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On the Nekrasov partition function of gauged Argyres-Douglas theories

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Abstract: We study SU(2) gauge theories coupled to (A_1, D_N) theories with or without a fundamental hypermultiplet. For even N, a formula for the contribution of (A_1, D_N) to the Nekrasov partition function was recently obtained by us with Y. Sugawara and T. Uetoko. In this paper, we generalize it to the case of odd N in the classical limit, under the condition that the relevant couplings and vacuum expectation values of Coulomb branch operators of (A_1, D_N) are all turned off. We apply our formula to the (A_2, A_5) theory to find that its prepotential is related to that of the SU(2) gauge theory with four fundamental flavors by a simple change of variables.

Keywords: Supersymmetric Gauge Theory, Supersymmetry and Duality

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Contents

1 Introduction

The Nekrasov's instanton partition function [1] for 4D $\mathcal{N} = 2$ gauge theories has uncovered various non-perturbative phenomena in these theories. For instance, the Seiberg-Witten prepotential was derived from the path integral [1, 2], a relation to integrable systems was discovered [3], and a novel 2d/4d correspondence called the AGT correspondence was discovered [4, 5].

A generalization of the above success to theories coupled to a strongly-coupled superconformal field theories (SCFTs) has partially been studied. In particular, the AGT correspondence has been generalized in [6, 7] to gauge theories coupled to Argyres-Douglas

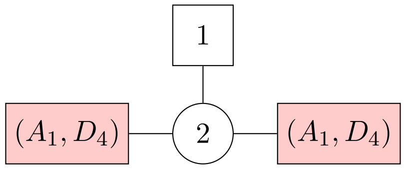

Figure 1. The (A_3, A_3) theory is identical to the conformal SU(2) gauge theory coupled to two (A_1, D_4) theories and a fundamental hypermultiplet of SU(2). Here, the middle circle with 2 inside stands for an SU(2) vector multiplet, and the top box with 1 inside stands for a fundamental hypermultiplet.

(AD) theories. We call these gauge theories "gauged AD theories." Since AD theories have no weak-coupling limit, supersymmetric localization is not available for these theories. As a result, the generalized AGT correspondence has been the only promising way of evaluating the instanton partition function of these theories.

One restriction of the generalized AGT correspondence was, however, that it was only applied to non-conformally gauged AD theories.[1] The reason for this is that conformally gauged AD theories have no known realization from 6d (2,0) A_1 theory, and therefore the AGT correspondence is not directly applied to them. As a result, until recently, the instanton partition function of conformally gauged AD theories was not evaluated.

A first idea of computing the instanton partition function of conformally gauged AD theories has been provided in [8]. A key ingredient is the U(2) version of the generalized AGT correspondence, which is stated in terms of irregular states of the direct sum of Virasoro and Heisenberg algebra $Vir \oplus H$. For instance, let us consider SU(2) gauge theory coupled to a fundamental hypermultiplet and two copies of AD theory called (A_1, D_4) (figure 1). Here, the "matter" sector is precisely chosen so that the beta function of the SU(2) gauge coupling vanishes. This coupled theory is also known as the "(A_3, A_3) theory." While the AGT correspondence cannot be directly applied to the (A_3, A_3) theory, one can apply it to a factor in the following decomposition of the partition function:

$$\mathcal{Z} = \mathcal{Z}_{\text{pert}} \sum_{Y_1, Y_2} q^{|Y_1|+|Y_2|} \mathcal{Z}^{\text{vec}}_{Y_1,Y_2}(a) \mathcal{Z}^{\text{fund}}_{Y_1,Y_2}(a, M) \prod_{i=1}^{2} \mathcal{Z}^{(A_1,D_4)}_{Y_1,Y_2}(a, m_i, d_i, u_i)\,, \tag{1.1}$$

where a is the vacuum expectation value (VEV) of the Coulomb branch operator in the vector multiplet, q is the exponential of the gauge coupling, the sum runs over pairs of Young diagrams (Y_1, Y_2), $|Y|$ stands for the number of boxes in a Young diagram Y, and $\mathcal{Z}^{\text{vec}}_{Y_1,Y_2}$ and $\mathcal{Z}^{\text{fund}}_{Y_1,Y_2}$ are the contributions from the vector and hypermultiplets.[2] The factor $\mathcal{Z}^{(A_1,D_4)}_{Y_1,Y_2}$ is the contribution from an (A_1, D_4) theory, which is hard to evaluate via localization but can be evaluated via the U(2)-version of the generalized AGT correspondence [8].

The reason for the "U(2)-version" is that the decomposition (1.1) is possible only when the gauge group is U(2) instead of SU(2). The difference between U(2) and SU(2) gives rise to a prefactor of the partition function, known as the U(1)-factor. By factoring

[1]Here, by "non-conformally gauged," we mean that the beta function of the gauge coupling is asymptotically free.

[2]Here $\mathcal{Z}_{\text{pert}}$ is a prefactor that makes the q-series start with 1.

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out the U(1)-factor, one can read off the partition function and the prepotential of the (A_3, A_3) theory from (1.1). As discussed in [8], when dimensionful parameters are turned off except for a, the prepotential $\mathcal{F}_{(A_3,A_3)}(q;a)$ of the (A_3, A_3) theory read off as above is in a surprising relation to the prepotential $\mathcal{F}^{N_f=4}_{\mathrm{SU}(2)}(q;a)$ of SU(2) gauge theory with four fundamental flavors, i.e.,

$$2\mathcal{F}_{(A_3,A_3)}(q;a) = \mathcal{F}^{N_f=4}_{\mathrm{SU}(2)}(q^2;a)\,. \tag{1.2}$$

This remarkable relation was then used to read off how the S-duality of (A_3, A_3) acts on the UV gauge coupling q.

While the above U(2)-version of the generalized AGT correspondence provides a novel way of evaluating the instanton partition function of conformally gauged AD theories, one of its restrictions is that the formula provided in [8] is only for (A_1, D_{even}) theories. The reason for this is that only irregular states of *integer* ranks were constructed in [8], and those of *half-integer* ranks are still to be identified.[3]

In this paper, we extend the result of [8] to the case of (A_1, D_{odd}) theories, under the condition that all couplings and VEVs of Coulomb branch operators in (A_1, D_{odd}) are turned off. This is done by explicitly identifying the action of $Vir \times H$ on irregular states of half-integer ranks. This action turns out to be very simple in the classical limit $\epsilon_1, \epsilon_2 \to 0$, when the above condition is satisfied.

As an application of our extension, we evaluate the prepotential of the (A_2, A_5) theory, which is the conformal SU(2) gauge theory coupled to a fundamental hypermultiplet and AD theories called (A_1, D_6) and (A_1, D_3) (figure 2).[4] To compute the partition function $\mathcal{Z}_{(A_2,A_5)}$ of this theory, one needs to know the contribution of the (A_1, D_3) and (A_1, D_6) theories at each fixed point on the instanton moduli space, i.e., $\mathcal{Z}^{(A_1,D_3)}_{Y_1,Y_2}$ and $\mathcal{Z}^{(A_1,D_6)}_{Y_1,Y_2}$. While the latter can be evaluated via the method of [8], computing the former needs a prescription that we develop in this paper. We then read off from $\mathcal{Z}_{(A_2,A_5)}$ an expression for the prepotential $\mathcal{F}_{(A_2,A_5)}$ of the (A_2, A_5) theory, which turns out to be in a surprising relation to $\mathcal{F}^{N_f=4}_{\mathrm{SU}(2)}$:

$$3\mathcal{F}_{(A_2,A_5)}(q;a) = \mathcal{F}^{N_f=4}_{\mathrm{SU}(2)}(q^3;a)\,. \tag{1.3}$$

Note that this relation is quite similar to (1.2) but different. From this relation, we read off how the S-duality group acts on the UV gauge coupling q of the (A_2, A_5) theory. A generalization of our result to the case of all dimensionful parameters turned on is left for future work.

The organization of this paper is the following. In section 2, we review the generalized AGT correspondence and its U(2)-version. In section 3, we consider the generalization of the U(2)-version to (A_1, D_{odd}). In section 4, we apply a formula developed in section 3 to the (A_2, A_5) theory and show that the prepotential of (A_2, A_5) is related to that of SU(2) superconformal QCD by a change of variables. In section 5, we show that the prepotential relation found in section 4 is consistent with the Seiberg-Witten curve.

[3] As explained in the next section, the rank of an irregular state $|I\rangle$ is defined by the maximal $n \in \mathbb{N}/2$ such that $L_{2n}|I\rangle \neq 0$.

[4] See [9] for a recent discussion on the conformal manifold of (A_n, A_m) theories.

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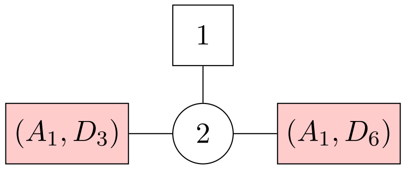

Figure 2. The (A_2, A_5) theory is an $\mathcal{N} = 2$ SCFT, which is identical to a conformal SU(2) gauging of (A_1, D_3) and (A_1, D_6) theories together with a fundamental hypermultiplet of SU(2).

2 U(2)-version of generalized AGT for (A_1, D_{even})

In this section, we give a brief review of the U(2)-version of the generalized AGT correspondence for (A_1, D_N) theories with even N.

2.1 Generalized AGT correspondence

We first review the original generalized AGT correspondence. Recall that the (A_1, D_N) theory for a positive integer $N \geq 2$ is realized by compactifying the 6d (2,0) A_1 theory on a sphere with two punctures, one of which is a regular puncture and the other is an irregular puncture of rank $N/2$ [6, 7, 10]. These punctures specify how the Higgs field $\Phi(z)$ in the corresponding Hitchin system behaves around them; $\Phi(z)$ has a simple pole at a regular puncture while it behaves as $\Phi(z) \sim 1/z^{N/2+1}$ around an irregular puncture of rank $N/2$, where we take $z = 0$ as the locus of the puncture.

According to the generalized AGT correspondence [6, 7], the regular puncture corresponds to a Virasoro primary state $|a\rangle$, and the irregular puncture corresponds to an irregular state $|I^{(N/2)}\rangle$ of Virasoro algebra at central charge $c = 1 + 6Q^2$. While there are two different characterizations of $|I^{(N/2)}\rangle$, we will use the one discussed in [7]. Here, the irregular state is not a primary state but a simultaneous eigen state of L_k for $k \geq \lceil N/2 \rceil$, with vanishing eigenvalues for $k > N$. Therefore, an irregular state $|I^{(N/2)}\rangle$ satisfies

$$L_k |I^{(N/2)}\rangle = \begin{cases} 0 & \text{for} \quad N < k \\ \lambda_k |I^{(N/2)}\rangle & \text{for} \quad \left\lceil \frac{N}{2} \right\rceil \leq k \leq N \end{cases}, \tag{2.1}$$

for a set of eigenvalues $\{\lambda_{\lceil N/2 \rceil}, \cdots, \lambda_N\}$.[5] This characterization of the irregular state is such that

$$x^2 = -\frac{\langle a|T(z)|I^{(N/2)}\rangle}{\langle a|I^{(N/2)}\rangle} \tag{2.2}$$

is equivalent to the Seiberg Witten (SW) curve of the 4d theory. Indeed, from (2.1), we see that (2.2) is evaluated as

$$x^2 = -\frac{\lambda_N}{z^{N+2}} - \frac{\lambda_{N-1}}{z^{N+1}} - \cdots - \frac{a(Q-a)}{z^2}, \tag{2.3}$$

which is identical to the SW curve of the (A_1, D_N) theory.

[5] While $\lceil N/2 \rceil = N/2$ for even N, we here write things so that they can be easily generalized to odd N in the next section.

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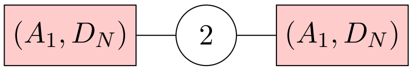

Figure 3. SU(2) gauge theory coupled to two (A_1, D_N) theories.

Given the above regular state $|a\rangle$ and the irregular state $|I^{(N/2)}\rangle$, the generalized AGT correspondence states that

$$\mathcal{Z}_{(A_1,D_N)} = \langle a|I^{(N/2)}\rangle\,, \tag{2.4}$$

is identified with the Nekrasov partition function of the (A_1, D_N) theory. Note here that, since no weakly-coupled description is known for this theory, the above partition function cannot be evaluated by supersymmetric localization.

Similarly, 4d $\mathcal{N} = 2$ SU(2) gauge theory coupled to two copies of (A_1, D_N) (figure 3) is constructed by compactifying 6d (2,0) A_1 theory on sphere with two irregular singularities of rank $N/2$. The generalized AGT correspondence then implies that the Nekrasov partition function of this theory is given by

$$\mathcal{Z}^{2\times(A_1,D_N)}_{\mathrm{SU}(2)} = \langle I^{(N/2)}|I^{(N/2)}\rangle\,. \tag{2.5}$$

Note here that the characterization (2.1) does not fix the irregular state $|I^{(N/2)}\rangle$. In particular, the actions of $L_0, \cdots, L_{\lfloor N/2\rfloor}$ are not specified there. When N is even, these actions are expressed in terms of differential operators with respect to $(N/2+1)$ parameters, $c_0, \cdots, c_{N/2}$ [7]:

$$L_k|I^{(N/2)}\rangle = \begin{cases} 0 \qquad \text{for} \qquad N < k \\ \lambda_k|I^{(N/2)}\rangle \qquad \text{for} \quad \frac{N}{2} \leq k \leq N \\ \left(\lambda_k + \sum_{\ell=1}^{N/2-k} \ell\, c_{\ell+k}\frac{\partial}{\partial c_\ell}\right)|I^{(N/2)}\rangle \qquad \text{for} \quad 0 \leq k < \frac{N}{2} \end{cases}, \tag{2.6}$$

where the non-vanishing eigenvalues, λ_k, of $L_{N/2}, \cdots, L_N$ are fixed by $c_0, \cdots, c_{N/2}$ as

$$\lambda_k = \begin{cases} -\sum_{\ell=k-N/2}^{N/2} c_\ell c_{k-\ell} \qquad \text{for} \quad \frac{N}{2} < k \leq N \\ -\sum_{\ell=0}^{k} c_\ell c_{k-\ell} + (k+1)Qc_k \qquad \text{for} \quad k \leq \frac{N}{2} \end{cases}. \tag{2.7}$$

The above actions of $L_0, \cdots, L_{N/2}$ follow from the construction of $|I^{(N/2)}\rangle$ for even N in a colliding limit of regular primary operators. Similar colliding limit is not known for odd N, and therefore the actions of $L_0, \cdots, L_{\frac{N-1}{2}}$ have not been identified in the literature.[6]

2.2 U(2)-version for even N

In this sub-section, we discuss the U(2)-version of the generalized AGT correspondence. Here, we focus on irregular states $|I^{(N/2)}\rangle$ for even N, and therefore on (A_1, D_{even}) theories.

