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Radiation-reaction in the Effective Field Theory approach to Post-Minkowskian dynamics

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Abstract: We extend the Post-Minkowskian (PM) effective field theory (EFT) approach to incorporate conservative and dissipative radiation-reaction effects in a unified framework. This is achieved by implementing the Schwinger-Keldysh "in-in" formalism and separating conservative and non-conservative terms according to the formulation in [1], which we show promotes Feynman's $i0$-prescription and *cutting* rules to a prominent role at the classical level. The resulting integrals, involving both Feynman and retarded propagators, can be bootstrapped to all orders in the velocity via differential equations with boundary conditions including potential and radiation modes. As a paradigmatic example we provide an *ab initio* derivation of the classical solution to the scattering problem in general relativity to $\mathcal{O}(G^3)$. For the sake of completeness, we also reproduce the leading order radiation-reaction effects in classical electrodynamics.

Keywords: Classical Theories of Gravity, Effective Field Theories, Scattering Amplitudes

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1 Introduction

The successes of current gravitational wave (GW) detectors [2], together with the expected reach of future observatories [3–9], has reinvigorated various efforts to tackle the perturbative regime of the two-body dynamics in gravity stemming off of "traditional"[10–12] as well as those inspired by effective field theory (EFT) techniques [13–17]. In a concerted effort with various other theoretical frameworks, e.g. [18–32], EFT-based approaches — both in the Post-Newtonian (PN) [33–70] and Post-Minkowskian (PM) regimes [71–87]—in parallel with amplitudes-based methodologies, e.g. [88–116], have led to major breakthroughs in our understanding of the (classical) relativistic problem in general relativity. However, albeit with some notable exceptions, e.g. [49, 50, 68–70, 87], the vast majority of the EFT-rooted results have occurred in the conservative sector of PN/PM theory, culminating in the recent derivation of the state-of-the-art knowledge for potential [54, 55, 61, 78, 109] and radiation-reaction tail effects [56, 57, 62, 79, 110] to 5PN and 4PM order for non-spinning bodies, respectively. Following pioneering developments in the realm of PN expansions [1, 45–50], the purpose of this paper is to extend the EFT approach in order to incorporate conservative and dissipative effects in the PM regime within a unified framework.

In principle, radiative observables at infinity, such as the (source) radiated energy, can be computed by matching the stress-energy tensor using the trajectories in the conservative PM

EFT [74] and subsequently squaring the (on-shell) one-point function [85–87]. Mimicking the simplifications also found in PN theory [41, 67–70], this piecewise derivation—*one scale at a time*—thus becomes simpler than constructing the full gravitational integrand, entailing (for instance for the case of *instantaneous* interaction) only a single "radiation graviton" coupled to potential exchanges. This is even more striking if we resort to an adiabatic approach within a PN multipole expansion [41, 42, 44]. In this case so-called *moment relations* (from Ward identities) allow us to trade non-linearities for derivatives of the multipoles that can then be computed using the conservative potential, e.g. [67–70]. Instead, for reasons that will become clear momentarily, in this paper we develop a formalism which allows us to simultaneously derive both conservative and dissipative effects.

One of the major bottlenecks in computations in the PM regime has been the existence of intricate families of relativistic integrals, e.g. [78, 79, 109, 110]. Remarkably, following key ideas in particle physics [117–145], a very powerful tool has emerged as a weapon-of-choice to tackle multi-loop relativistic integration, namely the use of differential equations [75, 113]. One of its main virtues is that only through their boundary values the solution may be sensitive to whether potential and/or radiation modes are considered, allowing us to tackle the entire (classical) *soft* region at once. This was exploited in amplitude-based approaches to compute the relativistic impulse at 3PM order [105, 112, 114]. In these methodologies one relies on taking the classical limit of a quantum amplitude, similarly to the derivations in [99, 100, 109, 110]. However, as we demonstrate here, we can bypass these manipulations and directly compute the classical total relativistic impulse within a worldline EFT approach.

The use of differential equations has been already implemented in the EFT approach to solve the conservative dynamics to 4PM order, first using only the potential region for the boundary conditions [75, 78] and later on incorporating radiation-reaction effects [79]. Yet, in all cases the standard "in-out" *vaccum-to-vaccum* amplitude was utilised, together with Feynman's prescription for the propagators, which captures conservative-only contributions. However, the existence of such a powerful machinery in the form of differential equations — encoding in principle both conservative and radiative contributions to the dynamics — begs us for a more general framework where all dynamical effects may be incorporated in unison: *feeding two birds with one scone*. Fortunately, the Schwinger-Keldysh "in-in" formalism [146–150] is tailor-made for this purpose. Contrary to in-out boundary conditions, the in-in approach systematically accounts for the correct causal propagation, while at the same time the prescription in [1] provides us with a natural separation between conservative and dissipative terms at the level of the effective action (already implemented in [48, 53] to obtain unambiguous results at 4PN). As we shall see, the procedure promotes Feynman's $i0$-prescription and *cutting* rules to a prominent role also at the classical level.

The full integrand including dissipative effects turns out to be a tad more involved than matching for the stress-energy tensor; however, as we show here it is practically equivalent to building the in-out counterpart. Hence, the difference between conservative and/or dissipative terms ultimately arises through the choice of propagators. Nevertheless, as we shall see, the type of Green's functions does have an impact in the application of standard integration tools, notably the use of 'symmetry relations', which is essentially the part of the computation which requires a careful treatment. Yet, bootstrapping the solution

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through differential equations yields powerful constraints, in particular in the radiation sector, which allow us to read off much of the final answer without the need to compute all of the boundary constants [79]. As a paradigmatic example we explicitly derive the full solution of the scattering problem to 3PM order, reproducing previous results in the literature [75, 85, 99, 105, 112, 114]. For completeness, we also reproduce the leading radiation-reaction effects in classical electrodynamics [151, 152]. The derivation of the hitherto unknown total relativistic impulse at 4PM order is discussed in [153].

Throughout this paper we use the following conventions: we use the mostly minus signature: $\eta_{\mu\nu} = \mathrm{diag}(+,-,-,-)$. The Minkowski product between four-vectors is denoted as $k \cdot x = \eta_{\mu\nu} k^\mu x^\nu$, and we use $\boldsymbol{k} \cdot \boldsymbol{x} = \delta^{ij} \boldsymbol{k}^i \boldsymbol{x}^j$ for the Euclidean version, with bold letters representing **3**-vectors. We use the shorthand notation $\int_k \equiv \int \mathrm{d}^D k/\pi^{D/2}$ and $\hat{\delta}^{(n)}(x) \equiv 2\pi\delta^{(n)}(x)$ for the n-th derivative of Dirac-δ functions. We handle divergent integrals through dimensional regularization (dim. reg.), with $D = 4 - 2\epsilon$. We define $M_{\rm Pl}^{-1} \equiv \sqrt{32\pi G}$ for the Planck mass in $\hbar = c = 1$ units. We denote $M = m_1 + m_2$ the total mass, $\mu = m_1 m_2/M$ the reduced mass, $\nu \equiv \mu/M$ the symmetric mass ratio, and $\Delta_m \equiv (m_1 - m_2)/M = \sqrt{1-4\nu}$ for the (dimensionless) mass difference, with $m_1 \geq m_2$.

2 Schwinger-Keldysh worldline theory

The initial value formulation of non-conservative classical dynamics at the level of Hamilton's action principle was broadly discussed in [1], and more concretely in [45–50] in the context of the EFT formalism with multipole expanded PN sources introduced in [33]. In this section we adapt the same formalism to the worldline EFT approach in the PM regime and encourage the reader to consult [1, 45–50] for further details.

2.1 Basic formalism

The main idea is to compute the effective action imposing only vacuum boundary conditions for the gravitational field at initial time. This is achieved by performing a 'closed-time-path' integral over the field variables, e.g. [148, 149]. To this end it is convenient to introduce auxiliary fields, both for the metric and worldline sources. In our case at hand, the effective action for the two-body problem then takes the form (schematically)

$$\exp\left(i\mathcal{S}_{\rm eff}[x_{a(1)}, x_{a(2)}]\right) = \int \mathcal{D}h_1 \mathcal{D}h_2 \exp\left(iS_{\rm EH}[h_1] - iS_{\rm EH}[h_2] + iS_{\rm pp}[h_1, x_{a,1}] - iS_{\rm pp}[h_2, x_{a,2}]\right), \tag{2.1}$$

with a doubling of degrees of freedom both for the gravitational field as well as the particles ($a = 1, 2$), with the latter treated as external sources. The Einstein-Hilbert action is given by,

$$S_{\rm EH} = -2M_{\rm Pl}^2 \int d^4x \sqrt{-g}\, R\,, \tag{2.2}$$

while the point-particle action is of the Polyakov form [74]

$$S_{\rm pp} = -\sum_a \frac{m_a}{2} \int \mathrm{d}\tau_a \left(v_a^2 + \frac{h_{\mu\nu}}{M_{\rm Pl}} v_a^\mu v_a^\nu \right)\,, \tag{2.3}$$

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with $v^\mu \equiv dx^\mu/d\tau$ the four-velocity. A suitable gauge-fixing term must be added to (2.2) [74]. The path integral in (2.1) over the metric degrees of freedom is then computed via the saddle point approximation, without including (quantum) 'loops' of the gravitational field(s) [74].

