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Celestial OPEs and $\mathrm{w}_{1+\infty}$ algebra from worldsheet in string theory

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Abstract: Celestial operator product expansions (OPEs) arise from the collinear limit of scattering amplitudes and play a vital role in celestial holography. In this paper, we derive the celestial OPEs of massless fields in string theory from the worldsheet. By studying the worldsheet OPEs of vertex operators in worldsheet CFT and further examining their behaviors in the collinear limit, we find that new vertex operators for the massless fields in string theory are generated and become dominant in the collinear limit. Mellin transforming to the conformal basis yields exactly the celestial OPEs in celestial CFT. We also derive the celestial OPEs from the collinear factorization of string amplitudes and the results derived in these two different methods are in perfect agreement with each other. Our final formulae of celestial OPEs are applicable to general dimensions, corresponding to Einstein-Yang-Mills theory supplemented by some possible higher derivative interactions. Specializing to 4D, we reproduce all the celestial OPEs for gluon and graviton in the literature. We consider various string theories, including the open and closed bosonic string, as well as the closed superstring theory with $\mathcal{N}=1$ and $\mathcal{N}=2$ worldsheet supersymmetry. In the case of $\mathcal{N}=2$ string, we also derive all the $\overline{\mathrm{SL}(2,\mathbb{R})}$ descendant contributions in the celestial OPE; the soft sector of such OPE just yields the $w_{1+\infty}$ algebra after rewriting in terms of chiral modes. Our stringy derivation of celestial OPEs thus initiates the first step towards the microscopic realization of celestial CFT dual to string theory in flat spacetime.

Keywords: Conformal and W Symmetry, Gauge-gravity correspondence, Scattering Amplitudes, Superstrings and Heterotic Strings

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1 Introduction

The quest for quantum gravity is one of the most fundamental questions in theoretical physics. Although the quantization of quantum gravity is notoriously hard, string theory has provided us with explicit examples of non-perturbative theories of quantum gravity. Moreover, string theory enables us to learn some profound aspects of quantum gravity, in particular the holographic nature. Although the holographic principle was first proposed based on black hole entropy [1, 2], such an idea was very vague until a concrete model was realized in string theory [3]. In string theory, such a holographic duality, now known as the AdS/CFT correspondence, naturally arises from the duality between open and closed strings. After more than two decades of intensive study, the AdS/CFT correspondence has been tested very precisely and has also taught us even more profound aspects of quantum gravity, like entanglement. Considering the very beautiful and successful story of AdS/CFT correspondence, it is natural to wonder whether we can study quantum gravity beyond AdS. One interesting and natural generalization is flat spacetime, where the boundary is null and even non-smooth. This turns out to be very difficult and very little progress was made in the past. On the other hand, the last two decades also witnessed fruitful achievements in scattering amplitude both computationally and conceptually. In particular, the idea of on-shellness, locality, unitarity and causality has been playing a crucial role. Interestingly, scattering amplitudes are just the observables of quantum field theories in flat spacetime. Bridging the ideas from two seemingly unrelated areas together, a promising program towards flat holography, called celestial holography, starts to emerge in recent years [4, 5]. The precursor of celestial holography comes from noticing the equivalence between asymptotic symmetries and soft theorems [6, 7]. Since then various interesting progress has been made. See [8, 9] for recent reviews. In spite, the study of celestial holography has been mostly focusing on symmetries and using the bottom-up approach. This is drastically different from the AdS/CFT correspondence which has various top-down concrete realizations in string theory. It is then natural to ask whether we can also find a concrete realization for celestial holography in string theory.[1] This is particularly important for several reasons. First of all, the celestial amplitudes seem to be very UV sensitive and it is only well-defined in theories, like string theory, where the UV behavior is soft enough. Secondly, string theory may be a very promising candidate for establishing an exact microscopic model for celestial holography. Once we have an exact model for celestial holography, we can then test various salient ideas there. In AdS/CFT, string theory has helped us to discover the correspondence between $\mathcal{N} = 4$ SYM and type IIB string on $AdS_5 \times S^5$, which is believed to be exact and has been tested precisely.

The goal of this paper is to initiate the first step towards the microscopic realization of celestial holography in string theory. More specifically, we will derive the celestial operator production expansion (OPE) from string theory. Although we have not been able to construct explicitly the celestial CFT for string theory, it turns out that the worldsheet of

[1]For previous studies of celestial holography related to (ambitwistor) string theory, see [10] for celestial string amplitudes, and [11, 12] for conformally soft theorems and celestial double copy in ambitwistor string theory.

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string theory already implies something nontrivial about celestial holography. In particular, we can derive the celestial OPEs from the worldsheet OPEs. Our derivations show that there are indeed some universal connections between celestial sphere and string worldsheet.

The celestial OPEs characterize the behavior of two operators in the coincident limit in celestial CFT (CCFT). They can be obtained from the collinear limit of scattering amplitude by performing the Mellin transformation. Since the collinear factorization is a universal property of scattering amplitude, the celestial OPE is supposed to also play an important role in CCFT. Moreover, the soft sector of celestial OPEs also encodes the underlying symmetry of CCFT and the bulk scattering amplitude. In particular, starting from celestial OPEs, an infinite dimensional holographic symmetry algebra has been discovered recently [13, 14].[2] In the case of gravity, this symmetry algebra is just the $w_{1+\infty}$ algebra [14]. The supersymmetric extension and the infinite Ward identities associated with this algebra were studied in [15]. See [16, 17] for other aspects of celestial OPEs.

The celestial OPEs can be obtained in several methods. The most direct way is to consider the collinear limit of scattering amplitude and then perform the Mellin transformation [18–20]. Alternatively, one can bootstrap celestial OPEs using conformal symmetry as well as the input from soft theorems [21]. Using these methods, the general formula of celestial OPEs for spinning massless particles with cubic interactions in 4D has been recently derived in [15, 22].