[6]There is another characterization of the irregular state $|I^{(N/2)}\rangle$ [6], where $|I^{(N/2)}\rangle$ has an explicit expression and is an eigen state of L_N and L_1 for even and odd N. In this paper, we use the one discussed in [7] since it can easily be extended to the U(2)-version that we will review in the next sub-section. It would be an interesting open problem to consider the U(2)-version of the one discussed in [6].

Such a U(2)-version was considered in [8] in order to compute the instanton partition function of the (A_3, A_3) theory. Here, the (A_3, A_3) theory is an $\mathcal{N} = 2$ superconformal SU(2) gauge theory coupled to two (A_1, D_4) theories and a fundamental hypermultiplet (figure 1). When the fundamental hypermultiplet is absent, one can compute the partition function via the generalized AGT correspondence as in (2.5), but its generalization to the (A_3, A_3) theory is not straightforward. The reason for this is that (A_3, A_3) has no known realization from 6d (2,0) A_1 theory.

Therefore, a more indirect route was taken in [8] to compute the partition function of the (A_3, A_3) theory. First, the generalized AGT correspondence was extended to the case of U(2) gauge group. Corresponding to the extra U(1) part of the gauge group, the Virasoro algebra on the 2d side is now accompanied with an extra Heisenberg algebra [11]. The Virasoro irregular state $|I^{(N/2)}\rangle$ is then promoted to an irregular state $|\widehat{I}^{(N/2)}\rangle$ of the direct sum of Virasoro and Heisenberg algebras $Vir \oplus H$, which This state is generally decomposed as

$$|\widehat{I}^{(N/2)}\rangle = |I^{(N/2)}\rangle \otimes |I_H^{(N/2)}\rangle\,, \tag{2.8}$$

where $|I^{(N/2)}\rangle$ is the Virasoro irregular state satisfying (2.6), and $|I_H^{(N/2)}\rangle$ is an irregular state of the Heisenberg algebra characterized by

$$a_k|I_H^{(N/2)}\rangle = \begin{cases} 0 & \text{for} \quad N/2 < k\,, \\ -ic_k|I^{(N/2)}\rangle & \text{for} \quad 1 \leq k \leq n\,. \end{cases}, \tag{2.9}$$

where a_k is the basis of the Heisenberg algebra such that $\{a_k, a_\ell\} = \frac{k}{2}\delta_{k+\ell,0}$. Given the irregular state $|\widehat{I}^{(N/2)}\rangle$ of $Vir \oplus H$, the partition function of U(2) gauge theory coupled to two (A_1, D_N) theories is identified as

$$\mathcal{Z}_{\mathrm{U}(2)}^{2\times(A_1,D_N)} = \langle\widehat{I}^{(N/2)}|\widehat{I}^{(N/2)}\rangle\,, \tag{2.10}$$

which is a natural generalization of (2.5) to the U(2) gauge group.

A nice feature of this generalization is that the highest-weight module of $Vir \oplus H$ has an orthogonal basis $|a; Y_1, Y_2\rangle$ labeled by two Young diagrams, Y_1 and Y_2, that satisfies [11]

$$\mathbf{1} = \sum_{Y_1,Y_2} \mathcal{Z}_{Y_1,Y_2}^{\mathrm{vec}}(a)\, |a; Y_1, Y_2\rangle\langle a; Y_1, Y_2|\,, \tag{2.11}$$

where $\mathcal{Z}_{Y_1,Y_2}^{\mathrm{vec}}(a)$ is the contribution from a U(2) vector multiplet to the Nekrasov partition function, at the fixed point corresponding to (Y_1, Y_2) on the moduli space of U(2) instantons. Here $|a; Y_1, Y_2\rangle$ is a linear combination of states of the form $L_{-n_1}^{p_1} \cdots L_{-n_k}^{p_k} a_{-m_1}^{q_1} \cdots a_{-m_\ell}^{q_\ell}|a\rangle$, and $\langle a; Y_1, Y_2|$ is obtained by replacing each of these states with $\langle a|a_{m_\ell}^{q_\ell} \cdots a_1^{q_1} L_{n_k}^{p_k} \cdots L_{n_1}^{p_1}$ *without changing the coefficients of the linear combination.*

One can use (2.11) to decompose (2.10) as

$$\mathcal{Z}_{\mathrm{U}(2)}^{2\times(A_1,D_N)} = \mathcal{Z}_{\mathrm{pert}} \sum_{Y_1,Y_2} \Lambda^{b_0(|Y_1|+|Y_2|)} \mathcal{Z}_{Y_1,Y_2}^{\mathrm{vec}}(a) \mathcal{Z}_{Y_1,Y_2}^{(A_1,D_N)}(a, m, \boldsymbol{d}, \boldsymbol{u}) \widetilde{\mathcal{Z}}_{Y_1,Y_2}^{(A_1,D_N)}(a, \widetilde{m}, \widetilde{\boldsymbol{d}}, \widetilde{\boldsymbol{u}})\,, \tag{2.12}$$

where $\mathcal{Z}_{\text{pert}} \equiv \langle \widehat{I}^{(N/2)}|a\rangle\langle a|\widehat{I}^{(N/2)}\rangle$, $\Lambda \equiv -\zeta^2 c_{N/2}\widetilde{c}^*_{N/2}$, $b_0 \equiv 4/N$ and

$$\mathcal{Z}^{(A_1,D_N)}_{Y_1,Y_2}(a,m,\boldsymbol{d},\boldsymbol{u}) \equiv (\zeta c_{N/2})^{-\frac{2(|Y_1|+|Y_2|)}{N}} \frac{\langle a;Y_1,Y_2|\widehat{I}^{(N/2)}\rangle}{\langle a|\widehat{I}^{(N/2)}\rangle}\,, \tag{2.13}$$

$$\widetilde{\mathcal{Z}}^{(A_1,D_N)}_{Y_1,Y_2}(a,\widetilde{m},\widetilde{\boldsymbol{d}},\widetilde{\boldsymbol{u}}) \equiv (-\zeta \widetilde{c}^*_{N/2})^{-\frac{2(|Y_1|+|Y_2|)}{N}} \frac{\langle \widehat{I}^{(N/2)}|a;Y_1,Y_2\rangle}{\langle \widehat{I}^{(N/2)}|a\rangle}\,. \tag{2.14}$$

Here, m, $\boldsymbol{d} \equiv (d_1,\cdots,d_{\frac{N}{2}-1})$ and $\boldsymbol{u} \equiv (u_1,\cdots,u_{\frac{N}{2}-1})$ are respectively, a mass parameter, relevant couplings and the VEVs of Coulomb branch operators. These are related to two-dimensional parameters by

$$d_k = \sum_{\ell=\frac{N}{2}-k}^{\frac{N}{2}} \frac{c_\ell c_{N-k-\ell}}{(c_{\frac{N}{2}})^{2-\frac{2k}{N}}}\,, \qquad m = \sum_{\ell=0}^{\frac{N}{2}} \frac{c_\ell c_{\frac{N}{2}-\ell}}{c_{\frac{N}{2}}}\,, \tag{2.15}$$

$$u_k = \sum_{\ell=0}^{\frac{N}{2}-k} \frac{c_\ell c_{\frac{N}{2}-k-\ell}}{(c_{\frac{N}{2}})^{1-\frac{2k}{N}}} - \sum_{\ell=1}^{k} \ell \frac{c_{\frac{N}{2}+\ell-k}}{(c_{\frac{N}{2}})^{1-\frac{2k}{N}}} \frac{\partial \mathcal{F}_{(A_1,D_N)}}{\partial c_\ell}\,, \tag{2.16}$$

where $\mathcal{F}_{(A_1,D_N)} \equiv \lim_{\epsilon_i\to 0}\left(-\epsilon_1\epsilon_2 \log\langle a|I^{(N/2)}\rangle\right)$ is the prepotential of the (A_1,D_N) theory. The parameter, ζ, is a free parameter that can be absorbed or emerged by rescaling the dynamical scale Λ.

Given the expression (2.12), the factors (2.13) and (2.14) are interpreted as the contribution of the (A_1,D_N) theories at the fixed point corresponding to (Y_1,Y_2) on the U(2) instanton moduli space. Note that the gauge group is now U(2) instead of SU(2), and the difference between (2.13) and (2.14) is how the $\mathrm{U}(1) \subset \mathrm{U}(2)$ is coupled to the (A_1,D_N) theory.

An advantage of the expression (2.12) is that one can easily introduce an extra fundamental hypermultiplet by multiplying $\mathcal{Z}^{\text{fund}}_{Y_1,Y_2}(a,M)$ to the summand, where M is the mass of the hypermultiplet. In particular, setting $N=4$ in (2.12) and introducing an extra fundamental hypermultiplet, the partition function is now

$$\mathcal{Z}_{\mathrm{U}(2)} = \mathcal{Z}_{\text{pert}} \sum_{Y_1,Y_2} q^{|Y_1|+|Y_2|} \mathcal{Z}^{\text{vec}}_{Y_1,Y_2}(a)\mathcal{Z}^{(A_1,D_4)}_{Y_1,Y_2}(a,b,u)\widetilde{\mathcal{Z}}^{(A_1,D_4)}_{Y_1,Y_2}(a,\widetilde{b},\widetilde{u})\mathcal{Z}^{\text{fund}}_{Y_1,Y_2}(a,M)\,, \tag{2.17}$$

where Λ^{b_0} is now replaced by q since the SU(2) gauge coupling is exactly marginal. This is almost equivalent to the instanton partition function of the (A_3,A_3) theory. The only difference from the (A_3,A_3) is that the SU(2) gauge group in figure 1 is replaced by U(2), which gives rise to an extra prefactor, $\mathcal{Z}_{\mathrm{U}(1)}$, of the partition function. Therefore, the partition function of the (A_3,A_3) theory is evaluated as

$$\mathcal{Z}_{(A_3,A_3)} = \frac{\mathcal{Z}_{\mathrm{U}(2)}}{\mathcal{Z}_{\mathrm{U}(1)}}\,. \tag{2.18}$$

3 U(2)-version of generalized AGT for (A_1, D_{odd})

In this section, we will extend the U(2)-version of the generalized AGT correspondence reviewed in section 2 to the case of (A_1,D_N) theories for odd N. Specifically, we will generalize (2.13) to the case of odd N.[7]

[7]The same generalization is possible for (2.14), but we will focus on generalizing (2.13) here to make our argument concise.

Even when N is odd, the (A_1, D_N) theory is still realized by compactifying 6d (2,0) A_1 theory on sphere with an irregular and a regular puncture. Therefore, exactly the same discussion as in section 2.1 leads us to identifying

$$\mathcal{Z}_{(A_1,D_N)} = \langle a | I^{(N/2)} \rangle \,, \tag{3.1}$$

as the partition function of the (A_1, D_N) theory. From the equivalence between (2.2) and (2.3), we see that the non-vanishing eigenvalues, $\lambda_N, \cdots, \lambda_{\frac{N+1}{2}}$, in (2.1) appear as the coefficients of the first $\frac{N-1}{2}$ non-trivial terms in the SW curve:[8]

$$x^2 = \frac{1}{z^{N+2}} - \frac{\lambda_{N-1}}{(-\lambda_N)^{\frac{N-1}{N}}} \frac{1}{z^{N+1}} - \frac{\lambda_{N-2}}{(-\lambda_N)^{\frac{N-2}{N}}} \frac{1}{z^{N}} - \cdots - \frac{\lambda_{\frac{N+1}{2}}}{(-\lambda_N)^{\frac{N+1}{2N}}} \frac{1}{z^{\frac{N+5}{2}}} + \cdots \,, \tag{3.2}$$

which are identified as the relevant couplings of (A_1, D_N) for odd N [10, 12]. Therefore the relevant couplings of (A_1, D_N) theories are all encoded in the eigenvalues of $L_{\frac{N+1}{2}}, \cdots, L_{N-2}$ and L_{N-1} (normalized by that of L_N). This is a straightforward generalization of what we reviewed in section 2.1 to odd N.

One difficulty for odd N is, however, the irregular state $|I^{(N/2)}\rangle$ cannot be obtained in a colliding limit of regular primary operators. As such, any result derived via the colliding limit for even N is not available for odd N. For instance, while λ_k are translated into c_k through (2.7) for even N, a similar translation is not available for odd N. As a result, an explicit expression for the action of $L_1, \cdots, L_{\frac{N-1}{2}}$ on $|I^{(N/2)}\rangle$ has not been identified for odd N.

The lack of a colliding-limit construction gives rise to another difficulty when considering the U(2)-version of the generalized AGT correspondence. Generalizing the argument in section 2.2, it is natural to expect that there exists an irregular state $|\widehat{I}^{(N/2)}\rangle$ of $Vir \oplus H$ such that

$$\mathcal{Z}^{(A_1,D_N)}_{Y_1,Y_2} \sim \frac{\langle a; Y_1, Y_2 | \widehat{I}^{(N/2)} \rangle}{\langle a | \widehat{I}^{(N/2)} \rangle} \,, \tag{3.3}$$

is identified, even for odd N, as the contribution from an (A_1, D_N) sector at each fixed point on the U(2) instanton moduli space for the gauge theory described by the quiver in figure 3. Here, the irregular state $|\widehat{I}^{(N/2)}\rangle$ is decomposed as $|\widehat{I}^{(N/2)}\rangle = |I^{(N/2)}\rangle \otimes |I_H^{(N/2)}\rangle$, where $|I^{(N/2)}\rangle$ is the irregular state of Virasoro algebra discussed in the previous two paragraphs, and $|I_H^{(N/2)}\rangle$ is a rank-$\frac{N}{2}$ irregular state of Heisenberg algebra. For even N, $|I_H^{(N/2)}\rangle$ is completely characterized by (2.9), which was derived via the colliding-limit construction of $|I_H^{(N/2)}\rangle$. However, for odd N, the lack of a colliding-limit construction makes it difficult to find a similar characterization of $|I_H^{(N/2)}\rangle$.

