arXiv:1803.00136](#)] [[INSPIRE](#)].

- [25] D.S. Freed, *Dirac charge quantization and generalized differential cohomology*, [hep-th/0011220](#) [[INSPIRE](#)].
- [26] M.J. Hopkins and I.M. Singer, *Quadratic functions in geometry, topology, and M theory*, *J. Diff. Geom.* **70** (2005) 329 [[math/0211216](#)] [[INSPIRE](#)].
- [27] D.S. Freed, G.W. Moore and G. Segal, *Heisenberg Groups and Noncommutative Fluxes*, *Annals Phys.* **322** (2007) 236 [[hep-th/0605200](#)] [[INSPIRE](#)].
- [28] C. Córdova, D.S. Freed, H.T. Lam and N. Seiberg, *Anomalies in the Space of Coupling Constants and Their Dynamical Applications I*, *SciPost Phys.* **8** (2020) 001 [[arXiv:1905.09315](#)] [[INSPIRE](#)].
- [29] C.-T. Hsieh, Y. Tachikawa and K. Yonekura, *Anomaly Inflow and p-Form Gauge Theories*, *Commun. Math. Phys.* **391** (2022) 495 [[arXiv:2003.11550](#)] [[INSPIRE](#)].
- [30] J.H.C. Whitehead, *On simply connected, 4-dimensional polyhedra*, *Comment. Math. Helv.* **22** (1949) 48.
- [31] O. Aharony, N. Seiberg and Y. Tachikawa, *Reading between the lines of four-dimensional gauge theories*, *JHEP* **08** (2013) 115 [[arXiv:1305.0318](#)] [[INSPIRE](#)].
- [32] F. Benini, C. Córdova and P.-S. Hsin, *On 2-Group Global Symmetries and their Anomalies*, *JHEP* **03** (2019) 118 [[arXiv:1803.09336](#)] [[INSPIRE](#)].
- [33] C. Córdova, D.S. Freed, H.T. Lam and N. Seiberg, *Anomalies in the Space of Coupling Constants and Their Dynamical Applications II*, *SciPost Phys.* **8** (2020) 002 [[arXiv:1905.13361](#)] [[INSPIRE](#)].
- [34] T. Banks and N. Seiberg, *Symmetries and Strings in Field Theory and Gravity*, *Phys. Rev. D* **83** (2011) 084019 [[arXiv:1011.5120](#)] [[INSPIRE](#)].
- [35] E. Witten, *AdS / CFT correspondence and topological field theory*, *JHEP* **12** (1998) 012 [[hep-th/9812012](#)] [[INSPIRE](#)].
- [36] S. Gukov, E. Martinec, G.W. Moore and A. Strominger, *Chern-Simons gauge theory and the AdS(3) / CFT(2) correspondence*, in the proceedings of the *From Fields to Strings: Circumnavigating Theoretical Physics: A Conference in Tribute to Ian Kogan*, Oxford, U.K., January 8–10 January (2004), p. 1606–1647 [[DOI:10.1142/9789812775344_0036](#)] [[hep-th/0403225](#)] [[INSPIRE](#)].
- [37] P.-S. Hsin, H.T. Lam and N. Seiberg, *Comments on One-Form Global Symmetries and Their Gauging in 3d and 4d*, *SciPost Phys.* **6** (2019) 039 [[arXiv:1812.04716](#)] [[INSPIRE](#)].
- [38] C. Córdova and K. Ohmori, *Noninvertible Chiral Symmetry and Exponential Hierarchies*, *Phys. Rev. X* **13** (2023) 011034 [[arXiv:2205.06243](#)] [[INSPIRE](#)].
- [39] A. Kapustin and N. Seiberg, *Coupling a QFT to a TQFT and Duality*, *JHEP* **04** (2014) 001 [[arXiv:1401.0740](#)] [[INSPIRE](#)].
- [40] J. Polchinski, *Dirichlet Branes and Ramond-Ramond charges*, *Phys. Rev. Lett.* **75** (1995) 4724 [[hep-th/9510017](#)] [[INSPIRE](#)].
- [41] J. Polchinski, S. Chaudhuri and C.V. Johnson, *Notes on D-branes*, [hep-th/9602052](#) [[INSPIRE](#)].
- [42] M.R. Douglas, *Branes within branes*, *NATO Sci. Ser. C* **520** (1999) 267 [[hep-th/9512077](#)] [[INSPIRE](#)].