In this paper, we offer another derivation of celestial OPEs from the string worldsheet perspective.[3] The general strategy is as follows. In celestial holography, the celestial amplitudes are defined as the Mellin transformation of momentum space scattering amplitude[4] and can be regarded as the correlation functions of celestial operators $\langle \mathcal{O}_{\Delta_1}(x_1)\mathcal{O}_{\Delta_2}(x_2)\cdots\rangle$ in some putative CCFT on the celestial sphere living at boundary null infinity [5, 24]. This is very similar to the scattering amplitudes in string theory which are computed by the correlator of vertex operators in worldsheet CFT. One can make the relation more precise by introducing the so-called conformal vertex operators, which are defined as the Mellin transformation of the standard vertex operators. The celestial string amplitude can then be alternatively regarded as the correlator of conformal vertex operators $\langle \mathcal{V}_{\Delta_1}(x_1)\mathcal{V}_{\Delta_2}(x_2)\cdots\rangle$ in worldsheet CFT. The fact that the celestial amplitude can be computed in two different ways thus suggests a map between worldsheet CFT and CCFT, and correspondingly a map between conformal vertex operators $\mathcal{V}_\Delta$ and celestial operators $\mathcal{O}_\Delta$. Then the derivation of OPEs of celestial operators, namely $\lim_{x_1\to x_2}\mathcal{O}_{\Delta_1}(x_1)\mathcal{O}_{\Delta_2}(x_2)$, boils down to the computation of OPEs of conformal vertex operators $\lim_{x_1\to x_2}\mathcal{V}_{\Delta_1}(x_1)\mathcal{V}_{\Delta_2}(x_2)$. It turns out that the latter can be obtained by first computing worldsheet OPEs of two vertex operators and then performing the Mellin transformation.[5] More precisely, the vertex operator is given by

[2] It is more precise to refer to this algebra as holographic chiral algebra as this algebra governs the chiral subsector of soft particles with positive helicity only and all the symmetries are generated by chiral currents.

[3] The celestial OPEs in 4D can also be derived from ambitwistor string [23].

[4] In this paper, we only consider massless fields, which just correspond to the gluon and graviton/dilaton/Kalb-Ramond field in string theory.

[5] In principle, one may compute the worldsheet OPE of two conformal vertex operators directly, and then take the collinear limit. We will not pursue this approach in this paper, and leave it to the future.

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some operator $V_p(z)$ in the worldsheet CFT integrating over the worldsheet.[6] The celestial OPE, namely $\lim_{x_1 \to x_2} \mathcal{V}_{\Delta_1}(x_1)\mathcal{V}_{\Delta_2}(x_2)$ can ultimately be recovered from the worldsheet OPE for a pair of vertex operators $V_{p_1}(z_1)$ and $V_{p_2}(z_2)$. Explicit computation shows that worldsheet OPE in the collinear limit $p_1 \cdot p_2 \to 0$, which is equivalent to the celestial coincident limit $x_1 \to x_2$, localizes to a delta-function $\delta^2(z_1 - z_2)$ on the worldsheet, and thus produces another vertex operator after integrating over the worldsheet. This thus fulfills our derivation of celestial OPEs from the worldsheet perspective. And the computation essentially reduces to computing OPEs of free fields on the worldsheet.

We will consider various string theories, including open and closed bosonic string, and closed superstring with $\mathcal{N} = 1$ and $\mathcal{N} = 2$ worldsheet supersymmetry. For bosonic string, we compute the worldsheet OPE of two vertex operators for massless gluons in open string and obtain the vertex operators for various open string fields including tachyon, gluon, etc. In spite, we show that in the collinear limit, only the gluon in the OPE is dominant and the rest can be ignored, up to some subtle boundary contact terms. The boundary contact terms arise when the two vertex operators in the OPE hit another vertex operator. They can partially be attributed to the remnant contribution from tachyon. On the other hand, the gluon contribution in the worldsheet OPE of two gluon vertex operators comes with a delta-function in the collinear limit. Doing a worldsheet integral and performing the Mellin transformation thus give us the celestial OPEs for gluons. The same derivation is also done for massless fields in the closed string, namely graviton, Kalb-Ramond two-form field and dilaton, and we obtain the celestial OPEs for these fields. We also discuss the OPEs between open and closed string massless fields. This is a bit different; at tree level the open string vertex operator sits only at the boundary of the disk, while the closed string vertex operator sits in the interior of the disk. Nevertheless, we manage deriving the corresponding celestial OPEs, namely the fusion of gluon and graviton to another gluon. However, it is known that graviton can also appear in the celestial OPE of two gluons [21]. The derivation of this OPE from the open-closed string setup is not clear, as one needs to produce a graviton vertex operator in the interior of the disk from two gluon vertex operators sitting on the boundary of the disk. We sidestep this problem by considering heterotic string where one can realize both gluon and graviton/dilaton/Kalb-Ramond field from the closed string. Using similar techniques in the bosonic string, we are able to derive all the celestial OPEs involving gluon and graviton/dilaton/Kalb-Ramond field, including the fusion of two gluons into one graviton. We also discuss the OPEs of NS-NS massless fields in type I and type IIA/IIB string theory.

To further corroborate our celestial OPEs, we study string amplitudes in the collinear limit and then perform the Mellin transformation, which offers another derivation of celestial OPEs. As in the generic theories of gauge and gravity, the three-point amplitude is just enough to determine the leading behavior of celestial OPEs. After performing these computations for string amplitudes, we find that the final results agree exactly with those derived from worldsheet. Our derivation is for string theory in critical dimensional flat spacetime, namely 26 dimensions for bosonic string and 10 dimensions for superstring.

[6] Here to be explicit, we only discuss the integrated form of vertex operators in closed string. The discussions for open string and the unintegrated form of vertex operators are the same.

Nevertheless, the final formulae of celestial OPEs are supposed to be applicable to general dimensions, corresponding to Einstein-Yang-Mills theory with some possible higher derivative corrections. Specializing to four dimensional spacetime, we recover all the gluon and graviton OPEs obtained in the literature [15, 18, 21, 22].

Last but not least, we also generalize our discussions to the $\mathcal{N}=2$ string theory [25–27]. The $\mathcal{N}=2$ string theory has critical dimension four and is consistent in $(2,2)$ signature instead of Minkowski signature. The simplest version of $\mathcal{N}=2$ theory has a massless degree of freedom and the low energy effective action for $\mathcal{N}=2$ string is described by some kind of self-dual gravity. Following the same approach in other string theories, we study the worldsheet OPE of two vertex operators in $\mathcal{N}=2$ string theory and derive the corresponding celestial OPE. The derivation is very similar to other string theories with less supersymmetry. However, now we can go further beyond the previous derivations. It turns out that in the spacetime with $(2,2)$ signature, we can even derive all the $\overline{\mathrm{SL}(2,\mathbb{R})}$ descendants in the OPE. We find that this essentially comes from the momentum conservation,[7] and the fact that in $(2,2)$ signature we can vary two celestial coordinates independently and thus have more freedom to realize the collinear limit. These two features, $\overline{\mathrm{SL}(2,\mathbb{R})}$ descendants and independent celestial coordinates, are just the crucial ingredients in the derivation of $w_{1+\infty}$ symmetry. Indeed, focusing on the soft sector of our OPE with descendants and then performing the mode expansion, we recover the $w_{1+\infty}$ algebra [13, 14]. We thus provide a stringy derivation of the $w_{1+\infty}$ symmetry, although it is indirect.[8] In a more direct derivation, one should be able to construct the chiral generators in $w_{1+\infty}$ algebra directly from $\mathcal{N}=2$ string worldsheet. We will make some comments and defer the direct construction to the future.