The above discussions imply that, due to the lack of colliding-limit, we do not know how $L_1, \cdots, L_{\frac{N-1}{2}}$ and $a_{k>0}$ act on the irregular state $|\widehat{I}^{(N/2)}\rangle = |I^{(N/2)}\rangle \otimes |I_H^{(N/2)}\rangle$ when N is odd. Without knowing these actions, one cannot compute

$$\frac{\langle a | a_{m_\ell}^{q_\ell} \cdots a_1^{q_1} L_{n_k}^{p_k} \cdots L_{n_1}^{p_1} | \widehat{I}^{(N/2)} \rangle}{\langle a | \widehat{I}^{(N/2)} \rangle} \,, \tag{3.4}$$

[8] Here we absorbed λ_N in front of z^{N-2} by rescaling z and x so that xdz is kept fixed. The fact that we can absorb λ_N this way reflects the conformal invariance of (A_1, D_N).

for $n_i > 0$ and $m_i > 0$. This generically makes it hard to compute (3.3) since $\langle a; Y_1, Y_2|$ is a linear combination of vectors of the form $\langle a|a_{m_\ell}^{q_\ell} \cdots a_1^{q_1} L_{n_k}^{p_k} \cdots L_{n_1}^{p_1}$. In the next four sub-sections, however, we will argue that this difficulty can be overcome when we focus on the classical limit $\epsilon_1, \epsilon_2 \to 0$ and turn off relevant couplings and the VEVs of Coulomb branch operators in the (A_1, D_N)-sector.

3.1 Classical limit as the commutative limit

While the irregular state $|I^{(N/2)}\rangle$ is an eigen state of $L_{\frac{N+1}{2}}, \cdots, L_N$ with non-vanishing eigenvalues, it is not an eigen state of $L_1, \cdots, L_{\frac{N-1}{2}}$. Indeed, the Virasoro algebra

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{n(n^2-1)}{12}\delta_{n+m,0} \tag{3.5}$$

forbids $L_1, \cdots, L_N$ to have non-vanishing eigenvalues when $N > 2$. This is the main reason that (2.6) (which is only for even N) involves differential operators on the r.h.s. for $0 \leq k < \frac{N}{2}$.

However, when computing the matrix element (3.4) in the classical limit $\epsilon_1, \epsilon_2 \to 0$, the sub-algebra formed by $\{L_{n>0}\}$ reduces to a commutative algebra. The reason for this is the following. First, in the context of the generalized AGT correspondence, the SW curve (2.3) of a 4d theory is identified as (2.2) on the 2d side. This and the fact that the SW 1-form, xdz, has scaling dimension 1 imply that z and $T(z)$ in (2.2) has four-dimensional scaling dimensions $\Delta_{4\mathrm{d}}(z) = -2/N$ and $\Delta_{4\mathrm{d}}(T(z)) = \Delta_{4\mathrm{d}}(x^2) = 2(1+2/N)$, respectively. Since the stress tensor is expanded as

$$T(z) = \sum_{n\in\mathbb{Z}} \frac{L_n}{z^{n+2}}\,, \tag{3.6}$$

this implies that, when acting on $|I^{(N/2)}\rangle$, L_n is associated with four-dimensional scaling dimension

$$\Delta_{4\mathrm{d}}(L_n) = 2\left(1 - \frac{n}{N}\right)\,. \tag{3.7}$$

Recall here that, in the AGT correspondence, the 4d scaling dimensions are invisible since we set $\epsilon_1\epsilon_2 = 1$, as explained around eq. (3.2) of [4]. To recover the correct scaling dimensions, we need to multiply every quantity of dimension $\Delta_{4\mathrm{d}}$ by $(\epsilon_1\epsilon_2)^{\Delta_{4\mathrm{d}}/2}$. This particularly means the replacement $L_n \to (\epsilon_1\epsilon_2)^{1-\frac{n}{N}} L_n$, and therefore

$$[L_n, L_m] = (n-m)(\epsilon_1\epsilon_2)L_{n+m}\,, \tag{3.8}$$

for $m, n > 0$. This implies that, when focusing on the leading term in the limit $\epsilon_1, \epsilon_2 \to 0$, the sub-algebra formed by $\{L_{n>0}\}$ reduces to a commutative algebra. Therefore, in the computation of (3.4) in the classical limit, one can regard all L_n and a_m as commutative and simultaneously diagonalizable.

This suggests the following conjecture: *in the classical limit $\epsilon_1, \epsilon_2 \to 0$, the irregular states $|I^{(N/2)}\rangle$ approaches to a simultaneous eigen state of $\{L_{n>0}\}$ and $\{a_{m>0}\}$*. As seen in (2.6), this is indeed the case when N is even; in the third line of the r.h.s. of (2.6), $\sum_{\ell=1}^{N/2-k} \ell\, c_{\ell+k} \frac{\partial}{\partial c_\ell}$ is sub-leading in the classical limit, and therefore $|I^{(N/2)}\rangle$ approaches to a simultaneous eigen state of $\{L_k\}$ and $\{a_k\}$ in the classical limit. We here assume that the above conjecture is also satisfied for odd N. Then the matrix element (3.4) can be evaluated in the classical limit as

$$\frac{\langle a|a_{m_\ell}^{q_\ell}\cdots a_1^{q_1} L_{n_k}^{p_k}\cdots L_{n_1}^{p_1}|\widehat{I}^{(N/2)}\rangle}{\langle a|\widehat{I}^{(N/2)}\rangle} = \left(\prod_{i=1}^{\ell}(\mathfrak{a}_{m_i})^{q_i}\right)\left(\prod_{j=1}^{k}(\mathfrak{b}_{n_j})^{p_j}\right), \tag{3.9}$$

where $\mathfrak{a}_i$ and $\mathfrak{b}_i$ are defined by

$$\mathfrak{a}_m \equiv \frac{\langle a|a_m|\widehat{I}^{(N/2)}\rangle}{\langle a|\widehat{I}^{(N/2)}\rangle}, \qquad \mathfrak{b}_n \equiv \frac{\langle a|L_n|\widehat{I}^{(N/2)}\rangle}{\langle a|\widehat{I}^{(N/2)}\rangle}, \tag{3.10}$$

for $m, n > 0$.[9] Note here that, from (2.1), we see that $\mathfrak{b}_n = 0$ for $n > N$. Therefore, (3.9) is a function of $\mathfrak{b}_n$ for $n = 1, \cdots, N$ and $\mathfrak{a}_m$ for $m > 0$. Note also that, for $\frac{N+1}{2} \leq n \leq N$, $\mathfrak{b}_n$ is identical to the eigenvalues λ_n in (2.1).

3.2 4d scaling dimensions of 2d parameters

Here we evaluate the 4d scaling dimensions of the parameters $\{\mathfrak{a}_m\}$ and $\{\mathfrak{b}_n\}$ defined above. We will use them in the next sub-section to argue that, when all the couplings and VEVs of Coulomb branch operators of (A_1, D_N) are turned off, one has $\mathfrak{b}_n = \mathfrak{a}_m = 0$ for all $n \neq N$ and $m > 0$.

To that end, we first see from (3.7) that

$$\Delta_{4\mathrm{d}}(\mathfrak{b}_n) = 2\left(1 - \frac{n}{N}\right), \tag{3.11}$$

which implies that $\mathfrak{b}_n$ for $n > N$ are of negative dimensions and therefore irrelevant in the infrared. Since the Nekrasov partition function is the quantity defined in the infrared, (3.3) must be independent of such parameters. This is consistent with the condition $\mathfrak{b}_n = 0$ for $n > N$.

Let us now turn to the scaling dimensions of $\mathfrak{a}_m$. To evaluate them, one needs to use explicit expressions for the basis $|a; Y_1, Y_2\rangle$ of the highest weight module of $Vir \oplus H$. As shown in [11], the state $|a; Y_1, Y_2\rangle$ is generally a linear combination of descendants of the highest weight state $|a\rangle$ of degree $(|Y_1| + |Y_2|)$. Here, the degree is defined by the sum of the degrees in the sense of Virasoro and Heisenberg algebras; for instance, the degree of

[9] The reduction of (3.4) to (3.9) was explicitly observed in the case of $N = 4$ in section 5.1 of [8].

$(L_{-1})^2 a_{-5}|a\rangle$ is evaluated as seven. A few examples of $|a;Y_1,Y_2\rangle$ are shown below:

$$|a;\emptyset,\emptyset\rangle = |a\rangle\,, \tag{3.12}$$

$$|a;\square,\emptyset\rangle = \Big(-i\,(\epsilon_1+\epsilon_2+2a)\,a_{-1} - L_{-1}\Big)|a\rangle\,, \tag{3.13}$$

$$\begin{aligned}|a;\square\!\square,\emptyset\rangle = \Big(&-i\epsilon_1(\epsilon_1+\epsilon_2+2a)(2\epsilon_1+\epsilon_2)a_{-2} - (\epsilon_1+\epsilon_2+2a)(2\epsilon_1+\epsilon_2+2a)a_{-1}^2\,,\\ &+2i(2\epsilon_1+\epsilon_2+2a)a_{-1}L_{-1} - \epsilon_1(\epsilon_1+\epsilon_2+2a)L_{-2} + L_{-1}^2\Big)|a\rangle\,,\end{aligned} \tag{3.14}$$

$$\begin{aligned}|a;\square,\square\rangle = \Big(&-i(\epsilon_1+\epsilon_2)a_{-2} - (\epsilon_1^2+\epsilon_2^2+\epsilon_1\epsilon_2-4a^2)a_{-1}^2 + 2i(\epsilon_1+\epsilon_2)a_{-1}L_{-1}\\ &- L_{-2} + L_{-1}^2\Big)|a\rangle\,,\end{aligned} \tag{3.15}$$

where we recovered the complete ϵ_i-dependence. In the context of the AGT correspondence, the highest weight a of $|a\rangle$ is identified as the mass of the W-boson that arises on the Coulomb branch of SU(2) gauge theory, and therefore has scaling dimension one. Similarly the Ω-deformation parameters ϵ_i have scaling dimension one, i.e.,

$$\Delta_{\rm 4d}(a) = \Delta_{\rm 4d}(\epsilon_1) = \Delta_{\rm 4d}(\epsilon_2) = 1\,. \tag{3.16}$$

Combining this with the expressions for $|a;Y_1,Y_2\rangle$ shown in (3.12)–(3.15), one can read off the 4d scaling dimensions of $\mathfrak{a}_m = \langle a|a_m|\widehat{I}^{(N/2)}\rangle/\langle a|\widehat{I}^{(N/2)}\rangle$.

For instance, we see from (3.12) and (3.10) that $\mathcal{Z}^{(A_1,D_N)}_{\square,\emptyset} \sim \langle a;\square,\emptyset|\widehat{I}^{(N/2)}\rangle/\langle a|\widehat{I}^{(N/2)}\rangle$ is evaluated as[10]

$$\mathcal{Z}^{(A_1,D_N)}_{\square,\emptyset} \sim -i(\epsilon_1+\epsilon_2+2a)\mathfrak{a}_1 - \mathfrak{b}_1\,, \tag{3.17}$$

Since the two terms in (3.17) must have the same scaling dimensions, we see that

$$\Delta_{\rm 4d}(\mathfrak{a}_1) = \Delta_{\rm 4d}(\mathfrak{b}_1) - 1 = 1 - \frac{2}{N}\,. \tag{3.18}$$

The same analysis for $\mathcal{Z}^{(A_1,D_N)}_{\square\!\square,\emptyset}$ implies

$$\Delta_{\rm 4d}(\mathfrak{a}_2) = 2\Delta_{\rm 4d}(\mathfrak{b}_1) - 3 = 1 - \frac{4}{N}\,. \tag{3.19}$$

It is straightforward to do the same analysis for all $\mathcal{Z}^{(A_1,D_N)}_{Y_1,\emptyset}$ with $Y_1 = [1,\cdots,1]$. As shown in [11], the state $|a;Y_1,\emptyset\rangle$ is concisely expressed as

$$|a;Y_1,\emptyset\rangle = \Omega_{Y_1}(a)\mathbf{J}^{(-\epsilon_2^2)}_{Y_1}(x)|a\rangle\,, \tag{3.20}$$

where $\Omega_{Y_1}(a) \equiv (-\epsilon_1)^{|Y_1|}\prod_{(j,k)\in Y_1}(2a+j\epsilon_1+k\epsilon_2)$, and $\mathbf{J}^{(1/g)}_{Y_1}(x)$ is the normalized Jack polynomial of variables $x\equiv(x_1,x_2,\cdots)$.[11] Here, the variables $(x_1,x_2,\cdots)$ are related to the $\{L_n\}$ and $\{a_m\}$ as follows. First, write the Virasoro generators $L_{n\neq 0}$ as

$$L_n = \sum_{k\neq 0,n} c_k c_{k-n} + i(nQ-2a)c_n \tag{3.21}$$

[10] Here, we recall that $\langle a;Y_1,Y_2|$ is obtained by expanding it as a linear combinations of $L_{-n_1}^{p_1}\cdots L_{-n_k}^{p_k}a_{-m_1}^{q_1}\cdots a_{-m_\ell}^{q_\ell}|a\rangle$ and replacing each of these vectors with $\langle a|a_{m_\ell}^{q_\ell}\cdots a_1^{q_1}L_{n_k}^{p_k}\cdots L_{n_1}^{p_1}$ with the expansion coefficients kept fixed.

[11] Note that we have $g=-\epsilon_2^2$ here.

