

Photon orbits and phase transition for non-linear charged anti-de Sitter black holes

Yun-Zhi Du, Huai-Fan Li, Fang Liu and Li-Chun Zhang

*Department of Physics, Shan xi Da tong University,
Da tong 037009, China*

E-mail: duyzh22@sxdttdx.edu.cn, huaifan999@163.com,
sapphire513@126.com, zhlc2969@163.cn

ABSTRACT: In this work, we investigate the relation between the photon sphere radius and the first-order phase transition for the charged Einstein-power-Yang-Mills AdS black hole. Through the analysis, we find with a certain condition there exist the non-monotonic behaviors between the photon sphere radius, the impact parameter, the non-linear Yang-Mills charge parameter, temperature, and pressure. And both the changes of photon sphere radius and impact parameter before and after phase transition can be regarded as the order parameter, their critical exponents near the critical point are equal to the same value $1/2$, just like the ordinary thermal systems. These indicate that there maybe exists a universal relation of gravity nearby the critical point for a black hole thermodynamical system. Furthermore, the effect of impact parameter on the deflect angle is also investigated.

KEYWORDS: Black Holes, Classical Theories of Gravity

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1 Introduction

Since the Hawking-Page phase transition [1] of an AdS spacetime was proposed by Hawking and Page in 1983, which was explained to a confinement/deconfinement phase transition in gauge theory in ref. [2] and could also be understood as a solid/liquid phase transition [3] by regarding the cosmological constant as pressure $P = -\frac{\Lambda}{8\pi} = \frac{(n-1)(n-2)}{16\pi l^2}$, black hole thermodynamics has attracted lots of attention. Subsequently the phase transition in the extended AdS/dS phase space has been widely considered [4–31]. Based on the similarity of thermodynamic behaviors for black holes and ordinary systems, the microstructure of black holes system was investigated by the Ruppeiner scalar curvature [20] following the Ruppeiner geometry [32]. The number density of the speculative black hole molecules was introduced to examine its phase transition and microstructure. This method was quickly generalized to other black holes [33, 34]. Furthermore from the view point of the Gibbs free energy landscape, the dynamic properties of various black hole phase transitions were studied in refs. [35–39]. Since the phase transition of black holes is more important in black hole thermodynamics, there exists a natural question how to probe the information of the black hole phase transition.

So far although experimental evidences of black hole thermodynamics have not been found, people still expect that black hole thermodynamics should have some observational signatures, for example the quasinormal modes (QNMs), which may be detected from astronomical observations in the future. Recently, the authors in refs. [40–50] had found that QNMs indeed exhibit characteristic behaviours along the black hole phase transition. Thus QNMs can be use to probe the phase transition of black holes in AdS/dS spacetime. Especially there is a dramatic change of the slope of QNMs along with the black hole phase transition [45]. These results indicate that there is a close connection between the dynamical and thermodynamical aspects of black holes. In addition, as we well known that QNMs are intimately linked to the unstable photon orbits [51–53], then they are playing an important role in strong gravitational phenomena (such as the shadow, lensing, and gravitational

waves) [54–62]. In short, QNMs can be regarded as the vibration frequencies of photon orbits and can be derived by the photon orbits [63]. Based on these observations, it is natural to construct the relation between the photon orbits and black hole phase transitions and to check out whether properties of photon orbits can signal the black hole phase transition.

Firstly, people proposed that the photon orbits signals a possible York-Hawking-Page phase transition in refs. [64, 65]. Then the more clearly relation between photon orbits and phase transitions of Reissner-Nordstrom-AdS black hole and rotating Kerr-AdS black hole were shown in refs. [66, 67]. Namely, the photon orbit radius and the minimum impact parameter along with the temperature are both exhibiting non-monotonic behaviours for the low pressure, whose behaviours are indeed the characters of the phase transition existence. Particularly, the changes of those two quantities near the critical point could be treated as the order parameters of phase transition, and they have an universal exponent $1/2$. Subsequently, this issue has been extended to other black holes [68] and gravity theory (Gauss-Bonnet case [69], massive gravity case [70], Born-Infeld case [71, 72]). A general proof of the existence for these relations between photon orbits and phase transitions were shown in ref. [73].

The linear charged black holes in AdS spacetime [74, 75] within a second-order phase transition show a scaling symmetry: at the critical point the state parameters scale respects to charge q , i.e., $S \sim q^2$, $P \sim q^{-2}$, $T \sim q^{-1}$ [76, 77]. It is naturally to guess whether there exists the scaling symmetry in the non-linear charged AdS black holes. As a generalization of the charged AdS Einstein-Maxwell black holes, it is interesting to explore new non-linear charged systems. Due to infinite self-energy of point like charges in Maxwell’s theory [78–82], Born and Infeld proposed a generalization when the field is strong, bringing in non-linearities [83, 84]. An interesting non-linear generalization of charged black holes involves a Yang-Mills field exponentially coupled to Einstein gravity, i.e., the Einstein-power-Yang-Mills (EPYM) theory, which possesses the conformal invariance and is easy to construct the analogues of the four-dimensional Reissner-Nordström black hole solutions in higher dimensions. Additionally several features of the EPYM gravity in extended thermodynamics have recently been studied [30, 85, 86].

Inspired by these, we mainly investigate the relation between photon orbits and phase transitions of the charged EPYM AdS black hole. This work is organized as follows. In section 2, we would like to briefly review the thermodynamic quantities and the second-order phase transition of charged EPYM AdS black hole. The null geodesics of this system and the deflection angle are also analyzed. Furthermore we will qualitatively analyze the relation of the hawking temperatures as a function of different coordinates (the event horizon radius and photon sphere radius) respectively. In section 3, we present the behaviors of the photon orbit radius and minimum impact parameter with the temperature nearby the phase transition point and explore the influence of non-linear YM charge parameter on the phase transition. A universal exponent for the changes of photon orbit radius and minimum impact parameter before and after phase transition is also obtained. A brief summary is given in section 4.

