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BCF anomaly and higher-group structure in the low energy effective theories of mesons

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Abstract: We discuss the BCF anomaly of massless QCD-like theories, first obtained by Anber and Poppitz, from the viewpoint of the low energy effective theories. We assume that the QCD-like theories exhibit spontaneous chiral symmetry breaking due to a quark bilinear condensate. Using the 't Hooft anomaly matching condition for the BCF anomaly, we find that the low energy effective action is composed of a chiral Lagrangian and a Wess-Zumino-Witten term together with an interaction term of the η' meson with the background gauge field for a discrete one-form symmetry. It is shown that the low energy effective action cancels the quantum inconsistencies associated with η' due to an ambiguity of how to uplift the action to a five-dimensional spacetime with a boundary. The η' term plays a substantial role in exploring the emergent higher-group structure at low energies.

Keywords: Anomalies in Field and String Theories, Effective Field Theories, Global Symmetries

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Contents

1 Introduction

It has been established that higher-form symmetries [1] (see also refs. [2–12]) provides us with a powerful tool for studying nonperturbative aspects of quantum field theory [13–22]. A first attempt to utilize higher-form symmetries to quantum chromodynamics(QCD) with massless flavor degrees of freedom is made in [23] on the basis of a careful analysis of faithful group action of the symmetries for the case of a certain ratio of the number of the flavors to that of the colors. The paper [24] studies QCD-like theories of gauge group SU(N_c) with the quark fields belonging to an irreducible representation R. It is found there that the QCD-like theories admit discrete one-form symmetries for generic R and flavors as well. Furthermore, turning on the background gauge fields for the flavor and one-form symmetries, the associated 't Hooft anomaly is worked our and called the baryon-color-flavor(BCF) anomaly. For recent studies of the BCF anomaly, see [25–28]. In particular, by coupling the QCD-like theories with a neutral and complex Higgs field, the paper [25] focuses on an RG flow to an axion system that is triggered by the Higgs vev. It is shown that the low energy effective theory is given by a BF-type action and reproduces the BCF anomaly as expected from the 't Hooft anomaly matching condition. It is also pointed out that three-group structure is manifested as a Green-Schwarz(GS) transformation [29] of a dynamical three-form gauge potential under a discrete one-form gauge transformation. It is known that the mathematical structure of higher-form symmetries is naturally formulated in terms of higher-group. Roughly speaking, the higher-group is a set of the groups that describe higher-form symmetries including correlations among them. For recent developments in understanding the QFT dynamics from the viewpoint of higher-group structure, see [30–42] for instance.

It is a long-standing problem to determine the phase structures of the QCD-like theories such as spontaneous chiral symmetry breaking, unbroken chiral symmetry leading to massless baryons, conformal window, etc. In this paper, we assume that the QCD-like theories exhibit spontaneous chiral symmetry breaking. Then, the low energy effective theory is given by nonlinear σ-models that are constructed from nonlinear realization of the associated chiral symmetry breaking. The 't Hooft anomaly matching condition for the chiral flavor symmetry requires that the Wess-Zumino-Witten(WZW) term [43, 44] be added to the nonlinear σ-model. It is found that a naive generalization of the WZW term to the cases in the presence of the BCF background fields is insufficient because the BCF anomaly contains a mixed 't Hooft anomaly between a discrete axial symmetry and a discrete one-form symmetry. It is argued that the mixed anomaly can be reproduced by adding an interaction term between a U(1) meson and a background two-form gauge field. This term is not left unchanged under a one-form gauge symmetry transformation. It is found that the one-form symmetry is restored by introducing a background three-form gauge field that makes a GS-type transformation under the one-form gauge symmetry transformation. This is a manifestation of three-group structure in the low energy effective action of the QCD-like theories.

We also examine a quantum inconsistency associated with the U(1) meson, which is realized as an ambiguity of how the low energy effective action is uplifted to a five-dimensional one that has manifest gauge invariance. We call it an operator-valued ambiguity. This is required to vanish for the consistency of the quantum QCD-like theories. It is discussed that in the presence of the BCF background gauge fields, both the WZW term and the U(1) meson term suffer an operator-valued ambiguity. However, no ambiguity arises in the total effective action because of cancellation of the ambiguities from the two terms.

The organization of this paper is as follows. In section 2, we review the BCF anomalies in the QCD-like theories. Section 3 is devoted to a derivation of the low energy effective action of the QCD theories that reproduces the BCF anomaly, assuming that a quark bilinear condensate gives rise to spontaneous chiral symmetry breaking. We also show cancellation of the operator-valued ambiguity associated with the U(1) meson. We end this paper with discussions in section 4. In appendix A, we demonstrate in detail how to obtain the low energy effective action when the quark fields belong to a real representation of $\mathrm{SU}(N_c)$.

2 Review of BCF anomaly

In this section, we review the BCF anomaly [24].

We consider a QCD-like $\mathrm{SU}(N_c)$ gauge theory coupled with massless fermions in (3+1)-dimensional spacetime. The matter content of the QCD-like theory we analyze reads

$$\begin{array}{cccc} & \mathrm{SU}(N_c) & \mathrm{U}(N_f)_L & \mathrm{U}(N_f)_R \\ \psi^{ai}_{L\alpha} & R & \Box_1 & \mathbf{1}_0 \\ \psi^{\dot{\alpha}ai'}_{R} & R & \mathbf{1}_0 & \Box_1 \\ a & \textbf{adj.} & \mathbf{1}_0 & \mathbf{1}_0 \end{array} \tag{2.1}$$

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Here, $(\psi_{L\alpha}^{ai}, \psi_R^{\dot{\alpha}ai'})$ are fermions, and $a = a_\mu dx^\mu$ is an $\mathfrak{su}(N_c)$-valued 1-form gauge field. R denotes a representation of $\mathrm{SU}(N_c)$ to which the fermions belong. **adj**. means the adjoint representation of $\mathrm{SU}(N_c)$. The undotted and dotted indices $\alpha, \dot{\alpha}$ are the left- and right-handed spinor indices for the Lorentz symmetry, $\alpha, \dot{\alpha} = 1, 2$. The index a runs over the dimension of the gauge group $\mathrm{SU}(N_c)$, $a = 1, 2, \cdots, \dim R$. The indices $i, i' = 1, \ldots, N_f$ are the chiral flavor indices of $\mathrm{U}(N_f)_L$ and $\mathrm{U}(N_f)_R$, respectively. The action is given by

$$S = \int d^4x \left(-\frac{1}{2g^2} \operatorname{tr} f_{\mu\nu} f^{\mu\nu} + \bar{\psi}(i\partial_\mu + a_\mu^R)\psi + \frac{\theta}{32\pi^2} \epsilon^{\mu\nu\rho\sigma} \operatorname{tr} f_{\mu\nu} f_{\rho\sigma} \right), \quad \psi = \begin{pmatrix} \psi_{L\alpha} \\ \psi_R^{\dot{\alpha}} \end{pmatrix}. \tag{2.2}$$

Here, a^R is the dynamical $\mathrm{SU}(N_c)$ gauge field that couples with the matter fields in the representation R. It may be complex (e.g., fundamental) or real (e.g., adjoint) representation. When R is real, the chiral flavor symmetry enhances as

$$\mathrm{U}(N_f)_L \times \mathrm{U}(N_f)_R \to \mathrm{U}(2N_f)\,. \tag{2.3}$$

This is because we can define an undotted spinor Ψ as

$$\Psi_\alpha^{aI} = (\psi_{L\alpha}^{ai},\, \bar{\psi}_{R\alpha i'}^{a})\,, \quad (I = 1, 2, \cdots, 2N_f) \tag{2.4}$$

where

$$\bar{\psi}_{Ri'}^{\alpha a} = \left(\psi_R^{\dot{\alpha}ai'}\right)^* = \epsilon^{\alpha\beta}\, \bar{\psi}_{R\beta i'}^{a}\,. \tag{2.5}$$

This fermion Ψ belongs to the defining representation of $\mathrm{U}(2N_f)$. $\mathrm{U}(1)_V$ is identified with the Cartan part of $\mathrm{SU}(2N_f)$ with the generator given by $I_{N_f} \oplus (-I_{N_f})$. $\mathrm{U}(1)_A$ is the $\mathrm{U}(1)$ subgroup of $\mathrm{U}(2N_f)$ that is orthogonal to the semi-simple $\mathrm{SU}(2N_f)$.

2.1 ABJ and mixed 't Hooft anomalies

We turn on the background gauge fields for the flavor subgroup $\mathrm{SU}(N_f)_V \times \mathrm{U}(1)_B \times \mathrm{U}(1)_A$, which are denoted by A, A_B and χ, respectively. Here, $\mathrm{SU}(N_f)_V$ is the vector-like subgroup of $\mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R$, and $\mathrm{U}(1)_L \times \mathrm{U}(1)_R = (\mathrm{U}(1)_B \times \mathrm{U}(1)_A)/\mathbb{Z}_2$ with $\mathbb{Z}_2$ acting as

$$\mathbb{Z}_2 :\ (e^{i\alpha_V}, e^{i\gamma_5\alpha_A}) \in \mathrm{U}(1)_V \times \mathrm{U}(1)_A \to -(e^{i\alpha_V}, e^{i\gamma_5\alpha_A})\,.$$

Then, the fermionic part of the Lagrangian becomes

$$\begin{aligned} \mathcal{L}_f &= \bar{\psi}\gamma^\mu \left(i\partial_\mu + a_\mu^R \otimes I_{N_f} + I_{d(R)} \otimes A_\mu - (A_{B\mu} - \gamma_5 \chi_\mu)\, I_{d(R)} \otimes I_{N_f} \right) \psi \\ &= \bar{\psi}\gamma^\mu\, (i\partial_\mu + \mathcal{L}_\mu)\, P_L\, \psi + \bar{\psi}\gamma^\mu\, (i\partial_\mu + \mathcal{R}_\mu)\, P_R\, \psi\,, \end{aligned} \tag{2.6}$$

where $P_{L,R} = (1 \mp \gamma_5)/2$ and

$$\begin{aligned} \mathcal{L} &= a^R \otimes I_{N_f} + I_{d(R)} \otimes A - (A_B + \chi)\, I_{d(R)} \otimes I_{N_f} = a^R \otimes I_{N_f} + I_{d(R)} \otimes \bar{A}_L\,, \\ \mathcal{R} &= a^R \otimes I_{N_f} + I_{d(R)} \otimes A - (A_B - \chi)\, I_{d(R)} \otimes I_{N_f} = a^R \otimes I_{N_f} + I_{d(R)} \otimes \bar{A}_R\,. \end{aligned} \tag{2.7}$$