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in terms of $\{c_k\}$ such that $[c_k, c_\ell] = \frac{k}{2}\delta_{k+\ell,0}$. Then $x = (x_1, x_2, \cdots)$ is related to $\{c_k\}$ and $\{a_m\}$ by the identifications

$$a_{-n} - c_{-n} = -i\epsilon_1 p_n(x)\,, \tag{3.22}$$

where $p_n(x) \equiv \sum_{i=1}^{|Y_1|} x_i^n$. Therefore, to express (3.20) in terms of $\{a_m\}$ and $\{L_m\}$, one first needs to write $\mathbf{J}_{Y_1}^{(-\epsilon_2^2)}(x)$ in terms of $\{p_n(x)\}$, and then replace $p_n(x)$ with $i(a_{-n} - c_{-n})/\epsilon_1$. When $Y_1 = [1, \cdots, 1]$, the Jack polynomial is simply $\mathbf{J}_{Y_1}^{(1/g)}(x) = |Y_1|! \prod_{i=1}^{|Y_1|} x_i$. Rewriting this in terms of $p_n(x) = i(a_{-n} - c_{-n})/\epsilon_1$ for $n \in \mathbb{N}$, one finds that the expression (3.20) for $Y_1 = [1, \cdots, 1]$ is of the form

$$\begin{aligned}|a;\, Y_1 = [1, \cdots, 1],\, \emptyset\rangle = \Bigg(\mathcal{N}(Y_1)\, \epsilon_1^{|Y_1|-1} \left(\prod_{j=1}^{|Y_1|}(2a + j\epsilon_1 + \epsilon_2)\right) a_{-|Y_1|} \\ + (-L_{-1})^{|Y_1|} + \cdots\Bigg)|a\rangle\,,\end{aligned} \tag{3.23}$$

where $\mathcal{N}(Y_1)$ is a numerical factor independent of ϵ_1 and ϵ_2. Note here that the presence of $(-L_{-1})^{|Y_1|}$ on the right-hand side of (3.23) is already stressed in [11]. The expression (3.23) implies that, for $Y_1 = [1, \cdots, 1]$,

$$\mathcal{Z}^{(A_1,D_N)}_{Y_1=[1,\cdots,1],\emptyset} \sim \mathcal{N}(Y_1)\, \epsilon_1^{|Y_1|-1} \prod_{j=1}^{|Y_1|}(2a + j\epsilon_1 + \epsilon_2)\mathfrak{a}_{|Y_1|} + (-\mathfrak{b}_1)^{|Y_1|} + \cdots\,. \tag{3.24}$$

For the first two terms on the right-hand side to be of the same scaling dimension, we must have

$$\Delta_{4\text{d}}(\mathfrak{a}_m) = m\Delta_{4\text{d}}(\mathfrak{b}_1) - 2m + 1 = 1 - \frac{2m}{N}\,. \tag{3.25}$$

3.3 Computation of matrix elements for odd N

In the rest of this paper, we focus on the classical limit $\epsilon_1, \epsilon_2 \to 0$ so that (3.9) is valid. In this case, it is sufficient to identify the values of (3.10) for the computation of (3.3). Here we argue that, when the relevant couplings and VEVs of Coulomb branch operators of (A_1, D_N) are all turned off, the only non-vanishing parameter among (3.10) is $\mathfrak{b}_N$ and therefore (3.9) reduces to

$$\frac{\langle a|a_{m_\ell}^{q_\ell} \cdots a_1^{q_1} L_{n_k}^{p_k} \cdots L_{n_1}^{p_1} |\widehat{I}^{(N/2)}\rangle}{\langle a|\widehat{I}^{(N/2)}\rangle} = \begin{cases} 1 & \text{for} \quad \ell = k = 0 \\ \delta_{n_1,N}(\mathfrak{b}_N)^{p_1} & \text{for} \quad \ell = 0,\ \ k = 1 \\ 0 & \text{for the others} \end{cases}. \tag{3.26}$$

To derive (3.26), we first note that all parameters of (A_1, D_N) on the Coulomb branch are encoded in the SW curve (2.2). Through the equivalence of (2.2) and (2.3), these are related to a and the non-vanishing components of $\mathfrak{b}_n$. The interpretation of non-vanishing $\mathfrak{b}_n$ in four dimensions is as follows. From (3.11), we see that $\mathfrak{b}_1, \cdots, \mathfrak{b}_{\frac{N-1}{2}}$ are identified as the VEVs of Coulomb branch operators since they have scaling dimensions larger than one [13–15]. Similarly, $\mathfrak{b}_{\frac{N+1}{2}}, \cdots, \mathfrak{b}_{N-1}$ are identified as relevant couplings since their dimensions are smaller than one. Note that, the (A_1, D_N) theory has no exactly marginal coupling, and therefore the dimensionless parameter $\mathfrak{b}_N$ has no counterpart in four dimensions. This implies that the final result must be independent of $\mathfrak{b}_N$, as discussed in the next sub-section.

Since the Coulomb branch of (A_1, D_N) is completely characterized by $\{\mathfrak{b}_n\}$ and a, any physical quantity of the (A_1, D_N) theory (on the Coulomb branch) should be determined by these parameters. In particular, $\mathfrak{a}_m$ must be a function of $\{\mathfrak{b}_\mathfrak{n}\}$ and a. When N is even, this function was identified in [8] via the colliding-limit construction of $|\widehat{I}^{(N/2)}\rangle$, where $\mathfrak{a}_m$ turned out to be independent of a. Here we assume this independence to hold for odd N as well, and therefore $\mathfrak{a}_m$ is a function only of $\{\mathfrak{b}_n\}$.[12]

While it is beyond the scope of this paper to compute $\mathfrak{a}_m$ for generic values of $\{\mathfrak{b}_n\}$, one can easily compute it when all the relevant couplings and VEVs of Coulomb branch operators are turned off in the (A_1, D_N) theory. Indeed, turning off these couplings and VEVs implies that

$$\mathfrak{b}_n = 0\,, \qquad \text{for} \qquad n \neq N\,. \tag{3.27}$$

Note that this is equivalent to the condition that $\mathfrak{b}_n = 0$ unless $\Delta_{4\text{d}}(\mathfrak{b}_n) = 0$. Since $\mathfrak{a}_m$ is assumed to be a function only of $\{\mathfrak{b}_n\}$, this implies that $\mathfrak{a}_m = 0$ unless $\Delta_{4\text{d}}(\mathfrak{a}_m) = 0$.[13] From (3.25), we see that $\Delta_{4\text{d}}(\mathfrak{a}_m) = 0$ occurs if and only if $m = N/2$, but this condition is never satisfied for odd N. Hence, we conclude that

$$\mathfrak{a}_m = 0\,, \tag{3.28}$$

for all m, when the relevant couplings and the VEVs of Coulomb branch operators of (A_1, D_N) are turned off. The above discussion implies that the matrix element (3.9) reduces to (3.26) when focusing on the classical limit $\epsilon_1, \epsilon_2 \to 0$ and turning off all the relevant couplings and VEVs of Coulomb branch operators of the (A_1, D_N) theory.

[12]We believe this assumption is natural for the following reason. First, note that there is an RG-flow from (A_1, D_N) to (A_1, D_{N-1}) triggered by a relevant perturbation. Since this flow preserves the SU(2) flavor symmetry gauged in the quiver in figure 3, this induces an RG-flow from the SU(2) gauge theory coupled to two (A_1, D_N) theories to that coupled to two (A_1, D_{N-1}). At the level of partition function, this is a flow from $\langle I^{(N/2)}|I^{(N/2)}\rangle$ to $\langle I^{((N-1)/2)}|I^{((N-1)/2)}\rangle$, or a flow from $\langle \widehat{I}^{(N/2)}|\widehat{I}^{(N/2)}\rangle$ to $\langle \widehat{I}^{((N-1)/2)}|\widehat{I}^{((N-1)/2)}\rangle$ when the gauge group is replaced by U(2). This strongly suggests that there is a flow between the two irregular states $|\widehat{I}^{(N/2)}\rangle$ and $|\widehat{I}^{((N-1)/2)}\rangle$, corresponding to the RG-flow discussed above.

Now, let us consider the composition $RG_2 \circ RG_1$ of the flows $RG_1 : |\widehat{I}^{(n)}\rangle \to |\widehat{I}^{(n-\frac{1}{2})}\rangle$ and $RG_2 : |\widehat{I}^{(n-\frac{1}{2})}\rangle \to |\widehat{I}^{(n-1)}\rangle$ for a positive integer n. As mentioned in the main text, $\mathfrak{a}_m \equiv \langle a|a_m|\widehat{I}^{(N/2)}\rangle / \langle a|\widehat{I}^{(N/2)}\rangle$ is independent of a when $N = 2n$ and $N = 2n-2$. Therefore $RG_2 \circ RG_1$ preserves the a-independence of $\mathfrak{a}_m$. This is natural since the relevant perturbations corresponding to RG_1 and RG_2 both preserve the SU(2) flavor symmetry associated with a. It is then extremely natural to expect that RG_1 and RG_2 separately preserve the a-independence of $\mathfrak{a}_m$. This suggests that $\mathfrak{a}_m \equiv \langle a|a_m|\widehat{I}^{(N/2)}\rangle / \langle a|\widehat{I}^{(N/2)}\rangle$ is always independent of a for even and odd N. We also provide another supporting evidence for the a-independence of $\mathfrak{a}_m$ in appendix C.

[13]Note here that, since we are already taking the classical limit $\epsilon_1, \epsilon_2 \to 0$, the only non-vanishing dimensionful parameter in the (A_1, D_N) sector is now a. Since $\mathfrak{a}_m$ is assumed to be independent of a, we see that $\mathfrak{a}_m = 0$ unless $\Delta_{4\text{d}}(\mathfrak{a}_m) = 0$.

3.4 Removing an unphysical degree of freedom

Suppose that we turn off all the relevant couplings and VEVs of Coulomb branch operators in (A_1, D_N). Then one can compute the r.h.s. of (3.3) using (3.3) and (3.26). From (3.26), we see that the result depends on $\mathfrak{b}_N$.

Note that (3.11) implies $\Delta_{4\mathrm{d}}(\mathfrak{b}_N) = 0$, and therefore $\mathfrak{b}_N$ must be an exactly marginal coupling if it is a physical degree of freedom. However, the (A_1, D_N) theory has no such coupling. This means that $\mathfrak{b}_N$, that appears on the r.h.s. of (3.11), is not a physical parameter in four dimensions. The fact that $\mathfrak{b}_N$ is unphysical can also be seen in the SW curve (3.2) of the (A_1, D_N) theory; $\lambda_N = \mathfrak{b}_N$ can be absorbed by a change of variables. Hence, to make the relation (3.11) more precise, one has to introduce a prefactor on the r.h.s. to remove this unphysical degree of freedom.[14]

As shown in [11], the basis $|a; Y_1, Y_2\rangle$ is a descendant at level $|Y_1| + |Y_2|$.[15] Combining this fact with (3.26), we find that $\langle a; Y_1, Y_2|\widehat{I}^{(N/2)}\rangle/\langle a|\widehat{I}^{(N/2)}\rangle$ is proportional to $(\mathfrak{b}_N)^{\frac{|Y_1|+|Y_2|}{N}}$. This means that the following expression is independent of $\mathfrak{b}_N$:

$$\mathcal{Z}^{(A_1,D_N)}_{Y_1,Y_2}(a) = (\xi\mathfrak{b}_N)^{-\frac{|Y_1|+|Y_2|}{N}} \frac{\langle a; Y_1, Y_2|\widehat{I}^{(N/2)}\rangle}{\langle a|\widehat{I}^{(N/2)}\rangle}\,, \tag{3.29}$$

where ξ is a numerical free parameter that can be absorbed or emerged by rescaling the dynamical scale. We therefore identify (3.29) as the precise expression for the contribution from (A_1, D_N) to the instanton partition function. Note that this is the "odd-N version" of (2.13). We will apply the above formula in the next section to the computation of the instanton partition function of the (A_2, A_5) theory.

4 Application to the (A_2, A_5) theory

In this section, we compute the instanton partition function of the (A_2, D_5) theory using our method described in the previous section.

4.1 Partition function

Recall that the (A_2, A_5) theory is SU(2) gauge theory described by the quiver diagram in figure 2. We first replace the gauge group with U(2), and then the partition function of the theory is evaluated as

$$\mathcal{Z}_{\mathrm{U}(2)} = \mathcal{Z}^{\mathrm{U}(2)}_{\text{pert}} \sum_{Y_1,Y_2} q^{|Y_1|+|Y_2|} \mathcal{Z}^{\text{vec}}_{Y_1,Y_2}(a) \mathcal{Z}^{\text{fund}}_{Y_1,Y_2}(a, M) \mathcal{Z}^{(A_1,D_3)}_{Y_1,Y_2}(a, d, u) \mathcal{Z}^{(A_1,D_6)}_{Y_1,Y_2}(a, m, \boldsymbol{d}, \boldsymbol{u})\,. \tag{4.1}$$

Here $\mathcal{Z}^{\text{vec}}_{Y_1,Y_2}$ and $\mathcal{Z}^{\text{fund}}_{Y_1,Y_2}$ are contributions respectively from the vector multiplet and fundamental hypermultiplet [1, 2], which have simple product expressions [16–18] as reviewed in

[14] This is exactly the same situation as in (2.13) for even N, where $(\zeta c_{N/2})^{-\frac{2(|Y_1|+|Y_2|)}{N}}$ removes a degree of freedom that has no physical meaning in the corresponding four-dimensional theory.

[15] Here, the level of a descendant means the sum of the level of the Virasoro descendant and that of a Heisenberg descendant. For instance, $L_{-1}a_{-3}|a\rangle$ is a descendant at level four.

(A.1) and (A.3) of [8]. On the other hand, $\mathcal{Z}^{(A_1,D_3)}_{Y_1,Y_2}$ and $\mathcal{Z}^{(A_1,D_6)}_{Y_1,Y_2}$ are contributions respectively from the (A_1, D_3) and (A_1, D_6) sectors in figure 2. Here, q is the exponential of the exactly marginal gauge coupling, d and u are respectively the relevant coupling and VEV of Coulomb branch operator in the (A_1, D_3) theory, and m, $\boldsymbol{d} = (d_1, d_2)$ and $\boldsymbol{u} = (u_1, u_2)$ are respectively the mass parameter, relevant couplings and VEVs of Coulomb branch operators in the (A_1, D_6) theory. The scaling dimensions of these parameters are as follows:

$$[q] = 0, \qquad [d_1] = \frac{1}{3}, \qquad [d] = [d_2] = \frac{2}{3}, \qquad [u] = [u_1] = \frac{4}{3}, \qquad [u_2] = \frac{5}{3}. \tag{4.2}$$

In the rest of this section, we set $d = u = 0$ so that our formula derived in the previous section is available. Using (3.29), we identify the contribution of the (A_1, D_3) theory as

$$\mathcal{Z}^{(A_1,D_3)}_{Y_1,Y_2}(a) = (\xi\mathfrak{b}_3)^{-\frac{|Y_1|+|Y_2|}{3}} \frac{\langle a; Y_1, Y_2|\widehat{I}^{(3/2)}\rangle}{\langle a|\widehat{I}^{(3/2)}\rangle}. \tag{4.3}$$

Since we turn off the relevant coupling and the VEV of the Coulomb branch operator, the r.h.s. of (4.3) can be computed via (3.26).