In these variables the matrix of propagators for the $h_{1,2}$ gravitational perturbation(s) becomes, see appendix A,

$$G_{AB}(x-y) = i \begin{pmatrix} \Delta_{\rm F}(x-y) & -\Delta_-(x-y) \\ -\Delta_+(x-y) & \Delta_{\rm D}(x-y) \end{pmatrix}, \tag{2.4}$$

with $A, B \in \{1, 2\}$.

The equations of motion follow from $S_{\rm eff}[x_{a(1)}, x_{a(2)}]$ via the standard Euler-Lagrange procedure, and subsequently identifying $x^\alpha_{a,1} \Leftrightarrow x^\alpha_{a,2}$.

An alternative representation, often used in the literature, is given by the following change of variables due to Keldysh [147]

$$\begin{aligned} h^-_{\mu\nu} &= \tfrac{1}{2}(h_{1\mu\nu} + h_{2\mu\nu}) \\ h^+_{\mu\nu} &= h_{1\mu\nu} - h_{2\mu\nu} \end{aligned} \quad \Leftrightarrow \quad \begin{aligned} h_{1\mu\nu} &= h^-_{\mu\nu} + \tfrac{1}{2}h^+_{\mu\nu} \\ h_{2\mu\nu} &= h^-_{\mu\nu} - \tfrac{1}{2}h^+_{\mu\nu}\,, \end{aligned} \tag{2.5}$$

and likewise to the external worldline sources,

$$\begin{aligned} x^\alpha_{a,+} &= \tfrac{1}{2}(x^\alpha_{a,1} + x^\alpha_{a,2}) \\ x^\alpha_{a,-} &= x^\alpha_{a,1} - x^\alpha_{a,2} \end{aligned} \quad \Leftrightarrow \quad \begin{aligned} x^\alpha_{a,1} &= x^\alpha_{a,+} + \tfrac{1}{2}x^\alpha_{a,-} \\ x^\alpha_{a,2} &= x^\alpha_{a,+} - \tfrac{1}{2}x^\alpha_{a,-}\,. \end{aligned} \tag{2.6}$$

In this basis the matrix of propagators takes the most familiar form dictated by causality,[1]

$$K^{AB}(x-y) = i \begin{pmatrix} 0 & -\Delta_{\rm adv}(x-y) \\ -\Delta_{\rm ret}(x-y) & \tfrac{1}{2}\Delta_H(x-y) \end{pmatrix}, \tag{2.7}$$

relax where this time $A, B \in \{+, -\}$. The $\Delta_H(x) \equiv \Delta_+(x) + \Delta_-(x)$ is the Hadamard propagator, see appendix A, which does not feature in the classical regime and may be ignored in the saddle point approximation. In this basis the equations of motion follow from the condition,

$$\left.\frac{\delta \mathcal{S}_{\rm eff}[x_{a,+}, x_{a,-}]}{\delta x^\mu_{b,-}(\tau)}\right|_{\substack{x_{a,-}\to 0 \\ x_{a,+}\to x_a}} = 0\,. \tag{2.8}$$

It is straightforward to show that the effective action is given by the general form

$$\begin{aligned} \mathcal{S}_{\rm eff}[x_{a,+}, x_{a,-}] &= \sum_a m_a \int {\rm d}\tau_a \left(-\frac{1}{2}v^2_{a,1}(\tau_a) + \frac{1}{2}v^2_{a,2}(\tau_a)\right) + \mathcal{S}_{\rm int}[x_{a,1}, x_{a,2}] \\ &= -\sum_a m_a \int {\rm d}\tau_a\, v_{a,+}\cdot v_{a,-} + \mathcal{S}_{\rm int}[x_{a,+}, x_{a,-}]\,, \end{aligned} \tag{2.9}$$

such that

$$m_b \frac{d}{d\tau} v^\mu_b(\tau) = -\eta^{\mu\nu} \left.\frac{\delta \mathcal{S}_{\rm int}[x_{a,+}, x_{a,-}]}{\delta x^\nu_{b,-}(\tau)}\right|_{\substack{x_{a,-}\to 0 \\ x_{a,+}\to x_a}}, \tag{2.10}$$

[1] Notice the difference between upper and lower indices.

and the total impulse, e.g. for particle 1, becomes

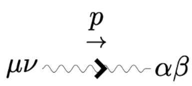
$$\begin{aligned}\Delta p_1^\mu &= m_1 \Delta v_1^\mu = m_1 \left(v_1^\mu(\infty) - v_1^\mu(-\infty)\right) \\ &= m_1 \int_{-\infty}^{\infty} \mathrm{d}\tau_1 \frac{d}{d\tau_1} v_1^\mu(\tau_1) = -\eta^{\mu\nu} \int_{-\infty}^{\infty} \mathrm{d}\tau_1 \left.\frac{\delta \mathcal{S}_{\text{int}}\left[x_{a,+}, x_{a,-}\right]}{\delta x_{1,-}^\nu(\tau_1)}\right|_{\substack{x_{a,-}\to 0 \\ x_{a,+}\to x_a}} .\end{aligned} \tag{2.11}$$

Our task now is to compute the interaction part of the effective action, $\mathcal{S}_{\text{int}}\left[x_{a,+}, x_{a,-}\right]$, using the Feynman rules that descend from (2.1). Despite the doubling of degrees of freedom in the in-in effective theory, we shall demonstrate next that the construction of the full integrand turns out to be remarkably similar to the computation of the impulse using the standard in-out Feynman rules [75, 78, 79].

2.2 Feynman rules for the *in-in*tegrand

The quantity of interest, once the graviton fields have been integrated out, is the variation of the effective action which appears in the impulse (2.11). Rather than the effective action, it turns out to be more efficient in this case to construct Feynman rules directly for its variation, which features both in the equations of motion and total impulse. The reason is two-fold. First of all, the physical limit $x_{a-} \to 0$ can be explicitly taken, and moreover one can readily identify the similarities with the in-out integrand prior to performing the integration. For notational simplicity throughout this section we suppress sums over particle's ($a = 1, 2$) labels.

Let us start with the propagators. To construct the integrand it turns out to be convenient to use the Keldysh representation. Hence, since we will be dealing with the retarded (or advanced) Green's functions in (2.7), we will depict the graviton propagator as



$$= \quad i P_{\mu\nu;\alpha\beta} \Delta_{\text{ret}}(p)\,, \tag{2.12}$$

where the arrow on the momentum is associated with the standard momentum routing whereas the arrow on the wavy line indicates the direction of the *time flow*. The advanced propagator, therefore, may be represented by the same arrow on the momentum p but with the time flowing in the opposite direction.

Next we move onto the interacting part of the point-particle action, which reads

$$\begin{aligned}\mathcal{S}_{\text{pp}}[x_\pm, h_\pm] \to -\frac{m}{2M_{\text{Pl}}} \int \mathrm{d}\tau \int_k \Bigg\{ \frac{h^+_{\mu\nu}(k)}{2} \Bigg[& e^{i\,k\cdot\left(x_+ + \frac{x_-}{2}\right)} \left(v_+^\mu + \frac{v_-^\mu}{2}\right)\left(v_+^\nu + \frac{v_-^\nu}{2}\right) \\ & + e^{i\,k\cdot\left(x_+ - \frac{x_-}{2}\right)} \left(v_+^\mu - \frac{v_-^\mu}{2}\right)\left(v_+^\nu - \frac{v_-^\nu}{2}\right)\Bigg] \\ & + h^-_{\mu\nu}(k) \Bigg[e^{i\,k\cdot\left(x_+ + \frac{x_-}{2}\right)} \left(v_+^\mu + \frac{v_-^\mu}{2}\right)\left(v_+^\nu + \frac{v_-^\nu}{2}\right) \\ & - e^{i\,k\cdot\left(x_+ - \frac{x_-}{2}\right)} \left(v_+^\mu - \frac{v_-^\mu}{2}\right)\left(v_+^\nu - \frac{v_-^\nu}{2}\right)\Bigg]\Bigg\} .\end{aligned} \tag{2.13}$$

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In general, a Feynman diagram will have several insertions of the worldline coupling to the gravitational field(s). In the computation of the variation of the effective action, however, only one such insertion at a time will be hit by the derivative in (2.11). Therefore, we can distinguish two types of worldline vertices. The first is obtained when no variation is acting on it. In this case we can set the physical condition already at the level of the effective action such that we have, in Fourier space,

$$\mathcal{S}_{\rm pp}|_{\substack{x_-\to 0\\ x_+\to x}} = -\frac{m}{2M_{\rm Pl}}\int {\rm d}\tau\, h^+_{\mu\nu}(k) e^{i\,k\cdot x} v^\mu v^\nu\,, \tag{2.14}$$

which we represent as

$\downarrow k$ 

$$= \quad -\frac{im}{2M_{\rm Pl}}\int {\rm d}\tau\, e^{i\,k\cdot x} v^\mu v^\nu\,, \tag{2.15}$$

with the arrows defined according to (2.12). Notice that this Feynman rule resembles the standard in-out scenario, with the Feynman propagator replaced by a retarded one. For reasons that will become clear momentarily we will refer to these vertices as *sources*.