Here,

$$\bar{A}_L = A - (A_B + \chi) I_{N_f} = \bar{A} - \chi I_{N_f}\,, \quad \bar{A}_R = A - (A_B - \chi) I_{N_f} = \bar{A} + \chi I_{N_f}\,, \tag{2.8}$$

with $\bar{A} = A - A_B I_{N_f}$ being the $\mathrm{SU}(N_f)_V \times \mathrm{U}(1)_B$ gauge field. I is an identity matrix of rank equal to the suffix and $d(R) = \mathrm{dim}R$. We regard a^R as a background gauge field for the moment and define the effective action

$$e^{i\Gamma[\mathcal{L},\mathcal{R}]} = \int D\psi D\bar{\psi}\, e^{i\int d^4x\, \mathcal{L}_f}\,.$$

The consistent chiral anomaly, which is written in the L-R form, reads

$$\delta\Gamma = \frac{-1}{24\pi^2}\int \mathrm{Tr}\left[\lambda_L\, d\left(\mathcal{L}d\mathcal{L} - \frac{i}{2}\mathcal{L}^3\right) - \lambda_R\, d\left(\mathcal{R}d\mathcal{R} - \frac{i}{2}\mathcal{R}^3\right)\right], \tag{2.9}$$

where the infinitesimal gauge transformations for $\mathcal{L}$ and $\mathcal{R}$ are given by

$$\delta\mathcal{L} = d\lambda_L - i\mathcal{L}\lambda_L + i\lambda_L\mathcal{L}\,,\quad \delta\mathcal{R} = d\lambda_R - i\mathcal{R}\lambda_R + i\lambda_R\mathcal{R}\,.$$

Here, the symbol "Tr" is a trace taken over the product space of the representation R of $\mathfrak{su}(N_c)$ and the fundamental representation of $\mathfrak{su}(N_f)$. The consistent anomaly can be rewritten into a covariant form by adding the local counter term [45, 46]

$$\mathcal{Y}[\mathcal{L},\mathcal{R}] = \frac{-1}{48\pi^2}\int \mathrm{tr}\left[(F_\mathcal{R} + F_\mathcal{L})(\mathcal{R}\mathcal{L} - \mathcal{L}\mathcal{R}) + i(\mathcal{R}^3\mathcal{L} - \mathcal{L}^3\mathcal{R}) - \frac{i}{2}\mathcal{R}\mathcal{L}\mathcal{R}\mathcal{L}\right], \tag{2.10}$$

with

$$F_\mathcal{L} = d\mathcal{L} - i\mathcal{L}^2\,,\quad F_\mathcal{R} = d\mathcal{R} - i\mathcal{R}^2\,.$$

This makes the vector-like flavor symmetry anomaly free. It is easy to verify that

$$\mathcal{Y}[\mathcal{L},\mathcal{R}] = \mathcal{Y}_c[a,\chi] + \mathcal{Y}_f[\bar{A},\chi]\,. \tag{2.11}$$

Here,

$$\mathcal{Y}_c = -\frac{2N_f T(R)}{24\pi^2}\int \chi\,\mathrm{tr}\left(4fa + ia^3\right),\quad \mathcal{Y}_f = -\frac{d(R)}{24\pi^2}\int \chi\,\mathrm{tr}\left(4\bar{F}\bar{A} + i\bar{A}^3\right). \tag{2.12}$$

a is the dynamical $\mathrm{SU}(N_c)$ gauge field for $R = \square$, and with

$$f = da - ia^2\,,\quad \bar{F} = d\bar{A} - i\bar{A}^2\,.$$

Here, the symbol "tr" is a trace taken over the fundamental representation of either $\mathfrak{su}(N_c)$ or $\mathfrak{su}(N_f)$. Throughout this paper, we assume $d\chi = 0$, which suffices for later purposes. With the local counter term $\mathcal{Y}$, the chiral anomaly takes the form

$$\delta(\Gamma + \mathcal{Y}) = \frac{1}{4\pi^2}\int\left[2T(R)\ \mathrm{tr}(\hat{\alpha})\,\mathrm{tr}(f\wedge f) + d(R)\ \mathrm{tr}(\hat{\alpha}\,\bar{F}\wedge\bar{F})\right]. \tag{2.13}$$

Here, $\hat{\alpha}$ is a $\mathrm{U}(N_f)_L \times \mathrm{U}(N_f)_R$ rotation angle given by

$$\frac{1}{2}(\lambda_R - \lambda_L) = I_{d(R)} \otimes \hat{\alpha}\,. \tag{2.14}$$

The $\mathrm{U}(1)_A$ rotation angle α_A is equal to the trace part of $\hat{\alpha}$:

$$\hat{\alpha} = \alpha_A I_{N_f} + \alpha\,,\qquad \mathrm{tr}\,\alpha = 0\,. \tag{2.15}$$

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The chiral anomaly for the $\mathrm{U}(1)_A$ transformation is given by

$$\begin{aligned}\delta_A(\Gamma+\mathcal{Y}) &= \frac{1}{4\pi^2}\int \alpha_A\Big[2N_f T(R)\ \mathrm{tr}(f\wedge f) + d(R)\ \mathrm{tr}(\bar{F}\wedge\bar{F})\Big] \\ &= \frac{1}{4\pi^2}\int \alpha_A\Big[2N_f T(R)\,\mathrm{tr}(f\wedge f) + d(R)\,\mathrm{tr}(F\wedge F) + N_f d(R)\, F_B\wedge F_B\Big]\,. \end{aligned} \tag{2.16}$$

It follows from the quantization condition

$$\frac{1}{8\pi^2}\int \mathrm{tr}(f\wedge f)\in\mathbb{Z}\,,$$

that the Adler-Bell-Jackiw(ABJ) anomaly [47, 48] leaves $\mathbb{Z}_{4N_fT(R)}\subset \mathrm{U}(1)_A$ unbroken such that

$$\alpha_A = \frac{2\pi n_A}{4N_f T(R)}\,, \tag{2.17}$$

with $n_A\in\mathbb{Z}$ mod $4N_fT(R)$. We also note that the subgroup $\mathbb{Z}_2\subset\mathbb{Z}_{4N_fT(R)}$ acts as

$$(\psi_L,\psi_R)\to -(\psi_L,\psi_R)\,, \tag{2.18}$$

which is an element of the vector-like $\mathrm{U}(1)_B$. As found above, $\mathcal{Y}$ makes the vector-like flavor group anomaly free so that the anomaly free $\mathrm{U}(1)_A$ subgroup is given by $\mathbb{Z}_{2N_fT(R)} = \mathbb{Z}_{4N_fT(R)}/\mathbb{Z}_2$ with $n_A\in\mathbb{Z}$ mod $2N_fT(R)$.

As we will see shortly, the BCF anomaly is obtained by incorporating the background gauge potentials for one-form symmetries into (2.16) in a manner consistent with the gauge symmetries.

2.2 BCF anomaly

We have discussed conventional anomalies between axial, baryon, and flavor symmetries. In addition, this system can have a $\mathbb{Z}_{N_c}$ center symmetry for the gauge group $\mathrm{SU}(N_c)$ if the fermions belong to a real representation of $\mathrm{SU}(N_c)$. The center symmetry for the gauge group can be recently understood as a one-form symmetry associated to a $\mathbb{Z}_{N_c}$ rotation of a Wilson loop in the fundamental representation. Furthermore, we can also have one-form symmetries even if the fermions belong to the (anti-)fundamental representation using a simultaneous rotation of the baryon symmetry. The one-form symmetry also have a mixed 't Hooft anomaly with the baryon and flavor symmetries, which is called the BCF anomaly.

To work out the anomaly, we need to specify faithful group action of the symmetries on the quark fields ψ_L and ψ_R. As noted in [24], it is given by

$$\frac{\mathrm{SU}(N_c)\times\mathrm{SU}(N_f)_V\times\mathrm{U}(1)_B}{\mathbb{Z}_{N_c}\times\mathbb{Z}_{N_f}}\,, \tag{2.19}$$

where the identifications of the elements in $\mathrm{SU}(N_c)\times\mathrm{SU}(N_f)_V\times\mathrm{U}(1)_B$ by the discrete group $\mathbb{Z}_{N_c}\times\mathbb{Z}_{N_f}$ are given by

$$\begin{aligned}\mathbb{Z}_{N_c} &: (g,g_V,g_B)\to(e^{2\pi i/N_c}g, g_V, e^{-2\pi i n/N_c}g_B)\,,\\ \mathbb{Z}_{N_f} &: (g,g_V,g_B)\to(g, e^{2\pi i/N_f}g_V, e^{-2\pi i/N_f}g_B)\,,\end{aligned} \tag{2.20}$$

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with

$$(g, g_V, g_B) \in \mathrm{SU}(N_c) \times \mathrm{SU}(N_f)_V \times \mathrm{U}(1)_B\,. \tag{2.21}$$

Here, the elements g and g_V are realized in the fundamental representation. The quantity n is the N-ality of the representation R. Let g^R be the $\mathrm{SU}(N_c)$ transformation matrix in the representation R. Then, $g^R \otimes g_V \otimes g_B$ is left invariant under $\mathbb{Z}_{N_c} \times \mathbb{Z}_{N_f}$, showing that the quotient (2.19) leads to faithful group action on ψ_L and ψ_R.

Now, we define the gauge fields for (2.19). They are obtained by starting with the direct product of the $\mathrm{SU}(N_c)/\mathbb{Z}_{N_c}$ and $\mathrm{SU}(N_f)_V/\mathbb{Z}_{N_f}$ bundles, and then requiring the constraints to be explained shortly. As discussed in [12], the $\mathrm{SU}(N_c)/\mathbb{Z}_{N_c}$ and $\mathrm{SU}(N_f)_V/\mathbb{Z}_{N_f}$ bundles are constructed by lifting a and A to a $\mathrm{U}(N_c)$ and a $\mathrm{U}(N_f)$ gauge fields

$$\hat{a} = a + \frac{1}{N_c} C I_{N_c}\,, \quad \hat{A} = A + \frac{1}{N_f} C_V\, I_{N_f}\,, \tag{2.22}$$

respectively. $\hat{a}$ and $\hat{A}$ are realized in the fundamental representations of $\mathfrak{u}(N_c)$ and $\mathfrak{u}(N_f)$, respectively. C and C_V are one-form gauge fields which are assumed to be properly normalized by the flux quantization conditions on a closed surface $\mathcal{S}$,

$$\int_{\mathcal{S}} dC\,, \quad \int_{\mathcal{S}} dC_V \in 2\pi\mathbb{Z}\,. \tag{2.23}$$

The field strengths for $\hat{a}$ and $\hat{A}$ are defined as

$$\hat{f} = d\hat{a} - i\hat{a}\wedge\hat{a}\,, \quad \hat{F} = d\hat{A} - i\hat{A}\wedge\hat{A}\,. \tag{2.24}$$

Note that a, $\frac{1}{N_c}C$, A, and $\frac{1}{N_f}C_V$ in (2.22) are not properly normalized one-forms, but $\hat{a}$ and $\hat{A}$ are properly normalized,

$$\frac{1}{2\pi}\int_{\mathcal{S}} \mathrm{tr}\,\hat{f} \in \mathbb{Z}\,, \quad \frac{1}{8\pi^2}\int \mathrm{tr}(\hat{f}\wedge\hat{f}) \in \mathbb{Z}\,, \quad \frac{1}{2\pi}\int_{\mathcal{S}} \mathrm{tr}\,\hat{F} \in \mathbb{Z}\,, \quad \frac{1}{8\pi^2}\int \mathrm{tr}(\hat{F}\wedge\hat{F}) \in \mathbb{Z}\,, \tag{2.25}$$

on a spin manifold.