The contribution of the (A_1, D_6) theory was already identified in [8] and have reviewed in (2.13); substituting $N = 6$ we find

$$\mathcal{Z}^{(A_1,D_6)}_{Y_1,Y_2}(a, m, \boldsymbol{d}, \boldsymbol{u}) = (\zeta c_3)^{-\frac{|Y_1|+|Y_2|}{3}} \frac{\langle a; Y_1, Y_2|\widehat{I}^{(3)}\rangle}{\langle a|\widehat{I}^{(3)}\rangle}, \tag{4.4}$$

where $\boldsymbol{d} = (d_1, d_2)$ and $\boldsymbol{u} = (u_1, u_2)$ are identified as in (2.15) and (2.16). We choose the free parameter ζ to be $\zeta = 2/\xi$ so that the expressions in the next sub-section are simple. Changing the value of ζ or ξ just corresponds to rescaling q. The r.h.s. of (4.4) can be evaluated by using $|\widehat{I}^{(3)}\rangle = |I^{(3)}\rangle \otimes |I^{(3)}_H\rangle$ and the following equations:

$$L_k|I^{(3)}\rangle = 0 \quad \text{for} \quad k \geq 7, \tag{4.5}$$
$$L_6|I^{(3)}\rangle = -c_3^2|I^{(3)}\rangle, \tag{4.6}$$
$$L_5|I^{(3)}\rangle = -2c_2c_3|I^{(3)}\rangle, \tag{4.7}$$
$$L_4|I^{(3)}\rangle = -\left(c_2^2 + 2c_3c_1\right)|I^{(3)}\rangle, \tag{4.8}$$
$$L_3|I^{(3)}\rangle = -2\left(c_1c_2 + c_3(c_0 - 2Q)\right)|I^{(3)}\rangle, \tag{4.9}$$
$$L_2|I^{(3)}\rangle = \left(c_3\frac{\partial}{\partial c_1} - c_2(2c_0 - 3Q) - c_1^2\right)|I^{(3)}\rangle, \tag{4.10}$$
$$L_1|I^{(3)}\rangle = \left(2c_3\frac{\partial}{\partial c_2} + c_2\frac{\partial}{\partial c_1} - 2c_1(c_0 - Q)\right)|I^{(3)}\rangle, \tag{4.11}$$

and

$$a_k|I^{(3)}_H\rangle = \begin{cases} -ic_k|I^{(3)}_H\rangle & \text{for} \quad k = 1, 2, 3 \\ 0 \quad \text{for} \quad k > 3 \end{cases}. \tag{4.12}$$

Using (4.1), (4.3) and (4.4), one can evaluate $\mathcal{Z}_{\mathrm{U}(2)}/\mathcal{Z}^{\mathrm{U}(2)}_{\mathrm{pert}}$ order by order in q.

Recall that we have replaced the SU(2) gauge group in figure 2 with U(2). This induces an extra prefactor of the partition function, $\mathcal{Z}_{\mathrm{U}(1)}$, that is called the "U(1)-factor." The partition function of the original (A_2, A_5) theory is then recovered by removing $\mathcal{Z}_{\mathrm{U}(1)}$ from $\mathcal{Z}_{\mathrm{U}(2)}$, i.e.,

$$\mathcal{Z}_{(A_2,A_5)} = \frac{\mathcal{Z}_{\mathrm{U}(2)}}{\mathcal{Z}_{\mathrm{U}(1)}} . \tag{4.13}$$

Since a is the VEV of a scalar field in the SU(2) vector multiplet, a is neutral under U(1). Therefore we expect that $\mathcal{Z}_{\mathrm{U}(1)}$ is independent of a. This means that, up to an a-independent prefactor, $\mathcal{Z}_{\mathrm{U}(2)}$ and $\mathcal{Z}_{(A_2,A_5)}$ are identical.

4.2 S-duality from the prepotential relation

We here focus on the prepotential of the (A_2, A_5) theory:

$$\mathcal{F}^{(A_2,A_5)} \equiv \lim_{\epsilon_i \to 0} \left(-\epsilon_1\epsilon_2 \log \mathcal{Z}_{(A_2,A_5)}\right) . \tag{4.14}$$

Up to the a-independent term $\lim_{\epsilon_i\to 0}(-\epsilon_1\epsilon_2 \log \mathcal{Z}_{\mathrm{U}(1)})$, this is identical to

$$\lim_{\epsilon_i \to 0} \left(-\epsilon_1\epsilon_2 \log \mathcal{Z}_{\mathrm{U}(2)}\right) . \tag{4.15}$$

The prepotential (4.14) is generally decomposed into the perturbative and instanton parts as

$$\mathcal{F}^{(A_2,A_5)} = \mathcal{F}^{(A_2,A_5)}_{\mathrm{pert}} + \mathcal{F}^{(A_2,A_5)}_{\mathrm{inst}} . \tag{4.16}$$

Again, up to a-independent terms affected by the U(1)-factor, the instanton part $\mathcal{F}^{(A_2,A_5)}_{inst}$ is identical to

$$\lim_{\epsilon_i \to 0} \left(-\epsilon_1\epsilon_2 \log \frac{\mathcal{Z}_{\mathrm{U}(2)}}{\mathcal{Z}^{\mathrm{U}(2)}_{\mathrm{pert}}}\right) , \tag{4.17}$$

which one can compute using the formula (4.1).[16] Below, we will compute this instanton part, and read off from it how the S-duality group acts on the UV gauge coupling of the (A_2, A_5) theory.

To study the S-duality of the theory, it is useful to turn off the couplings and VEVs in the (A_1, D_6) sector as well, i.e., $\boldsymbol{d} = (0, 0)$ and $\boldsymbol{u} = (0, 0)$ in (4.1). In this case, $\mathcal{F}^{(A_2,A_5)}_{\mathrm{inst}}$ is a function of q, a and two mass parameters M and m. Using (4.3) and (4.4), one obtains

$$\begin{aligned}\mathcal{F}^{(A_2,A_5)}_{\mathrm{inst}}(q; a, m, M) \sim \frac{1}{6}\left(a^2 + \frac{mM^3}{2}a^{-2}\right)q^3 \\ + \frac{1}{192}\Bigg[13a^2 + \left(\frac{3}{4}m^2M^2 + 8mM^3 + 3M^4\right)a^{-2} \\ - \left(\frac{9}{4}m^2M^4 + 3M^6\right)a^{-4} + \frac{5}{4}m^2M^6a^{-6}\Bigg]q^6 + \mathcal{O}(q^9) ,\end{aligned} \tag{4.18}$$

[16] Note here that we are setting $d = u = 0$ in (4.1), and therefore (4.17) can be unambiguously computed via (4.1) with (4.3) and (4.4).

where "$\sim$" means that the l.h.s. and r.h.s. are identical up to a-independent terms affected by the U(1)-factor. Remarkably, the above expression is in a striking resemblance to the instanton part $\mathcal{F}^{N_f=4}_{\text{inst}}$ of the prepotential of SU(2) gauge theory with four fundamental flavors. Indeed, comparing (4.18) with (A.8) in appendix A, we see that the relation

$$3\mathcal{F}^{(A_2,A_5)}_{\text{inst}}(q;a,m,M) = \mathcal{F}^{N_f=4}_{\text{inst}}\left(q^3;a,\frac{m}{2},M,M,M\right) \tag{4.19}$$

holds, at least up to $\mathcal{O}(q^9)$![17] Note that one of the four mass parameters on the r.h.s. is related to the mass parameter m in the (A_1,D_6) sector on the l.h.s. , while the remaining three masses on the r.h.s. are identified with the mass M of the single fundamental hypermultiplet on the l.h.s. . In the next sub-section, we will show that these mass relations are consistent with the SW curves of (A_2,A_5) and SU(2) gauge theory with four flavors.

In the same spirit as [8], we conjecture that the relation (4.19) extends to the full prepotential. This particularly implies that, when the mass parameters are also turned off, one finds

$$3\mathcal{F}^{(A_2,A_5)}(q;a) = \mathcal{F}^{N_f=4}(q^3,a)\,. \tag{4.20}$$

This prepotential relation is extremely powerful since one can study the S-duality of the (A_2,A_5) theory via that of SU(2) gauge theory with four flavors. To see this, first note that the prepotentials of the two theories must be written as

$$\mathcal{F}^{(A_2,A_5)}(q;a) = (\log q_{\text{IR}})\,a^2\,, \qquad \mathcal{F}^{N_f=4}(q;a) = (\log \widetilde{q}_{\text{IR}})\,a^2\,, \tag{4.21}$$

for dimensional reasons, where q_{IR} and $\widetilde{q}_{\text{IR}}$ are functions of the UV gauge coupling q. One can regard q_{IR} and $\widetilde{q}_{\text{IR}}$ as IR gauge couplings of these theories on the Coulomb branch. Indeed, in the weak coupling limit, both q_{IR} and $\widetilde{q}_{\text{IR}}$ coincide with the UV gauge coupling q.

For SU(2) gauge theory with four flavors, the IR and UV gauge couplings are known to be related by (A.4) in appendix A [19]. This theory is known to be invariant under an action of $\text{PSL}(2,\mathbb{Z})$. Its action on the IR gauge coupling is written as

$$T:\widetilde{\tau}_{\text{IR}} \to \widetilde{\tau}_{\text{IR}}+1\,, \qquad S:\widetilde{\tau}_{\text{IR}} \to -\frac{1}{\widetilde{\tau}_{\text{IR}}}\,, \tag{4.22}$$

where $\widetilde{\tau}_{\text{IR}} \equiv \frac{1}{\pi i}\log\widetilde{q}_{\text{IR}}$. Through (A.4), one can translate the above as

$$T: q \to \frac{q}{q-1}\,, \qquad S: q \to 1-q\,. \tag{4.23}$$

Similarly, the (A_2,A_5) theory is known to be invariant under $\text{PSL}(2,\mathbb{Z})$ [20–22]. Indeed, the SW curve of the (A_2,A_5) theory reduces to a genus-one curve when dimensionful parameters except for a are all turned off. One difference from the previous paragraph is that the action of $\text{PSL}(2,\mathbb{Z})$ on q has not been identified, since the relation between q and q_{IR} has been unclear for (A_2,A_5). However, from the prepotential relation we found above, one

[17] To be precise, we have only checked this relation up to $\mathcal{O}(q^9)$, and also up to terms affected by the U(1)-factor.

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can now identify the explicit relation between q and $q_{\rm IR}$ for the (A_2, A_5) theory. Specifically, we see from (4.20) that $\mathcal{F}^{(A_2,A_5)}(q;a)$ is obtained from $\mathcal{F}^{N_f=4}(q;a)$ by the replacement

$$q \longrightarrow q^3\,, \qquad \widetilde{q}_{\rm IR} \longrightarrow q_{\rm IR}^3\,. \tag{4.24}$$

Applying this replacement to (A.4), we find the following relation between the UV and IR gauge couplings of the (A_2, A_5) theory:

$$q^3 = \frac{\theta_2(q_{\rm IR}^3)^4}{\theta_3(q_{\rm IR}^3)^4}\,. \tag{4.25}$$

This suggests that the $\mathrm{PSL}(2,\mathbb{Z})$ acts on the IR gauge coupling $\tau_{\rm IR} \equiv \frac{3}{\pi}\log q_{\rm IR}$ as

$$T : \tau_{\rm IR} \to \tau_{\rm IR} + 1\,, \qquad S : \tau_{\rm IR} \to -\frac{1}{\tau_{\rm IR}}\,, \tag{4.26}$$

and on the UV gauge coupling as

$$T : q^3 \to \frac{q^3}{q^3-1}\,, \quad S : q^3 \to 1 - q^3\,. \tag{4.27}$$

Indeed, applying (4.24) to (4.22) and (4.23), one obtains (4.26) and (4.27).

Remarkably, the above $\mathrm{PSL}(2,\mathbb{Z})$-action on the (A_2, A_5) theory can be extended to a more non-trivial situation. Let us now turn on u_1 and u_2 in (4.1) while keeping d, d_1, d_2 and u vanishing. Then the resulting $\mathcal{F}_{\rm inst}^{(A_2,A_5)}$ is a function of a, m, M, u_1, u_2 and q. We find that this $\mathcal{F}_{\rm inst}^{(A_2,A_5)}$ is invariant under the following change of variables:

$$q \to \frac{e^{\frac{\pi i}{3}} q}{(1-q^3)^{\frac{1}{3}}}\,, \qquad m \to -m\,, \qquad u_1 \to e^{\frac{2\pi i}{3}} u_1\,, \qquad u_2 \to e^{\frac{\pi i}{3}} u_2\,, \tag{4.28}$$

where M and a are kept fixed. We checked this invariance up to $\mathcal{O}(q^6)$. Note that the transformation (4.28) is a natural extension of the T-transformation in (4.27). We believe this can be further extended to the case of non-vanishing d, d_1, d_2 and u. In particular, we believe the T-transformations for non-vanishing d, d_1 and d_2 involve a non-trivial q-dependence as in the case of (A_3, A_3) theory studied in [8]. We leave a careful study of it for future work.[18]

5 Consistency with the Seiberg-Witten curve

In this section, we show that the surprising relation (4.20) is consistent with the SW curve of the (A_2, A_5) theory. In particular, we will show that the relation between the two sets of mass parameters can also be seen in the SW curve. We also show that the T-transformation (4.28) corresponds to a symmetry of the curve.

[18] As discussed in section 3, our formula for $\mathcal{Z}^{(A_1,D_3)}_{Y_1,Y_2}$ is only for vanishing d and u. Therefore, our discussion on the S-duality here is limited to the case of $d = u = 0$. Since d and d_2 are of the same dimension, the T-transformation is expected to mix them, which is why we turn off d_2 as well in the main text.