The remaining case, which occurs when the variation with respect to x_- hits the vertex, turns out to be surprisingly simple. The key is to notice,

$$\begin{aligned}\left.\frac{\delta\mathcal{S}_{\rm pp}}{\delta x^\alpha_-(\sigma)}\right|_{\substack{x_-\to 0\\ x_+\to x}} &= -\frac{m}{2M_{\rm Pl}}\int {\rm d}\tau \int_k h^-_{\mu\nu}(k) e^{i\,k\cdot x}\left[i\,k^\alpha v^\mu v^\nu \delta(\tau-\sigma) + (\eta^{\mu\alpha}v^\nu + \eta^{\nu\alpha}v^\mu)\delta'(\tau-\sigma)\right] \\ &= -\frac{m}{2M_{\rm Pl}}\int {\rm d}\tau \int_k h^-_{\mu\nu}(k) e^{i\,k\cdot x}\delta(\tau-\sigma)\left[i\,k^\alpha v^\mu v^\nu - i\,k\cdot v\eta^{\mu\alpha}v^\nu - \right.\\ &\qquad\qquad \left.\eta^{\mu\alpha}\dot v^\nu - i\,k\cdot v\eta^{\nu\alpha}v^\mu - \eta^{\nu\alpha}\dot v^\mu\right]\,,\end{aligned} \tag{2.16}$$

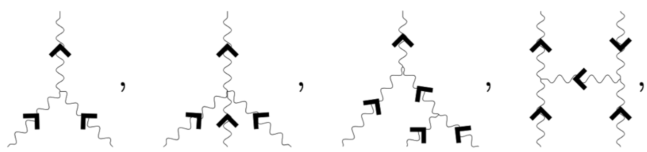
which means only one type of time flow — opposite to the previous case — must be considered. The associated Feynman rule for this particular vertex becomes (evaluated at $x^\alpha(\sigma)$)

$\downarrow k$ 

$$= \quad -\frac{im}{2M_{\rm Pl}} e^{i\,k\cdot x}\left[i\,k^\alpha v^\mu v^\nu - i\,k\cdot v\,\eta^{\mu\alpha}v^\nu - \eta^{\mu\alpha}\dot v^\nu - i\,k\cdot v\,\eta^{\nu\alpha}v^\mu - \eta^{\nu\alpha}\dot v^\mu\right]\,, \tag{2.17}$$

where the cross, which we will refer to as a *sink*, indicates this is the worldline vertex where the $\delta/\delta x^\alpha$ derivative of the action has been taken. This is the only rule which differs from the standard in-out treatment.

Adding to the mix the bulk-type interactions written in the Keldysh representation, it is straightforward now to construct the in-in integrand. In particular, it is clear that any graviton vertex that leads to more than one sink can be neglected. Hence, only non-linear couplings with sources attached to a single sink linked to e.g. particle 1, for instance



(2.18)

must be considered to compute the total Δp^α. This is also consistent with having a *causal* time ordering attached to each diagram, as expected. At the end of the day, this follows directly from the fact that we are solving classical field equations with iterated retarded Green's functions. Notice that, as long as loops of the gravitational field are neglected, this also precludes the appearance of the Hadamard propagator.

Let us conclude this part with a few useful remarks. First of all, because of the property we just discussed about sinks and sources and the fact that only tree-level diagrams of the gravitational field(s) are needed in the classical theory, we notice that non-linear terms involving more than one h^+ in the in-in expansion of the Einstein-Hilbert cannot contribute to the path integral in (2.1). This is obviously the case for even number of h^+'s (including zero) which identically vanish, and is also the case for odd terms greater than one simply because there is no K^{++} component in (2.7). This is a direct consequence of our in-in Feynman rules yielding a natural causal flow for each (classical) diagram.[2] Likewise, following our conventions in (2.5) and taking into account the relative minus sign in (2.1), it is also easy to show that the coefficient for the nth order term (schematically) $h^+[h^-]^{n-1}$ in the bulk action would become identical to the standard in-out counterpart, provided we ignore the $\pm$ labels. The only apparent difference at the level of the Feynman rules are symmetry factors. The h^+ and h^- are indistinguishable in the in-out approach, whereas these are two separate fields in the in-in framework. This means that, at the level of the in-out and in-in effective actions, we do not expect to arrive to the same result between the two by simply replacing propagators. However, at the level of the equations of motion, it is clear that we must also consider diagrams with sinks at all instances where e.g. particle 1 appears. Moreover, since in the classical theory we only include tree-level connected Feynman diagrams, it is straightforward to show that the additional Wick contractions over the metric field in the in-out computation of the impulse are compensated by the sum over sinks in the in-in formalism.[3]

From here we conclude that upon replacing $\Delta_{\rm ret} \to \Delta_F$ in the in-in integrand for the impulse we would directly recover the value computed with in-out boundary conditions. The converse is also manifestly true. After taking into account the time ordering consistently with the in-in rules, the integrand for the impulse may be equally computed directly from the in-out approach by replacing Feynman with retarded propagators consistently with a causal ordering. Another relevant comment has to do with the expression for the vertex in (2.17). Since the last four terms appear due to time derivatives in the Euler-Lagrange equations, they only matter for the equations of motion but vanish in the total impulse.

2.3 Conservative vs dissipative

In principle the interaction part of the effective theory, $\mathcal{S}_{\rm int}[x_{a,1}, x_{a,2}]$, includes both dissipative and conservative terms alike. It is clear, however, that any term in the effective action

[2] Notice this is no longer the case in the quantum theory once loops of the gravitational field(s) are allowed.

[3] This is also manifest at the level of PN computations of the effective action in terms of multipole moments, where different overall factors appear in the in-in and in-out approach but both coincide after the variations of the respective actions are considered, see e.g. [48].

which does not mix between the 1 and 2 variables, e.g. of the form

$$\mathcal{S}_{\rm int}[x_{a,1},x_{a,2}] \supset \int {\rm d}\tau_a \left(V_1[x_{a,1}] - V_2[x_{a,2}]\right) , \tag{2.19}$$

can be combined with the kinetic part yielding

$$\mathcal{S}_{\rm eff}[x_{a,1},x_{a,2}] = \sum_a m_a \int {\rm d}\tau_a \left(L_{\rm cons}[x_{a,1}] - L_{\rm cons}[x_{a,2}]\right) + \mathcal{S}_{\rm diss}[x_{a,1},x_{a,2}] , \tag{2.20}$$

with (up to terms that vanish in the limit $x_{a,-}\to 0$)

$$L_{\rm cons}[x_a] \equiv -\left(\sum_a \frac{m_a}{2} v_a^2 - \frac{V_1[x_a]+V_2[x_a]}{2}\right) , \tag{2.21}$$

such that the physics described by this Lagrangian — which can be equally described by a conserved Hamiltonian — naturally falls into the conservative sector. See [1] for a more detailed discussion.

Notice we do not make any assumption regarding the origin of $V_a[x_a]$, which in principle can (and will) receive contributions from both off-shell potential as well as on-shell radiation modes. Needless to say, if the Green's functions are expanded in the potential region around quasi-instantaneous interactions [33], the difference between Feynman, retarded (or advanced) propagators becomes obsolete. In such case the effective action becomes entirely of the form in (2.19), as expected. However, that is not the full story. In fact, radiation-reaction tail terms are known to contribute to the conservative sector through a term of the form in (2.19) [48].