$\mathrm{SU}(N_c)/\mathbb{Z}_{N_c}$ and $\mathrm{SU}(N_f)/\mathbb{Z}_{N_f}$ gauge fields are obtained by considering $\mathrm{U}(N_c)$ and $\mathrm{U}(N_f)$ gauge fields that are defined to obey the gauge transformation laws at double overlaps of the coordinate patches:

$$\hat{a}_j = \hat{g}_{ji}\,\hat{a}_i\,\hat{g}_{ji}^{-1} + i\hat{g}_{ji}\,d\hat{g}_{ji}^{-1} + \Lambda_{ji} I_{N_c}\,, \quad \hat{A}_j = \hat{g}_{Vji}\,\hat{A}_i\,\hat{g}_{Vji}^{-1} + i\hat{g}_{Vji}\,d\hat{g}_{Vji}^{-1} + \Lambda_{Vji} I_{N_f}\,. \tag{2.26}$$

This implies that C and C_V transform as

$$C \to C + N_c\Lambda\,, \quad C_V \to C_V + N_f\Lambda_V\,. \tag{2.27}$$

The gauge fields $\hat{a}$ and $\hat{A}$ can be regarded as $\mathrm{SU}(N_c)/\mathbb{Z}_{N_c}$ and $\mathrm{SU}(N_f)_V/\mathbb{Z}_{N_f}$ gauge fields respectively because (2.27) gauges away $\mathrm{U}(1)$ parts of the $\mathrm{U}(N_c)$ and $\mathrm{U}(N_f)$ gauge fields.

The transformation law (2.26) can be extended to that of a gauge potential for the representation R of $\mathfrak{u}(N_c)$:

$$\hat{a}_j^R = \hat{g}_{ji}^R\,\hat{a}_i\,\hat{g}_{ji}^{R-1} + i\hat{g}_{ji}^R d\hat{g}_{ji}^{R-1} + n\Lambda_{ji} I_{d(R)}\,, \tag{2.28}$$

with g^R being a $\mathrm{U}(N_f)$ gauge transformation for the representation R. For the purpose of constructing the gauge fields for (2.19), we consider

$$\widetilde{A} = \hat{a}^R \otimes I_{N_f} + I_{d(R)} \otimes \hat{A} - \hat{A}_B\, I_{d(R)} \otimes I_{N_f} \tag{2.29}$$

We require that $\widetilde{A}$ obey the gauge transformation law,

$$\widetilde{A}_j = (\hat{g}^R \hat{g}_V g_B)_{ji}\, \widetilde{A}_i\, (\hat{g}^R \hat{g}_V g_B)^{-1}_{ji} + i(\hat{g}^R \hat{g}_V g_B)_{ji}\, d(\hat{g}^R \hat{g}_V g_B)^{-1}_{ji}\,, \tag{2.30}$$

or equivalently,

$$\widetilde{A} \to (\hat{g}^R \hat{g}_V g_B)\widetilde{A}\,(\hat{g}^R \hat{g}_V g_B)^{-1} + i\hat{g}^R d\hat{g}^{R-1} + i\hat{g}_V d\hat{g}_V^{-1} + i g_B d g_B^{-1}\,. \tag{2.31}$$

This is achieved by setting

$$\hat{A}_B := A_B + \frac{n}{N_c} C + \frac{1}{N_f} C_V\,, \tag{2.32}$$

which shows that $\hat{A}_B$ transforms under (2.27) as

$$\hat{A}_B \to \hat{A}_B + n\Lambda + \Lambda_V\,. \tag{2.33}$$

$\widetilde{A}$ couples to the fermion ψ in a gauge invariant manner, and hence the group (2.19) is a symmetry of the theory (2.1).

Let (B, C) and (B_V, C_V) be the sets of two- and one-form gauge fields of $\mathbb{Z}_{N_c} \times \mathbb{Z}_{N_f}$ group that arises in (2.19). They satisfy by definition

$$N_c B = dC\,, \quad N_f B_V = dC_V\,. \tag{2.34}$$

Eq. (2.27) gives rise to the gauge transformations of B and B_V as

$$B \to B + d\Lambda\,, \quad B_V \to B_V + d\Lambda_V\,. \tag{2.35}$$

The origin of the two-form gauge fields can be understood as follows. We formally rewrite the field strengths f and F in terms of the gauge field $\hat{a}$ and $\hat{A}$:

$$\begin{aligned} f = da - ia \wedge a &= d\left(\hat{a} - \frac{1}{N_c} C I_{N_c}\right) - i\left(\hat{a} - \frac{1}{N_c} C I_{N_c}\right) \wedge \left(\hat{a} - \frac{1}{N_c} C I_{N_c}\right) \\ &= \hat{f} - \frac{1}{N_c} dC I_{N_c}, \end{aligned} \tag{2.36}$$

$$F = dA - iA \wedge A = d\hat{A} - i\hat{A} \wedge \hat{A} - \frac{1}{N_f} dC_V I_{N_f} = \hat{F} - \frac{1}{N_f} dC_V I_{N_f}. \tag{2.37}$$

By noting that (2.27) acts on the field strength $\hat{f}$ and $\hat{F}$ as

$$\hat{f} \to \hat{f} + d\Lambda I_{N_c}\,, \quad \hat{F} \to \hat{F} + d\Lambda I_{N_f}\,, \tag{2.38}$$

respectively, $\hat{f} - \frac{1}{N_c} dC I_{N_c}$ and $\hat{F} - \frac{1}{N_f} dC_V I_{N_f}$ are interpreted as Stückelberg couplings. Therefore, $\frac{1}{N_c} dC$ and $\frac{1}{N_f} dC_V I_{N_f}$ are naturally regarded as two-form gauge fields with the transformation law (2.35). The $\mathbb{Z}_{N_c} \times \mathbb{Z}_{N_f}$ gauge fields are characterized by the vanishing field strength and non-trivial Aharonov-Bohm phase. In fact, the field strength of the two-form B vanishes locally, $dB = \frac{1}{N_c} ddC = 0$ and B has $\mathbb{Z}_{N_c}$-valued Aharonov-Bohm phase, $\int_{\mathcal{S}} B = \frac{1}{N_c} \int_{\mathcal{S}} dC \in \frac{2\pi}{N_c}\mathbb{Z}$. Similar arguments hold for B_V.

Now we derive the BCF anomaly [24, 25]. This is obtained from the $\mathrm{U}(1)_A$ anomaly in (2.16) by replacing

$$\mathrm{tr}(f \wedge f) \to \mathrm{tr}((\hat{f} - B) \wedge (\hat{f} - B)) = \mathrm{tr}(\hat{f} \wedge \hat{f}) - N_c\, B \wedge B \tag{2.39}$$

$$\mathrm{tr}(F \wedge F) \to \mathrm{tr}((\hat{F} - B_V) \wedge (\hat{F} - B_V)) = \mathrm{tr}(\hat{F} \wedge \hat{F}) - N_f\, B_V \wedge B_V \tag{2.40}$$

$$F_B \to \hat{F}_B - nB - B_V\,, \tag{2.41}$$

where $\mathrm{tr}\,\hat{f} = dC = N_c B$ and $\mathrm{tr}\,\hat{F} = dC_V = N_f B_V$. It is found that

$$\begin{aligned} \delta_A(\Gamma + \mathcal{Y}) = &-2\pi N_c n_A \int \frac{1}{8\pi^2} B \wedge B \\ &+ \frac{2\pi\, d(R) n_A}{2N_f T(R)} \int \frac{1}{8\pi^2} \Big(\mathrm{tr}(\hat{F} \wedge \hat{F}) - N_f B_V \wedge B_V\Big) \\ &+ \frac{2\pi\, d(R) n_A}{2T(R)} \int \frac{1}{8\pi^2} \Big(\hat{F}_B - nB - B_V\Big)^2\,, \end{aligned} \tag{2.42}$$

mod $2\pi\mathbb{Z}$. The first term is a manifestation of the mixed 't Hooft anomaly between the axial $\mathbb{Z}_{2N_f T(R)} = \mathbb{Z}_{4N_f T(R)}/\mathbb{Z}_2$ and the one-form $\mathbb{Z}_{N_c}$ symmetry.

It is possible to remove the first term by adding the local counter term

$$\mathcal{Y}_B = \frac{4N_f T(R)}{2\pi}(1 + kN_c) \int \chi \wedge \Big(C_3 - \frac{1}{4\pi} B \wedge C\Big)\,, \tag{2.43}$$

with $k \in \mathbb{Z}$. Here, C_3 is a background three-form gauge field with a gauge transformation by a two-form gauge parameter Λ_2,

$$C_3 \to C_3 + d\Lambda_2. \tag{2.44}$$

The three-form gauge field is normalized by the flux quantization condition,

$$\int dC_3 \in 2\pi\mathbb{Z}\,. \tag{2.45}$$

Further, C_3 is defined to make a GS transformation

$$C_3 \to C_3 + \frac{N_c}{4\pi}\Big(2B \wedge \Lambda + \Lambda \wedge d\Lambda\Big)\,, \tag{2.46}$$

under the one-form $\mathbb{Z}_{N_c}$ gauge transformation

$$B \to B + d\Lambda\,, \quad C \to C + N_c \Lambda\,. \tag{2.47}$$

The field strength of C_3 can be defined as

$$G_4 := dC_3 - \frac{N_c}{4\pi} B \wedge B \tag{2.48}$$

This guarantees that $\mathcal{Y}_B$ is left invariant under the one-form $\mathbb{Z}_{N_c}$ gauge transformation. C_3 plays an essential role in exploring the higher-group structure in the low energy effective theory in the QCD-like theories. Hereafter, we set $k = 0$ for simplicity, since the term proportional to k does not contribute to cancel the anomaly.

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3 BCF anomaly from low energy effective action

In this subsection, we discuss how the BCF anomaly is reproduced by the low energy effective theories of pions in the QCD-like theories.

3.1 Vacuum structure and construction of chiral Lagrangian

We assume that the QCD-like theories develop a fermion bilinear condensate. Then, the low energy effective theory is described by a nonlinear σ-model constructed from nonlinear realization of the spontaneous chiral symmetry breaking. We refer to it as a chiral Lagrangian with the degrees of freedom given by the pions.