2 Null geodesics of non-linear charged AdS black holes

The action for four-dimensional Einstein-power-Yang-Mills (EPYM) gravity with a cosmological constant Λ was given by [86–89] ($8\pi G = 1$)

$$I = \frac{1}{2} \int d^4x \sqrt{g} (R - 2\Lambda - \mathcal{F}^\gamma) \quad (2.1)$$

with the Yang-Mills (YM) invariant \mathcal{F} and the YM field $F_{\mu\nu}^{(a)}$

$$\mathcal{F} = \text{Tr}(F_{\mu\nu}^{(a)} F^{(a)\mu\nu}), \quad (2.2)$$

$$F_{\mu\nu}^{(a)} = \partial_\mu A_\nu^{(a)} - \partial_\nu A_\mu^{(a)} + \frac{1}{2\xi} C_{(b)(c)}^{(a)} A_\mu^{(b)} A_\nu^{(c)}. \quad (2.3)$$

Here, $\text{Tr}(F_{\mu\nu}^{(a)} F^{(a)\mu\nu}) = \sum_{a=1}^3 F_{\mu\nu}^{(a)} F^{(a)\mu\nu}$, R and γ are the scalar curvature and a positive real parameter, respectively; $C_{(b)(c)}^{(a)}$ represent the structure constants of three-parameter Lie group G ; ξ is the coupling constant; and $A_\mu^{(a)}$ represents the $\text{SO}(3)$ gauge group Yang-Mills (YM) potentials defining by the Wu-Yang (WY) ansatz [90–92]. Variation of the action with respect to the spacetime metric $g_{\mu\nu}$ yields the field equations

$$G^\mu{}_\nu + \Lambda \delta^\mu{}_\nu = T^\mu{}_\nu, \quad (2.4)$$

$$T^\mu{}_\nu = -\frac{1}{2} \left(\delta^\mu{}_\nu \mathcal{F}^\gamma - 4\gamma \text{Tr} \left(F_{\nu\lambda}^{(a)} F^{(a)\mu\lambda} \right) \mathcal{F}^{\gamma-1} \right). \quad (2.5)$$

Variation with respect to the 1-form YM gauge potentials $A_\mu^{(a)}$ and implement the traceless yields the 2-forms YM equations

$$\mathbf{d} \left({}^\star \mathbf{F}^{(a)} \mathcal{F}^{\gamma-1} \right) + \frac{1}{\xi} C_{(b)(c)}^{(a)} \mathcal{F}^{\gamma-1} \mathbf{A}^{(b)} \wedge {}^\star \mathbf{F}^{(c)} = 0, \quad (2.6)$$

where $\mathbf{F}^{(a)} = \frac{1}{2} F_{\mu\nu}^{(a)} dx^\mu \wedge dx^\nu$, $\mathbf{A}^{(b)} = A_\mu^{(b)} dx^\mu$, and * stands for the duality. It is obviously that for the case of $\gamma = 1$ the EPYM theory reduces to the standard Einstein-Yang-Mills (EYM) theory [93, 94]. In this work our issue is paid on the role of the non-linear YM charge parameter γ .

Here we should point out that the non-Abelian property of the YM gauge field is expressed with its YM potentials

$$\mathbf{A}^{(b)} = \frac{q}{r^2} C_{(i)(j)}^{(a)} x^i dx^j, \quad r^2 = \sum_{j=1}^3 x_j^2, \quad (2.7)$$

and q is the YM charge, the indices (a, i, j) run the following ranges: $1 \leq a, i, j \leq 3$. The coordinates x_i take the following forms: $x_1 = r \cos \phi \sin \theta$, $x_2 = r \sin \phi \sin \theta$, $x_3 = r \cos \theta$. Since we have utilized the WY ansatz for the YM field, the invariant for this field takes the form [95–97]

$$\text{Tr}(F_{\mu\nu}^{(a)} F^{(a)\mu\nu}) = \frac{q^2}{r^4}. \quad (2.8)$$

This form leads to the disappearance of the structure constants which can be described the non-Abelian property of the YM gauge field. Therefore, under the condition of the WY

ansatz we may focus on the role of the non-linear YM charge parameter, instead of the non-Abelian character parameter.

For this system, the EPYM black hole solution with the negative cosmological constant Λ in the four-dimensional spacetime is obtained by adopting the following metric [98]:

$$ds^2 = -f(r)dt^2 + f^{-1}dr^2 + r^2d\Omega_2^2, \tag{2.9}$$

$$f(r) = 1 - \frac{2M}{r} - \frac{\Lambda}{3}r^2 + \frac{(2q^2)^\gamma}{2(4\gamma - 3)r^{4\gamma-2}}, \tag{2.10}$$

where $d\Omega_2^2$ is the metric on unit 2-sphere with volume 4π and q is the YM charge. Note that this solution is valid for the condition of the non-linear YM charge parameter $\gamma \neq 0.75$, and the power YM term holds the weak energy condition (WEC) for $\gamma > 0$ [87]. In the extended phase space, Λ was interpreted as the thermodynamic pressure $P = -\frac{\Lambda}{8\pi}$. For $\gamma = 1$, the YM charge term in the metric has the same form as the Maxwell charge term for the Einstein-Maxwell-Yang-Mills (EMYM) theory. Thus the contribution of the YM charge term on the thermodynamic property should be same as that of the Maxwell charge term in EMYM theory. The only difference is that they have different gauge groups: the YM field is of $SO(3)$, while the EM field is of $U(1)$. These comments are consistent with that in ref. [85]. The black hole event horizon locates at $f(r_+) = 0$. The parameter M represents the ADM mass of the black hole and it reads

$$M(S, q, P) = \frac{1}{6} \left[8\pi P \left(\frac{S}{\pi}\right)^{3/2} + 3 \left(\frac{S}{\pi}\right)^{\frac{3-4\gamma}{2}} \frac{(2q^2)^\gamma}{8\gamma - 6} + 3\sqrt{\frac{S}{\pi}} \right]. \tag{2.11}$$

And in our set up it is associated with the enthalpy of the system. The black hole temperature, entropy, and volume were given by [86]

$$T = \frac{1}{4\pi r_+} \left(1 + 8\pi P r_+^2 - \frac{(2q^2)^\gamma}{2r_+^{(4\gamma-2)}} \right), \quad S = \pi r_+^2, \quad V = \frac{4\pi r_+^3}{3}. \tag{2.12}$$

The YM potential Ψ was given by [80]

$$\Psi = \frac{\partial M}{\partial q^{2\gamma}} = \frac{r_+^{3-4\gamma} 2^{\gamma-2}}{4\gamma - 3}. \tag{2.13}$$