- [43] D.S. Freed and E. Witten, *Anomalies in string theory with D-branes*, *Asian J. Math.* **3** (1999) 819 [[hep-th/9907189](#)] [[INSPIRE](#)].

- [44] A. Kapustin, *D-branes in a topologically nontrivial B field*, *Adv. Theor. Math. Phys.* **4** (2000) 127 [[hep-th/9909089](#)] [[INSPIRE](#)].
- [45] R. Minasian and G.W. Moore, *K theory and Ramond-Ramond charge*, *JHEP* **11** (1997) 002 [[hep-th/9710230](#)] [[INSPIRE](#)].
- [46] E. Witten, *D-branes and K-theory*, *JHEP* **12** (1998) 019 [[hep-th/9810188](#)] [[INSPIRE](#)].
- [47] K. Roumpedakis, S. Seifnashri and S.-H. Shao, *Higher Gauging and Non-invertible Condensation Defects*, *Commun. Math. Phys.* **401** (2023) 3043 [[arXiv:2204.02407](#)] [[INSPIRE](#)].
- [48] I.R. Klebanov, P. Ouyang and E. Witten, *A gravity dual of the chiral anomaly*, *Phys. Rev. D* **65** (2002) 105007 [[hep-th/0202056](#)] [[INSPIRE](#)].
- [49] R.C. Myers, *Dielectric branes*, *JHEP* **12** (1999) 022 [[hep-th/9910053](#)] [[INSPIRE](#)].
- [50] J. Polchinski and M.J. Strassler, *The string dual of a confining four-dimensional gauge theory*, [hep-th/0003136](#) [[INSPIRE](#)].
- [51] F. Apruzzi et al., *On AdS₇ stability*, *JHEP* **07** (2020) 033 [[arXiv:1912.13491](#)] [[INSPIRE](#)].
- [52] R. Dijkgraaf and E. Witten, *Topological Gauge Theories and Group Cohomology*, *Commun. Math. Phys.* **129** (1990) 393 [[INSPIRE](#)].
- [53] A. Sen, *Tachyon condensation on the brane anti-brane system*, *JHEP* **08** (1998) 012 [[hep-th/9805170](#)] [[INSPIRE](#)].
- [54] A. Sen, *Stable nonBPS bound states of BPS D-branes*, *JHEP* **08** (1998) 010 [[hep-th/9805019](#)] [[INSPIRE](#)].
- [55] A. Sen, *NonBPS states and Branes in string theory*, in the proceedings of the *Advanced School on Supersymmetry in the Theories of Fields, Strings and Branes*, Santiago de Compostela, Spain, July 26–31 (1999), p. 187–234 [[hep-th/9904207](#)] [[INSPIRE](#)].
- [56] K. Olsen and R.J. Szabo, *Constructing D-branes from K theory*, *Adv. Theor. Math. Phys.* **3** (1999) 889 [[hep-th/9907140](#)] [[INSPIRE](#)].
- [57] E. Witten, *Overview of K theory applied to strings*, *Int. J. Mod. Phys. A* **16** (2001) 693 [[hep-th/0007175](#)] [[INSPIRE](#)].
- [58] D. Quillen, *Superconnections and the Chern character*, *Topology* **24** (1985) 89 [[INSPIRE](#)].
- [59] C. Kennedy and A. Wilkins, *Ramond-Ramond couplings on Brane-anti-Brane systems*, *Phys. Lett. B* **464** (1999) 206 [[hep-th/9905195](#)] [[INSPIRE](#)].
- [60] M. Alishahiha, H. Ita and Y. Oz, *On superconnections and the tachyon effective action*, *Phys. Lett. B* **503** (2001) 181 [[hep-th/0012222](#)] [[INSPIRE](#)].
- [61] T. Takayanagi, S. Terashima and T. Uesugi, *Brane-anti-brane action from boundary string field theory*, *JHEP* **03** (2001) 019 [[hep-th/0012210](#)] [[INSPIRE](#)].
- [62] R.J. Szabo, *Superconnections, anomalies and nonBPS brane charges*, *J. Geom. Phys.* **43** (2002) 241 [[hep-th/0108043](#)] [[INSPIRE](#)].
- [63] M.R. Garousi and E. Hatefi, *On Wess-Zumino terms of Brane-Antibrane systems*, *Nucl. Phys. B* **800** (2008) 502 [[arXiv:0710.5875](#)] [[INSPIRE](#)].
- [64] C.V. Johnson, *D-Branes*, Cambridge University Press (2023) [[DOI:10.1017/9781009401371](#)] [[INSPIRE](#)].