This paper is organized as follows. In section 2, we will first discuss the kinematics in general dimensions in terms of celestial variables, and then review the vertex operators in bosonic string theory. A map between conformal vertex operators and celestial operators will also be discussed. In section 3, we will derive the celestial OPEs from the worldsheet OPEs in bosonic string theory. In section 4, we will generalize the worldsheet derivation of celestial OPEs to superstring. In section 5, we will derive the celestial OPEs from the collinear factorization of string amplitude. The specialization of celestial OPEs to 4D will be presented. In section 6, we will derive the celestial OPE with descendants in $\mathcal{N}=2$ string theory, and then discuss the resulting $w_{1+\infty}$ algebra. In section 7, we will conclude and discuss possible future directions. The paper also has two appendices. In appendix A we will collect all the computation details of worldsheet OPEs of vertex operators in various string theories. In appendix B, we will derive the celestial OPE between gluon and graviton in the open-closed string setup from both the worldsheet perspective and the amplitude approach.

Notation. The spacetime dimension is $D+2$ and the corresponding celestial sphere is D-dimensional. We will use $\mu,\nu,\cdots=0,1,\cdots,D,D+1$ for spacetime indices and $a,b,c,\cdots=$

[7]Translational symmetry has also been used to constrain descendants in OPE purely at the level of CCFT [22, 28].

[8]Note that the $w_{1+\infty}$ algebra appeared before in self-dual gravity and $\mathcal{N}=2$ string, but in a different way [25, 29].

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$1, \cdots, D$ for celestial sphere indices. The spacetime metric is $\eta^{\mu\nu} = \text{diag}(-1, +1, \cdots, +1)$ and the celestial sphere metric is δ^{ab}, hence we will raise or lower the position of celestial sphere indices freely. Repeated indices are summed over. We use x_i^a to label the a-th coordinate of the i-th particle/operator. We use y and $z, \bar{z}$ for the open and closed string worldsheet coordinates, respectively, and $\mathsf{z}, \bar{\mathsf{z}}$ for the celestial sphere coordinates in four dimensional spacetime. The polarizations are denoted as $\zeta, \xi, e, \varepsilon$, while ϵ always refers to infinitesimal quantity.

2 Preliminary

In this preliminary section, we will introduce some tools and background knowledge that will be used in the later sections. We will first introduce the kinematics of massless fields in terms of celestial sphere variables in general dimensions. Then we will review the vertex operators in open and closed bosonic string theory. Finally we will introduce the notion of conformal vertex operators and their relation with celestial operators.

2.1 Kinematics in general dimension

We consider $D+2$-dimensional spacetime with D-dimensional celestial sphere at null infinity. A null momentum k can be parametrized as [30]

$$k^\mu(\omega_k, x) = \eta\omega_k \hat{k}^\mu(x)\,, \qquad \hat{k}^\mu(x) = \left(\frac{1+(x)^2}{2}\,, x^a, \frac{1-(x)^2}{2}\right), \qquad n^\mu = \Big(-1, 0^a, 1\Big)\,. \tag{2.1}$$

where $\omega_k > 0$, $\eta = \pm 1$ labels out-going/in-coming particles and $(x)^2 = \sum_a x^a x^a$. We also introduce the following basis of polarization vectors

$$\varepsilon_a^\mu(x) \equiv \varepsilon_a^\mu(k) = \partial_a \hat{k}^\mu(x) = (x^a, \delta^{ab}, -x^a)\,, \qquad \mu = (0, b, D+1)\,. \tag{2.2}$$

These D polarization vectors transform under the vector representation of the little group $\text{SO}(D)$ for massless particles. And we have

$$n \cdot n = \hat{k} \cdot \hat{k} = \varepsilon_a \cdot n = \varepsilon_a \cdot \hat{k} = 0\,, \qquad n \cdot \hat{k} = 1\,, \qquad \varepsilon_a \cdot \varepsilon_b = \delta_{ab}\,. \tag{2.3}$$

So the vectors $\left\{\frac{n+\hat{k}}{\sqrt{2}}, \varepsilon_a, \frac{n-\hat{k}}{\sqrt{2}}\right\}$ form a complete orthonormal basis. The little group rotates polarization vectors ε_a but leaves $n, \hat{k}$ invariant.

For different momenta, we further have the following identities

$$\hat{k}(x_i) \cdot \hat{k}(x_j) \equiv \hat{k}_i \cdot \hat{k}_j = -\frac{1}{2}(x_{ij})^2\,, \qquad \hat{k}_i \cdot \varepsilon_a(x_j) = x_{ij}^a\,, \qquad \varepsilon_a(x_i) \cdot \varepsilon_b(x_j) = \delta^{ab}\,, \tag{2.4}$$

where $x_{ij} = x_i - x_j$.

Taking the product of two polarizations, we get D^2 two-index tensors

$$\varepsilon_{ab}^{\mu\nu} = \varepsilon_a^\mu \varepsilon_b^\nu\,. \tag{2.5}$$

It is reducible under the little group SO(D) and can be decomposed into symmetric traceless, anti-symmetric and singlet representation:

$$\boldsymbol{D} \otimes \boldsymbol{D} = \frac{\boldsymbol{(D+2)(D-1)}}{\boldsymbol{2}} + \frac{\boldsymbol{D(D-1)}}{\boldsymbol{2}} + \boldsymbol{1}\,. \tag{2.6}$$