The obvious follow up question then becomes whether one can identify these pieces in advance. As we show next, that is indeed the case for most — although not necessarily all — of conservative terms.

Let us take the matrix in (2.4) and split it as

$$G_{AB} = G^{\rm F}_{AB} + G^{\rm cut}_{AB} , \tag{2.22}$$

with

$$G^{\rm F}_{AB} = i \begin{pmatrix} \Delta_{\rm F}(x-y) & 0 \\ 0 & \Delta_{\rm D}(x-y) \end{pmatrix} , \qquad G^{\rm cut}_{AB} = i \begin{pmatrix} 0 & -\Delta_-(x-y) \\ -\Delta_+(x-y) & 0 \end{pmatrix} . \tag{2.23}$$

Given that $G^{\rm F}_{AB}$ is diagonal, its contribution to both the $\mathcal{D}h_1$ and $\mathcal{D}h_2$ integrals in (2.1) can be computed independently of each other using the standard in-out rules. Moreover, since they come with opposite signs, it is straightforward to show that at the end of the day the contribution from $G^{\rm F}_{AB}$ to the effective action takes the form

$$\mathcal{S}_{\rm eff}[x_{a,1},x_{a,2}] \supset \int {\rm d}\tau_a \left(V_{\rm F}[x_{a,1}] - V_{\rm D}[x_{a,2}]\right) , \tag{2.24}$$

where we have $V_{\rm D}[x] = V^\star_{\rm F}[x]$ (which follows from the properties of the Feynman and Dyson propagators, see appendix A). Hence, according to (2.21), the in-in effective action contains a conservative piece given by

$$L_{\rm cons}[x_a] \supset \frac{1}{2}\int {\rm d}\tau_a \, \left(V_F[x_a] + V_D[x_a]\right) = \mathbb{R}\int {\rm d}\tau_a \, V_F[x_a] . \tag{2.25}$$

These manipulations then explicitly demonstrate that using the standard in-out Feynman rules with Feynman's $i0$-prescription and keeping only the real part of the resulting effective action produces a contribution to the dynamical equations which can be always included as part of a conservative sector. Notably, following our previous reasoning, we can also obtain this conservative contribution directly from the $\pm$-basis by replacing $\Delta_{\rm ret} \to \Delta_F$ in the integrand <u>*for the impulse*</u>, thus landing in the in-out result as we demonstrated before.

There are, of course, also *dissipative* effects. These are obtained by *at least* one insertion of $G^{\rm cut}$ in (2.1), which mixes the $x_{a,1}$ and $x_{a,2}$ variables. At the level of the $\pm$-basis, these terms appear from the mismatch between retarded/advanced and Feynman Green's functions, and therefore can be entirely written in terms of $\Delta_\pm$ propagators. They are proportional to $\delta(p^2)$ in Fourier space, arise due to gravitational radiation, and, as expected, are only present when on-shell modes are turned on.

A priori, other than those identified through (2.25), we cannot tell in advance whether the effective action contains more terms of the form in (2.19). In fact, it may not hold for all types of non-linear radiation-reaction contributions, in particular those involving several (non-vanishing) insertions of $\Delta_\pm$ which may lead to additional contributions of the form in (2.19) that are not captured by (2.25). In some paradigmatic examples the contribution from (2.25) does encapsulate the entire conservative sector. This is the case, for instance, for all potential interactions (for which $\Delta_\pm$ vanishes). It is also the case for tail-induced effects [48, 52, 53, 78, 79]. We will return to this point briefly in section 4, and in more detail elsewhere [154]. Fortunately, none of the above issues arise at leading order in the radiation-reaction at 3PM, which involves only a single non-vanishing insertion of Δ_-. In this case there is a clear separation into dissipative and conservative contributions, which we study next.

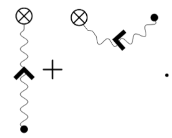
3 Scattering to 3PM

3.1 Building the integrand

We can now use the in-in diagrammatic rules to obtain the variation of the effective action necessary to compute the total impulse. For simplicity, we only display here the arrows which dictate — according to causality — whether retarded or advanced Green's functions appear in the graviton propagators.

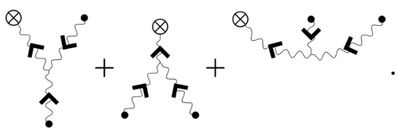
At 1PM order there are only two diagrams

$$\left.\frac{\delta \mathcal{S}_{\rm eff}[x_+, x_-]}{\delta x_-^\alpha}\right|^{\rm 1PM}_{\substack{x_-\to 0\\ x_+\to x}} = \tag{3.1}$$



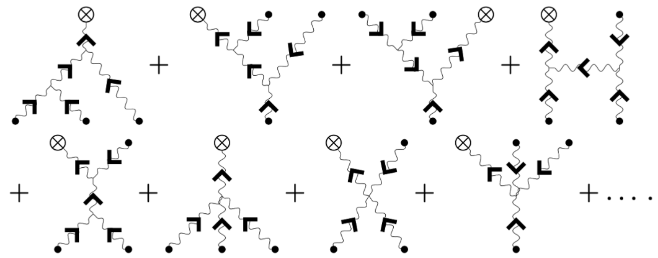
At 2PM, on the other hand, we have

$$\left.\frac{\delta \mathcal{S}_{\rm eff}[x_+, x_-]}{\delta x_-^\alpha}\right|^{\rm 2PM}_{\substack{x_-\to 0\\ x_+\to x}} = \tag{3.2}$$



Whereas, at 3PM order,

$$\frac{\delta \mathcal{S}_{\rm eff}[x_+, x_-]}{\delta x_-^\alpha}\bigg|^{\rm 3PM}_{\substack{x_-\to 0\\ x_+\to x}} = \tag{3.3}$$



The ellipsis in the last expression include the same type of *self-energy* diagrams (with all sources and sinks on the same worldline) shown above, which are in general needed to include radiation-reaction effects. Modulo the time routing of each diagram, all of these resemble the same type in the conservative sector to 3PM [74, 75], except for the self-energy terms which lead to scaleless integrals in the potential region that in dim. reg. can be set to zero. More on this below.

Once the integrand for the impulse is constructed via the above Feynman rules we still need to compute it on solutions to the equations of motion. This entails, as before [74], adding the *iterations* of lower order contributions to the effective action evaluated on the trajectories [74] to a given nPM order,

$$\begin{aligned} x_a^\mu(\tau_a) &= b_a^\mu + u_a^\mu \tau_a + \sum_{k=1}^{n} \delta^{(k)} x_a^\mu(\tau_a)\,, \\ v_a^\mu(\tau_a) &= u_a^\mu + \sum_{k=1}^{n} \delta^{(k)} v_a^\mu(\tau_a)\,, \end{aligned} \tag{3.4}$$

with u_a the incoming velocities and $b_1^\mu - b_2^\mu \equiv b^\mu$ the (space-like) impact parameter four-vector. These trajectories can be solved iteratively using (2.10) after the limit: $x_{a,-} \to 0$, $x_{a,+} \to x_a$, has been taken. Because they are obtained from the variation of the effective action, following our previous argument it is straightforward to show the resulting equations of motion in the in-in and in-out formalisms are equally related by replacing retarded and Feynman propagators, and vice-versa following a causal routing. Hence, the procedure to separate between conservative and dissipative terms we discussed before is unaltered by the inclusion of iterations. Moreover, since self-energy diagrams evaluated on unperturbed trajectories vanish in dim. reg., they can only contribute through iterations on each of the sources. Therefore, only the extra term in (3.1) is needed to 3PM order.