When R is complex, the order parameter reads

$$\begin{array}{ccccc} & \mathrm{SU}(N_f)_L & \mathrm{SU}(N_f)_R & \mathrm{U}(1)_V & \mathrm{U}(1)_A \\ \psi^{ai}_{L\alpha}\bar{\psi}^{\alpha}_{Rai'} & \square & \bar{\square} & 0 & 2 \end{array} \tag{3.1}$$

The bilinear condensate

$$\langle \psi_L \bar{\psi}_R \rangle = \Lambda^3 I_{N_f}\,, \tag{3.2}$$

gives rise to spontaneous symmetry breaking for the nonabelian chiral symmetry

$$\mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R \to \mathrm{SU}(N_f)_V\,. \tag{3.3}$$

Furthermore, the bilinear condensate breaks the discrete axial symmetry as

$$\mathbb{Z}_{4N_fT(R)} \to \mathbb{Z}_2\,, \tag{3.4}$$

where $\mathbb{Z}_2$ acts as $\psi_{L,R} \to -\psi_{L,R}$. Then, the vacuum manifold due to the bilinear condensate is given by

$$\frac{\mathbb{Z}_{4N_fT(R)}/\mathbb{Z}_2 \times (\mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R)/\mathrm{SU}(N_f)_V}{\mathbb{Z}_{N_f}}\,. \tag{3.5}$$

Here, $\mathbb{Z}_{N_f}$ is the center of either $\mathrm{SU}(N_f)_L$ or $\mathrm{SU}(N_f)_R$, which identifies two distinct phases of the condensate that are generated by the $\mathbb{Z}_{2N_fT(R)} = \mathbb{Z}_{4N_fT(R)}/\mathbb{Z}_2$ action. The low energy dynamics of the corresponding Nambu-Goldstone(NG) bosons are described by a chiral Lagrangian

$$\mathcal{L} = -\frac{f_\pi^2}{4}\,\mathrm{tr}(\hat{U}^\dagger \partial_\mu \hat{U})^2\,. \tag{3.6}$$

Here,

$$\hat{U}(\pi) = \exp\left(\frac{2i\pi}{f_\pi}\right) \in \mathrm{U}(N_f)\,, \tag{3.7}$$

is the exponentiated pion field, which transforms as

$$\mathrm{U}(N_f)_L \times \mathrm{U}(N_f)_R:\ \hat{U} \to g_L \hat{U} g_R^\dagger\,. \tag{3.8}$$

Note that the U(1) part of $\hat{U}$, the η' meson, is not massless because it is associated with the spontaneously breaking of the discrete symmetry $\mathbb{Z}_{2N_fT(R)} \subset \mathrm{U}(1)_A$. This fact is understood by a dynamically generated potential for the η' meson as discussed shortly.

In the presence of the background gauge fields for the gauge group (2.19) as well as $\mathbb{Z}_{4N_f T(R)} \subset \mathrm{U}(1)_A$, the derivatives in the chiral Lagrangian must be written in terms of the covariant derivative defined by

$$D\hat{U} = d\hat{U} - i\bar{A}_L\hat{U} + i\hat{U}\bar{A}_R \,, \tag{3.9}$$

where $\bar{A}_{L,R}$ is defined in (2.8) and rewritten in terms of the uplifted gauge fields as

$$\begin{aligned}\bar{A}_L &= \bar{A} - \chi I_{N_f} = \hat{A} - \left(\hat{A}_B - \frac{n}{N_c}C + \chi\right) I_{N_f} \,, \\ \bar{A}_R &= \bar{A} + \chi I_{N_f} = \hat{A} - \left(\hat{A}_B - \frac{n}{N_c}C - \chi\right) I_{N_f} \,.\end{aligned} \tag{3.10}$$

When R is real, a gauge invariant fermion bilinear is given by

$$\epsilon^{\alpha\beta}\,\Psi^{aI}_{\alpha}\Psi^{aJ}_{\beta} \,. \tag{3.11}$$

This is symmetric under the exchange $I \leftrightarrow J$. We assume that the QCD-like theory exhibits the nonabelian chiral symmetry breaking

$$\mathrm{SU}(2N_f) \to \mathrm{SO}(2N_f) \,, \tag{3.12}$$

due to the vacuum condensate

$$\langle \epsilon^{\alpha\beta}\,\Psi^{aI}_{\alpha}\Psi^{aJ}_{\beta}\rangle = \Lambda^3\delta^{IJ} \,. \tag{3.13}$$

As a consistency check for this assumption, this is in accord with the Vafa-Witten theorem [49], which states that vector-like global symmetry is unbroken for any vector-like gauge theory with the vacuum angle $\theta = 0$. We note that $\mathrm{SU}(N_f) \subset \mathrm{SO}(2N_f)$ is identified with $\mathrm{SU}(N_f)_V$. For recent studies of the chiral flavor symmetry breaking for the adjoint QCD, see [50] for instance.

We find that the bilinear condensate (3.13) leads to the vacuum manifold

$$\frac{\mathbb{Z}_{4N_f T(R)}/\mathbb{Z}_2 \times \mathrm{SU}(2N_f)/\mathrm{SO}(2N_f)}{\mathbb{Z}_{2N_f}} \,, \tag{3.14}$$

with $\mathbb{Z}_{2N_f}$ being the center of $\mathrm{SU}(2N_f)$. The low energy effective is given by the chiral Lagrangian associated with the spontaneous breaking $\mathrm{U}(2N_f) \to \mathrm{SO}(2N_f)$

$$\mathcal{L} = -\frac{f_\pi^2}{4}\,\mathrm{tr}(\hat{U}^\dagger\partial_\mu\hat{U})^2 \,. \tag{3.15}$$

Here, f_π is the decay constant for the QCD-like theory and

$$\hat{U}(\pi) \in \mathrm{U}(2N_f)/\mathrm{SO}(2N_f) \,. \tag{3.16}$$

We utilize the same notation of the decay constant and the exponentiated pion field as in the case of $R =$complex. $\hat{U}$ transforms under $\mathrm{U}(2N_f)$ as

$$\mathrm{U}(2N_f):\ \hat{U} \to g\,\hat{U}g^T \,. \tag{3.17}$$

For a construction of this action using the Callan-Coleman-Wess-Zumino (CCWZ) procedure, see [51]. The U(1) pion π^0 (referred to as the η' meson too) is massive because it is associated with the spontaneously broken chiral symmetry $\mathbb{Z}_{2N_fT(R)} \subset \mathrm{U}(1)_A$ as shown in (3.14). This fact is accounted for by a dynamically generated potential for the η' meson.

Let $\bar{\mathcal{A}}$ be the background gauge field for $\mathrm{U}(2N_f)$. In the presence of it, the covariant derivative of $\hat{U}$ reads

$$D\hat{U} = d\hat{U} - i\bar{\mathcal{A}}\hat{U} - i\hat{U}\bar{\mathcal{A}}^T . \tag{3.18}$$

$\bar{A}_L$ and $\bar{A}_R$, the background gauge fields for the $\mathrm{U}(N_f)_L \times \mathrm{U}(N_f)_R$ subgroup, are embedded into $\bar{\mathcal{A}}$ as

$$\bar{\mathcal{A}} = \begin{pmatrix} \bar{A}_L & 0 \\ 0 & -\bar{A}_R^* \end{pmatrix} . \tag{3.19}$$

Let us discuss the potential term of the η' meson for a generic representation R, which makes the η' meson massive. We note that the $\mathrm{U}(1)_A$ symmetry acts on the exponentiated pion field as

$$\hat{U} \to e^{-2i\alpha_A}\hat{U} , \tag{3.20}$$

where α_A is defined in (2.15). The U(1) pion is extracted from $\hat{U}$ as

$$\hat{U} = e^{2i\pi/f_\pi} = e^{2i\pi^{\hat{a}}T^{\hat{a}}/f_\pi} = e^{2i\eta' T^0/f_\pi} e^{2i\pi^a T^a/f_\pi} , \tag{3.21}$$

with

$$a = \begin{cases} 1, 2, \cdots, \dim \mathrm{SU}(N_f) , & (R = \text{complex}) \\ 1, 2, \cdots, \dim \mathrm{SU}(2N_f)/\mathrm{SO}(2N_f) . & (R = \text{real}) \end{cases} \tag{3.22}$$

Here, the generators $T^{\hat{a}}$ are normalized as

$$\mathrm{tr}(T^{\hat{a}}T^{\hat{b}}) = \frac{1}{2}\delta^{\hat{a}\hat{b}} , \tag{3.23}$$

with the U(1) generator given by

$$T^0 = \begin{cases} I_{N_f}/\sqrt{2N_f} , & (R = \text{complex}) \\ I_{2N_f}/\sqrt{4N_f} . & (R = \text{real}) \end{cases} . \tag{3.24}$$

With this choice, η' is a canonically normalized real scalar. It is also useful to rewrite the U(1) meson as

$$\hat{U} = e^{i\widetilde{\eta}} e^{2i\pi^a T^a/f_\pi} \equiv e^{i\widetilde{\eta}}\, U , \tag{3.25}$$

which gives

$$\widetilde{\eta} = \begin{cases} \sqrt{\frac{2}{N_f}}\, \frac{\eta'}{f_\pi} , & (R = \text{complex}) \\ \sqrt{\frac{1}{N_f}}\, \frac{\eta'}{f_\pi} . & (R = \text{real}) \end{cases} \tag{3.26}$$

Eq. (3.20) shows that $\mathrm{U}(1)_A$ acts as a shift of η

$$\widetilde{\eta} \to \widetilde{\eta} - 2\alpha_A . \tag{3.27}$$

We also note that the anomalous $\mathrm{U}(1)_A$ transformation induces the shift of the vacuum angle

$$\theta \to \theta + 4N_f T(R)\alpha_A . \tag{3.28}$$

Then, the potential term for $\widetilde{\eta}$, if exists, is required to depend on a single variable $\theta + 2N_f T(R)\widetilde{\eta}$, a $\mathrm{U}(1)_A$ invariant

$$V(\theta + 2N_f T(R)\widetilde{\eta})\,. \tag{3.29}$$

This potential is consistent with the Witten-Veneziano formula [52, 53] for the η' meson mass for large N_c QCD with $R = \Box$. The potential energy is of $\mathcal{O}(N_c^2)$ because it is dominated by the gluon loop effects. The large N_c limit should be defined by regarding the vacuum angle as of $\mathcal{O}(N_c)$. This is explained by noting the gauge theory action

$$S = \cdots + \theta \int \frac{1}{8\pi^2}\,\mathrm{tr}\, f \wedge f = N_c \left[\cdots + \frac{\theta}{N_c}\int \frac{1}{8\pi^2}\,\mathrm{tr}\, f \wedge f\right], \tag{3.30}$$

with θ/N_c held finite. We also note that $\widetilde{\eta}$ should be of $\mathcal{O}(N_c^0)$ because the chiral Lagrangian takes the form

$$-\frac{f_\pi^2}{4}\,\mathrm{tr}\left(\hat{U}^\dagger \partial_\mu \hat{U}\right)^2 = \frac{f_\pi^2}{4}\left[N_f(\partial_\mu \widetilde{\eta})^2 - \mathrm{tr}\left(U^\dagger \partial_\mu U\right)^2\right], \tag{3.31}$$

with $f_\pi^2 = \mathcal{O}(N_c)$. It then follows that the vacuum energy density as a function of θ and $\widetilde{\eta}$ takes the form

$$V = N_c^2\, h\left(\frac{\theta + 2N_f T(R)\widetilde{\eta}}{N_c}\right). \tag{3.32}$$

The periodicity of the vacuum angle is manifested with a multi-branched function [54]. For a generic value of θ of $\mathcal{O}(N_c^0)$, which is consistent with the large N_c limit, h can be approximated with a quadratic function for each branch because

$$\frac{\theta + 2N_f T(R)\widetilde{\eta}}{N_c} = \mathcal{O}(N_c^{-1})\,. \tag{3.33}$$

Then, we obtain

$$V = \frac{1}{2}\,\chi_g\,\min_{k\in\mathbb{Z}}\,(\theta + 2\pi k + 2N_f T(R)\widetilde{\eta})^2 + \mathcal{O}(N_c^{-1})\,. \tag{3.34}$$

Here, χ_g is the topological susceptibility, being of $\mathcal{O}(N_c^0)$. Rewriting it in terms of η', the canonically normalized field, the mass squared of the η' meson is given by

$$m_{\eta'}^2 = \frac{2N_f}{f_\pi^2}\,\chi_g\,. \tag{3.35}$$

This is the Witten-Veneziano formula [52, 53]. For a derivation of it using a holographic dual, see [55].