The above thermodynamic quantities satisfy the first law

$$dM = TdS + \Psi dq^{2\gamma} + VdP. \tag{2.14}$$

The equation of state $P(V, T)$ for canonical ensemble (fixed YM charge q) can be obtained from the expression of temperature as

$$P = \left(\frac{4\pi}{3V}\right)^{1/3} \left[\frac{T}{2} - \frac{1}{8\pi} \left(\frac{4\pi}{3V}\right)^{1/3} + \frac{(2q^2)^\gamma}{16\pi} \left(\frac{4\pi}{3V}\right)^{\frac{1-4\gamma}{3}} \right]. \tag{2.15}$$

It is known that this non-linear AdS black hole exhibits a vdW's-like phase transition, with the critical point given by [30, 99]

$$r_c^{4\gamma-2} = (2q^2)^\gamma f(1, \gamma), \quad f(1, \gamma) = \gamma(4\gamma - 1), \quad S_c = \pi (2q^2)^{\frac{\gamma}{2\gamma-1}} f^2(1, \gamma), \quad (2.16)$$

$$T_c = \frac{1}{\pi (2q^2)^{\gamma/(4\gamma-2)} f^{1/(4\gamma-2)}(1, \gamma)} \frac{2\gamma - 1}{4\gamma - 1}, \quad (2.17)$$

$$P_c = \frac{2\gamma - 1}{16\pi\gamma (2q^2)^{\gamma/(2\gamma-1)} f^{1/(2\gamma-1)}(1, \gamma)}. \quad (2.18)$$

where the YM charge parameter satisfies the condition $\frac{1}{2} < \gamma$. It is clear that this critical point closely depends on the YM charge and the non-linear YM charge parameter. We can obtain an interesting relation, which is only related with the non-linear YM charge parameter, from the above quantities as

$$S_c^2 T_c^2 P_c = \frac{(2\gamma - 1)^3}{16\pi\gamma(4\gamma - 1)}. \quad (2.19)$$

Now we will consider a free photon orbiting around a black hole on the equatorial hyperplane defined by $\theta = \pi/2$ and $p_\theta = 0$. The Lagrangian takes the form as

$$H = \frac{1}{2} g^{\mu\nu} p_\mu p_\nu = \frac{1}{2} \left(-f^{-1} p_t^2 + f p_r^2 + r^{-2} p_\phi^2 \right). \quad (2.20)$$

Note that the dot represents the derivative to the affine parameter and p_μ are the generalized momenta. For this black hole background, there are two Killing fields ∂_t and ∂_ϕ , which lead to two constants, the particle energy E and orbital angular momentum L along each geodesic

$$-E = p_t = -f(r)\dot{t}, \quad L = p_\phi = r^2\dot{\phi}. \quad (2.21)$$

The light rays are the solutions to Hamilton's equations

$$\dot{p}_\mu = -\frac{\partial H}{\partial x^\mu}, \quad \dot{x}^\mu = \frac{\partial H}{\partial p_\mu}, \quad (2.22)$$

which read

$$\begin{aligned} \dot{p}_t &= 0, & \dot{p}_\phi &= 0, & \dot{p}_\theta &= 0, \\ \dot{p}_r &= -\frac{1}{2} \left(\frac{f' p_t^2}{f^2} + f' p_r^2 + 2f p_r p_r' - \frac{2p_\phi^2}{r^3} \right), \\ \dot{t} &= -\frac{p_t}{f}, & \dot{r} &= f p_r, & \dot{\phi} &= \frac{p_\phi}{r^2}. \end{aligned} \quad (2.23)$$

With $H = 0$, we have

$$\dot{r}^2 + V_{\text{eff}} = 0, \quad V_{\text{eff}} = \frac{L^2 f}{r^2} - E^2. \quad (2.24)$$

As an example, we exhibit the effective potential in figure 1 for the EPYM AdS black hole with the fixed parameters $\gamma = 1.01$, $P = 0.003$, $q = 1.9$, $M = 2$, and different angular momenta L/E . Since the positive of \dot{r}^2 , there exists the condition: $V_{\text{eff}} < 0$. Thus the

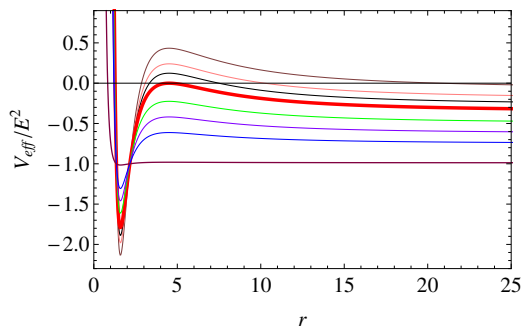


Figure 1. The effective potential for the EPYM AdS black hole with the parameters $\gamma = 1.01$, $P = 0.003$, $q = 1.9$, $M = 2$. The angular momentums L/E of the photon varies from 0.5 to 36 from bottom to top.

photon can only survive in the range of negative effective potential. When the conserved quantum L is small, the photon will fall into the black hole from somewhere with a larger value of r . While, for the larger value of L , the peak of effective potential will increase, that leads to the reflection of photon before it falls into black hole. Between these two cases, there exists a critical case which is described by the thickness red line, whose peak approaches zero at $r = 4.3122$ and at the same time the radial velocity of photon vanishes. This point just corresponds to the photon sphere because of the spherically symmetric static black hole. In the following, we mainly discuss the relation between the photon sphere radius and phase transition of the EPYM AdS black hole.

The unstable circular photon sphere is determined by

$$V_{\text{eff}} = 0, \quad \frac{dV_{\text{eff}}}{dr} = 0, \quad \frac{d^2V_{\text{eff}}}{dr^2} < 0. \quad (2.25)$$

From the second equation, we find that the radius r_{ph} satisfies

$$2f(r) \Big|_{r_{ph}} = r f'(r) \Big|_{r_{ph}}. \quad (2.26)$$

For given the metric function, we also obtain the radius from the above equation. Solving the first equation, the impact parameter or the angular momentum of the photon is

$$\mu_{ph} \equiv \frac{L}{E_c} = \frac{r}{\sqrt{f(r)}} \Big|_{r_{ph}}. \quad (2.27)$$

From this relation, we have the behaviors of the non-linear YM charge parameter and the impact parameter with the photon sphere radius, which are shown in figure 2. It is obviously that there exist the critical curves when given the pressure and mass of black hole. The corresponding critical values are exhibited in table 1. The critical impact parameter, radius, and non-linear YM charge parameter are both increasing with the YM charge.