- [65] J. Cheeger and J. Simons, *Differential characters and geometric invariants in Geometry and Topology* Springer Berlin, Heidelberg (1985), p. 50–80 [[DOI:10.1007/bfb0075216](#)].

- [66] E. Witten, *Baryons and branes in anti-de Sitter space*, *JHEP* **07** (1998) 006 [[hep-th/9805112](#)] [[INSPIRE](#)].
- [67] F. Apruzzi, M. van Beest, D.S.W. Gould and S. Schäfer-Nameki, *Holography, 1-form symmetries, and confinement*, *Phys. Rev. D* **104** (2021) 066005 [[arXiv:2104.12764](#)] [[INSPIRE](#)].
- [68] J.M. Maldacena and C. Nunez, *Towards the large N limit of pure $N = 1$ superYang-Mills*, *Phys. Rev. Lett.* **86** (2001) 588 [[hep-th/0008001](#)] [[INSPIRE](#)].
- [69] C.P. Herzog and I.R. Klebanov, *On string tensions in supersymmetric $SU(M)$ gauge theory*, *Phys. Lett. B* **526** (2002) 388 [[hep-th/0111078](#)] [[INSPIRE](#)].
- [70] S. Kachru, J. Pearson and H.L. Verlinde, *Brane / flux annihilation and the string dual of a nonsupersymmetric field theory*, *JHEP* **06** (2002) 021 [[hep-th/0112197](#)] [[INSPIRE](#)].
- [71] R. Bott and L.W. Tu, *Spectral Sequences and Applications*, in *Differential Forms in Algebraic Topology*, R. Bott and L.W. Tu eds, Springer New York, New York, (1982), p. 154–265 [[DOI:10.1007/978-1-4757-3951-0_4](#)].
- [72] G.W. Moore, *Anomalies, Gauss laws, and Page charges in M-theory*, *Comptes Rendus Physique* **6** (2005) 251 [[hep-th/0409158](#)] [[INSPIRE](#)].
- [73] D.S. Freed, G.W. Moore and G. Segal, *The Uncertainty of Fluxes*, *Commun. Math. Phys.* **271** (2007) 247 [[hep-th/0605198](#)] [[INSPIRE](#)].
- [74] A. Hanany and E. Witten, *Type IIB superstrings, BPS monopoles, and three-dimensional gauge dynamics*, *Nucl. Phys. B* **492** (1997) 152 [[hep-th/9611230](#)] [[INSPIRE](#)].
- [75] J.M. Maldacena, G.W. Moore and N. Seiberg, *D-brane charges in five-brane backgrounds*, *JHEP* **10** (2001) 005 [[hep-th/0108152](#)] [[INSPIRE](#)].
- [76] A. Kapustin and N. Saulina, *Topological boundary conditions in abelian Chern-Simons theory*, *Nucl. Phys. B* **845** (2011) 393 [[arXiv:1008.0654](#)] [[INSPIRE](#)].
- [77] J.M. Maldacena, *Wilson loops in large N field theories*, *Phys. Rev. Lett.* **80** (1998) 4859 [[hep-th/9803002](#)] [[INSPIRE](#)].
- [78] P. Putrov and J. Wang, *Categorical Symmetry of the Standard Model from Gravitational Anomaly*, [arXiv:2302.14862](#) [[INSPIRE](#)].
- [79] D. Gaiotto and J. Maldacena, *The gravity duals of $N = 2$ superconformal field theories*, *JHEP* **10** (2012) 189 [[arXiv:0904.4466](#)] [[INSPIRE](#)].
- [80] I. Bah, F. Bonetti, E. Leung and P. Weck, *The geometry of decoupling fields*, *JHEP* **09** (2022) 197 [[arXiv:2112.07796](#)] [[INSPIRE](#)].
- [81] L. Bhardwaj, L.E. Bottini, S. Schäfer-Nameki and A. Tiwari, *Non-invertible higher-categorical symmetries*, *SciPost Phys.* **14** (2023) 007 [[arXiv:2204.06564](#)] [[INSPIRE](#)].
- [82] L. Bhardwaj, S. Schäfer-Nameki and J. Wu, *Universal Non-Invertible Symmetries*, *Fortsch. Phys.* **70** (2022) 2200143 [[arXiv:2208.05973](#)] [[INSPIRE](#)].
- [83] L. Bhardwaj, S. Schäfer-Nameki and A. Tiwari, *Unifying constructions of non-invertible symmetries*, *SciPost Phys.* **15** (2023) 122 [[arXiv:2212.06159](#)] [[INSPIRE](#)].