3.2 Integration problem

After the integrand is assembled, say for the impulse of particle 1, and the tensor reduction performed following the same steps as in [75, 78, 79], we find the following families of integrals:

$$I^{\pm\pm;T_5\ldots T_9}_{\nu_1\ldots\nu_9} = \int_{\ell_1,\ell_2} \frac{\hat{\delta}^{(\nu_1-1)}(\ell_1\cdot u_a)\hat{\delta}^{(\nu_2-1)}(\ell_2\cdot u_b)}{(\pm\ell_1\cdot u_{\not a})^{\nu_3}(\pm\ell_2\cdot u_{\not b})^{\nu_4}} \prod_{j=5}^{9} \frac{1}{D_{j,T_j}^{\nu_j}}\,, \tag{3.5}$$

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where we use the same notation as in [75, 78, 79], except for the choice of $i0$-prescription, determined by sets of square propagators of type $T_j \in \{\text{ret}, \text{adv}\}$:

$$\begin{aligned} D_{5,\text{ret/adv}} &= (\ell_1^0 \pm i0)^2 - \boldsymbol{\ell}_1^2\,, & D_{6,\text{ret/adv}} &= (\ell_2^0 \pm i0)^2 - \boldsymbol{\ell}_2^2\,, \\ D_{7,\text{ret/adv}} &= (\ell_1^0 + \ell_2^0 + q^0 \pm i0)^2 - (\boldsymbol{\ell}_1 + \boldsymbol{\ell}_2 - \boldsymbol{q})^2\,, \\ D_{8,\text{ret/adv}} &= (\ell_1^0 - q^0 \pm i0)^2 - (\boldsymbol{\ell}_1 - \boldsymbol{q})^2\,, & D_{9,\text{ret/adv}} &= (\ell_2^0 - q^0 \pm i0)^2 - (\boldsymbol{\ell}_2 - \boldsymbol{q})^2\,. \end{aligned} \tag{3.6}$$

The integrand itself only contains factors with $\nu_1 = \nu_2 = 1$, i.e. no derivative. However, because the integrals in (3.5) turn out to be not all independent, derivatives of the Dirac-δ functions may be introduced by the standard *integration-by-parts* (IBP) relations [118, 119, 155]. We have used the combination of the automated programs `LiteRed` [145] and `FIRE6` [136] to relate the integrals appearing in the integrand to a set of $\mathcal{O}(100)$ masters.[4]

The reader will immediately notice the sharp contrast with the number of master integrals needed for the computation in [75]. The reason is one of the main advantages of Feynman propagators, and the use of the symmetry relations implemented by these programs, which are not present for the case of retarded/advanced Green's functions. Nevertheless, additional simplifications arise once we restrict ourselves to the conservative and dissipative sectors, respectively. First of all, for the former we can replace retarded/advanced by Feynman Green's functions, essentially returning to the analysis in [75], which we review below. For the latter, we notice only radiation modes contribute to $\Delta_\pm$. Hence, after isolating the different kinematical regions of integration via e.g. the *asy2.m* code included in the `FIESTA` package [124, 125, 143], we can show that only the 7th propagator in the above families has a non-zero radiation region, with all others entering only through potential modes. This means, as expected, that diagrams with a single insertion of Δ_- contribute to the dissipative sector at 3PM, with the others replaced by Feynman Green's functions. In such scenario, there is a subset of symmetry relations that reduce the number of master integrals to about $\mathcal{O}(50)$, which we computed via the method of differential equations [120, 121, 128, 129, 132, 133, 138].

Since there is only a single scale in the problem, it is convenient to choose the parameter x, defined through [113]

$$\gamma = (x^2+1)/2x\,, \qquad \text{with} \quad \gamma \equiv u_1 \cdot u_2\,. \tag{3.7}$$

The dependence on the parameter x is then extracted order by order in ϵ by iterated integrations. At this order it is straightforward to show the differential equations can be brought into canonical form [128]. For the problem at hand only three independent functions appear

$$\{\, \log(x), \log(1+x), \log(1-x) \}\,, \tag{3.8}$$

[4] There is one subtlety due to the presence of self-energy diagrams leading to iterations involving linear propagators with the same internal momentum but both $\pm i0$ prescriptions, e.g. $1/(\ell_1 \cdot u_2 + i0)(\ell_1 \cdot u_2 - i0)$. (Notice this cannot occur for quadratic propagators in the classical theory.) These are both treated on equal footing by the IBP programs, irrespectively of the signs. Hence, we would not know how to reassign the correct signs after IBP relations. Fortunately, we were able to construct a basis of master integrals for which these type of terms are absent (or appear in the numerator), thus avoiding this problem.

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up to polynomials in the x variable. This uniquely fixes the velocity dependence of the master integrals up to a series of appropriate boundary conditions, which we choose near the static limit $x \to 1$. Notably, the solution to the differential equations allow us to extract the scaling in $(1-x)$ for each individual integral in this limit [79, 144], which becomes extremely useful to set up a series of consistency conditions that allow us to simplify the number of boundary constants needed to read off the full answer.

Additional consistency requirements may be enforced to reduce the number of independent constants. The first is the lack of absorption, $\Delta m_a^2 = 0$, which implies the conservation of the on-shell condition $(p_a + \Delta p_a)^2 = p_a^2$, restricting the impulse to point in the direction orthogonal to u_a^μ [112, 114]. At a more technical level, we also impose the cancelation in the final answer of intermediate $1/\epsilon^k$ poles. Furthermore, terms proportional to $\log(1-x) \propto \log v$ are uniquely fixed by the universal connection to tail terms [73] (and therefore do not enter until 4PM order [79]). Finally, as anticipated e.g. [29], we can enforce continuity of the massless/high-energy limit at 3PM order, thus removing mass singularities.[5] Remarkably, all of these properties completely determine the radiation-reaction impulse in the direction of the impact parameter at 3PM, and only a handful of boundary integrals for the longitudinal direction are needed.

3.3 Boundary integrals

As we mentioned, once the solution of the differential equations are known, the last input are integration constants obtained from master integrals in the limit $x \to 1$. We first review the procedure for the conservative pieces before discussing radiative contributions.

3.3.1 Conservative

As it was computed in [75], and further established here from the in-in formalism, the conservative sector is obtained from the full integrand by replacing retarded/advanced propagators by Feynman counterparts and keeping the real part of the resulting impulse. In principle, at this stage both potential and radiation modes can be present. However, as we mentioned earlier, only a single propagator has support on radiation modes and therefore only the potential region contribute to the real part with time-symmetric boundary conditions (with an imaginary part consistent with the optical theorem). This also implies that all self-energy diagrams vanish in dim. reg. for the conservative sector at $\mathcal{O}(G^3)$.

The result for the potential boundary conditions was already computed in [75]. However, in order to relate the solution to the master integrals in [75] an additional set of (static) IBP relations must be used at $x = 1$. Decomposing the impulse as

$$\Delta p_1^\mu = \sum_n G^n \Delta^{(n)} p_1^\mu \,, \tag{3.9}$$

we find at 3PM order,

$$\begin{aligned}\Delta^{(3)} p^\mu_{1,\text{cons}} = &\frac{4M^4 \nu b^\mu}{3|b|^4 x^2 (x^2-1)^5} \left[c_1(x) + \nu c_2(x) + \nu c_3(x) \log(x)\right] \\ &- \frac{3\pi M^4 \nu c_4(x) \left((\Delta_m - 1)\check{u}_1^\mu + (\Delta_m + 1)\check{u}_2^\mu\right)}{8|b|^3 x^2 \left(x^2-1\right)^2} \,,\end{aligned} \tag{3.10}$$

[5] As we shall see in [153] this is no longer the case at 4PM order.

with $|b| = \sqrt{-b^\mu b_\mu}$, where we have introduced

$$\check{u}_1 = \frac{\gamma u_2 - u_1}{\gamma^2 - 1}\,, \quad \check{u}_2 = \frac{\gamma u_1 - u_2}{\gamma^2 - 1}\,, \tag{3.11}$$

obeying $\check{u}_a \cdot u_b = \delta_{ab}$. The c_i's are polynomials given by

$$\begin{aligned} c_1(x) &= -12x \left(x^{12} - 2x^{10} - x^8 - x^4 - 2x^2 + 1\right)\,, \\ c_2(x) &= -(x-1)^2\Big(5x^{12} - 14x^{11} - 88x^{10} - 114x^9 - 5x^8 + 128x^7 \\ &\qquad\qquad + 128x^6 + 128x^5 - 5x^4 - 114x^3 - 88x^2 - 14x + 5\Big)\,, \\ c_3(x) &= 6\left(x^8 - 8x^6 - 30x^4 - 8x^2 + 1\right)\left(x^2 - 1\right)^3\,, \\ c_4(x) &= \left(x^4 + 1\right)\left(5x^4 + 6x^2 + 5\right)\,. \end{aligned} \tag{3.12}$$

Using that, for $0<x<1$, $\log(x) = -\mathrm{arccosh}(\gamma)$, the above result can also be written directly in γ space, yielding

$$\begin{aligned} \Delta^{(3)} p^\mu_{1,\mathrm{cons}} = \frac{b^\mu}{|b|^4}\Big(&-\frac{2m_1 m_2\left(m_1^2 + m_2^2\right)}{\left(\gamma^2-1\right)^{5/2}}\left(16\gamma^6 - 32\gamma^4 + 16\gamma^2 - 1\right) \\ &- \frac{4m_1^2 m_2^2 \gamma}{3\left(\gamma^2-1\right)^{5/2}}\left(20\gamma^6 - 90\gamma^4 + 120\gamma^2 - 53\right) \\ &+ \frac{8m_1^2 m_2^2}{\gamma^2 - 1}\left(4\gamma^4 - 12\gamma^2 - 3\right)\mathrm{arccosh}\left(\gamma\right)\Big) \\ &+ \frac{3\pi}{2}\frac{\check{u}_1}{|b|^3} m_1 m_2^2\left(m_1 + m_2\right)\frac{\left(2\gamma^2-1\right)\left(5\gamma^2-1\right)}{\gamma^2-1} \\ &- \frac{3\pi}{2}\frac{\check{u}_2}{|b|^3} m_1^2 m_2\left(m_1 + m_2\right)\frac{\left(2\gamma^2-1\right)\left(5\gamma^2-1\right)}{\gamma^2-1}\,, \end{aligned} \tag{3.13}$$

in accordance with the result in [75].