Additionally, we will qualitatively analyze the relation of the phase transition both in the photon sphere radius coordinate and the event horizon radius coordinate in the following. The corresponding quantitative results will be presented in the next section.

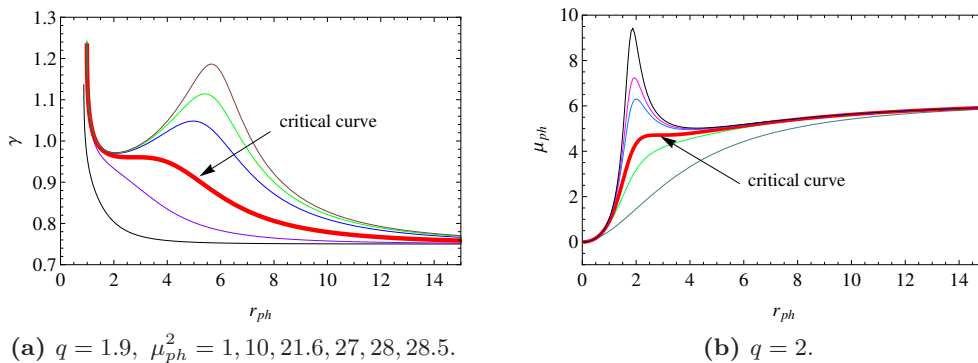


Figure 2. γ and μ_{ph} vs. r_{ph} with $P = 0.003$, $M = 2$. The non-linear YM parameter is set to 0.85, 0.95, 0.98, 1, 1.005, 1.01 in the $r_{ph} - \mu_{ph}$ diagram from the bottom to the top.

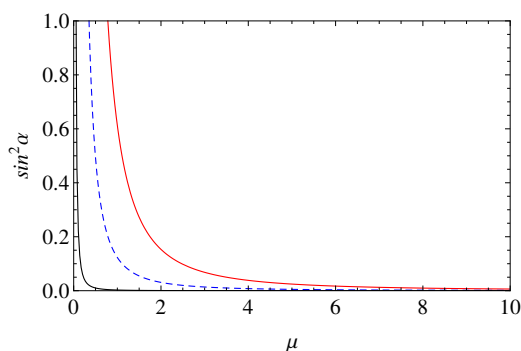


Figure 3. The angle as the function of impact parameter with the parameters $\gamma = 1$, $P = 0.0003$, $q = 1.9$, $M = 2$. The observer's position varies from 1 to 1000 from the red line to the black one.

Since the photon sphere radius meet the requirements in eqs. (2.26) and (2.27), so we have

$$r_{ph} = r_{ph}(r_+), \quad \text{or} \quad r_+ = r_+(r_{ph}), \tag{2.28}$$

$$f'(r_{ph}) = \frac{2r_{ph}}{\mu_{ph}^2} = \frac{df(r_{ph})}{dr_{ph}} = \frac{df(r_+)}{dr_+} \frac{dr_+}{dr_{ph}} > 0. \tag{2.29}$$

Because the temperature should be positive, we can obtain

$$\frac{df(r_+)}{dr_+} = \frac{T}{4\pi} > 0. \tag{2.30}$$

By combining above two expressions, we have

$$\frac{dr_+}{dr_{ph}} > 0, \tag{2.31}$$

which implies that there is a monotonously increasing relation between the event horizon radius and the photon sphere radius. For the van de Waals-like phase transitions for EPYM AdS black holes, the critical point is determined by the following equations

$$\left. \frac{dT(r_+)}{dr_+} \right|_{r_+=r_c} = \left. \frac{d^2T(r_+)}{dr_+^2} \right|_{r_+=r_c} = 0, \tag{2.32}$$

q	0.85	1	1.5	1.9	2
μ_{phc}^2	7.351515	10.18365	17.91095	21.5645	22.216
r_{phc}	1.06611	1.33639	2.18652	2.74	2.86303
γ_c	0.804019	0.821639	0.893342	0.960512	0.978177

Table 1. The critical quantities in $r_{ph} - \gamma$ diagram for different YM charges and $M = 2$.

where r_c is the critical event horizon radius of EPYM black holes. We just want to know whether this relation can be transferred to the photon sphere radius case. As

$$\left. \frac{dT(r_+)}{dr_+} \right|_{r_+=r_c} = \left. \frac{dT(r_{ph})}{dr_{ph}} \frac{dr_{ph}}{dr_+} \right|_{r_+=r_c} = 0, \quad (2.33)$$

we can know that there must exist a critical photon sphere radius r_{phc} that is of the following form

$$\left. \frac{dT(r_{ph})}{dr_{ph}} \right|_{r_{ph}=r_{phc}} = 0 \quad (2.34)$$

after considering eq. (2.31). Here the relation $T(r_+) = T(r_+(r_{ph})) = T(r_{ph})$ has been kept in mind. From the relation

$$\left. \frac{d^2T(r_+)}{dr_+^2} \right|_{r_+=r_c} = \left[\frac{d^2T(r_{ph})}{dr_{ph}^2} \frac{dr_{ph}}{dr_+} + \frac{dT(r_{ph})}{dr_{ph}} \frac{d}{dr_{ph}} \left(\frac{dr_{ph}}{dr_+} \right) \right] \times \left. \frac{dr_{ph}}{dr_+} \right|_{r_+=r_c} = 0 \quad (2.35)$$

together with the conditions eqs. (2.31) and (2.34), we can obtain

$$\left. \frac{dT(r_{ph})}{dr_{ph}} \right|_{r_{ph}=r_{phc}} = \left. \frac{d^2T(r_{ph})}{dr_{ph}^2} \right|_{r_{ph}=r_{phc}} = 0. \quad (2.36)$$

From eqs. (2.32) and (2.36), we can know that the critical point in the $r_+ - T$ diagram corresponds to the one in the $r_{ph} - T$ diagram. Apart from this, we can also know that

$$\left. \frac{dT(r_+)}{dr_+} \right|_{r_+=r_c} = \left. \frac{dT(r_{ph})}{dr_{ph}} \frac{dr_{ph}}{dr_+} \right|_{r_+=r_c} > 0 \quad \text{or} \quad < 0 \quad (2.37)$$

can correspond to

$$\left. \frac{dT(r_{ph})}{dr_{ph}} \right|_{r_{ph}=r_{phc}} > 0 \quad \text{or} \quad < 0, \quad (2.38)$$

respectively as we have the relation eq. (2.31), which means that the correspondence between $r_+ - T$ diagram and $r_{ph} - T$ diagram exists beyond the critical points.