When $R = \Box\Box$, $\begin{smallmatrix}\Box\\\Box\end{smallmatrix}$ and **adj.**, the vacuum energy density takes the same form as in (3.32). A distinction from the case of $R = \Box$ is that the quark field makes the $\mathcal{O}(N_c^2)$ contribution to V, which is comparable with the planar effects of the gluon. We have to note that for a generic θ

$$\frac{\theta + 2N_f T(R)\widetilde{\eta}}{N_c} = \mathcal{O}(N_c^0)\,, \tag{3.36}$$

because

$$T(R) = \mathcal{O}(N_c)\,. \tag{3.37}$$

This shows that no approximation of h is allowed, implying that no multi-branch structure of the potential is there. The η' mass is of $\mathcal{O}(N_c^0)$, and therefore not suppressed in the large N_c limit.

3.2 WZW term and η' term

The 't Hooft anomaly matching condition states that the low energy dynamics of the QCD-like theories must reproduce not only the BCF anomaly (2.42) but also the full nonabelian anomaly that is obtained for a generic rotation angle $\hat{\alpha}$ defined in (2.14). The anomaly matching condition for the BCF anomaly only is not powerful enough to determine the low energy effective theory uniquely, because it is a discrete anomaly. The 't Hooft anomaly matching for the nonabelian flavor symmetry requires that the WZW term be added to the low energy effective theory. With this action, the anomaly matching for the second and the third terms of the BCF anomaly is satisfied. It is argued that the first term of the BCF anomaly is reproduced by another interaction term of the η' meson. For a recent analysis of WZW terms in the presence of background fields for discrete one-form symmetries, see also [28]

We first discuss the cases for R complex. The WZW term reads [44]

$$S_{\rm WZW} = -i\frac{d(R)}{48\pi^2}\int_{M^4} Z - i\frac{d(R)}{240\pi^2}\int_{M^4\times\mathbb{R}} \mathrm{tr}(\hat{U}d\hat{U}^{-1})^5\,, \tag{3.38}$$

with

$$\begin{aligned} Z = {} & i\ \mathrm{tr}[(\bar{A}_R d\bar{A}_R + d\bar{A}_R\bar{A}_R - i\bar{A}_R^3)(\hat{U}^{-1}\bar{A}_L\hat{U} + i\hat{U}^{-1}d\hat{U}) - \text{p.c.}] \\ & - \mathrm{tr}[d\bar{A}_R d\hat{U}^{-1}\bar{A}_L\hat{U} - \text{p.c.}] - i\ \mathrm{tr}[\bar{A}_R(d\hat{U}^{-1}\hat{U})^3 - \text{p.c.}] \\ & - \frac{1}{2}\mathrm{tr}[(\bar{A}_R d\hat{U}^{-1}\hat{U})^2 - \text{p.c.}] - \mathrm{tr}[\hat{U}\bar{A}_R\hat{U}^{-1}\bar{A}_L d\hat{U}d\hat{U}^{-1} - \text{p.c.}] \\ & - i\ \mathrm{tr}[\bar{A}_R d\hat{U}^{-1}\hat{U}\bar{A}_R\hat{U}^{-1}\bar{A}_L\hat{U} - \text{p.c.}] + \frac{1}{2}\mathrm{tr}[(\bar{A}_R\hat{U}^{-1}\bar{A}_L\hat{U})^2]\,. \end{aligned} \tag{3.39}$$

Here, "p.c." represents the terms obtained by making the exchange $\bar{A}_L \leftrightarrow \bar{A}_R$ and $\hat{U} \leftrightarrow \hat{U}^{-1}$. As well-known, this yields the chiral anomaly of L-R form for a generic $\bar{A}_L$ and $\bar{A}_R$:

$$\delta S_{\rm WZW} = -\frac{d(R)}{24\pi^2}\int \mathrm{tr}\left[\alpha_L\, d\left(\bar{A}_L d\bar{A}_L - \frac{i}{2}\bar{A}_L^3\right) - \alpha_R\, d\left(\bar{A}_R d\bar{A}_R - \frac{i}{2}\bar{A}_R^3\right)\right]\,, \tag{3.40}$$

with $g_L = e^{i\alpha_L}$, $g_R = e^{i\alpha_R}$. Inserting $\hat{U} = e^{i\widetilde{\eta}}U$ and (3.10) into the WZW term gives

$$\begin{aligned} S_{\rm WZW} = {} & S^{(0)}_{\rm WZW} + i\frac{d(R)}{48\pi^2}\int (d\widetilde{\eta} + 2\chi)\ \mathrm{tr}\left[\bar{F}\left(U^{-1}DU + UDU^{-1}\right)\right] \\ & + \frac{d(R)}{48\pi^2}\int d\widetilde{\eta}\ \mathrm{tr}\left(6\bar{A}d\bar{A} - 4i\bar{A}^3\right) + \frac{d(R)}{48\pi^2}\int \chi\ \mathrm{tr}\left(8\bar{A}d\bar{A} - 6i\bar{A}^3\right)\,. \end{aligned} \tag{3.41}$$

Here,

$$S^{(0)}_{\rm WZW} = S_{\rm WZW}|_{\widetilde{\eta}=\chi=0}\,, \tag{3.42}$$

and

$$DU = dU - iA_L U + iUA_R\,, \quad DU^{-1} = dU^{-1} - iA_R U^{-1} + iU^{-1}A_L\,,$$

with $A_{L,R} = \bar{A} + A_B I_{N_f}$ being the background gauge fields for $\mathrm{SU}(N_f)_L$ and $\mathrm{SU}(N_f)_R$, respectively. It is found that the $\mathrm{U}(1)_A$ gauge transformation

$$\delta_A\widetilde{\eta} = -2\alpha_A\,, \quad \delta_A\chi = d\alpha_A\,, \tag{3.43}$$

with $(\alpha_R - \alpha_L)/2 = \alpha_A I_{N_f}$ leads to

$$\delta_A S_{\rm WZW} = -\frac{d(R)}{12\pi^2}\int \alpha_A \,{\rm tr}\left(\bar{F}^2 - \frac{i}{2}\bar{F}\bar{A}^2\right) \tag{3.44}$$

The local counter term $\mathcal{Y}_f$ given in (2.12) is written only by the background gauge fields so that it remains to be a local counter term in the IR as well. With this term added, the variation of the WZW action is changed as

$$\begin{aligned}\delta_A(S_{\rm WZW} + \mathcal{Y}_f) &= \frac{d(R)}{4\pi^2}\int \alpha_A \,{\rm tr}(\bar{F}\wedge\bar{F}) \\ &= \frac{d(R)}{4\pi^2}\int \alpha_A \left[{\rm tr}(\hat{F} - B_V)^2 + N_f\left(\hat{F}_B - nB - B_V\right)^2\right],\end{aligned} \tag{3.45}$$

reproducing the second and the third term of the BCF anomaly in (2.42) upon setting $\alpha_A = 2\pi n_A/4N_f T(R)$.

We note that, in the presence of the background gauge fields B and B_V, $S_{\rm WZW}$ suffers an operator-valued ambiguity of how to extend it to a five-dimensional local action. To see this, suppose that $S_{\rm WZW}$ is defined on five-manifolds X_1 and X_2 with a common boundary equal to the four-manifold where the QCD-like theory is defined. The difference of the actions is obtained by evaluating the action on the compact manifold $X_1 \cup \bar{X}_2$:

$$\begin{aligned}&-\frac{d(R)}{8\pi^2}\int_{X_1\cup\bar{X}_2} d\widetilde{\eta}\wedge\left({\rm tr}\,\bar{F}^2 + \frac{i}{6}\,d\,{\rm tr}\Big[\bar{F}\left(U^{-1}DU + UDU^{-1}\right)\Big]\right) \\ &\qquad = -\frac{d(R)}{8\pi^2}\int_{X_1\cup\bar{X}_2} d\widetilde{\eta}\wedge{\rm tr}\,\bar{F}^2 \notin 2\pi\mathbb{Z}\,.\end{aligned} \tag{3.46}$$

The integral of $d\,{\rm tr}\Big[\bar{F}\left(U^{-1}DU + UDU^{-1}\right)\Big]$ is trivial because ${\rm tr}\Big[\bar{F}\left(U^{-1}DU + UDU^{-1}\right)\Big]$ is gauge invariant. As will be shown, this ambiguity is eliminated automatically by adding an interaction term between the η' meson and the background B in such a way that the first term of the BCF anomaly (2.42) can be reproduced in the low energy effective action.

The WZW term for R real is derived in the appendix A. It exhibits the same operator-valued ambiguity (3.46) as in the case for R complex. This is easy to verify by inserting (3.19) and (3.10) into the WZW term (A.13). This ambiguity can be cancelled as well by requiring the anomaly matching of the full BCF anomaly.

Now that the WZW term is found to reproduce the second and the third terms of the BCF anomaly given in (2.42), we discuss how the first term of the BCF anomaly is matched in terms of the effective action of the pions. We argue that the anomaly matching is achieved by adding to the effective action[1]

$$S_{\eta BB} = -\frac{2N_f T(R)}{2\pi}(1+\ell N_c)\int \widetilde{\eta}\left(dC_3 - \frac{N_c}{4\pi}B\wedge B\right). \tag{3.47}$$

Here, ℓ is an integer that is not fixed from the discrete anomaly matching condition we are employing. C_3 is the background three-form gauge potential, which is defined in (2.43). To see why, we first note that the discrete axial symmetry $\mathbb{Z}_{2N_f T(R)} = \mathbb{Z}_{4N_f T(R)}/\mathbb{Z}_2$ acts as

$$\mathbb{Z}_{2N_f T(R)}:\ \widetilde{\eta}\to\widetilde{\eta} - \frac{2\pi}{2N_f T(R)}\,. \tag{3.48}$$

[1]The local counterterm composed of η' and $B\wedge B$ is considered also in [56].