The SW curve of the (A_2, A_5) theory can be written as [10, 12]

$$\begin{aligned} 0 &= x^3 + z^6 - \frac{1}{\mathsf{q}} x^2 z^2 - \mathsf{q} x z^4 + c_{20} x^2 + x(c_{11} z + c_{10}) \\ &\quad + c_{05} z^5 + c_{04} z^4 + c_{03} z^3 + c_{02} z^2 + c_{01} z - c_{00} \,, \end{aligned} \tag{5.1}$$

where q corresponds to the exactly marginal gauge coupling, and is a non-trivial function of q_{IR}. The SW 1-form is given by $\lambda = x dz$. Since the mass of a BPS state is given by $\oint \lambda$, the 1-form λ has scaling dimension one, which fixes the dimensions of the parameters in (5.1) as

$$[x] = \frac{2}{3} \,, \qquad [z] = \frac{1}{3} \,, \qquad [c_{ij}] = 2 - \frac{2i + j}{3} \,, \qquad [\mathsf{q}] = 0 \,. \tag{5.2}$$

The coefficients c_{ij} with $0 < [c_{ij}] < 1$ are regarded as relevant couplings, while those with $[c_{ij}] > 1$ are regarded as the VEV of Coulomb branch operators. The remaining parameters, c_{11} and c_{03}, are two mass parameters.

5.1 Three sectors in the (A_2, A_5) theory

We first show that the curve (5.1) splits into three sectors in the weak gauge coupling limit $\mathsf{q} \to 0$. To see this, let us study the behavior of the curve for $\mathsf{q} \sim 0$. As discussed in [23], the coefficients c_{ij} of the curve must be renormalized so that as many periods as possible are kept finite in the limit $\mathsf{q} \to 0$. We find that the correctly renormalized coefficients are as follows:

$$C_{ij} \equiv \mathsf{q}^{\frac{[c_{ij}]}{2}} c_{ij} \quad \text{for} \quad i \neq j \,, \qquad C_{11} \equiv \mathsf{q} c_{11} \,, \qquad C_{00} \equiv \mathsf{q} c_{00} \,. \tag{5.3}$$

In terms of these renormalized parameters, the curve (5.1) is written as

$$\begin{aligned} 0 &= x^3 + z^6 - \frac{1}{\mathsf{q}} x^2 z^2 - \mathsf{q} x z^4 + \mathsf{q}^{-\frac{1}{3}} C_{20} x^2 + x(\mathsf{q}^{-1} C_{11} z + \mathsf{q}^{-\frac{2}{3}} C_{10}) \\ &\quad + \mathsf{q}^{-\frac{1}{6}} C_{05} z^5 + \mathsf{q}^{-\frac{1}{3}} C_{04} z^4 + \mathsf{q}^{-\frac{1}{2}} C_{03} z^3 + \mathsf{q}^{-\frac{2}{3}} C_{02} z^2 + \mathsf{q}^{-\frac{5}{6}} C_{01} z^1 - \mathsf{q}^{-1} C_{00} \,. \end{aligned} \tag{5.4}$$

One can show that the curve (5.4) splits into the following three sectors when we take $\mathsf{q} \to 0$ with C_{ij} kept finite.

- In the region $|z/x| \sim \mathsf{q}^{-1/3}$, one has the curve

  $$0 = -\widetilde{x}^2 \widetilde{z}^2 + \widetilde{z}^6 + C_{11} \widetilde{x} \widetilde{z} + C_{05} \widetilde{z}^5 + C_{04} \widetilde{z}^4 + C_{03} \widetilde{z}^3 + C_{02} \widetilde{z}^2 + C_{01} \widetilde{z} - C_{00} \,, \tag{5.5}$$

  where we defined $\widetilde{x} = \mathsf{q}^{-\frac{1}{6}} x$ and $\widetilde{z} = \mathsf{q}^{\frac{1}{6}} z$. One can shift $\widetilde{x}$ as $\widetilde{x} \to \widetilde{x} + C_{11}/(2\widetilde{z})$ so that the curve coincides with a known expression for the (A_1, D_6) theory:

  $$\widetilde{x}^2 = \widetilde{z}^4 + C_{05} \widetilde{z}^3 + C_{04} \widetilde{z}^2 + C_{03} \widetilde{z} + C_{02} + \frac{C_{01}}{\widetilde{z}} - \frac{C_{00} - \frac{C_{11}^2}{4}}{\widetilde{z}^2} \,. \tag{5.6}$$

  Note that the above shift of $\widetilde{x}$ preserves the SW 1-form up to exact terms. Here, we see that C_{05} and C_{04} are relevant couplings, C_{02} and C_{01} are the VEVs of Coulomb branch operators, and C_{03} and $\sqrt{C_{00} - C_{11}^2/4}$ are mass parameters of the (A_1, D_6) theory. In particular, $\sqrt{C_{00} - C_{11}^2/4}$ is associated with the SU(2) flavor sub-group that is gauged by the SU(2) vector multiplet in figure 2.

- In the region $|z/x| \sim \mathsf{q}^{2/3}$, the curve reduces to

$$0 = \widetilde{x}^3 - \widetilde{x}^2\widetilde{z}^2 + C_{20}\widetilde{x}^2 + \widetilde{x}(C_{11}\widetilde{z} + C_{10}) - C_{00}\,, \tag{5.7}$$

where we defined $\widetilde{x} = \mathsf{q}^{-\frac{1}{3}}x$ and $\widetilde{z} = \mathsf{q}^{\frac{1}{3}}z$. By shifting and rescaling the coordinates, this curve is further rewritten as

$$0 = X^2 + Z^4 + 2^{\frac{1}{3}}C_{20}Z^2 + 4\sqrt{C_{00} - \frac{C_{11}^2}{4}}\,Z - 2^{\frac{2}{3}}\left(C_{10} - \frac{C_{20}^2}{4}\right), \tag{5.8}$$

where we defined $X \equiv 2^{\frac{1}{3}}i(\widetilde{x} + \frac{1}{2}(\widetilde{z}^2 - C_{20}))$ and $Z \equiv -2^{-\frac{1}{3}}i\widetilde{z}$. We note that this coincides with the curve of the (A_1, D_3) theory. In particular, C_{20} is the relevant coupling, $(C_{10} - C_{20}^2/4)$ is the VEV of the Coulomb branch operator, and $\sqrt{C_{00} - C_{11}^2/4}$ is the mass parameter associated with the SU(2) flavor symmetry.

- In the region $|z/x| \sim 1$, the curve reduces to

$$0 = -x^2z^2 + C_{11}xz - C_{00}\,, \tag{5.9}$$

which describes a weak coupling limit of the SU(2) superconformal QCD as discussed in [23]. In particular, C_{11} is identified as the mass parameter of a fundamental hypermultiplet.

As seen above, in the limit $\mathsf{q} \to 0$, the curve of the (A_2, A_5) theory splits into the curves of the three sectors shown in figure 2. Moreover, we have seen physical meanings of C_{ij} in these three sectors, which leads to the following identification of parameters in (4.1) in terms of those in the SW curve (5.4):[19]

$$\begin{aligned} &d_1 = C_{05}\,, \qquad d_2 = C_{04}\,, \qquad m = -\frac{C_{03}}{6}\,, \qquad u_1 = C_{02}\,, \qquad u_2 = C_{01}\,, \\ &d = C_{20}\,, \qquad u = C_{10} - \frac{C_{20}^2}{4}\,, \qquad M = -\frac{C_{11}}{12}\,. \end{aligned} \tag{5.10}$$

5.2 S-duality from the curve

We now show that the T-transformation (4.28) that we identified in section 4.2 corresponds to a symmetry of the SW curve (5.1). We first note that the curve (5.1) is invariant under the following transformation:[20]

$$\begin{aligned} &\mathsf{q} \to e^{\frac{2\pi i}{3}}\mathsf{q}\,, \qquad c_{10} \to e^{-\frac{4\pi i}{9}}c_{10}\,, \qquad c_{11} \to e^{-\frac{2\pi i}{3}}c_{11}\,, \qquad c_{20} \to e^{-\frac{2\pi i}{9}}c_{20}\,, \\ &c_{01} \to -e^{\frac{\pi i}{9}}c_{01}\,, \qquad c_{02} \to -e^{-\frac{\pi i}{9}}c_{02}\,, \qquad c_{03} \to e^{\frac{2\pi i}{3}}c_{03}\,, \\ &c_{04} \to e^{\frac{4\pi i}{9}}c_{04}\,, \qquad c_{05} \to e^{\frac{2\pi i}{9}}c_{05}\,, \qquad c_{00} \to -e^{\frac{\pi i}{3}}c_{00}\,. \end{aligned} \tag{5.12}$$

[19] Here, numerical factors in front of C_{03} and C_{11} are not physical. They are introduced here just to avoid unimportant numerical coefficients below.

[20] At the same time, we take the change of coordinates in the curve (5.1)

$$(x, z) \to (e^{-\frac{2\pi i}{9}}x, e^{\frac{2\pi i}{9}}z)\,. \tag{5.11}$$

In the weak coupling limit $\mathsf{q} \to 0$, one can translate the above transformation into a transformation of parameters in the three sectors. Indeed, (5.3) and (5.10) imply that (5.12) is equivalent to

$$\begin{aligned} &\mathsf{q} \to e^{\frac{2\pi i}{3}}\mathsf{q}\,, \qquad && d_1 \to -e^{\frac{2\pi i}{3}} d_1\,, \qquad && d_2 \to -e^{\frac{\pi i}{3}} d_2\,, \qquad && m \to -m\,, \\ &u_1 \to e^{\frac{2\pi i}{3}} u_1\,, && u_2 \to e^{\frac{\pi i}{3}} u_2\,, \end{aligned} \tag{5.13}$$

in the weak coupling limit. Note that this is in perfect agreement with our T-transformation (4.28).[21] This means that our T-transformation (4.28) corresponds to a symmetry of the SW curve.

One can show that the above symmetry transformation (5.12) coincides with an S-duality transformation of the theory. To see this, let us turn off c_{ij} except for c_{00}. In this case, the curve is written as

$$0 = (x - \sqrt{\mathsf{q}}z^2)(x + \sqrt{\mathsf{q}}z^2)\left(x - \frac{z^2}{\mathsf{q}}\right) - c_{00}\,. \tag{5.14}$$

By changing the coordinates,[22] the curve is expressed as

$$y^2 = (\widetilde{x}^2 - \widetilde{u})^2 - f\widetilde{x}^4\,, \tag{5.18}$$

where f is defined by $f \equiv 1 - \mathsf{q}^3$ and the SW 1-form is now written as $\frac{i\widetilde{u}}{3}\frac{d\widetilde{x}}{y}$ up to exact terms. This is a standard expression for the curve of SU(2) conformal QCD. As discussed in [24, 25], there is an S-duality transformation involving

$$\sqrt{1-f} \to -\sqrt{1-f}\,, \qquad \widetilde{u} \to \widetilde{u}\,, \tag{5.19}$$

which is equivalent in our case to

$$\mathsf{q} \to e^{\frac{2\pi i}{3}}\mathsf{q}\,, \qquad c_{00} \to -e^{\frac{\pi i}{3}} c_{00}\,. \tag{5.20}$$

Since this is precisely the action of (5.12) on q and c_{00}, we conclude that our T-transformation (5.12) (or equivalently (4.28)) is an extension of this S-duality transformation to the case of generic values of c_{ij}.

[21] Recall that we have set $d_2 = 0$ in section 4.2 and therefore consistent with $d_2 \to -e^{\frac{\pi i}{3}} d_2$.

[22] In terms of $w = x/z^2$ and $v = z^3$, the curve (5.14) is written as

$$v^2 = \frac{c_{00}}{(w^2 - \mathsf{q})\left(w - \frac{1}{\mathsf{q}}\right)} \tag{5.15}$$

We consider the following change of variables:

$$w \to \frac{w\mathsf{q}^{\frac{1}{2}}\sqrt{1+\sqrt{f}} + \mathsf{q}^{\frac{1}{2}}\sqrt{\frac{1-\sqrt{f}}{1+\sqrt{f}}}}{w\sqrt{1-\sqrt{f}}+1}\,, \quad v \to \frac{\sqrt{1+\sqrt{f}}}{2\mathsf{q}^{\frac{1}{2}}\sqrt{f}} v\left(w\sqrt{1-\sqrt{f}}+1\right)^2\,, \tag{5.16}$$

where f is defined by $f \equiv 1 - \mathsf{q}^3$. The curve is now written as

$$v^2 = \frac{\widetilde{u}}{(w^2+1) - fw^4}\,, \tag{5.17}$$

where $\widetilde{u}$ is defined by $\widetilde{u} \equiv \frac{2(1-f)^{\frac{1}{3}}}{\sqrt{1+\sqrt{f}}}c_{00}$. The SW curve is now written as $\frac{1}{3}wdv$. In terms of $\widetilde{x} \equiv i\sqrt{\widetilde{u}}w$ and $y \equiv \widetilde{u}^{\frac{3}{2}}/v$, the curve (5.17) is expressed as (5.18).

5.3 Relation between mass parameters

We have shown in (4.19) that the prepotential of (A_2, A_5) and that of SU(2) gauge theory with four flavors are in a surprising relation. In particular, one of the four mass parameters of the latter is identified with the mass of the fundamental hypermultiplet of the former, and the other three masses of the latter are identified with the mass in the (A_1, D_6) sector. In this sub-section, we rederive this mass relation from the SW curve.

As seen above, the curve of the (A_2, A_5) theory is identical to that of SU(2) conformal QCD when $c_{ij} = 0$ except for c_{00}. This can be generalized to the case of non-vanishing mass parameters. When we turn on the two mass parameters c_{03} and c_{11}, the curve (5.14) of the (A_2, A_5) theory is slightly modified. In terms of $w \equiv x/z^2$ and $v \equiv z^3$, the modified curve is written as

$$0 = v^2 \left(w - \sqrt{\mathsf{q}}\right) \left(w + \sqrt{\mathsf{q}}\right) \left(w - \frac{1}{\mathsf{q}}\right) + v \left(c_{03} + c_{11} w\right) - c_{00} \,. \tag{5.21}$$

Defining $P_3(w) \equiv (w - \sqrt{\mathsf{q}})(w + \sqrt{\mathsf{q}})(w - 1/\mathsf{q})$ and shifting v as $v \to v - (c_{03} + c_{11} w)/(2P_3(w))$, we can rewrite the above as

$$v^2 = \frac{c_{00}}{P_3(w)} + \frac{\left(c_{03} + c_{11} w\right)^2}{4P_3(w)^2} \,, \tag{5.22}$$

where the SW 1-form is now $\lambda = -\frac{1}{3} v dw$ up to exact terms.