Next, we will investigate the effect of the angular momentum on the observer's angle, especially the critical angle with the maximum impact parameter. Considering light rays sent from an observer at radius coordinate r_0 into the past. These light rays can be divided into two classes: light rays of the first class go to infinity after being deflected by the black hole. Light rays of the second one go towards the horizon of the black hole. If there are no light sources between the observer and the black hole, initial directions of the second class

correspond to darkness on the observer's sky. This dark circular disk on the observer's sky is called the shadow of the black hole. The boundary of the shadow is determined by the initial directions of light rays that asymptotically spiral towards the outermost photon sphere. Note that the light rays in the photon sphere are unstable with respect to radial perturbations.

We consider a light ray that is sent from the observer's position at r_0 into the past under a deflection angle α with respect to the radial direction. The deflection angle is given by

$$\cot \alpha = \sqrt{\frac{g_{rr}}{g_{\phi\phi}}} \frac{dr}{d\phi} \Big|_{r_0} = \sqrt{\frac{1}{f(r)r^2}} \frac{dr}{d\phi} \Big|_{r_0}. \quad (2.39)$$

From eqs. (2.20) and (2.23), we have the orbit equation

$$\frac{dr}{d\phi} = \sqrt{f(r)r^2 \sqrt{\frac{r^2\mu^2}{f(r)} - 1}} \quad (2.40)$$

with the impact parameter or the angular momentum $\mu \equiv \frac{L}{E}$. Thus the deflection angle α satisfies

$$\sin^2 \alpha = \frac{f(r_0)}{r_0^2 \mu^2}. \quad (2.41)$$

Especially, the critical angle reads $\sin^2 \alpha_c = \frac{f(r_0)}{\mu_{ph}^2 r_0^2}$. That indicates that μ has a close relation with the critical deflection angle and the observer's position, which is just the phenomenon of the black hole lensing. For the light ray with the large impact parameter, its deflection angle is small. Gradually decreasing μ to μ_{ph} , the deflection angle will be larger and larger, until to the bounded one α_c . These behaviors are presented in figure 3. That is consistent with that showed in ref. [54].

As what we have shown in the above, there are two important quantities r_{ph} and μ_{ph} for a photon sphere. In the following, we will probe the detailed behaviors of these two quantities near phase transition, and discuss whether they carry the phase transition information.

3 Behaviors of the photon sphere radius and the minimum impact parameter near phase transition point

For the EPYM AdS black hole with the non-linear source, the condition of the first-order phase transition was presented in our work [30] as

$$\chi \frac{2\gamma - 1}{\gamma^{1/(4\gamma-2)}(4\gamma - 1)^{(4\gamma-1)/(4\gamma-2)}} = \frac{1}{f^{1/(4\gamma-2)}(x, \gamma)} \left(1 + x - \frac{1 - x^{4\gamma}}{2f(x, \gamma)(1 - x)x^{4\gamma-2}} \right),$$

$$\frac{(2q^2)^\gamma}{r_2^{4\gamma-2}} = \frac{1}{f(x, \gamma)} \quad (3.1)$$

with

$$f(x, \gamma) = \frac{(3 - 4\gamma)(1 + x)(1 - x^{4\gamma}) + 8\gamma x^2(1 - x^{4\gamma-3})}{2x^{4\gamma-2}(3 - 4\gamma)(1 - x)^3}, \quad x = \frac{r_1}{r_2}. \quad (3.2)$$

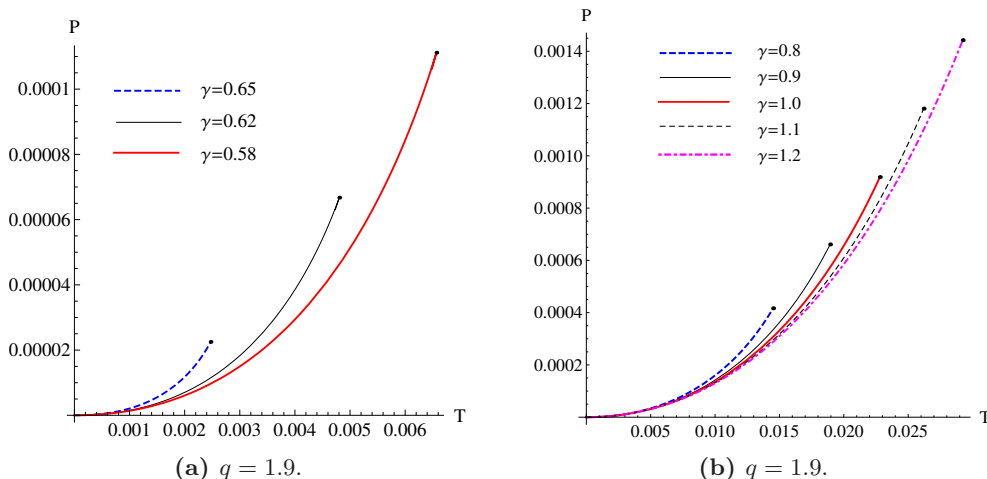


Figure 4. The coexistent curve of $P - T$.

And we also investigated the effect of the non-linear YM charge parameter γ on the phase transition from two different viewpoints. Firstly, from the viewpoint of the coexistent curve ($P - T$), when fixed the YM electric q we found that for a given temperature ($T < T_c$) or a given pressure ($P < P_c$), the pressure or the temperature and the radius of the coexistent big black hole phase increase with the increasing of γ ; while the radius of the coexistent small black hole phase decreases with γ . Here the two-phases coexistent curves in the $T - P$ plane with different values of the non-linear YM charge parameter γ are shown in figure 4. Secondly, from the viewpoint of the geometry scalar curvature, the absolute values of the scalar curvature for the coexistent big and small black holes in the range of $3/4 < \gamma \leq 1$ both decrease with the increasing of γ , while their behavior of the scalar curvature are just opposite in the range of $1 \leq \gamma$. In the following, we only exhibit the behaviors of the photon sphere radius as the function of temperature in two cases of $\gamma = 1$ (the analytical solution) and $\gamma = 1.5$ (the numerical Solution), respectively. Since the solutions of the photon sphere radius with other values of γ are difficult to present, so we cannot give the relevant analysis and conclusions.