This shift reproduces the mixed 't Hooft anomaly between the $\mathbb{Z}_{2N_fT(R)}$ and the one-form $\mathbb{Z}_{N_c}$ symmetry modulo $2\pi\mathbb{Z}$, because C_3 gives no contribution to the 't Hooft anomaly thanks to the normalization condition (2.45). Furthermore, $S_{\eta BB}$ is left invariant under the one-form $\mathbb{Z}_{N_c}$ gauge transformation (2.47) together with the GS-type transformation law (2.46). Hereafter, we set $\ell = 0$ for simplicity.

We remark that instead of (3.47), the mixed 't Hooft anomaly may be obtained from an interaction term between $\widetilde{\eta}$ and the gluon,

$$S_{\eta ff} = -\frac{2N_fT(R)}{8\pi^2}\int \widetilde{\eta}\,\mathrm{tr}((\hat{f}-B)\wedge(\hat{f}-B))\,. \tag{3.49}$$

This term is studied in [57, 58] to derive the Witten-Veneziano formula for the η' mass. In this scenario, the background gauge field C_3 plays no role in achieving the anomaly matching condition. Instead, it is assumed that the gluon fields are confined in the low energy regime. C_3 is interpreted as the background gauge field of an emergent two-form symmetry whose charged object is an η' vortex. To see this, we note that η' can have a winding number,

$$\int_{\mathcal{C}} d\widetilde{\eta} \in 2\pi\mathbb{Z}\,, \tag{3.50}$$

since it may be understood as a (pseudo-)NG boson of the $\mathrm{U}(1)_A$ symmetry. The integrand $d\widetilde{\eta}$ obeys the analog of the Bianchi identity,

$$dd\widetilde{\eta} = 0\,. \tag{3.51}$$

Therefore, the integral (3.50) is topological under a small deformation $\mathcal{C} \to \mathcal{C} \cup \partial\mathcal{S}_0$,

$$\int_{\mathcal{C}\cup\partial\mathcal{S}_0} d\widetilde{\eta} - \int_{\mathcal{C}} d\widetilde{\eta} = \int_{\partial\mathcal{S}_0} d\widetilde{\eta} = \int_{\mathcal{S}_0} dd\widetilde{\eta} = 0\,, \tag{3.52}$$

where $\mathcal{S}_0$ is a surface with a boundary. We thus have a unitary topological object parameterized by $e^{i\gamma} \in \mathrm{U}(1)$,

$$U_2(\mathcal{C}, e^{i\gamma}) = e^{\frac{i\gamma}{2\pi}\int_{\mathcal{C}} d\widetilde{\eta}}\,. \tag{3.53}$$

The topological object is regarded as a symmetry generator for a U(1) two-form symmetry, because the generator acts on the vortex string operator $V_\eta(\mathcal{S})$ with the worldsheet $\mathcal{S}$ as

$$U_2(\mathcal{C}, e^{i\beta})V_\eta(\mathcal{S}) = e^{i\gamma\,\mathrm{Link}(\mathcal{S},\mathcal{C})}V_\eta(\mathcal{S})\,, \tag{3.54}$$

where $\mathrm{Link}(\mathcal{S},\mathcal{C})$ denotes the linking number between $\mathcal{S}$ and $\mathcal{C}$. The conserved current for the two-form symmetry is identified with $\frac{1}{2\pi}d\widetilde{\eta}$ and couples minimally to C_3.

3.3 Cancellation of the operator-valued ambiguity

$S_{\eta BB}$ is written in the form of the integral of a local functional on a five-manifold with the boundary

$$S_{\eta BB} = -\frac{2N_fT(R)}{2\pi}\int_{X_1} d\widetilde{\eta}\wedge\left(dC_3 - \frac{N_c}{4\pi}B\wedge B\right)\,. \tag{3.55}$$

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This depends on how the action is extended to five dimensions because

$$-2N_f T(R) \int_{X_1 \cup \bar{X}_2} \frac{N_c}{8\pi^2}\, d\widetilde{\eta} \wedge B \wedge B \,\notin\, 2\pi\mathbb{Z}\,. \tag{3.56}$$

This is the manifestation of another operator-valued ambiguity in addition to that from the WZW term.

The net operator-valued ambiguity associated with the η' meson reads

$$\int_{X_1 \cup \bar{X}_2} d\widetilde{\eta} \wedge \left[-2N_f T(R)\, \frac{N_c}{8\pi^2} B \wedge B + \frac{d(R)}{8\pi^2}\ \mathrm{tr}\, \bar{F}^2 \right]. \tag{3.57}$$

Cancellation of the operator-valued ambiguity requires that this take values in $2\pi\mathbb{Z}$. As $\widetilde{\eta}$ is a 2π-periodic boson, this condition is stated in terms of the background gauge fields only

$$\int_X \left[-2N_f T(R)\, \frac{N_c}{8\pi^2} B \wedge B + \frac{d(R)}{8\pi^2}\ \mathrm{tr}\, \bar{F}^2 \right] \in \mathbb{Z}\,. \tag{3.58}$$

X is any compact spin four-manifold. This integral is always integer because this is equal to the index of the Dirac operator (mod $\mathbb{Z}$) whose gauge field is $\widetilde{A}$ satisfying the proper normalization.

This can be verified by taking $X = T^2 \times T^2$ and turning on the 't Hooft fluxes on it, whose explicit form is given in [24]. Let (x^1, x^2) and (x^3, x^4) be the coordinates of the two two-tori with $x^{1,2,3,4} \sim x^{1,2,3,4} + 2\pi$. Then,

$$B = \frac{1}{2\pi N_c} \left(m_{12}\, dx^1 \wedge dx^2 + m_{34}\, dx^3 \wedge dx^4 \right). \quad (m_{12}, m_{34} \in \mathbb{Z} \text{ mod } N_c) \tag{3.59}$$

$$\hat{F} = \frac{1}{2\pi} \left(m^f_{12}\, dx^1 \wedge dx^2 + m^f_{34}\, dx^3 \wedge dx^4 \right) \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix},$$

$$B_V = \frac{1}{2\pi N_f} \left(m^f_{12}\, dx^1 \wedge dx^2 + m^f_{34}\, dx^3 \wedge dx^4 \right). \quad (m^f_{12}, m^f_{34} \in \mathbb{Z} \text{ mod } N_f) \tag{3.60}$$

$$F_B = \frac{1}{2\pi} \left(m^B_{12}\, dx^1 \wedge dx^2 + m^B_{34}\, dx^3 \wedge dx^4 \right). \quad (m^B_{12}, m^B_{34} \in \mathbb{Z}) \tag{3.61}$$

As a consistency check, we note $\hat{F} - B_V \in \mathfrak{su}(N_f)$. It is found that, modulo $\mathbb{Z}$

$$\begin{aligned} &\int_{T^2 \times T^2} \left[-2N_f T(R)\, \frac{N_c}{8\pi^2} B \wedge B + \frac{d(R)}{8\pi^2}\ \mathrm{tr}\, \bar{F}^2 \right] \\ &= -2N_f T(R) \frac{m_{12} m_{34}}{N_c} \\ &\quad + \frac{d(R)}{N_c} \left[n(m^f_{12} m_{34} + m^f_{34} m_{12}) - n N_f (m^B_{12} m_{34} + m^B_{34} m_{12}) + \frac{n^2 N_f}{N_c} m_{12} m_{34} \right]. \end{aligned} \tag{3.62}$$

It is easy to see that this is always an integer.

3.4 Higher-group structure

It is found that the low energy effective action of the QCD-like theory is given by

$$S_{\mathrm{LEET}} = \int d^4x \left[-\frac{f_\pi^2}{4} \operatorname{tr} \left(\hat{U}^\dagger D_\mu \hat{U} \right)^2 - V(\theta + 2N_f T(R) \widetilde{\eta}) \right] + S_{\mathrm{WZW}} + \mathcal{Y}_f + S_{\eta BB} \,. \quad (3.63)$$

In this subsection, we specify the higher-group structure of the background gauge fields that is encoded in (3.63) and show that this is identified with a semistrict three-group (two-crossed module).

A semistrict three-group is defined by the following ingredients [59–62].

1. A triple of groups, (G, H, L).
2. Maps between groups, $\partial_1 : H \to G$, and $\partial_2 : L \to H$. They are homomorphism with respect to the group composition, $\partial_1(h_1 h_2) = (\partial_1 h_1)(\partial_1 h_2)$, and $\partial_1(l_1 l_2) = (\partial_1 l_1)(\partial_1 l_2)$ for $h_1, h_2 \in H$ and $l_1, l_2 \in L$, respectively.
3. Action $\triangleright$ of G on G, H, L: $g \triangleright g' := g g' g^{-1} \in G$, $g \triangleright h \in H$, and $g \triangleright l \in L$. This operation is a generalization of the adjoint transformation.
4. Peiffer lifting, $\{h_1, h_2\} \in L$ for $h_1, h_2 \in H$.
5. Consistency between the above operations, e.g., $g \triangleright \{h_1, h_2\} = \{g \triangleright h_1, g \triangleright h_2\}$.

For more detail, see e.g., [39].

From the above data, we can construct a gauge theory for the three-group, which consists of zero-, one-, and two-form gauge field which are valued on the Lie algebra of G, H, and L, respectively. For the Lie groups G, H, L, the Lie algebra of the three-group can be introduced as follows.

1. A triple of Lie algebra $(\mathfrak{g}, \mathfrak{h}, \mathfrak{l})$ of the triple of Lie groups (G, H, L).
2. Maps between groups, $\partial_1 : \mathfrak{h} \to \mathfrak{g}$, and $\partial_2 : \mathfrak{l} \to \mathfrak{h}$. They are homomorphism with respect to the Lie bracket, $\partial_1([\underline{h}_1, \underline{h}_2]) = [\partial_1 \underline{h}_1, \partial_1 \underline{h}_2]$, and $\partial_2([\underline{l}_1, \underline{l}_2]) = [\partial_1 \underline{l}_1, \partial_1 \underline{l}_2]$ for $\underline{h}_1, \underline{h}_2 \in \mathfrak{h}$ and $\underline{l}_1, \underline{l}_2 \in \mathfrak{l}$, respectively. They are boundary maps satisfying $\partial_1 \circ \partial_2 \, l = 1 \in G$ for all $l \in L$.
3. Action $\triangleright$ of $\mathfrak{g}$ on $\mathfrak{g}, \mathfrak{h}, \mathfrak{l}$: $\underline{g} \triangleright \underline{g}' := [\underline{g}, \underline{g}'] \in \mathfrak{g}$, $\underline{g} \triangleright \underline{h} \in \mathfrak{h}$, and $\underline{g} \triangleright \underline{l} \in \mathfrak{l}$. This operation is a generalization of the adjoint transformation.
4. Peiffer lifting, $\{\underline{h}_1, \underline{h}_2\} \in \mathfrak{l}$ for $\underline{h}_1, \underline{h}_2 \in \mathfrak{h}$.
5. Consistency between the above operations, e.g., $\underline{g} \triangleright \{\underline{h}_1, \underline{h}_2\} = \{\underline{g} \triangleright \underline{h}_1, \underline{g} \triangleright \underline{h}_2\}$.