They just correspond to the polarizations of graviton, Kalb-Ramond (KR) 2-form field and dilaton, respectively:

$$\varepsilon^{\mu\nu}_{(ab)} = \frac{1}{2}\Big(\varepsilon^{\mu}_{a}\varepsilon^{\nu}_{b} + \varepsilon^{\nu}_{a}\varepsilon^{\mu}_{b}\Big) - \frac{1}{D}\delta_{ab}\Pi^{\mu\nu}\,, \qquad \delta^{ab}\varepsilon^{\mu\nu}_{ab} = \eta_{\mu\nu}\varepsilon^{\mu\nu}_{ab} = 0\,, \qquad \varepsilon^{\mu\nu}_{(ab)} = \varepsilon^{\mu\nu}_{(ba)} = \varepsilon^{\nu\mu}_{(ab)}\,, \tag{2.7}$$

$$\varepsilon^{\mu\nu}_{[ab]} = \frac{1}{2}\Big(\varepsilon^{\mu}_{a}\varepsilon^{\nu}_{b} - \varepsilon^{\nu}_{a}\varepsilon^{\mu}_{b}\Big) = -\varepsilon^{\nu\mu}_{[ab]} = -\varepsilon^{\mu\nu}_{[ba]}\,, \tag{2.8}$$

$$\varepsilon^{\mu\nu} = \frac{1}{D}\varepsilon^{\mu}_{a}\varepsilon^{\nu}_{a} = \frac{1}{D}\Pi^{\mu\nu}\,, \tag{2.9}$$

where we also introduced

$$\Pi^{\mu\nu}(x) \equiv \delta^{ab}\varepsilon^{\mu}_{a}(x)\varepsilon^{\nu}_{b}(x) = \eta^{\mu\nu} - n^{\mu}\hat{k}^{\nu}(x) - n^{\nu}\hat{k}^{\mu}(x)\,. \tag{2.10}$$

We will also frequently use the abstract polarization tensors which satisfy various properties. In particular, the polarization vector of gluon e^{μ} satisfies

$$e \cdot e = 1\,, \qquad k \cdot e = 0\,, \qquad e^{\mu} \simeq e^{\mu} + \lambda k^{\mu}\,, \tag{2.11}$$

where the equivalence in the last equation is guaranteed by the gauge invariance. The last property allows us to choose a gauge where

$$e \cdot n = 0\,. \tag{2.12}$$

We can then decompose any gluon polarization vector as

$$e^{\mu} \simeq \sum_{a} c_a \varepsilon^{\mu}_{a}\,, \tag{2.13}$$

up to a pure gauge.

Similarly, for closed string massless spectra, we will use $e^{\mu\nu}$ to denote the polarization tensors. Let us first consider graviton and KR 2-form field, whose polarization tensors satisfy

$$e^{\mu\nu}e_{\mu\nu} = 1\,, \quad k_{\mu}e^{\mu\nu} = k_{\nu}e^{\mu\nu} = \eta_{\mu\nu}e^{\mu\nu} = 0\,, \quad e^{\mu\nu} = s e^{\nu\mu}\,, \quad e^{\mu\nu} \simeq e^{\mu\nu} + k^{\mu}\zeta^{\nu} + s\zeta^{\mu}k^{\nu}\,, \tag{2.14}$$

where $k \cdot \zeta = 0$, and $s = \pm 1$ for graviton and 2-form field, respectively. They can be expanded in terms of basis (2.7), (2.8) that we constructed before:

$$e^{\mu\nu} \simeq \sum_{\substack{a,b=1 \\ a<b \text{ or } a=b<D}}^{D} c_{ab}\varepsilon^{\mu\nu}_{(ab)}\,, \qquad e^{\mu\nu} \simeq \sum_{\substack{a,b=1 \\ a<b}}^{D} c_{ab}\varepsilon^{\mu\nu}_{[ab]}\,, \tag{2.15}$$

for graviton and Kalb-Ramond 2-form field, respectively.

For dilaton, we have

$$e^{\mu\nu}e_{\mu\nu} = 1\,, \qquad e^{\mu\nu} = e^{\nu\mu}\,, \qquad k_\mu e^{\mu\nu} = n_\mu e^{\mu\nu} = 0\,, \tag{2.16}$$

where n is another null direction defined such that $n \cdot n = 0, n \cdot \hat{k} = 1$. This fixes the dilaton polarization to be (2.9)

$$e^{\mu\nu} = \frac{1}{D}\Big(\eta^{\mu\nu} - n^\mu \hat{k}^\nu - n^\nu \hat{k}^\mu\Big) = \varepsilon^{\mu\nu} = \sum_a \frac{1}{D}\varepsilon^{\mu\nu}_{aa}\,. \tag{2.17}$$

One easily check that $e^{\mu\nu}\eta_{\mu\nu} = 1$.

In four dimensional spacetime with $D = 2$, it is convenient to introduce complex coordinates on the celestial sphere

$$x^1 = \frac{\mathsf{z} + \bar{\mathsf{z}}}{2}\,, \qquad x^2 = \frac{-i(\mathsf{z} - \bar{\mathsf{z}})}{2}\,, \qquad \mathsf{z} = x^1 + ix^2\,, \qquad \bar{\mathsf{z}} = x^1 - ix^2\,. \tag{2.18}$$

Then we can represent the momentum as $k = \eta\omega_k \hat{k}^\mu$ where

$$\hat{k}^\mu = \frac{1}{2}\Big(1 + (x)^2, 2x^1, 2x^2, 1 - (x)^2\Big) = \frac{1}{2}\Big(1 + \mathsf{z}\bar{\mathsf{z}}, \mathsf{z} + \bar{\mathsf{z}}, -i(\mathsf{z} - \bar{\mathsf{z}}), 1 - \mathsf{z}\bar{\mathsf{z}}\Big)\,. \tag{2.19}$$

The two polarization vectors are $\varepsilon_1^\mu = (x^1, 1, 0, -x^1)$, $\varepsilon_2^\mu = (x^2, 0, 1, -x^2)$. It is more convenient to define the polarization vectors in the helicity basis:

$$\varepsilon_+^\mu = \frac{1}{\sqrt{2}}\Big(\varepsilon_1^\mu - i\varepsilon_2^\mu\Big) = \frac{1}{\sqrt{2}}(\bar{\mathsf{z}}, 1, -i, -\bar{\mathsf{z}})\,, \quad \varepsilon_-^\mu = \frac{1}{\sqrt{2}}\Big(\varepsilon_1^\mu + i\varepsilon_2^\mu\Big) = \frac{1}{\sqrt{2}}(\mathsf{z}, 1, i, -\mathsf{z})\,, \tag{2.20}$$

satisfying $\varepsilon_+ \cdot \varepsilon_+ = \varepsilon_- \cdot \varepsilon_- = 0$, $\varepsilon_+ \cdot \varepsilon_- = 1$.