For $\gamma = 1$, the photon sphere radius of this system can be expressed as

$$r_{ph} = \frac{1}{2} \left(3M + \sqrt{9M^2 - 8q^2} \right). \tag{3.3}$$

It is clear that this result is exactly equal to that of the asymptotically flat charged black hole, in which the orbit radius depends on the pressure. However, we need to note that the mass of eq. (2.11) is related with the pressure. Substituting it into above equation, we have the photon sphere radius

$$r_{ph} = \frac{3\pi q^2 + S(3 + 8PS) + \sqrt{[3\pi q^2 + S(3 + 8PS)]^2 - 32\pi q^2 S}}{4\sqrt{\pi S}}. \tag{3.4}$$

At the critical point given in eqs. (2.16) and (2.18), the critical radius reads

$$r_{phc} = (2 + \sqrt{6})q. \tag{3.5}$$

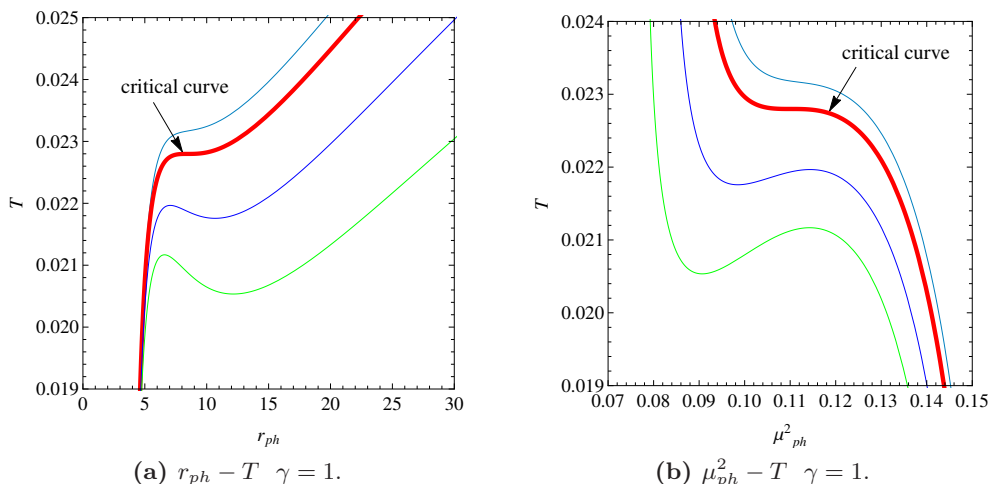


Figure 5. The YM charge set to $q = 1.9$. The pressure varies from $P_c - 0.0002$ to $P_c + 0.00004$ from bottom to top.

The temperature in eq. (2.12) can be rewritten as

$$T(S, P, q) = \frac{1}{4\sqrt{\pi S}} \left(1 + 8PS - \frac{\pi q^2}{S} \right). \quad (3.6)$$

Combining eqs. (3.4) and (3.6), we give the behavior of temperature as a function of the photon sphere radius with the fixed pressure, which is shown in figure 5a. It is obviously that for $P < P_c$, there will be a non-monotonic behavior. However, the temperature will become only a monotone increasing function of the photon sphere radius when the pressure is bigger than the critical one. At the critical pressure, there is a deflection point at $r_{ph} = r_{phc}$. That behavior is very similar to the isobar of the vdW's system in $T - S$ plane, which means there exists the first-order phase transition. Thought constructing the Maxwell's equal-area law, we also obtain the phase transition point.

The behavior of the temperature as a function of the impact parameter square in Figs 5b. Interestingly, it also confirms a non-monotonic property for $P < P_c$. And it becomes monotonous for $P \leq P_c$. These non-monotonic behaviors in $r_{ph} - T$ and $\mu_{ph}^2 - T$ diagrams indicate the existence of a first-order phase transition. So are these phase transitions $r_{ph} - T$ and $\mu_{ph}^2 - T$ diagrams just the ones between the high-potential black hole phase and low-potential black hole phase? We will illustrate this by depicting phase diagrams in the $r_+ - T$ and $r_{ph} - T$ planes under the same condition (see figure 6a). From the figures, we can see that the $r_+ - T$ and $r_{ph} - T$ diagrams share the synchronized variation trend. The correspondence between $T'(r_+)$ and $T'(r_{ph})$, as well as between $T''(r_+)$ and $T''(r_{ph})$, shown in last part has been conspicuously corroborated. That means that the correspondence between $r_+ - T$ and $r_{ph} - T$ diagrams exists beyond the critical points. Thus the first-order phase transition temperature both in $r_+ - T$ and $r_{ph} - T$ curves are the same, which is caused by the coordinate transformation between the horizon radius and the photon sphere radius. From the physical perspective, this phenomenon is equivalent to the situation: the observer is a static observer at infinity, and the object of study is EPYM

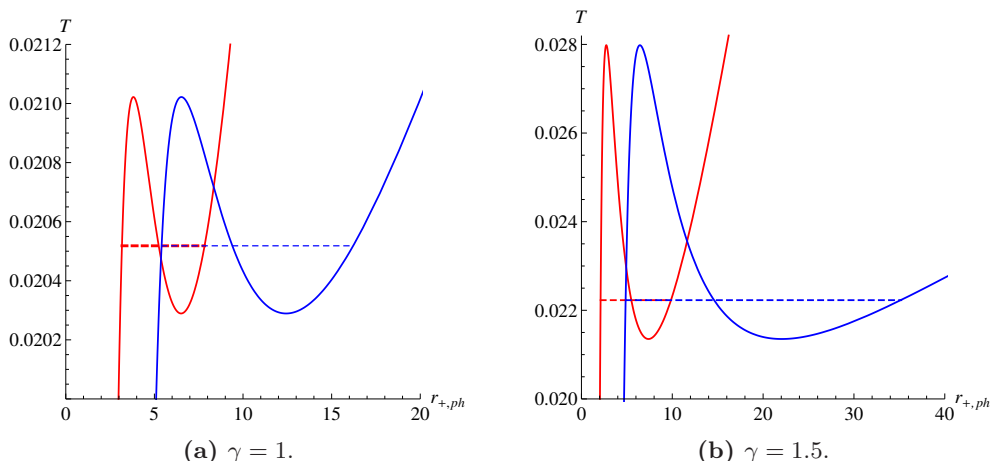


Figure 6. The parameters set to $q = 1.9$ and $P = P_c - 0.0002$. The red lines stand for the curves with the event horizon radius coordinate, and the blue lines are the curves with the photon sphere radius coordinate.