We then introduce the gauge fields for the three-group, which consist of $\mathfrak{g}$-, $\mathfrak{h}$-, and $\mathfrak{l}$-valued one-, two-, and three-form fields $A = A^\alpha u_\alpha$, $B = B^a v_a$, and $C = C^A w_A$, satisfying the

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following gauge transformation laws,[2]

$$
\begin{aligned}
&A \to A' = g \triangleright A + g d g^{-1} + \partial_1 \underline{h}, \\
&B \to B' = g \triangleright B + d\underline{h} - \underline{h} \wedge \underline{h} + A' \triangleright \underline{h} + \partial_2 \underline{l}, \\
&C \to C' = g \triangleright C + d\underline{l} + A' \triangleright \underline{l} + \{\partial_2 \underline{l}, \underline{h}\} - \{B', \underline{h}\} - \{\underline{h}, g^{-1} \triangleright B\},
\end{aligned} \tag{3.64}
$$

and field strengths,

$$
\begin{aligned}
F &= dA + A \wedge A, \\
H &= dB + A \triangleright B, \\
G &= dC + A \triangleright C + \{B, B\}.
\end{aligned} \tag{3.65}
$$

Here, g, $\underline{h} = \underline{h}^a v_a$, and $\underline{l} = \underline{l}^A w_A$ are G-, $\mathfrak{h}$-, and $\mathfrak{l}$-valued zero-, one-, and two-form parameters. We have taken the bases of $\mathfrak{g}$, $\mathfrak{h}$, and $\mathfrak{l}$ as $\{u_\alpha\}$, $\{v_a\}$, and $\{w_A\}$, respectively. The operations ∂_1, ∂_2, $\triangleright$, and $\{\}$ on the differential forms are given by the wedge products with the operations for the Lie algebra:

$$g \triangleright A = A^\alpha g \triangleright u_\alpha = A^\alpha g u_\alpha g^{-1}, \tag{3.66}$$

$$\partial_1 \underline{h} = \underline{h}^a (\partial_1 v_a), \tag{3.67}$$

$$\underline{h} \wedge \underline{h} = \frac{1}{2} \underline{h}^a \wedge \underline{h}^b [v_a, v_b], \tag{3.68}$$

$$\{\partial_2 \underline{l}, \underline{h}\} = \underline{l}^A \wedge \underline{h}^a \{(\partial_2 w_A), v_a\}, \tag{3.69}$$

and so on. It is possible to take the following conditions,

$$F - \partial_1 B = 0, \quad H - \partial_2 C = 0, \tag{3.70}$$

which are called fake curvature vanishing conditions.

Conversely, we can specify the three-group structure of the background gauge fields in an appropriate class of a gauge theory with given one-, two-, and three-form gauge fields. We first regard the group $\mathrm{SU}(N_c)$ as a global symmetry group. The background gauge fields $\hat{a}$, $\hat{A}$ and $\hat{A}_B$, B and B_V, and C_3 are identified with those for the gauge groups

$$G_0 = \mathrm{U}(N_c) \times \mathrm{U}(N_f)_V \times \mathrm{U}(1)_B \cong \frac{\mathrm{SU}(N_c) \times \mathrm{U}(1)}{\mathbb{Z}_{N_c}} \times \frac{\mathrm{SU}(N_f) \times \mathrm{U}(1)}{\mathbb{Z}_{N_f}} \times \mathrm{U}(1)_B, \tag{3.71}$$

$$H = \mathbb{Z}_{N_c} \times \mathbb{Z}_{N_f}, \tag{3.72}$$

$$L = \mathrm{U}(1). \tag{3.73}$$

respectively. For convenience, we write the elements of G_0 as

$$(g, e^{i\alpha_c}, g_V, e^{i\alpha_f}, e^{i\alpha_B}) \in \mathrm{SU}(N_c) \times \mathrm{U}(1) \times \mathrm{SU}(N_f) \times \mathrm{U}(1) \times \mathrm{U}(1)_B, \tag{3.74}$$

with the identifications by $\mathbb{Z}_{N_c}$ and $\mathbb{Z}_{N_f}$,

$$(I_{N_c}, e^{2\pi i m_c/N_c}, I_{N_f}, 1, 1) \sim (e^{2\pi i m_c/N_c} I_{N_c}, 1, I_{N_f}, 1, 1), \tag{3.75}$$

$$(I_{N_c}, 1, I_{N_f}, e^{2\pi i m_f/N_f}, 1) \sim (I_{N_c}, 1, e^{2\pi i m_f/N_f} I_{N_f}, 1, 1). \tag{3.76}$$

[2] In this subsection, the gauge fields are taken to be anti-Hermitian.

We determine the structure of the three-group from the gauge transformation laws and field strengths. Identification of the gauge transformation laws in (2.26), (2.33) and (2.35) with (3.64) shows that the map $\partial_1 : H \to G_0$ should read

$$\begin{aligned}&\partial_1(e^{2\pi i m_c/N_c}, e^{2\pi i m_f/N_f})\\&\quad= (I_{N_c}, e^{2\pi i m_c/N_c}, I_{N_f}, e^{2\pi i m_f/N_f}, e^{-2\pi i n m_c/N_c} e^{-2\pi i m_f/N_f}).\end{aligned} \tag{3.77}$$

Meanwhile, the map $\partial_2 : L \to H$ is trivial,

$$\partial_2(e^{i\gamma}) = (1, 1) \in \mathbb{Z}_{N_c} \times \mathbb{Z}_{N_f}, \tag{3.78}$$

for $e^{i\gamma} \in L$. This is because the two-form gauge fields B and B_V are not transformed under the gauge transformation in (2.44). We remark that the constraint $\mathrm{tr}(\hat{f}) = dC = N_c B$ can be understood as the fake curvature vanishing condition imposed on the U(1) sector of $\mathrm{U}(N_c)$, i.e., $0 = \mathrm{tr}(\hat{f}) - \mathrm{tr}(\partial_1 B) = \mathrm{tr}(\hat{f}) - \mathrm{tr}(B I_{N_c})$. Identification of the four-form field strength G_4 (2.48) with G in (3.65) leads to the Peiffer lifting,

$$\{(e^{2\pi i m_c/N_c}, e^{2\pi i m_f/N_f}), (e^{2\pi i m'_c/N_c}, e^{2\pi i m_f/N_f})\} = e^{-i\frac{N_c}{4\pi}\cdot\frac{2\pi m_c}{N_c}\cdot\frac{2\pi m'_c}{N_c}} = e^{-i\pi m_c m'_c/N_c}, \tag{3.79}$$

where $(e^{2\pi i m_c/N_c}, e^{2\pi i m_f/N_f}) \in \mathbb{Z}_{N_c} \times \mathbb{Z}_{N_f}$.[3]

Now we promote the $\mathrm{SU}(N_c)$ gauge field to dynamical degrees of freedom. $\mathrm{SU}(N_c) \subset G_0$ should be treated as a gauge redundancy, i.e., the elements of $\mathrm{SU}(N_c)$ in G_0 are regarded as the identity. In particular, the gauging of $\mathrm{SU}(N_c)$ gives rise to further identifications in (3.75) and (3.77):

$$(I_{N_c}, e^{2\pi i m_c/N_c}, I_{N_f}, 1, 1) \sim (e^{2\pi i m_c/N_c} I_{N_c}, 1, I_{N_f}, 1, 1) \sim (I_{N_c}, 1, I_{N_f}, 1, 1), \tag{3.80}$$

and

$$\begin{aligned}&\partial_1(e^{2\pi i m_c/N_c}, e^{2\pi i m_f/N_f})\\&\quad= (I_{N_c}, e^{2\pi i m_c/N_c}, I_{N_f}, e^{2\pi i m_f/N_f}, e^{-2\pi i n m_c/N_c} e^{-2\pi i m_f/N_f})\\&\quad\sim (I_{N_c}, 1, I_{N_f}, e^{2\pi i m_f/N_f}, e^{-2\pi i n m_c/N_c} e^{-2\pi i m_f/N_f}),\end{aligned} \tag{3.81}$$

respectively. We find that the kernel of ∂_1 is given by

$$\mathrm{Ker}\,\partial_1 = \{(e^{2\pi i m_c/N_c}, 1) \in \mathbb{Z}_{N_c} \times \mathbb{Z}_{N_f} | e^{2\pi i m_c/N_c} \in \mathbb{Z}_n\} \simeq \mathbb{Z}_{\mathrm{gcd}(N_c,n)}\,, \tag{3.82}$$

ending up with a one-form symmetry group $\mathbb{Z}_{\mathrm{gcd}(N_c,n)}$. This is identical to the subgroup of $\mathbb{Z}_{N_c}$ that acts nontrivially on Wilson loop operators in QCD-like theories [24].

[3] This definition of the Peiffer lifting has an ambiguity under $m_c \to m_c + N_c$. In order to define the Peiffer lifting in an unambiguous way, we may need to include the spin structure of the spacetime. We leave this issue as future work.

3.5 Integrating out the η' meson

We now focus on the low energy scales where the η' meson is heavy enough to be integrated out. Then, the potential (3.29) gives rise to the spontaneous breaking of the discrete axial symmetry $\mathbb{Z}_{2N_fT(R)}$. The low energy effective action for η' is given by replacing the kinetic term of $\widetilde{\eta}$ with a BF action associated with the spontaneous $\mathbb{Z}_{2N_fT(R)}$ breaking [12]:

$$\frac{2N_fT(R)}{2\pi}\int(d\widetilde{\eta}+2\chi)\wedge c_3\,, \tag{3.83}$$

where c_3 is a dynamical three-form gauge field with the normalization condition given by

$$\int dc_3\in 2\pi\mathbb{Z}\,. \tag{3.84}$$

The resultant low energy effective action is given by

$$S=-\frac{f_\pi^2}{4}\int d^4x\ \mathrm{tr}\left(U^\dagger D_\mu U\right)^2+S_{\mathrm{WZW}}+\mathcal{Y}_f+S_{\mathrm{BF}}\,. \tag{3.85}$$

Here,

$$\begin{aligned}S_{\mathrm{BF}}&=\frac{2N_fT(R)}{2\pi}\int(d\widetilde{\eta}+2\chi)\wedge c_3+S_{\eta BB}\\&=\frac{2N_fT(R)}{2\pi}\int(d\widetilde{\eta}+2\chi)\wedge\left(c_3+C_3-\frac{1}{4\pi}B\wedge C\right)-\mathcal{Y}_B\,.\end{aligned} \tag{3.86}$$

It is useful to shift the dynamical gauge field as $c_3\to c_3-C_3$, which keeps the normalization condition unchanged. Then, the resultant c_3 makes a GS transformation under the one-form $\mathbb{Z}_{N_c}$ gauge symmetry transformation

$$c_3\to c_3+\frac{N_c}{4\pi}\left(2\Lambda\wedge B+\Lambda\wedge d\Lambda\right)\,. \tag{3.87}$$

Summarizing the terms that involve the η' meson, we find

$$\begin{aligned}&S_{\mathrm{WZW}}+\mathcal{Y}_f+S_{\mathrm{BF}}\\&=-\frac{2N_fT(R)}{2\pi}\int(d\widetilde{\eta}+2\chi)\Bigg[dc_3-\frac{N_c}{4\pi}B\wedge B\\&\qquad\qquad+\frac{d(R)}{8\pi N_fT(R)}\left\{\mathrm{tr}(\bar{F}^2)+\frac{i}{6}d\,\mathrm{tr}\left[\bar{F}\wedge\left(U^{-1}DU+UDU^{-1}\right)\right]\right\}\Bigg]\\&\quad+\frac{2N_fT(R)}{\pi}\int\chi\wedge\left[dC_3-\frac{N_c}{4\pi}B\wedge B+\frac{d(R)}{8\pi N_fT(R)}\mathrm{tr}\,\bar{F}^2\right]+S^{(0)}_{\mathrm{WZW}}\,.\end{aligned} \tag{3.88}$$

This reproduces the BCF anomaly evidently. It is interesting to note that part of this action is equivalent to the effective action of an axion that is constructed in [25].