We see that (5.22) is precisely of the same form as the mass-deformed curve of SU(2) conformal QCD with four flavors [26]:

$$v^2 = \frac{U}{P_3(w)} + \frac{M_4(w)}{P_3(w)^2} \,, \tag{5.23}$$

where U stands for a coordinate of the Coulomb branch, and $M_4(w)$ is a fourth-order polynomial of w and related to the mass parameters of the theory. Since there exists one constraint on the coefficients of $M_4(w)$, there are four independent coefficients of $M_4(w)$. These four independent degrees of freedom are encoded in the residues of the SW 1-form at $w = \pm\sqrt{\mathsf{q}}$, $1/\mathsf{q}$ and ∞. These residues are known to be identified with the following linear combinations of the mass parameters, $m_1, \cdots, m_4$, of fundamental hypermultiplets:

$$m_1 \pm m_2 \,, \qquad m_3 \pm m_4 \,. \tag{5.24}$$

Comparing (5.22) and (5.23), we see that $(c_{03} + c_{11} w)^2$ in (5.22) is identified with $M_4(w)$ in (5.23). This implies that the four mass parameters of the latter theory are related to the two mass parameters of the former.

To see more concretely the relation between the mass parameters, let us compute the residues of the SW 1-form of the (A_2, A_5) theory. From (5.22), we see that the residues of the 1-form $\lambda = -\frac{1}{3} v dw$ at $w = \pm\sqrt{\mathsf{q}}, 1/\mathsf{q}$ and ∞ are respectively

$$-\frac{c_{03} \pm c_{11}\sqrt{\mathsf{q}}}{12(\mathsf{q} - 1/\sqrt{\mathsf{q}})} \,, \qquad -\frac{c_{03} + \frac{c_{11}}{\mathsf{q}}}{6\left(\frac{1}{\mathsf{q}} - \sqrt{\mathsf{q}}\right)\left(\frac{1}{\mathsf{q}} + \sqrt{\mathsf{q}}\right)} \,, \qquad 0 \,, \tag{5.25}$$

which reduce in the weak-coupling limit $\mathsf{q} \to 0$ to

$$\frac{m \pm 2M}{2}\,, \qquad 2M\,, \qquad 0\,. \tag{5.26}$$

We see that these residues coincide with (5.24) if we identify

$$m_1 = \frac{m}{2}\,, \qquad m_2 = m_3 = m_4 = M\,. \tag{5.27}$$

This implies that the mass-deformed SW curve of (A_2, A_5) is identical to that of the SU(2) conformal QCD when the four mass parameters of the latter are restricted as in (5.27). Note here that the restriction (5.27) of mass parameters is precisely equivalent to the one observed in the relation (4.19) for the prepotentials of these theories![23] This is a very non-trivial consistency check of (4.18) and our formula for $\mathcal{Z}^{(A_1,D_3)}_{Y_1,Y_2}$ that we developed in section 3.

6 Conclusion and discussions

In this paper, we have considered the U(2)-version of the generalized AGT correspondence for (A_1, D_N) theories for odd N, in terms of irregular states of the direct sum of Virasoro and Heisenberg algebras $Vir \oplus H$. In contrast to the (A_1, D_{even}) case, the action of $Vir \oplus H$ on the irregular state cannot be obtained in a colliding limit of primary operators, which makes it very difficult to compute the (normalized) inner product of the form in (3.3). However, we have shown that, when the relevant couplings and the VEVs of Coulomb branch operators of the (A_1, D_N) theory are turned off, one can compute the inner product as in (3.29).

Using the formula (3.29), we have computed the instanton partition function of the (A_2, A_5) theory, i.e., the coupled system of an SU(2) vector multiplet, a fundamental hypermultiplet, (A_1, D_6) and (A_1, D_3) as described in figure 2. Our result implies a surprising relation (4.19) between the prepotential of the (A_2, A_5) theory and that of the SU(2) superconformal QCD. A similar relation was found in [8] for the (A_3, A_3) theory. Using the relation (4.19), we have read off how the S-duality group acts on parameters including the UV gauge coupling. We have also checked in section 5 that the relation (4.19) is consistent with the Seiberg-Witten curves of the (A_2, A_5) theory and the SU(2) superconformal QCD.

One can also apply our formula for $\mathcal{Z}^{(A_1,D_{\text{odd}})}_{Y_1,Y_2}$ to other gauged AD theories. For instance, let us consider the SU(2) gauge theory coupled to three copies of the (A_1, D_3) theory. As in the case of (A_2, A_5), the SU(2) gauge coupling of this theory is exactly marginal. Using our formula for $\mathcal{Z}^{(A_1,D_3)}_{Y_1,Y_2}$, one can then compute the prepotential of this theory, up to terms affected by the U(1)-factor, at least when the relevant coupling and the VEV of Coulomb branch operator of the (A_1, D_3) sectors are turned off. We have done this computation and checked up to $\mathcal{O}(q^6)$ that the resulting prepotential has no instanton

[23] The coincidence of the numerical factor 1/2 in front of m is a consequence of our identification (5.10), and therefore is not non-trivial. What is non-trivial here is the coincidence that, both in (4.19) and (5.27), three of the four mass parameters of the SU(2) conformal QCD are equal and proportional to M, and the remaining one is proportional to m.

correction at all. Note that the same situation occurs for the prepotential of $\mathcal{N}=4$ super Yang-Mills theories (SYMs). Indeed, a peculiar connection between the SU(2) gauge theory coupled to three (A_1, D_3) theories and $\mathcal{N}=4$ SU(2) SYM has already been pointed out in [27]; the Schur index of these two theories are related by a simple change of variables.

Let us denote by T_{p_1,p_2,p_3} the SU(2) gauge theory coupled to (A_1, D_{p_1}), (A_1, D_{p_2}) and (A_1, D_{p_3}). Given the results discussed in the previous two paragraphs, we see that $T_{3,3,3}$ has similarity with $\mathcal{N}=4$ SU(2) SYM while $T_{2,4,4}$ and $T_{2,3,6}$ have similarity with SU(2) superconformal QCD with four flavors. These similarities can be universally described from the string theory viewpoint. Indeed, since (A_1, D_p) is identical to the $D_p(\mathrm{SU}(2))$ theory [28], the three theories $T_{3,3,3}$, $T_{2,4,4}$ and $T_{2,3,6}$ are respectively identical to the $N=2$ cases of $(E_6^{1,1}, \mathrm{SU}(N))$, $(E_7^{1,1}, \mathrm{SU}(N))$ and $(E_8^{1,1}, \mathrm{SU}(N))$ theories described in eq. (B.2) of [20]. As discussed in section 3.3 of [20], these theories are engineered by type IIA string theory on $(T^2 \times \mathbb{C}^2)/\mathsf{G}$ with G being an abelian subgroup of U(2) containing $\mathbb{Z}_N$ and $\mathbb{Z}_{2p}$. Here, $p=3,4$ and 6 for $(E_6^{1,1}, \mathrm{SU}(N))$, $(E_7^{1,1}, \mathrm{SU}(N))$ and $(E_8^{1,1}, \mathrm{SU}(N))$, respectively.[24] In terms of N and p, we see that the similarity to $\mathcal{N}=4$ SYM occurs when $N=2$ and p is odd, while the similarity to superconformal QCD occurs when $N=2$ and p is even. Moreover, focusing on the Schur index, the former similarity to $\mathcal{N}=4$ SYM was generalized to all coprime N and p in [27]. This suggests that different similarities occur depending on whether N and p are coprime. It would be interesting to study the reason for this phenomenon.

There are clearly many future directions. One of the most important directions is to understand the reason for the peculiar relation (4.19) for the prepotentials. Another interesting direction is to study the Nekrasov-Shatashvili limit of the instanton partition function [3], which should be combined with the recent results on the quantum periods of AD theories [29–32]. The uplift of our formula (3.29) to five dimensions would also be an interesting direction. It would also be interesting to search for a matrix model description of the instanton partition function of (A_2, A_5), generalizing the ones studied in [33–38].

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A S-duality of SU(2) conformal QCD

Here we give a brief review of an expression for the prepotential of SU(2) superconformal QCD, following [4]. When the mass parameters are turned off, the prepotential must be

[24] The remaining case $(D_4^{1,1}, \mathrm{SU}(2))$ from (B.2) of [20] is identical to SU(2) superconformal QCD with four flavors, and corresponds to the case of $p=2$.

written as

$$\mathcal{F}^{N_f=4} = (\log \widetilde{q}_{\rm IR})a^2 \,, \tag{A.1}$$

for dimensional reasons, where a is the VEV of the adjoint scalar in the vector multiplet, and $\widetilde{q}_{\rm IR}$ is a function of the UV gauge coupling q. The above prepotential is written as the sum of the following perturbative and instanton parts

$$\mathcal{F}^{N_f=4}_{\rm pert}(a) = (\log q - \log 16)a^2 \,, \tag{A.2}$$

$$\mathcal{F}^{N_f=4}_{\rm inst}(q;a) = \left(\frac{1}{2}q + \frac{13}{64}q^2 + \frac{23}{192}q^3 + \cdots\right) \,, \tag{A.3}$$

from which the following relation between q and $\widetilde{q}_{\rm IR}$ [19]:

$$q = \frac{\theta_2\left(\widetilde{q}_{\rm IR}\right)^4}{\theta_3\left(\widetilde{q}_{\rm IR}\right)^4} \,. \tag{A.4}$$

This relation implies that there are S-dual transformations S and T such that

$$T : \tau_{\rm IR} \to \ \tau_{\rm IR} + 1 \,, \qquad S : \tau_{\rm IR} \to -\frac{1}{\tau_{\rm IR}} \,, \tag{A.5}$$

where τ_{IR} is defined by

$$\tau_{IR} \equiv \frac{1}{i\pi}\log \widetilde{q}_{\rm IR} = \frac{\theta_{\rm IR}}{\pi} + \frac{8\pi i}{g^2_{\rm IR}} \,. \tag{A.6}$$

Note that the T-transformation corresponds to $\theta_{\rm IR} \to \theta_{\rm IR} + \pi$. In terms of q, the above S-dual transformations are written as

$$T : q \to \frac{q}{q-1} \,, \qquad S : q \to 1 - q \,. \tag{A.7}$$

Let us now turn on all the mass parameters. Then the instanton part of the prepotential is modified as

$$\begin{aligned}
&\mathcal{F}^{N_f=4}_{\rm inst}(q;a,m_i) \\
&\quad = \frac{1}{2}(a^2 + m_1m_2m_3m_4a^{-2})q \\
&\qquad + \frac{1}{64}\Big(13a^2 + (16m_1m_2m_3m_4 + m_3^2m_4^2 + m_2^2(m_3^2+m_4^2) + m_1^2(m_2^2+m_3^2+m_4^2))a^{-2} \\
&\qquad\quad - 3(m_2^2m_3^2m_4^2 + m_1^2(m_3^2m_4^2 + m_2^2(m_3^2+m_4^2)))a^{-4} + 5m_1^2m_2^2m_3^2m_4^2a^{-6}\Big)q^2 + \cdots \,.
\end{aligned} \tag{A.8}$$

The S-dual transformations (A.7) are now accompanied with the SO(8) triality [24].

B Decoupling a fundamental matter from (A_2, A_5)

Here, we consider the decoupling limit of the fundamental hypermultiplet from the (A_2, A_5) theory at the level of the SW curve. Recall that the (A_2, A_5) theory is described by the quiver in figure 2. When the fundamental hypermultiplet is decoupled, the resulting theory is described by the quiver in figure 4. This theory is called the "$\widehat{A}_{3,6}$ theory" in [6].

To see this decoupling at the level of the SW curve, we take the mass of the fundamental hypermultiplet to infinity, i.e., $C_{11} \to \infty$ in (5.4). For the periods of the curves to be finite, one needs to keep

$$\Lambda \equiv -\frac{1}{2}\sqrt{\mathsf{q}}C_{11} \tag{B.1}$$

finite in this limit. The finite constant Λ is then identified as a dynamical scale of the resulting theory. In terms of

$$X \equiv -\frac{\Lambda}{\sqrt{\mathsf{q}}}\left(\frac{x}{z^2}\right)^{\frac{1}{3}} + z^3\left(\frac{x}{z^2}\right)^{\frac{2}{3}}, \qquad Z \equiv \left(\frac{z^2}{x}\right)^{\frac{1}{3}}, \tag{B.2}$$

the curve in the limit $C_{11} \to \infty$ is written as

$$\begin{aligned} X^2 = {}& \Lambda^{5/3}C_{05}Z^3 + \Lambda^{4/3}C_{04}Z^2 + \Lambda C_{03}Z + \Lambda^{2/3}C_{02} + \frac{\Lambda^{1/3}C_{01}}{Z} \\ & - \frac{C_{00} + 2\Lambda^2}{Z^2} + \frac{\Lambda^{2/3}C_{10}}{Z^3} + \frac{\Lambda^{4/3}C_{20}}{Z^4} + \frac{\Lambda^2}{Z^5}, \end{aligned} \tag{B.3}$$

and the SW 1-form is written as $\lambda = XdZ$ up to exact terms. We see that the above curve is precisely identical to that of the $\widehat{A}_{3,6}$ theory [6].

Note that, by standard arguments, the gauge coupling of the conformal theory on the Coulomb branch, $\exp\left(i\theta_{\mathrm{IR}} - \frac{8\pi^2}{g_{\mathrm{IR}}^2}\right)$, is related to the dynamical scale of the mass-deformed theory, Λ, by

$$\frac{\Lambda}{C_{11}} = \exp\left(i\theta_{\mathrm{IR}} - \frac{8\pi^2}{g_{\mathrm{IR}}^2}\right). \tag{B.4}$$

Combining this with (B.1), we see that

$$\mathsf{q} \propto \exp\left(2i\theta_{\mathrm{IR}} - \frac{16\pi^2}{g_{\mathrm{IR}}^2}\right). \tag{B.5}$$

Recall here that our T-transformation (5.12) involves $\mathsf{q} \to e^{\frac{2\pi i}{3}}\mathsf{q}$. Using the above relation, one can translate this into

$$\theta_{\mathrm{IR}} \to \theta_{\mathrm{IR}} + \frac{\pi}{3}, \tag{B.6}$$

which implies that the T-transformation exchanges the minimal magnetic monopole with a dyonic particle whose electric charge is 1/6 of that of the W-boson, which is consistent with the fact that $\mathrm{PSL}(2,\mathbb{Z})$ naturally acts on a modified electro-magnetic charge lattice [21].