3.3.2 Dissipative

Non-conservative terms are obtained by systematically including the factors of Δ_- accounting for the mismatch between Feynman and retarded propagators. Unlike Feynman Green's functions, the Δ_- vanishes for off-shell modes and therefore we just need to consider the single propagator which can have support on the radiation region. Furthermore, as we discussed in [78, 79], each radiation mode scales with a power of $(1-x)^{-2\epsilon}$. This already allows us to individually target radiative contributions.

Surprisingly, most of the integration constants for the dissipative sector can be determined by the same type of consistency constraints already implemented in [78, 79] to study radiation-reaction effects in the conservative sector at 4PM. In particular, the contribution in the direction of the impact parameter b^μ turns out to be uniquely fixed. Furthermore, for the longitudinal term, which must be proportional to $\check{u}_2^\mu$, the master integrals have already been computed in the literature [85, 112, 114] via Cutkosky rules [156] (see e.g. appendix C in [85]), which we confirm here in terms of solutions of the differential equations.

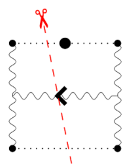
As an example, consider the H-type master integral resulting after IBP relations

$$I_{\rm H} = \int_{\ell_1,\ell_2} \frac{\hat\delta'(\ell_1\cdot u_a)\hat\delta(\ell_2\cdot u_b)}{\ell_1^2\ell_2^2(((\ell_1+\ell_2-q)^0+i0)^2-(\boldsymbol{\ell}_1+\boldsymbol{\ell}_2+\boldsymbol{q})^2)(\ell_1-q)^2(\ell_2-q)^2}\,. \tag{3.14}$$

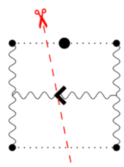
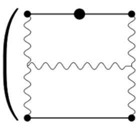
As we mentioned, only one leg may be taken as a full retarded propagator, with the other ones having support only on potential modes. After replacing $\Delta_{\rm ret}\to\Delta_{\rm F}+\Delta_-$ and identifying the conservative piece from the real part of the Feynman-only computation,[6] the dissipative term is obtained by replacing $\Delta_{\rm ret}\to\Delta_-$, yielding for this integral

$$I_{\rm H}^{\rm cut} = \int_{\ell_1,\ell_2} \frac{\hat\delta'(\ell_1\cdot u_a)\hat\delta(\ell_2\cdot u_b)\hat\delta\left((\ell_1+\ell_2-q)^2\right)\Theta\left(-\ell_1^0-\ell_2^0+q^0\right)}{\ell_1^2\ell_2^2(\ell_1-q)^2(\ell_2-q)^2}\,, \tag{3.15}$$

which (up to a sign) corresponds to the master integral f_4 in [85] after performing the shift $\ell_i\leftrightarrow-\ell_i$ and $q\leftrightarrow -q$. This integral was computed by resorting to Cutkosky's rules [156], including also a cut over hypothetical linear 'worldline propagators,' which account for the Dirac-δ functions already present in $I_{\rm H}$, and conspicuously everywhere in the worldline formalism. We may describe the procedure diagrammatically as follows

$I_{\rm H}^{\rm cut} =$ , (3.16)

where the dotted lines represent the $\delta^{(n)}$ functions as cut linear propagators, with the big dot on the upper line denoting the case $\nu_1=2$ (i.e. a derivative) for one of the δ's. The cut over the retarded propagator denotes the replacement $\Delta_{\rm ret}(p)\to 2\pi i\,\delta(p^2)\Theta(-p^0)$ in momentum space.[7] By Cutkosky's rules, we can relate the cut integral to the imaginary part of its *uncut* version,[8]

 $= 2{\rm Im}$ $\Bigg($  $\Bigg)$, (3.17)

[6]Remarkably, the resulting imaginary part of the Feynman integrals entering the impulse in the $\check u_2$ direction vanish. Consequently, the contribution from the remaining cut integrals are real functions such that the cancelation of imaginary parts becomes trivial in this case.

[7]Because of the $\Theta(-p^0)$, we have flipped the arrow on the retarded propagator to illustrate that positive energy is flowing towards the right of the diagram, as it is standard lore.

[8]The only subtlety here is the cut over the linear propagator, which relies on the identity

$$\int\frac{{\rm d}^4\ell}{(2\pi)^4}\frac{e^{-i\,\ell\cdot x}}{\ell\cdot u+i0} = -2\pi i\int\frac{{\rm d}^4\ell}{(2\pi)^4}e^{-i\,\ell\cdot x}\delta(\ell\cdot u)\left(\Theta(u^0)\Theta(x^0)+\Theta(-u^0)\Theta(-x^0)\right)\,,$$

such that the cut propagator is given by $2\pi i\,\delta(\ell\cdot u)\Theta(u^0)$, as expected. (Note this differs from the version given in [114] in terms of $\Theta(\ell^0)$ which, however, we believe to be a typo.) In this fashion, the $\delta(\ell\cdot u)$ from the time-integration of the worldline sources — naturally obeying $u^0>1$ in the classical theory — can then be equally described by cut linear propagators.

with the dotted and straight lines representing the cut and uncut linear propagators, including higher derivatives of Dirac-δ functions corresponding to higher powers of linear propagators. The uncut integral, which is now entirely written in terms of Feynman propagators, can then be computed in the near-static limit $x \to 1$ by direct integration. See [85] for more details.

Combining all these results we find at 3PM,

$$\begin{aligned}\Delta^{(3)} p^{\mu}_{1,\mathrm{diss}} = & \frac{M^4 \nu^2 (1+x^4)^2 b^{\mu}}{3|b|^4 x^2 (x^2-1)^5} \left[d_1(x) + d_2(x)\log(x)\right] \\ & + \frac{\pi M^4 \nu^2 \check{u}_2^{\mu}}{192|b|^3 x^3 (x^2-1)^4} \left[d_3(x) + d_4(x) \log\left(\frac{x+1}{2}\right) + d_5(x)\log(x)\right] ,\end{aligned} \tag{3.18}$$

for the leading order contribution to the dissipative part, with

$$\begin{aligned} & d_1(x) = 4\left(5x^6 - 27x^4 + 27x^2 - 5\right), \quad d_2(x) = -24\left(x^6 - 3x^4 - 3x^2 + 1\right), \\ & d_3(x) = 105x^{12} - 552x^{11} + 1308x^{10} - 6408x^9 + 29471x^8 - 69840x^7 + 93368x^6 - 69840x^5 \\ & \qquad + 29471x^4 - 6408x^3 + 1308x^2 - 552x + 105, \\ & d_4(x) = -6\left(x^2-1\right)^3 \left(35x^8 + 120x^7 - 460x^6 + 968x^5 - 1070x^4 + 968x^3 - 460x^2 + 120x + 35\right), \\ & d_5(x) = 6x\Big(35x^{13} + 60x^{12} - 325x^{11} + 304x^{10} + 198x^9 - 788x^8 \\ & \qquad + 446x^7 - 889x^5 + 788x^4 - 217x^3 - 304x^2 + 240x - 60\Big). \end{aligned} \tag{3.19}$$