For graviton, the polarization tensors for two helicities are

$$\varepsilon_\pm^{\mu\nu} = \varepsilon_\pm^\mu \varepsilon_\pm^\nu = \frac{1}{2}\Big(\varepsilon_{11}^{\mu\nu} - \varepsilon_{22}^{\mu\nu} \mp i\varepsilon_{12}^{\mu\nu} \mp i\varepsilon_{21}^{\mu\nu}\Big) = \varepsilon_{(11)}^{\mu\nu} \mp i\varepsilon_{(12)}^{\mu\nu}\,. \tag{2.21}$$

2.2 Vertex operator in bosonic string

In this subsection, we review the vertex operators in open and closed bosonic string theory.

2.2.1 Open string vertex operator

At tree level, open string amplitudes are computed by the correlator of vertex operators inserted at the boundary of the disk, or equivalently the boundary of upper half plane, which we parametrize by $y \in \mathbb{R}$.

There are two types of vertex operators. The integrated vertex operators have the following general structure[9,10]

$$\mathcal{V}^{A,\ell}(k) = \int dy\; V_k^{A,\ell}(y) = \int dy\; \mathscr{V}^{A,\ell}(y) e^{ik\cdot X(y)}\,, \tag{2.22}$$

[9] Note that the product of operators at coincident points should always be understood as normal ordered product.

[10] We will sometimes abuse the terminology of vertex operators and call both $\mathcal{V}$ and V vertex operators. The exact meaning should be clear from the context.

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while the unintegrated vertex operators take the form

$$\mathcal{V}^{A,\ell}(k) = c(y)V_k^{A,\ell}(y) = c(y)\mathscr{V}^{A,\ell}(y)e^{ik\cdot X(y)}\,. \tag{2.23}$$

where c is the ghost and the position y in the unintegrated vertex operator is arbitrary. Here ℓ is the number of derivatives in $\mathscr{V}_k^{A,\ell}$ (e.g. $\partial_y X(y)$ has $\ell = 1$), and A are the extra quantum numbers labelling the vertex operators, including the polarization vectors, Chan-Paton factors, etc.

The spectra of fields in open string are given by

$$-k\cdot k = M^2 = \frac{1}{\alpha'}\left(N - \frac{D}{24}\right) = \frac{1}{\alpha'}\left(N-1\right), \tag{2.24}$$

where $N \in \mathbb{Z}_{\geq 0}$ is the level and in the last equality we set $D = 24$ for the consideration of critical bosonic string.

When the momentum becomes on-shell, namely $k^2 \to -M^2 = (1-N)/\alpha'$, the vertex operator becomes BRST invariant and should has weight 1,

$$h(V_k^{A,\ell}) = h(\mathscr{V}_k^{A,\ell}) + h(e^{ik\cdot X}) = \ell + \alpha' k^2 = 1\,, \tag{2.25}$$

Since on-shell momentum satisfies $k^2 = (1-N)/\alpha'$, we see that ℓ is essentially the level N, namely $N = \ell$.

At level 0, we have tachyon whose vertex operator is given by[11]

$$V_k^0 = e^{ik\cdot X}\,, \qquad M^2 = -\frac{1}{\alpha'}\,. \tag{2.26}$$

At level 1, we have gluons whose vertex operators are [31]

$$V_k^1 = e_\mu \dot{X}^\mu e^{ik\cdot X} t^A\,, \qquad M^2 = 0\,. \tag{2.27}$$

where $\dot{X} \equiv \frac{\partial}{\partial y}X$. The polarization e_μ satisfies (2.11)

$$k^2 = k\cdot e = 0\,, \qquad e\cdot e = 1\,. \tag{2.28}$$

For the gluon vertex operator, we also have the Chan-Paton factor t^A, which is just the gauge algebra generator with color index A. The corresponding structure constant, denoted as f^{ABC}, is fully anti-symmetric.

2.2.2 Closed string vertex operator

At tree level, the closed string amplitudes are computed as the correlator of vertex operators on the sphere, or equivalently the complex plane, which is parametrized by coordinates $z, \bar{z}$.

The integrated vertex operator has the structure

$$\mathcal{V}^{A,\ell}(k) = \int d^2z\; V_k^{A,\ell}(z,\bar{z}) = \int d^2z\; \mathscr{V}^{A,\ell}(z,\bar{z})e^{ik\cdot X(z,\bar{z})}\,, \tag{2.29}$$

[11] For simplicity we will set all the coefficients in front of the vertex operators to be 1. Meanwhile, in the final formulae of celestial OPEs, we will normalize the overall coefficient properly to make the result as simple as possible. All the couplings can be easily restored, as we keep track of the overall factors in the intermediate steps.

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while the unintegrated vertex operator takes the form

$$\mathcal{V}^{A,\ell}(k) = c(z)\tilde{c}(\bar{z})V_k^{A,\ell}(z,\bar{z}) = c(z)\tilde{c}(\bar{z})\mathscr{V}^{A,\ell}(z,\bar{z})e^{ik\cdot X(z,\bar{z})}\,, \tag{2.30}$$

where $c, \tilde{c}$ are the ghosts and the position $z, \bar{z}$ in the unintegrated vertex operator is arbitrary. Here ℓ is the number of holomorphic derivatives in $\mathscr{V}_k^{A,\ell}$ (which is also the number of anti-holomorphic derivatives), and A are the extra quantum numbers labelling the vertex operators, such as the polarization tensors.

The closed string spectra are given by

$$-k\cdot k = M^2 = \frac{2}{\alpha'}\Big(N + \tilde{N} - 2\frac{D}{24}\Big) = \frac{4}{\alpha'}\Big(N-1\Big)\,, \tag{2.31}$$

where we use the level matching condition $N = \tilde{N} \in \mathbb{Z}_{\geq 0}$, and in the last equality we set $D = 24$ for critical bosonic string.

In order to be BRST invariant, the on-shell vertex operator should have holomorphic and anti-holomorphic weights 1, namely

$$h(V_k^{A,\ell}) = h(\mathscr{V}_k^{A,\ell}) + h(e^{ik\cdot X}) = \ell + \frac{\alpha'}{4}k^2 = 1\,, \tag{2.32}$$

and the same for $\bar{h}$. Therefore, on-shell momentum should satisfy $k^2 = 4(1-\ell)/\alpha'$. Comparing with the mass-shell condition in (2.31), we again have $N = \ell$.