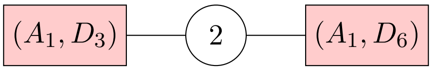

Figure 4. The mass deformed theory of the (A_2, A_5) theory.

C The a independence of $\mathfrak{a}_m$

Here, we give another supporting evidence for the a-independence of $\mathfrak{a}_m$ that we have defined in (3.10). Since its definition involves $\langle a|$, *a priori* it might depend on a. However, as we will show below, a natural 2D interpretation of the U(1)-factor strongly suggests that $\mathfrak{a}_m$ is indeed independent of a.

To see this, we first consider the theory described by the quiver diagram in figure 3. The partition functions of this theory for U(2) and SU(2) gauge groups are related by

$$\mathcal{Z}_{\mathrm{U}(2)}(a) = \mathcal{Z}_{\mathrm{U}(1)}\, \mathcal{Z}_{\mathrm{SU}(2)}(a)\,, \tag{C.1}$$

where the U(1)-factor $\mathcal{Z}_{\mathrm{U}(1)}$ is independent of a, since a is the VEV of a scalar field in the SU(2) vector multiplet and therefore neutral under the U(1). Recall here that the U(2)-version of the generalized AGT correspondence implies $\mathcal{Z}_{\mathrm{U}(2)} = \langle \widehat{I}^{(N/2)}|\widehat{I}^{(N/2)}\rangle$, where $\langle \widehat{I}^{(N/2)}|$ and $|\widehat{I}^{(N/2)}\rangle$ are characterized by the two (A_1, D_N) theories in figure 3. Here and below, we set all the parameters of the two (A_1, D_N) theories to be equal so that $\langle \widehat{I}^{(N/2)}|$ is indeed the conjugate of the state $|\widehat{I}^{(N/2)}\rangle$. Then one can write

$$\mathcal{Z}_{\mathrm{U}(2)}(a) = \left||\widehat{I}^{(N/2)}\rangle\right|^2 = \left||I^{(N/2)}\rangle\right|^2 \left||I_H^{(N/2)}\rangle\right|^2, \tag{C.2}$$

where we used that fact that $|\widehat{I}^{(N/2)}\rangle = |I^{(N/2)}\rangle \otimes |I_H^{(N/2)}\rangle$. Since the original (SU(2)-version of) the generalized AGT correspondence implies $\mathcal{Z}_{\mathrm{SU}(2)}(a) = \left||I^{(N/2)}\rangle\right|^2$, we identify the U(1)-factor as

$$\mathcal{Z}_{\mathrm{U}(1)} = \left||I_H^{(N/2)}\rangle\right|^2. \tag{C.3}$$

Note that $\mathcal{Z}_{\mathrm{SU}(2)}(a)$ and $\mathcal{Z}_{\mathrm{U}(1)}$ are expanded in powers of the dynamical scale Λ as

$$\mathcal{Z}_{\mathrm{U}(1)} = \sum_{k=0}^{\infty} \Lambda^{b_0 k} Z_k^{\mathrm{U}(1)}\,, \qquad Z_{\mathrm{SU}(2)}(a) = \sum_{\ell=0}^{\infty} \Lambda^{b_0 \ell} Z_\ell^{\mathrm{SU}(2)}(a)\,, \tag{C.4}$$

where $b_0 \equiv 4/N$ is the one-loop beta function coefficient of the gauge coupling. The identification (C.3) then implies that the expansion coefficients $Z_k^{\mathrm{U}(1)}$ are somehow related to $|I_H^{(N/2)}\rangle$. Below, we will argue that $Z_k^{\mathrm{U}(1)}$ are written in terms of $\langle a_H|I_H^{(N/2)}\rangle$ and

$$\mathfrak{a}_m \equiv \frac{\langle a|a_m|\widehat{I}^{(N/2)}\rangle}{\langle a|\widehat{I}^{(N/2)}\rangle} = \frac{\langle a_H|a_m|I_H^{(N/2)}\rangle}{\langle a_H|I_H^{(N/2)}\rangle}\,, \tag{C.5}$$

where we decompose the highest weight state $|a\rangle$ of $Vir \times H$ as $|a\rangle = |a_V\rangle \otimes |a_H\rangle$. As we will see below, this and the a-independence of $\mathcal{Z}_{\mathrm{U}(1)}$ then strongly suggest that $\mathfrak{a}_m$ are all independent of a.

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To see how $\mathfrak{a}_m$ are related to $Z_k^{\mathrm{U}(1)}$, let us first expand $\mathcal{Z}_{\mathrm{U}(2)}$ as

$$\mathcal{Z}_{\mathrm{U}(2)}(a) = \sum_{Y_1,Y_2} \Lambda^{b_0(|Y_1|+|Y_2|)} \mathcal{Z}^{\mathrm{vec}}_{Y_1,Y_2}(a) \langle \widehat{I}^{(N/2)}|a; Y_1, Y_2\rangle\langle a; Y_1, Y_2|\widehat{I}^{(N/2)}\rangle\,. \tag{C.6}$$

This is obtained by inserting (2.11) into $\mathcal{Z}_{\mathrm{U}(2)}(a) = \langle \widehat{I}^{(N/2)}|\widehat{I}^{(N/2)}\rangle$. Using (C.1) and (C.4), one also finds

$$\mathcal{Z}_{\mathrm{U}(2)}(a) = \sum_{k,\ell=0} \Lambda^{b_0(k+\ell)} Z_k^{\mathrm{SU}(2)}(a)\ Z_\ell^{\mathrm{U}(1)}\,. \tag{C.7}$$

By comparing (C.6) and (C.7) order by order, one can read off how $Z_k^{\mathrm{SU}(2)}(a)$ and $Z_k^{\mathrm{U}(1)}$ are related to $|I^{(N/2)}\rangle$ and $|I_H^{(N/2)}\rangle$. Below, we will perform this comparison explicitly at $\mathcal{O}(\Lambda^0)$, $\mathcal{O}(\Lambda^{b_0})$ and $\mathcal{O}(\Lambda^{2b_0})$.

Comparing the terms of $\mathcal{O}(\Lambda^0)$ in (C.6) and (C.7), we find

$$Z_0^{\mathrm{SU}(2)}(a)\, Z_0^{\mathrm{U}(1)} = |\langle a_V|I^{(N/2)}\rangle|^2\, |\langle a_H|I_H^{(N/2)}\rangle|^2\,, \tag{C.8}$$

where we used the fact that $|a;\emptyset,\emptyset\rangle = |a\rangle = |a_V\rangle\otimes|a_H\rangle$ and $|\widehat{I}^{(N/2)}\rangle = |I^{(N/2)}\rangle \otimes |I_H^{(N/2)}\rangle$. Note that, according to the (SU(2)-version of) generalized AGT correspondence, we already have the identification $Z_0^{\mathrm{SU}(2)}(a) = |\langle a_V|I^{(N/2)}\rangle|^2$ [6, 7].[25] This and (C.8) then imply that

$$Z_0^{\mathrm{U}(1)} = |\langle a_H|I_H^{(N/2)}\rangle|^2\,. \tag{C.9}$$

We then see that the a-independence of $|\langle a_H|I_H^{(N/2)}\rangle|$ follows from that of $\mathcal{Z}_0^{\mathrm{U}(1)}$.

We now compare the terms of $\mathcal{O}(\Lambda^{b_0})$ in (C.6) and (C.7). Hereafter, we set $\epsilon_1 = 1/\epsilon_2 = i$ for simplicity. The relevant equation is

$$\begin{aligned} &Z_1^{\mathrm{SU}(2)}(a)\, Z_0^{\mathrm{U}(1)} + Z_0^{\mathrm{SU}(2)}(a)\, Z_1^{\mathrm{U}(1)} \\ &\quad = \mathcal{Z}^{\mathrm{vec}}_{\square,\emptyset}\langle \widehat{I}^{(N/2)}|a;\square,\emptyset\rangle\langle a;\square,\emptyset|\widehat{I}^{(N/2)}\rangle + \mathcal{Z}^{\mathrm{vec}}_{\emptyset,\square}\langle \widehat{I}^{(N/2)}|a;\emptyset,\square\rangle\langle a;\emptyset,\square|\widehat{I}^{(N/2)}\rangle\,. \end{aligned} \tag{C.10}$$

Note that, using $\mathcal{Z}_0^{\mathrm{SU}(2)} = |\langle a_V|I^{(N/2)}\rangle|^2$, (C.9) and the explicit expressions for $|a;\square,\emptyset\rangle$ and $|a;\emptyset,\square\rangle$, one can rewrite the right-hand side as

$$-\frac{1}{2a^2}|\langle a_V|L_1|I^{(N/2)}\rangle|^2 \mathcal{Z}_0^{\mathrm{U}(1)} + 2\mathcal{Z}_0^{\mathrm{SU}(2)}(a)\,|\langle a_H|a_1|I_H^{(N/2)}\rangle|^2\,. \tag{C.11}$$

Therefore, the natural identification

$$Z_1^{\mathrm{SU}(2)}(a) = -\frac{1}{2a^2}|\langle a_V|L_1|I^{(N/2)}\rangle|^2\,, \qquad Z_1^{\mathrm{U}(1)} = 2|\langle a_H|a_1|I_H^{(N/2)}\rangle|^2\,. \tag{C.12}$$

solves the equation (C.10). Under the above identification, the a-independence of $|\langle a_H|a_1|I_H^{(N/2)}\rangle|$ follows from the fact that $Z_1^{\mathrm{U}(1)}$ is independent of a.

[25] One (A_1, D_N) in figure 3 gives rise to $\langle a_V|I^{(N/2)}\rangle$, and the other gives its conjugate. The perturbative contribution from the SU(2) vector multiplet is omitted here.

One can continue this procedure to higher orders of Λ. At $\mathcal{O}(\Lambda^{2b_0})$, we find

$$\begin{aligned}
&Z_2^{\mathrm{SU}(2)}(a)\, Z_0^{\mathrm{U}(1)} + Z_1^{\mathrm{SU}(2)}(a)\, Z_1^{\mathrm{U}(1)} + Z_0^{\mathrm{SU}(2)}(a)\, Z_2^{\mathrm{U}(2)} \\
&\quad = \mathcal{Z}^{\mathrm{vec}}_{\square\square,\emptyset} |\langle a; \square\square, \emptyset | \widehat{I}^{(N/2)}\rangle|^2 + \mathcal{Z}^{\mathrm{vec}}_{\substack{\square\\\square},\emptyset} |\langle a; \substack{\square\\\square}, \emptyset | \widehat{I}^{(N/2)}\rangle|^2 + \mathcal{Z}^{\mathrm{vec}}_{\square,\square} |\langle a; \square, \square | \widehat{I}^{(N/2)}\rangle|^2 \\
&\qquad + \mathcal{Z}^{\mathrm{vec}}_{\emptyset,\square\square} |\langle a; \emptyset, \square\square | \widehat{I}^{(N/2)}\rangle|^2 + \mathcal{Z}^{\mathrm{vec}}_{\emptyset,\substack{\square\\\square}} |\langle a; \emptyset, \substack{\square\\\square} | \widehat{I}^{(N/2)}\rangle|^2 \,. \qquad \text{(C.13)}
\end{aligned}$$

Using $Z_0^{\mathrm{SU}(2)}(a) = |\langle a_V | I^{(N/2)}\rangle|^2$, (C.9) and (C.12), we see that

$$\begin{aligned}
Z_2^{\mathrm{SU}(2)} &= \frac{1}{(1+4a^2)^2}\Bigg[4(-1+2a^2)^2\, |\langle a_V|L_2|I^{(N/2)}\rangle|^2 + \frac{-1+8a^2}{4a^4}\frac{|\langle a_V|L_1|I^{(N/2)}\rangle|^4}{|\langle a_V|I^{(N/2)}\rangle|^4} \qquad \text{(C.14)} \\
&\quad - 3\left(\frac{\langle I^{(N/2)}|L_{-2}|a_V\rangle\langle a_V|L_1|I^{(N/2)}\rangle^2}{\langle a_V|I^{(N/2)}\rangle} + \frac{\langle I^{(N/2)}|(L_{-1})^2|a_V\rangle\langle a_V|L_2^{(N/2)}\rangle}{\langle I^{(N/2)}|a_V\rangle}\right)\Bigg] \\
Z_2^{\mathrm{U}(1)} &= |\langle a_H|a_2|I_H^{(N/2)}\rangle|^2 + 2\frac{|\langle a_H|a_1|I_H^{(N/2)}\rangle|^4}{|\langle a_H|I_H^{(N/2)}\rangle|^2}\,. \qquad \text{(C.15)}
\end{aligned}$$

solve the equation (C.13). Since $Z_2^{\mathrm{U}(1)}$, $|\langle a_H|I_H^{(N/2)}\rangle|$ and $|\langle a_H|a_1|I_H^{(N/2)}\rangle|$ are independent of a, we see from (C.15) that $|\langle a_H|a_2|I_H^{(N/2)}\rangle|$ is also independent of a.

Given the a-independence of $|\langle a_H|I_H^{(N/2)}\rangle|$, $\langle a_H|a_1|I_H^{(N/2)}\rangle|$ and $|\langle a_H|a_2|I_H^{(N/2)}\rangle|$, it is straightforward to show that

$$|\mathfrak{a}_m| = \left|\frac{\langle a_H|a_m|I_H^{(N/2)}\rangle}{\langle a_H|I_H^{(N/2)}\rangle}\right| \qquad \text{(C.16)}$$

is independent of a at least for $m = 1, 2$. We expect that, via the same procedure, one can argue that $|\mathfrak{a}_m|$ are all independent of a for $m = 1, 2, 3, \cdots$. Indeed, we have checked this for $m = 1, \cdots, 4$. While this discussion is only about $|\mathfrak{a}_m|$, it is a strong supporting evidence for the a-independence of $\mathfrak{a}_m$ for all m.

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