AdS black hole undergoing a phase transition; and neither the observer nor the object of research has changed, so the thermodynamical properties of the system will not change.

Furthermore, when $\gamma = 1.5$ we also present the numerical results of the temperature with the photon sphere radius and impact parameter for different pressures in figure 7. The phase transition relationship from two different viewpoints of the photon sphere radius and the horizon radius is also exhibited in figure 6b. Those behaviors in this case are similar to that in the case of $\gamma = 1$. In other words, in both two cases of $\gamma = 1$ and $\gamma = 1.5$ r_{ph} and μ_{ph}^2 have the non-monotonic behavior and exhibit the first-order phase transition under a certain condition. And the photon sphere radius of two coexistent phases of the system are both larger than them from the viewpoint of the horizon radius. These results are the same in these two cases. While the differences shown in figure 8 are that: the phase transition temperature with $\gamma = 1.5$ is high than the one with $\gamma = 1$; the photon sphere radii and impact parameter of two coexistent phases are different in this two cases.

In order to probe the changes of the photon sphere radius and the impact parameter during the phase transition process, we present the temperature as a function of the photon sphere radius with different pressure ($P < P_c$) in figure 9. The dashed lines stand for the coexistent phases, the corresponding phase transition temperature is determined by eq. (3.6). With the increasing of photon sphere radius, the phase transitions will emerge at the dashed lines. And there existences the sudden change of photon sphere radius. Further increasing the pressure or temperature to the critical values, the sudden change of photon sphere radius disappears: $\Delta r_{ph} \equiv r_{ph2} - r_{ph1} = 0$. Thus if the photon sphere radius has a sudden change, the black hole must be undergoing a first-order phase transition. The similar conclusions hold for the case between the impact parameter and temperature. With the method of numerical drawing, we exhibit the changes of the reduced photon sphere radius and reduced impact parameter ($\Delta r_{ph}/r_{phc}$ and $\Delta \mu_{ph}/\mu_{phc}$) as the functions of the reduced temperature (T/T_c) with the parameters $\gamma = 1$, $q = 1.9$ in figure 10. It

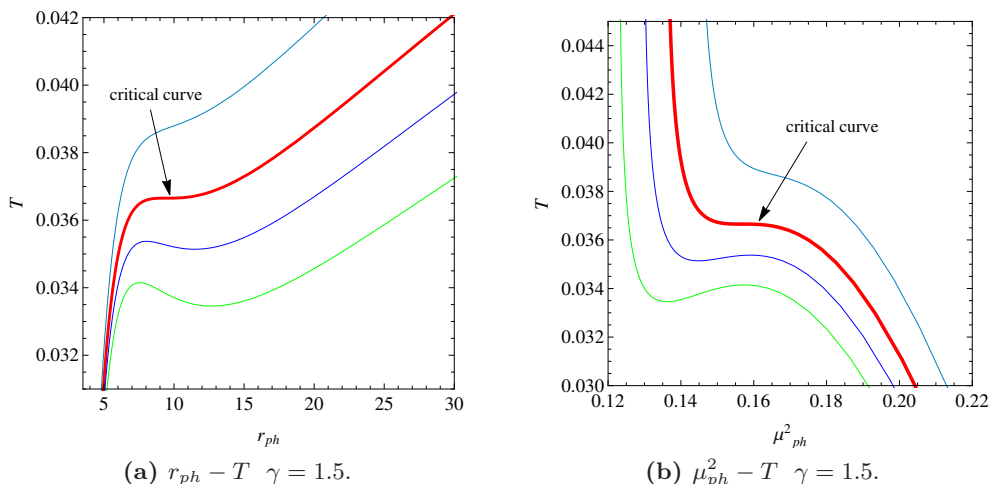


Figure 7. The YM charge set to $q = 1.9$. The pressure varies from $P_c - 0.0004$ to $P_c + 0.00003$ from bottom to top.

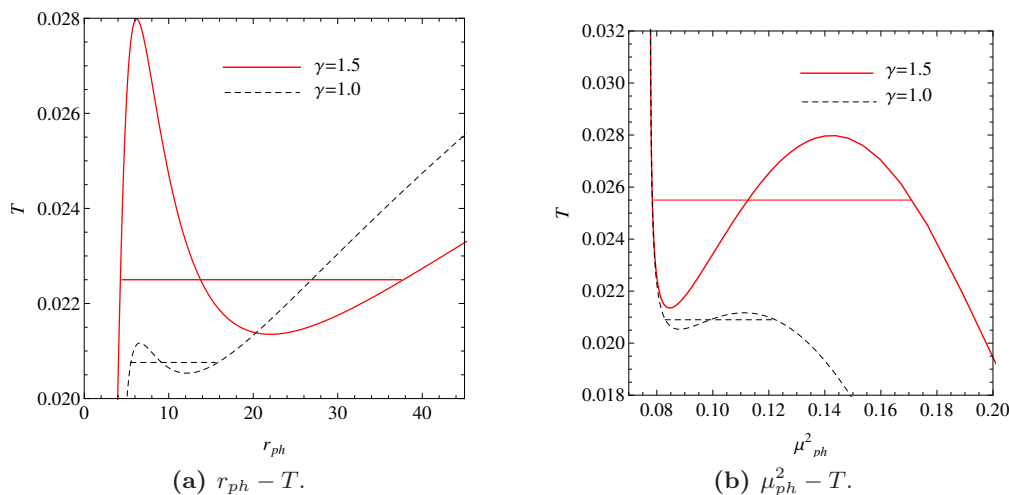


Figure 8. The pressures set to $P_c - 0.0002$ and $q = 1.9$.

is obviously that both the values of $\Delta r_{ph}/r_{phc}$ and $\Delta \mu_{ph}/\mu_{phc}$ decrease with the reduced temperature, and they are approach to zero until $\frac{T}{T_c} = 1$. At the same time the first-order phase transition becomes the second-order one. Therefore the changes of reduced photon sphere radius and reduced impact parameter can be regarded the order parameters for the black hole phase transition.