At level 0, we have closed string tachyon whose vertex operator is given by

$$V_k^0 = e^{ik\cdot X}\,, \qquad M^2 = -\frac{4}{\alpha'}\,. \tag{2.33}$$

At level 1, we have massless fields whose vertex operators are given by [31]

$$V_k^1 = e_{\mu\nu}\partial X^\mu \bar{\partial} X^\nu e^{ik\cdot X}\,, \qquad M^2 = 0\,. \tag{2.34}$$

Depending on the structure of polarization tensor $e_{\mu\nu}$, the vertex operator can represent different fields, either graviton, or KR 2-form field, or dilaton. See (2.14)–(2.17) for the properties and explicit forms of polarization tensors corresponding to these different fields.

2.3 CFT on the worldsheet and on the celestial sphere

Given the vertex operators, one can then compute the string amplitudes. In general, the string amplitude is schematically computed by the following formula [31]

$$\mathcal{A}_n(k_1, k_2, \cdots, k_n) \sim \sum_{\text{topologies}} e^{-\lambda\chi} \int_{\mathcal{M}_{g,n}} dm\, \langle \mathcal{V}^{A_1,\ell_1}(k_1)\mathcal{V}^{A_2,\ell_2}(k_2)\cdots\mathcal{V}^{A_n,\ell_n}(k_n)\rangle_{\text{WS}}\,, \tag{2.35}$$

where we need to sum over all the topologies for the string worldsheet, and for each topology, we need to incorporate the contribution from various ghosts properly and integrate over the moduli space of Riemann surface. The world-sheet correlator is evaluated through the path integral of Polyakov action with operator insertions. The computation of string

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amplitude is very hard as one goes to higher loops, but it simplifies dramatically at tree level. At tree level, the topology is fixed and there are no moduli to integrate over.

As a consequence, the string amplitude at tree level is given by

$$\mathcal{A}_n(k_1, k_2, \cdots, k_n) = \langle \mathcal{V}^{A_1,\ell_1}(k_1)\mathcal{V}^{A_2,\ell_2}(k_2)\cdots\mathcal{V}^{A_n,\ell_n}(k_n)\rangle_{\mathrm{WS}}\,. \tag{2.36}$$

In the case of open string, the worldsheet is given by the disk, which is conformally equivalent to the upper half complex plane. The vertex operators are inserted on the boundary of the disk. In particular, among n vertex operators, three of them should take the unintegrated form (2.23) and the rest in the integrated form (2.22), in order to soak up the zero modes. For gluon amplitudes, we should take a trace over the product of Chan-Paton factors t^A, which are ordered on the boundary according to their positions. So different ways of inserting the vertex operators on the disk boundary give rise to different Chan-Paton factors, and one needs to include all orderings of insertions. For closed string amplitude, the worldsheet is given by the sphere, which is conformally equivalent to the whole complex plane. Again to soak up the zero modes, we should choose three vertex operators in the unintegrated form (2.23) and the rest in the integrated form (2.29).

On the other hand, the celestial amplitudes for massless particles are given by the Mellin transformation of momentum space amplitudes [4, 5, 24][12]

$$\mathcal{M}_n(x_1, \cdots, x_n) = \prod_{j=1}^{n}\int_0^\infty d\omega_j\;\omega_j^{\Delta_j-1}\mathcal{A}_n(\eta_i\omega_i\hat{k}_i)\,, \qquad \mathcal{A}_n(k_i) = A_n(k_i)\;\delta^{D+2}\left(\sum_{i=1}^{n}k_i\right). \tag{2.37}$$

In contrast to the momentum space amplitude which has manifest translational invariance, the celestial amplitude is designed to make the Lorentz symmetry manifest. Indeed, in $D+2$ dimensional Minkowski spacetime we have Lorentz group $\mathrm{SO}(1, D+1)$, which is also the conformal group of CFT in D dimensions. The place supporting such a conformal group is just the celestial sphere, which sits at the boundary null infinity of spacetime. The celestial amplitude defined above can thus be regarded as the correlator of some celestial operators in a putative celestial conformal field theory living on the celestial sphere

$$\mathcal{M}_n(x_1, \cdots, x_n) = \langle \mathcal{O}_{\Delta_1}(x_1)\cdots\mathcal{O}_{\Delta_n}(x_n)\rangle_{\mathrm{CS}}\,. \tag{2.38}$$

Note that for simplicity of notation we have stripped off all the extra labels of the operators except for their positions and dimensions. The representation of bulk scattering amplitude in terms of boundary correlator is just reminiscent of the holographic principle.

So the string amplitude can be in principle obtained in two different ways, either (2.36) or (2.38). To make the connection more precise, we can also define the vertex operators in the conformal basis through a Mellin transformation

$$\mathcal{V}^{A,\ell,\eta}_{\Delta}(x) = \int_0^\infty d\omega\;\omega^{\Delta-1}\mathcal{V}^{A,\ell}(\eta\omega\hat{k})\,. \tag{2.39}$$

[12]We focus on massless fields in this paper. For massive fields, the Mellin transformation needs to be modified, see [5, 24, 32].

We will refer to it as the *conformal vertex operators* in this paper. Then the celestial string amplitude is essentially given by the correlator of conformal vertex operators evaluated in worldsheet CFT

$$\mathcal{M}_n(x_1,\cdots,x_n)=\langle \mathcal{V}_{\Delta_1}(x_1)\cdots\mathcal{V}_{\Delta_n}(x_n)\rangle_{\mathrm{WS}}\,. \tag{2.40}$$

The very similar structure between (2.38) and (2.40) suggests a map $\mathscr{F}$ from worldsheet CFT (WSCFT) to celestial CFT (CCFT), such that the two Hilbert spaces are related as follows

$$\mathscr{F}:\qquad H_{\mathrm{WSCFT}}\to H_{\mathrm{CCFT}}\,. \tag{2.41}$$

In particular, there is a one-to-one map between string conformal vertex operators and celestial operators

$$\mathscr{F}:\qquad \mathcal{V}_\Delta\mapsto\mathcal{O}_\Delta\,. \tag{2.42}$$

Our goal in this paper is to derive the celestial OPEs

$$\mathcal{O}_{\Delta_1}(x_1)\mathcal{O}_{\Delta_2}(x_2)\sim\sum_j C^{\Delta_j}_{\Delta_1\Delta_2}(x_1,x_2)\mathcal{O}_{\Delta_j}(x_2)\,. \tag{2.43}$$