- [84] T. Bartsch, M. Bullimore, A.E.V. Ferrari and J. Pearson, *Non-invertible Symmetries and Higher Representation Theory I*, [arXiv:2208.05993](#) [[INSPIRE](#)].
- [85] T. Bartsch, M. Bullimore, A.E.V. Ferrari and J. Pearson, *Non-invertible Symmetries and Higher Representation Theory II*, [arXiv:2212.07393](#) [[INSPIRE](#)].

- [86] C. Copetti, M. Del Zotto, K. Ohmori and Y. Wang, *Higher Structure of Chiral Symmetry*, [arXiv:2305.18282](#) [INSPIRE].
- [87] L. Kong, X.-G. Wen and H. Zheng, *Boundary-bulk relation in topological orders*, *Nucl. Phys. B* **922** (2017) 62 [[arXiv:1702.00673](#)] [INSPIRE].
- [88] W. Ji and X.-G. Wen, *Categorical symmetry and noninvertible anomaly in symmetry-breaking and topological phase transitions*, *Phys. Rev. Res.* **2** (2020) 033417 [[arXiv:1912.13492](#)] [INSPIRE].
- [89] A. Chatterjee and X.-G. Wen, *Symmetry as a shadow of topological order and a derivation of topological holographic principle*, *Phys. Rev. B* **107** (2023) 155136 [[arXiv:2203.03596](#)] [INSPIRE].
- [90] L. Kong et al., *Algebraic higher symmetry and categorical symmetry — a holographic and entanglement view of symmetry*, *Phys. Rev. Res.* **2** (2020) 043086 [[arXiv:2005.14178](#)] [INSPIRE].
- [91] C. Zhang and C. Córdova, *Anomalies of $(1+1)D$ categorical symmetries*, [arXiv:2304.01262](#) [INSPIRE].
- [92] I.R. Klebanov and E. Witten, *Superconformal field theory on three-branes at a Calabi-Yau singularity*, *Nucl. Phys. B* **536** (1998) 199 [[hep-th/9807080](#)] [INSPIRE].
- [93] P. Candelas and X.C. de la Ossa, *Comments on Conifolds*, *Nucl. Phys. B* **342** (1990) 246 [INSPIRE].
- [94] D. Cassani and A.F. Faedo, *A supersymmetric consistent truncation for conifold solutions*, *Nucl. Phys. B* **843** (2011) 455 [[arXiv:1008.0883](#)] [INSPIRE].
- [95] I. Bah, F. Bonetti, R. Minasian and E. Nardoni, *Class \mathcal{S} Anomalies from M-theory Inflow*, *Phys. Rev. D* **99** (2019) 086020 [[arXiv:1812.04016](#)] [INSPIRE].
- [96] I. Bah, F. Bonetti, R. Minasian and E. Nardoni, *Anomaly Inflow for M5-branes on Punctured Riemann Surfaces*, *JHEP* **06** (2019) 123 [[arXiv:1904.07250](#)] [INSPIRE].
- [97] I. Bah and F. Bonetti, *Anomaly Inflow, Accidental Symmetry, and Spontaneous Symmetry Breaking*, *JHEP* **01** (2020) 117 [[arXiv:1910.07549](#)] [INSPIRE].
- [98] L.A. Pando Zayas and A.A. Tseytlin, *3-branes on resolved conifold*, *JHEP* **11** (2000) 028 [[hep-th/0010088](#)] [INSPIRE].
- [99] A. Hatcher, *Algebraic topology*, Cambridge University Press, Cambridge (2000) [ISBN: 9780521795401].
- [100] J. McCleary, *A User's Guide to Spectral Sequences*, second edition, Cambridge University Press (2000) [DOI:10.1017/CB09780511626289].
- [101] D.-E. Diaconescu, G.W. Moore and E. Witten, *$E(8)$ gauge theory, and a derivation of K theory from M theory*, *Adv. Theor. Math. Phys.* **6** (2003) 1031 [[hep-th/0005090](#)] [INSPIRE].
- [102] E. Witten, *Duality relations among topological effects in string theory*, *JHEP* **05** (2000) 031 [[hep-th/9912086](#)] [INSPIRE].
- [103] Y. Choi et al., *Non-invertible Condensation, Duality, and Triality Defects in 3+1 Dimensions*, *Commun. Math. Phys.* **402** (2023) 489 [[arXiv:2204.09025](#)] [INSPIRE].