It is well known that the critical exponent of the order parameter for a ordinary thermodynamic system nearby the critical point equals to $1/2$, whether these two order parameters have the same value of critical exponent. Using the numerical fitting method near T_c , we find

$$\Delta r_{ph}/r_{phc} \sim 3.935\sqrt{1 - T/T_c}, \tag{3.7}$$

$$\Delta \mu_{ph}/\mu_{phc} \sim 1.1423\sqrt{1 - T/T_c}. \tag{3.8}$$

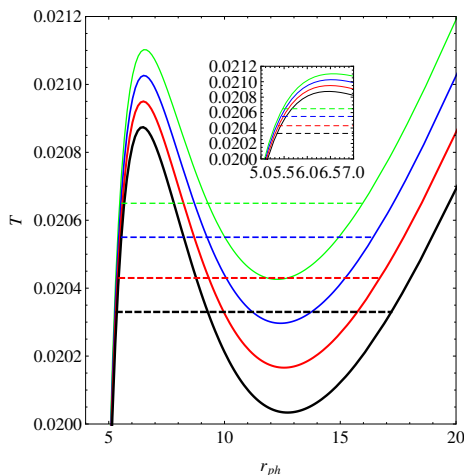


Figure 9. The parameters set to $q = 1.9$ and $\gamma = 1$. The pressure varies from $P_c - 0.00023$ to $P_c + 0.0002$ from bottom to top.

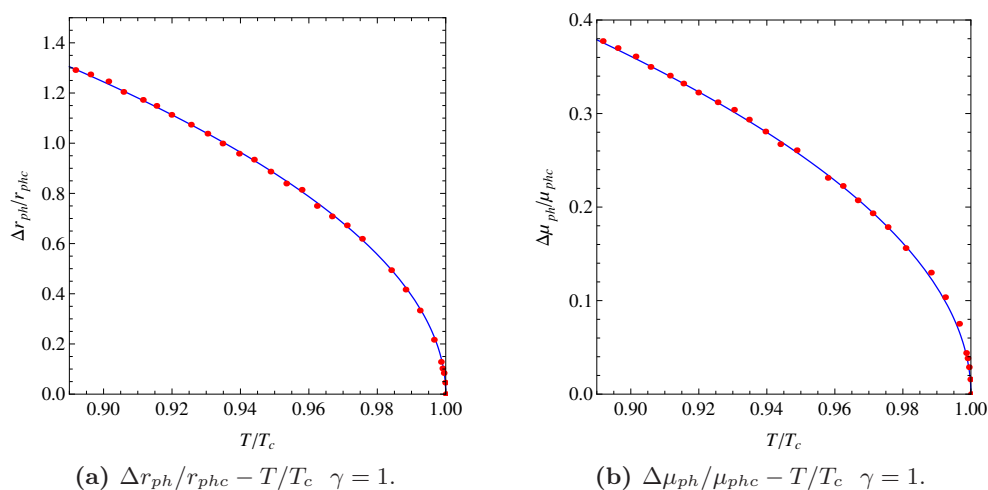


Figure 10. The YM charge set to $q = 1.9$.

This result exactly confirms that both the critical exponents of $\Delta r_{ph}/r_{phc}$ and $\Delta \mu_{ph}/\mu_{phc}$ equal to $1/2$ at the critical point. That indicates from the viewpoint of thermodynamic black holes have more similar behaviors as the ordinary thermal systems, and they maybe also have the similar microstructures from the perspective of the statistics [20]. Those conclusions strongly imply there exist a relationship between the photon sphere radius and phase transition of black holes.

4 Discussions and conclusions

In this manuscript, we examined the relationship between the photon sphere radius the thermodynamic phase transition for the four-dimensional charge Einstein-power-Yang-Mills (EPYM) AdS black hole. This will provide a new way to realize the link between the gravity and thermodynamics of black holes.

First, we presented the characters of EPYM AdS black hole phase transition: the critical point depend on the YM charge and non-linear YM charge parameter with the condition $\gamma > 1/2$ and $\gamma \neq 3/4$. Since the critical point stands for the boundary of the coexistent phases, it is keys to probe the black hole phase structure. Then we investigated the null geodesics of a photon in the equatorial plane of EPYM AdS black hole background. By analyzing the effective potential of photon orbits with the certain parameters, we obtained the photon sphere radius and impact parameter (angular momentum of the photon sphere). The results showed that the non-monotonic behaviors appear in $\gamma - r_{ph}$ and $\mu_{ph} - r_{ph}$ diagrams. That indicated that there exists a certain relationship between the photon sphere radius, impact parameter, and EPYM AdS black hole phase transitions. In addition, we also presented the influence of impact parameter on the reflected angle.

Finally, we further explored the relation between the unstable photon sphere radius, impact parameter, and thermodynamic phase transition. It mainly contains three aspects as the following:

- For the isobar and isotherm processes of this system, the photon sphere radius and impact parameter have the non-monotonic behaviors with the certain parameters. When the pressure or temperature is less than the corresponding critical value, these two quantities both have two extremal points. Until the pressure or temperature approaches to the critical one, two extremal points coincide with each other. When the pressure or temperature is more than the critical one, there is no extremal point. Thus, the photon sphere radius and impact parameter are the non-monotonic functions with the pressure or temperature. Those behaviors of the photon sphere radius and impact parameter are consistent with that of the black hole thermodynamics. Therefore, the behavior of photon sphere can be regarded a probe to reveal the thermodynamic phase transition information of black holes;
- No matter the qualitative analysis and quantitative calculation, the behaviours both in the $r_+ - T$ and $r_{ph} - T$ diagrams have the synchronized variation trend. The correspondence between $T'(r_+)$ and $T'(r_{ph})$, as well as between $T''(r_+)$ and $T''(r_{ph})$ have been conspicuously corroborated. In other words, the phase transition information (including the first-order transition and the critical point) of the EPYM AdS black hole can be reflected by its surrounding particle's circular orbit (photon sphere orbit);
- At the first-order phase transition point, the photon sphere radius and impact parameter both have a sudden change. The changes of them before and after first-order phase transition can act as order parameters to describe EPYM AdS black hole phase transitions. More importantly, at the critical point, the fitting result displays that they have a same critical exponent of $1/2$.

Those analyze firmly enhance our conjecture that there exists the relationship between the null geometry and thermodynamic phase transition for EPYM AdS black holes. It further supports the link between the gravity and thermodynamics of black holes, and also provides a possible way to describe the strong gravitational effect from the thermodynamic side.

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