Due to the map $\mathscr{F}$, it is sufficient to compute

$$\mathcal{V}_{\Delta_1}(x_1)\mathcal{V}_{\Delta_2}(x_2)\sim\sum_j C^{\Delta_j}_{\Delta_1\Delta_2}(x_1,x_2)\mathcal{V}_{\Delta_j}(x_2)\,. \tag{2.44}$$

We would like to have several remarks here. In spite of the similarity between (2.38) and (2.40), the celestial CFT and worldsheet CFT are very different in many aspects. The worldsheet CFT is always two dimensional, while the dimension of celestial CFT depends on the bulk spacetime dimension. Moreover, the worldsheet CFT and thus its correlator (2.36) are explicit and well-defined, while for celestial CFT which has many unusual features, our understanding remains poor and is mostly about symmetry. The map $\mathscr{F}$ suggests that these two are supposed to be related in some way. In particular, since the integrated conformal vertex operator $\mathcal{V}_\Delta(x)$ arises after doing the worldsheet integration, it is reasonable to speculate that the celestial CFT arises from worldsheet CFT by some kind of projection of worldsheet coordinates and going to the labelling space, namely momentum k or Δ, x space. On the other hand, the similarity between (2.38) and (2.40) is for tree level amplitude. At loop level, the string amplitude in (2.35) is more complicated and only admits perturbative expansion. The CCFT correlator (2.38) is however defined formally but exactly. To reconcile the difference, the celestial CFT is supposed to also admit some expansion, and then one may compare the two sides order by order. This is indeed the case in AdS/CFT where the gauge theory in the CFT side admits large-N expansion and can be compared with the loop expansion of string theory. Understanding these points may be useful for us to establish a concrete and exact model of celestial holography in string theory.

3 Celestial OPE from worldsheet OPE in bosonic string

In this section, we will discuss the derivation of celestial OPEs from worldsheet OPEs following the strategy outlined in the introduction. The string theory we will consider in this section is bosonic strings, either open or closed. As a result, we are able to derive the gluon OPEs and graviton/dilaton/KR field OPEs from open and closed strings, respectively. The mixed OPEs involving both gluon and graviton/dilaton/KR field can be derived from the open-closed string setup, which will be deferred to appendix B. All the OPEs involving gluon, graviton/dilaton/KR as well as their mixture will also be discussed in the heterotic superstring in the next section. In section 5, we will also confirm our celestial OPEs through the collinear factorization of string amplitude.

3.1 OPE in open string

Let us now discuss how to derive the celestial gluon OPEs from the worldsheet OPEs in open string theory. As we described before, it is sufficient to compute the OPE (2.44). For this aim, in principle we need to know $V_{\Delta_1}(x_1,y_1)V_{\Delta_2}(x_2,y_2)$ for arbitrary y_1,y_2, even if they are very far away from each other on the worldsheet. Here $V_\Delta(x,y)$ is essentially the Mellin transformation of the integrand $V_k(y)$ in (2.22). Since we are only interested in the collinear limit $x_1\to x_2$, it is reasonable to speculate that the celestial OPE is determined by the worldsheet OPE.[13] This will be our worldsheet approach to deriving the celestial OPEs. In particular, the celestial gluon OPE can be derived by computing[14]

$$\mathcal{V}^A(p,\zeta)\mathcal{V}^B(q,\xi)=\int dy_1dy_2\theta(y_1-y_2)V^A_{p,\zeta}(y_1)V^B_{q,\xi}(y_2)+\int dy_1dy_2\theta(y_2-y_1)V^B_{q,\xi}(y_2)V^A_{p,\zeta}(y_1) \tag{3.1}$$

$$\begin{aligned}&=\int dy_1dy_2\theta(y_1-y_2)\zeta\cdot\dot{X}e^{ip\cdot X}(y_1)\xi\cdot\dot{X}e^{iq\cdot X}(y_2)t^At^B\\&\quad+\int dy_1dy_2\theta(y_2-y_1)\xi\cdot\dot{X}e^{iq\cdot X}(y_2)\zeta\cdot\dot{X}e^{ip\cdot X}(y_1)t^Bt^A,\end{aligned} \tag{3.2}$$

where we used the explicit form of gluon vertex operators in (2.27) and ζ,ξ are the polarization vectors. We will first compute the worldsheet OPE of two V's in the integrand; then we will exam the behavior of this worldsheet OPE in the collinear limit. As we will see, in the collinear limit, the worldsheet OPE actually localizes to a delta-function, which just gives another vertex operator. Performing the Mellin transformation then leads us to the celestial OPEs.

Before going to the details, let us add some comments about the formula above (3.2). Since we are interested in the worldsheet OPE, the two vertex operators should be next to each other and there is no other operator insertion between them. Nevertheless, we have two different contributions in the formula above, corresponding to two different orderings of vertex operators on the boundary. On the other hand, the formula above should be understood in the correlator where other operator insertions indeed appear; this gives rise

[13] To be clear, the worldsheet OPE refers to the OPE of two operators approaching each other on the worldsheet, namely $z_1\to z_2$, while the celestial OPE refers to the OPE of two celestial operators approaching each other on the celestial sphere, namely $x_1\to x_2$. The celestial coincident limit $x_1\to x_2$ is equivalent to the collinear limit.

[14] Instead of considering two vertex operators in the integrated forms (2.22), one can also choose one in the integrated form (2.22) and the other in the unintegrated form (2.23). Their OPE leads to another unintegrated vertex operator. The final results for celestial OPEs obtained in these two ways are the same.

to the upper and lower limits of integration for the integrals. As we will see, this brings the subtle issue of boundary contact terms.

Now we can start the derivation of celestial OPEs. As the first step, we need to compute the worldsheet OPE for the integrand in (3.2). This can be done as the fields X are essentially free scalars. All the detailed computations of worldsheet OPEs are given in appendix A. In particular, the worldsheet OPE for the integrand in (3.2) is given by (A.18):

$$\zeta\cdot\dot{X}e^{ip\cdot X}(y_1)\xi\cdot\dot{X}e^{iq\cdot X}(y_2)\sim -2\alpha' y_{12}^{2\alpha' p\cdot q-2}e^{i(p+q)\cdot X}\Big[\Big(\zeta\cdot\xi-2\alpha'\zeta\cdot q\,\xi\cdot p\Big)$$
$$+iy_{12}\Big(\zeta\cdot q\,\xi\cdot\dot{X}-\xi\cdot p\,\zeta\cdot\dot{X}+(\zeta\cdot\xi-2\alpha'\zeta\cdot q\,\xi\cdot p)p\cdot\dot{X}\Big)+\cdots\Big](y_2)\,, \tag{3.3}$$

where we assume $y_1>y_2$.