4 Discussions

In this paper, we have derived the low energy effective theories of the QCD-like theories by requiring the 't Hooft anomaly matching condition for the BCF anomaly. This result is based on the assumption that the QCD-like theories exhibit spontaneous chiral symmetry breaking due to the quark bilinear condensate. Use of the BCF anomaly matching seems not to be powerful enough to specify uniquely the phase structures for a given R and N_f. However, it might give us an important clue to a classification of the phase structures by applying the methods employed in this paper. As an example, we assume instead that the QCD-like theories gives rise to an exotic chiral symmetry breaking due to the condensate of a gauge invariant operator other than the quark bilinear.[4] It would be interesting to examine if we can find the low energy effective action of the Nambu-Goldstone bosons such that it satisfies the BCF anomaly matching condition and furthermore any type of operator-valued ambiguity disappears.

As pointed out before, the effective action (3.88) is reminiscent of the axion effective action derived in [25]. It would be nice to clarify the relation of the two actions in more detail. For this purpose, we may couple the QCD-like theory with a neutral and complex Higgs field as performed in [25] with a difference being that the Higgs coupling is tuned so that the Higgs vev gives a light quark mass. The light mass is proportional to an axion phase and generates a mass term of the pions. It is expected that integrating out the massive pions leads to the axion effective action.

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A Low energy effective action for R real

We derive the low energy effective action of the QCD-like theory for R real in an assumption that it exhibits the quark bilinear condensate.

A.1 Chiral Lagrangian

This chiral Lagrangian is constructed by using the Callan-Coleman-Wess-Zumino(CCWZ) procedure [51]. We first divide the generator of $\mathrm{SU}(2N_f)$ into the unbroken and broken ones respectively.

$$\begin{aligned} &H^a \in \mathfrak{so}(2N_f)\,, \;\; (a = 1, 2, \cdots \dim \mathrm{SO}(2N_f)) \\ &X^\alpha \in \mathfrak{u}(2N_f) - \mathfrak{so}(2N_f)\,, \end{aligned} \tag{A.1}$$

with the normalization condition given by

$$\mathrm{tr}(H^a H^b) = \frac{1}{2}\delta^{ab}\,, \quad \mathrm{tr}(X^\alpha X^\beta) = \frac{1}{2}\delta^{\alpha\beta}\,. \tag{A.2}$$

[4] Such a scenario has been discussed recently in [27, 28, 63] for instance.

Since the coset $\mathrm{U}(2N_f)/\mathrm{SO}(2N_f)$ is a symmetric space, these obey the commutation relations of the form

$$[H^a, H^b] = if^{abc}H^c\,, \quad [H^a, X^\alpha] = (t^a)^{\alpha\beta}X^\beta\,, \quad [X^\alpha, X^\beta] = if^{\alpha\beta c}H^c\,. \tag{A.3}$$

Here, t^a is the generators of $\mathrm{SO}(2N_f)$ in the $\square\square$ representation. Consider an element of the coset $\mathrm{U}(2N_f)/\mathrm{SO}(2N_f)$

$$\xi(\pi) = e^{i\pi^\alpha X^\alpha/f_\pi}\,. \tag{A.4}$$

The left action of $\mathrm{U}(2N_f)$ on ξ leads to

$$\mathrm{U}(2N_f):\ \xi(\pi) \to g\,\xi(\pi) = \xi(\pi')h(\pi, g)\,, \quad {}^\exists h \in H\,. \tag{A.5}$$

This defines the transformation law of the pion field π to π'. The exponentiated pion field $\hat{U}(\pi)$ is given by

$$\hat{U} = \xi\xi^T\,. \tag{A.6}$$

It is easy to verify that

$$\hat{U}(\pi') = g\,\hat{U}(\pi)g^T\,, \tag{A.7}$$

and furthermore $\mathrm{U}(\pi)$ is independent of the choice of the representatives of ξ.

A.2 WZW term

The WZW term for R real is obtained as follows. We start with the WZW term for R complex with N_f flavors. Replace $N_f \to 2N_f$ yields the NG boson field

$$\hat{U} \in \mathrm{U}(2N_f)\,. \tag{A.8}$$

We impose the condition

$$\hat{U} = \hat{U}^T\,, \tag{A.9}$$

which is consistent with the CCWZ construction as reviewed before. Let $\bar{\mathcal{A}}_L$ and $\bar{\mathcal{A}}_R$ be the background gauge field for $\mathrm{U}(2N_f)_L$ and $\mathrm{U}(2N_f)_R$, respectively. Then $\bar{\mathcal{A}}$, that for the chiral flavor symmetry $\mathrm{U}(2N_f)$, is obtained by setting

$$\bar{\mathcal{A}}_L = \bar{\mathcal{A}}\,, \quad \bar{\mathcal{A}}_R = -\bar{\mathcal{A}}^T\,. \tag{A.10}$$

To see this, we note that with this embedding, the covariant derivative for R complex (3.9) becomes that for R real (3.18). The consistency condition of (A.10) with the gauge transformation of $\bar{A}_{L,R}$

$$\delta\bar{A}_L = d\alpha_L + i\alpha_L\bar{A}_L - i\bar{A}_L\alpha_L\,, \quad \delta\bar{A}_R = d\alpha_R + i\alpha_R\bar{A}_R - i\bar{A}_R\alpha_R\,,$$

demands

$$\alpha_L = \alpha\,, \quad \alpha_R = -\alpha^T\,,$$

with α identified with the infinitesimal gauge transformation parameter for the chiral $\mathrm{U}(2N_f)$ symmetry. Then, the WZW term for R real is given by

$$S^{R=\text{real}}_{\text{WZW}} = \frac{1}{2}\,S^{R=\text{cpx}}_{\text{WZW}}\Big|_{N_f\to 2N_f,\ (A.9),\ (A.10)}\,.$$

This is because the chiral anomaly of L-R form (3.40) is evaluated as

$$\delta S_{\rm WZW}^{R=\rm cpx}\Big|_{N_f\to 2N_f,\ (A.9),\ (A.10)} = -\frac{2\,d(R)}{24\pi^2}\int {\rm tr}\left[\alpha\, d\left(\bar{\mathcal{A}}d\bar{\mathcal{A}}-\frac{i}{2}\bar{\mathcal{A}}^3\right)\right].$$

Here, tr is taken for the fundamental representation of $\mathfrak{u}(2N_f)$.

It is easy to verify that

$$S_{\rm WZW}^{R=\rm real} = -i\frac{d(R)}{48\pi^2}\int Z^{\rm r} - i\frac{d(R)}{480\pi^2}\int {\rm tr}(\hat{U}d\hat{U}^{-1})^5\,, \tag{A.11}$$

where

$$\begin{aligned}
Z^{\rm r} = {}& i\ {\rm tr}\left[(\bar{\mathcal{A}}d\bar{\mathcal{A}}+d\bar{\mathcal{A}}\bar{\mathcal{A}}-i\bar{\mathcal{A}}^3)(\hat{U}^{-1}\bar{\mathcal{A}}^T\hat{U}-i\hat{U}d\hat{U}^{-1})\right]\\
&-\frac{1}{2}\ {\rm tr}\left[d\bar{\mathcal{A}}\left(\hat{U}\bar{\mathcal{A}}^Td\hat{U}^{-1}+d\hat{U}\bar{\mathcal{A}}^T\hat{U}^{-1}\right)\right]+i\ {\rm tr}\left[\bar{\mathcal{A}}(d\hat{U}\hat{U}^{-1})^3\right]\\
&+\frac{1}{2}\ {\rm tr}\left(\bar{\mathcal{A}}\,d\hat{U}\hat{U}^{-1}\right)^2+\frac{1}{2}\ {\rm tr}\left[\left(\bar{\mathcal{A}}^T\hat{U}^{-1}\bar{\mathcal{A}}\hat{U}-\hat{U}^{-1}\bar{\mathcal{A}}\hat{U}\bar{\mathcal{A}}^T\right)d\hat{U}^{-1}d\hat{U}\right]\\
&-i\ {\rm tr}\left(\bar{\mathcal{A}}d\hat{U}\hat{U}^{-1}\bar{\mathcal{A}}\hat{U}\bar{\mathcal{A}}^T\hat{U}^{-1}\right)+\frac{1}{4}\ {\rm tr}\left(\bar{\mathcal{A}}^T\hat{U}^{-1}\bar{\mathcal{A}}\hat{U}\right)^2.
\end{aligned} \tag{A.12}$$

Factorizing the η' meson as

$$\hat{U}=e^{i\widetilde{\eta}}U\,,\quad U\in {\rm SU}(2N_f)\,,$$

and extracting from $\bar{\mathcal{A}}$ the $\mathbb{Z}_{4N_fT(R)}\subset {\rm U}(1)_A$ gauge field as

$$\bar{\mathcal{A}}=\mathcal{A}-\chi I_{2N_f}\,,\quad \mathcal{A}\in\mathfrak{su}(2N_f)\,,$$

a bit lengthy computation shows that

$$Z^{\rm r}=Z^{\rm r}\big|_{\chi=\widetilde{\eta}=0}-\int(d\widetilde{\eta}+2\chi)\,{\rm tr}\left(\mathcal{F}UDU^{-1}\right)+i\int d\widetilde{\eta}\left(3\mathcal{A}d\mathcal{A}-2i\mathcal{A}^3\right)+i\int\chi\left(4\mathcal{A}d\mathcal{A}-3i\mathcal{A}^3\right). \tag{A.13}$$

Here, $\mathcal{F}=d\mathcal{A}-i\mathcal{A}^2$ and

$$DU^{-1}=dU^{-1}+iU^{-1}\mathcal{A}+i\mathcal{A}^TU^{-1}\,,$$

is the covariant derivative for the $\mathrm{SU}(2N_f)$ gauge group.

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