Translating to γ space we have,

$$\begin{aligned}\Delta^{(3)} p^{\mu}_{1,\mathrm{diss}} = & \frac{4 m_1^2 m_2^2 \, b^{\mu}}{3|b|^4} \frac{\left(2\gamma^2-1\right)^2}{\left(\gamma^2-1\right)^{5/2}} \left(\sqrt{\gamma^2-1}\left(5\gamma^2 - 8\right) + 3\gamma\left(3 - 2\gamma^2\right) \operatorname{arccosh}(\gamma)\right) \\ & + \frac{\pi m_1^2 m_2^2 \, \check{u}_2}{48\left(\gamma^2-1\right)^{3/2}|b|^3} \Bigg[-3\gamma\left(70\gamma^6 - 165\gamma^4 + 112\gamma^2 - 33\right)\frac{\operatorname{arccosh}(\gamma)}{\sqrt{\gamma^2-1}} \\ & + 6\left(35\gamma^6 + 60\gamma^5 - 185\gamma^4 + 16\gamma^3 + 145\gamma^2 - 76\gamma + 5\right)\log\left(\frac{\gamma+1}{2}\right) \\ & + \left(-210\gamma^6 + 552\gamma^5 - 339\gamma^4 + 912\gamma^3 - 3148\gamma^2 + 3336\gamma - 1151\right)\Bigg] ,\end{aligned} \tag{3.20}$$

which (up to an overall sign) reproduces the result in [114]. From here one can extract the total radiated energy in hyperbolic-like motion in the center-of-mass frame,

$$\Delta E_{\mathrm{hyp}} = -(\Delta p_1 + \Delta p_2) \cdot \frac{(m_1 u_1 + m_2 u_2)}{|m_1 u_1 + m_2 u_2|} , \tag{3.21}$$

that can then be further transformed via the boundary-to-bound dictionary [73]

$$\Delta E_{\mathrm{ell}}(J) = \Delta E_{\mathrm{hyp}}(J) - \Delta E_{\mathrm{hyp}}(-J) , \tag{3.22}$$

with J the (total) angular momentum, to obtain the radiated energy over one period of a bound elliptic-like orbit.

4 Discussion & outlook

Building upon the formalism introduced in [1, 45–50], we developed the in-in framework to study scattering processes within the worldline EFT approach. Despite the apparent increase in complexity, we have shown how the computation of the relativistic integrand for the total impulse can be mapped to the in-out counterpart, and vice-versa, provided a consistent flow (of time) is assigned to each diagram following the in-in Feynman rules. After reducing the problem to a series of master integrals involving retarded Green's functions, these can then be computed to all orders in velocity through differential equations. In comparison with standard Feynman propagators, only the issue of symmetry relations (under $p \to -p$) must be reevaluated.[9] The final solution then follows up to a set of boundary integrals in the near-static limit. Notably, consistency conditions imposed by the full solution can be further exploited to bootstrap the final answer up to a handful of constants.

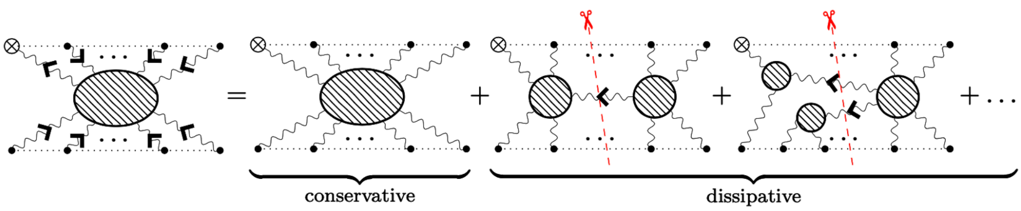
In order to identify the different terms, the identity $\Delta_{\rm ret} = \Delta_{\rm F} + \Delta_{-}$ allows us to split the total in-in impulse as

$$\Delta p^\mu_{\rm tot} = \Delta p^\mu_{\rm F} + \Delta p^\mu_{\rm cut}\,, \tag{4.1}$$

where $\Delta p^\mu_{\rm F}$ is obtained by replacing $\Delta_{\rm ret}$ with $\Delta_{\rm F}$ in the in-in integrand for the total impulse. As we showed, this procedure agrees with the in-out derivation. The remaining $\Delta p^\mu_{\rm cut}$ then involves the insertions of Δ_{-}'s where the time ordering plays an important role. Following [1] we demonstrated that $\Delta p^\mu_{\rm cons} \equiv \mathbb{R}\Delta p^\mu_{\rm F}$ can be obtained from a conservative Lagrangian/Hamiltonian, such that $\Delta p^\mu_{\rm diss} \equiv \mathbb{R}\Delta p^\mu_{\rm cut}$ includes all the dissipative effects, yielding for the total impulse a decomposition

$$\Delta p^\mu_{\rm tot} = \Delta p^\mu_{\rm cons} + \Delta p^\mu_{\rm diss}\,, \tag{4.2}$$

which is independent of the nature of the interaction. As expected, the imaginary parts cancel out in the sum, which must add up to a real value for the impulse. The conservative piece of the impulse is thus built up entirely in terms of (unoriented) Feynman propagators, whereas the dissipative part is obtained by systematically replacing the full $\Delta_{\rm ret}$ with the Δ_{-} Green's functions. Diagrammatically, following our previous conventions and up to iterations from lower order equations of motion, the computation for the variation of the effective action can then be represented as follows



[9] In principle, because of the $\Theta(\pm p^0)$ in advance/retarded propagators, it may seem these would introduce extra terms once they are hit by derivatives w.r.t. the momentum. However, one can also think of the $i0$-prescription as a choice of contour integration for Green's function of the *box* operator. Hence, we do not expect the ultimate choice to alter the form of the differential equations other than through the boundary conditions. We thank Ruth Britto for a discussion on this point.

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Let us mention again that, although consistent with the analysis in [1], the separation through Feynman's prescription may, however, not be complete. In fact, we cannot rule out the possibility that additional conservative-like terms, descending from interactions of the form in (2.19), could remain hidden within the 'dissipative' part. This is particularly relevant for non-linear radiative effects, starting at $\mathcal{O}(G^4)$ and beyond, which include even-in-time hereditary contributions, such a tail and memory effects.[10] For instance, the complete result for the impulse at 4PM reported in [153] suggests that conservative-like hereditary terms may be present — from the point of view of the *relative motion*—beyond the Feynman-only prescription, see also [62, 157] for related work in the PN regime. We will return to this issue in more detail elsewhere [154].

After implementing the methodology of differential equations to compute the resulting master integrals, the remaining task is reduced to obtaining the necessary boundary conditions, both in the conservative and dissipative sectors, which may be then further decomposed using the method of regions [33]. As we have seen, the contribution from radiation modes to $\Delta p^\mu_{\rm F}$ produces only imaginary terms at 3PM so that the conservative sector is dominated by the potential-only region at this order. The impulse was already computed in [75] and reproduced here from the full integrand. Radiation modes do enter in the conservative sector, yet at higher orders. For instance, as discussed in [78, 79, 153], the above procedure yields a conservative impulse at 4PM which includes both potential and radiation-reaction effects. On the other hand, because of the presence of $\delta(p^2)$, radiation modes are necessary for a non-zero dissipative term.

The associated boundary integrals with the insertion of Δ_- propagators can be obtained either by subtracting from the computation with retarded Green's functions the radiative (Feynman-only) conservative part, see [153]; or directly, using Cutkosky's rules [156], as it was first implemented in [114] and discussed in here. For the latter one must generalize the standard cutting rules, such that the Dirac-δ functions for the sources in the worldline EFT approach may be described in terms of cuts over linear propagators. Once again, in addition to the incursion of Feynman's $i0$-prescription dictated by the path integral, here we find yet another standard tool from quantum field theory, i.e. unitary relations, playing a key role at the classical level. Even though all the integral cuts at 3PM may be computed following the analysis in [85, 112, 114], we have shown a dramatic simplification occurs after applying a set of consistency conditions to the full answer [79]. Similar simplifications occur at higher PM orders [153]. This is another instance in which the use of differential equations turn out to be a powerful tool to solve the entire dynamics.

Let us comment on some important points regarding the above manipulations, and in particular with regards to the different choices of basis in the in-in formalism, namely either

[10] Let us emphasize that the distinction between different types of non-linear radiative effects, i.e., tails (scattering off of the geometry), memory (scattering off of the earlier emission), or second order in the radiation-reaction force, is only meaningful from the standpoint of the radiation (or far) zone, see e.g. [48, 62]. On the other hand, in a PM scheme as the one here, only the overall scaling associated with the number of radiation modes involved has a significant meaning [79, 153]. This makes the connection between PM computations and the more standard derivations of tail and memory effects in the PN literature, e.g. [11], somewhat less straightforward, with complete results perhaps as the only path to meaningful comparisons, see e.g. [157].

using (2.4) or (2.7). On the one hand, the former provides us with a natural identification of conservative terms via Feynman Green's functions within the standard in-out prescription; whereas, on the other hand, the latter makes causality explicit in the form of retarded propagators. Yet, as we demonstrated here, the map between the in-in and in-out results for the impulse allows us to implement the split in (4.1) even though, in principle, the integrand was constructed in the $\pm$-basis. In turn, the choice of basis influences the way the real part of the impulse computed with Feynman propagators arises. While, as we discussed, this is straightforward in the 1/2-basis, the cancelation of imaginary parts in the $\pm$-basis becomes manifest only after (A.3) and re-writing $\Delta_{\rm ret}$ as the average of Feynman and Dyson terms.

Let us conclude with a few remarks regarding the connection between the in-in worldline formalism and the so-called KMOC approach introduced in [97]. Following unitarity of the S-matrix, the KMOC prescription to compute the impulse is split into two contributions, denoted as $I_{(1)}$ and $I_{(2)}$ in [97]. The former is constructed in terms of the scattering amplitude in the standard in-out approach, whereas the latter appears in the form of a sum over cuts. As in our case, the sum of these two terms must add up to the (real) total impulse, and therefore intermediate imaginary parts must cancel out. Let us concentrate first on the conservative sector. Following momentum conservation, the impulse (in general relativity) can be uniquely determined at any PM order from $I_\perp$, the perpendicular part introduced in [114]. Hence, after taking the classical limit and subtracting *super-classical* terms, the real part of the (time-symmetric) contribution linear in the scattering amplitude ought to be identified with a conservative contribution.[11] Modulo the lack of subtractions in our formalism, this part then resembles the first term in our decomposition of the total impulse — provided massive lines are further reduced and combined into Dirac-δ functions using reversed unitarity [113].

For dissipative terms, on the other hand, both the perpendicular and longitudinal contributions matter. Moreover, since the real part of the Feynman piece is invariant under $t \to -t$, only the cuts contribute to the dissipative sector. The latter therefore must include, among other things, the remaining terms in our decomposition. However, because of super-classical pieces and the appearance of cuts over massive lines needed to account for the conservative impulse in the longitudinal direction (consistently with momentum conservation), the direct comparison between terms is less transparent. The map also becomes less straightforward at higher orders once we include iterations of lower order deflections that include radiation-reaction effects. Hence, although the KMOC and in-in worldline formalism are clearly tightly related, the identification of terms requires further study. Yet, understanding the map between the different frameworks, and the relation to the eikonal phase [112] (see also [101]), can ultimately allow us to freely incorporate techniques and simplifications that can help us not only fine-tune the computational machinery, but also help us elucidate the role of quantum-based tools from the theory of scattering amplitudes in classical computations.

[11]Notice that due to the pollution from super-classical terms additional subtractions, also in the form of cuts, may still be needed even with potential-only modes, see e.g. eq. 6.18 in [114], resembling the iterated terms in [99]. However, these cuts also contain physical information needed to include non-conservative effects.

Note added. While the results in this paper were prepared for submission we became aware of the work in [158] which has some overlap with the derivations in our paper. We thank the authors for sharing a draft prior to submission.

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A Green's functions

Throughout this paper we use the following conventions for Feynman/Dyson, retarded/advanced and Wightman Green's functions:

$$\begin{aligned}
\Delta_{\mathrm{F}}(x) &= \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{e^{-i\,p\cdot x}}{p^2 + i0}\,, & \Delta_{\mathrm{D}}(x) &= \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{e^{-i\,p\cdot x}}{p^2 - i0}\,, \\
\Delta_{\mathrm{ret}}(x) &= \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{e^{-i\,p\cdot x}}{(p^0 + i0)^2 - \boldsymbol{p}^2}\,, & \Delta_{\mathrm{adv}}(x) &= \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{e^{-i\,p\cdot x}}{(p^0 - i0)^2 - \boldsymbol{p}^2}\,, \\
\Delta_{\pm}(x) &= 2\pi i \int \frac{\mathrm{d}^4 p}{(2\pi)^4} e^{-i\,p\cdot x} \delta(p^2)\Theta(\pm p^0)\,,
\end{aligned} \tag{A.1}$$

obeying, in position space,

$$\begin{aligned}
\Delta_{\mathrm{F}}(x) &= -\Delta_+(x)\Theta(x^0) - \Delta_-(x)\Theta(-x^0)\,, \\
\Delta_{\mathrm{D}}(x) &= \Delta_-(x)\Theta(x^0) + \Delta_+(x)\Theta(-x^0)\,, \\
\Delta_{\mathrm{ret}}(x) &= -(\Delta_+(x) - \Delta_-(x))\Theta(x^0), \\
\Delta_{\mathrm{adv}}(x) &= (\Delta_+(x) - \Delta_-(x))\Theta(-x^0)\,,
\end{aligned} \tag{A.2}$$

implying the relations

$$\begin{aligned}
\Delta_{\mathrm{ret}}(x) &= \Delta_{\mathrm{F}}(x) + \Delta_-(x) = \Delta_{\mathrm{D}}(x) - \Delta_+(x)\,, \\
\Delta_{\mathrm{adv}}(x) &= \Delta_{\mathrm{F}}(x) + \Delta_+(x) = \Delta_{\mathrm{D}}(x) - \Delta_-(x)\,,
\end{aligned} \tag{A.3}$$

B Classical electrodynamics

It is straightforward to use the formalism developed in this paper to study the case of classical electromagnetism, controlled by the worldline action

$$S = -\frac{1}{4}\int \mathrm{d}^4 x\, F_{\mu\nu}F^{\mu\nu} - \sum_a \int \mathrm{d}\tau_a \left(\frac{m_a}{2} v_a^2 + e\, q_a v_a^\mu A_\mu(x_a^\alpha)\right)\,, \tag{B.1}$$

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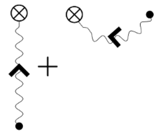
where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, A_μ is the four-vector potential, and $q_{a=1,2}$ are the respective charges. The computation simplifies drastically compared to the gravity case. The main reason is the absence of bulk interactions, which implies only two diagram are needed for the complete, all-order variation of the effective action:

$$\left.\frac{\delta \mathcal{S}_{\text{eff}}[x_+, x_-]}{\delta x_-^\alpha}\right|_{\substack{x_- \to 0 \\ x_+ \to x}} = \quad + \quad . \tag{B.2}$$



Hence, all higher order contributions to the deflection are exclusively coming from iterations via the equations of motion. The rest of the calculation is unchanged, such that the result depends on a subset of the master integrals for the gravity case. Expanding the impulse in powers of $\alpha = e^2/4\pi$,

$$\Delta p_1^\mu = \sum_{n=1}^{\infty} \Delta^{(n)} p_1^\mu \alpha^n \,, \tag{B.3}$$

we find for the conservative contribution

$$\begin{aligned}\Delta^{(3)} p^\mu_{1,\text{cons}} = & \frac{q_1^3 q_2^3 b^\mu}{8\pi^2 |b|^4 M^2 \nu^2 (x^2-1)^5} \left[4\left(x^2+1\right)x^4 + \nu\left(x^9 - 8x^6 + 14x^5 - 8x^4 + x\right)\right] \\ & + \frac{q_1^3 q_2^3 x\,(x^2+1)\,((\Delta_m - 1)\check{u}_1^\mu + (\Delta_m+1)\check{u}_2^\mu)}{4\pi |b|^3 M^2 (\Delta_m^2 - 1)^2 (x^2-1)^2} \,,\end{aligned} \tag{B.4}$$

whereas the dissipative part becomes

$$\begin{aligned}\Delta^{(3)} p^\mu_{1,\text{diss}} = & \frac{q_1^2 q_2^2 b^\mu}{96\pi^3 |b|^4 M^2 (x^2-1)^2 \nu^2} \Bigg[\left(x^2+1\right)^2 \left(q_1^2(\Delta_m + 2\nu - 1) - q_2^2(\Delta_m - 2\nu + 1)\right) \\ & \qquad + \frac{12\nu q_1 q_2 x\,(x^2+1)^2\,(x^4 - 1 - 4x^2\log(x))}{(x^2-1)^3}\Bigg] \\ & + \frac{q_1^2 q_2^2 \check{u}_2^\mu}{768\pi^2 |b|^3 M^2}\Bigg[\frac{(x^2+3)\,(3x^2+1)\,((\Delta_m-1)^2 q_1^2\,(x^2+1) + 2(\Delta_m+1)^2 q_2^2 x)}{(\Delta_m^2-1)^2\, x^2 (x^2-1)} \\ & \qquad - \frac{3q_1 q_2\,(3x^6 - 8x^5 + 45x^4 - 48x^3 + 45x^2 - 8x + 3)}{\nu\,(x^2-1)^3} \\ & \qquad + \frac{12 q_1 q_2 x^2\,(3x^4 + 10x^2 + 3)\log(x)}{\nu\,(x^2-1)^4}\Bigg] \,,\end{aligned} \tag{B.5}$$

which agrees with the result in [151, 152].

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