We will take p,q on-shell, namely $p^2=q^2=0$, then the polarization vectors ζ,ξ satisfy the properties in (2.11). We are interested in the collinear limit where p,q are almost parallel and thus $p\cdot q\to 0$, implying that $p+q$ is almost on-shell. However, in order to gain information from the OPE, we can not take the strict collinear limit. Instead, we will denote $K\equiv p+q$ and choose a nearby null momentum $k=\omega_k\hat{k}$ such that $K=k+\epsilon v$, where v is a generic momentum of order one. Then we have

$$K^2=2p\cdot q=k^2+2\epsilon k\cdot v+\epsilon^2 v\cdot v=2\epsilon k\cdot v+\mathscr{O}(\epsilon^2)\,. \tag{3.4}$$

Hence ϵ has the same order of magnitude with $p\cdot q$ in the collinear limit. Without loss of generality, we will define $\epsilon=2\alpha' p\cdot q$. Then $K=p+q=k+\mathscr{O}(\epsilon)$. In the following discussions, we will try to simplify (3.3) in such a collinear limit.

Given the null momentum k, we can define the basis for polarization vectors $\varepsilon_a(k)$ (2.2), which satisfy (2.10)

$$\varepsilon_a^\mu(k)\varepsilon_a^\nu(k)=\eta^{\mu\nu}-n^\mu\hat{k}^\nu-n^\nu\hat{k}^\mu\,. \tag{3.5}$$

Contracting with $p^\mu\dot{X}^\nu$, we get

$$p\cdot\varepsilon_a\,\varepsilon_a\cdot\dot{X}=p\cdot X-p\cdot n\,\hat{k}\cdot\dot{X}-\dot{X}\cdot n\,\hat{k}\cdot p\,. \tag{3.6}$$

Using $n\cdot\hat{p}=1$ (2.3) and hence $n\cdot p=\omega_p n\cdot\hat{p}=\omega_p$,[15] the above equation can be written as

$$p\cdot\dot{X}=p\cdot\varepsilon_a\,\varepsilon_a\cdot\dot{X}+\frac{1}{\omega_k}k\cdot p\,\dot{X}\cdot n+\frac{\omega_p}{\omega_k}k\cdot\dot{X} \tag{3.7}$$
$$=p\cdot\varepsilon_a\,\varepsilon_a\cdot\dot{X}+\frac{1}{\omega_k}p\cdot q\,\dot{X}\cdot n+\frac{\omega_p}{\omega_k}k\cdot\dot{X}+\mathscr{O}(\epsilon)\,, \tag{3.8}$$

where we used $k\cdot p=(p+q+\mathscr{O}(\epsilon))\cdot p=p\cdot q+\mathscr{O}(\epsilon)$ as $p^2=0$.

Similarly, we can contract (3.5) with $\zeta^\mu\dot{X}^\nu$, yielding

$$\zeta\cdot\varepsilon_a\,\varepsilon_a\cdot\dot{X}=\zeta\cdot X-\zeta\cdot n\,\hat{k}\cdot\dot{X}-\dot{X}\cdot n\,\hat{k}\cdot\zeta=\zeta\cdot X-\frac{1}{\omega_k}\dot{X}\cdot n\,q\cdot\zeta+\mathscr{O}(\epsilon)\,, \tag{3.9}$$

[15] We will only consider the celestial OPEs of celestial operators corresponding to out-going particles in this paper, so $\eta=1$.

where we used $k\cdot\zeta = (p+q+\mathscr{O}(\epsilon))\cdot\zeta = q\cdot\zeta+\mathscr{O}(\epsilon)$ and $\zeta\cdot n = 0$ due to (2.11) and (2.12).[16] As a result, we have

$$\zeta\cdot\dot{X}\,\xi\cdot p = \zeta\cdot\varepsilon_a\,\varepsilon_a\cdot\dot{X}\,\xi\cdot p + \frac{1}{\omega_k}\dot{X}\cdot n\,q\cdot\zeta\,p\cdot\xi + \mathscr{O}(\epsilon)\,, \tag{3.10}$$

and similarly

$$\xi\cdot\dot{X}\,\zeta\cdot q = \xi\cdot\varepsilon_a\,\varepsilon_a\cdot\dot{X}\,\zeta\cdot q + \frac{1}{\omega_k}\dot{X}\cdot n\,p\cdot\xi\,q\cdot\zeta + \mathscr{O}(\epsilon)\,. \tag{3.11}$$

Taking the difference, we find

$$\xi\cdot\dot{X}\,\zeta\cdot q - \zeta\cdot\dot{X}\,\xi\cdot p = \xi\cdot\varepsilon_a\,\varepsilon_a\cdot\dot{X}\,\zeta\cdot q - \zeta\cdot\varepsilon_a\,\varepsilon_a\cdot\dot{X}\,\xi\cdot p + \mathscr{O}(\epsilon)\,. \tag{3.12}$$

With (3.8) and (3.12), we can simplify the terms in the square bracket of (3.3)

$$\begin{aligned}
&\Big(\zeta\cdot\xi - 2\alpha'\zeta\cdot q\,\xi\cdot p\Big) + i\,y_{12}\Big(\zeta\cdot q\,\xi\cdot\dot{X} - \xi\cdot p\,\zeta\cdot\dot{X} + (\zeta\cdot\xi - 2\alpha'\zeta\cdot q\,\xi\cdot p)p\cdot\dot{X}\Big) \\
&= \zeta\cdot h\cdot\xi + i\,y_{12}\bigg(\zeta\cdot q\,\xi\cdot\varepsilon_a\,\varepsilon_a\cdot\dot{X} - \xi\cdot p\,\zeta\cdot\varepsilon_a\,\varepsilon_a\cdot\dot{X} + \zeta\cdot h\cdot\xi\,p\cdot\varepsilon_a\,\varepsilon_a\cdot\dot{X} \\
&\qquad\qquad + \zeta\cdot h\cdot\xi\left(\frac{1}{\omega_k}p\cdot q\,\dot{X}\cdot n + \frac{\omega_p}{\omega_k}k\cdot\dot{X}\right)\bigg) + \mathscr{O}(\epsilon)\,,
\end{aligned} \tag{3.13}$$

where we have introduced

$$h^{\mu\nu} = \eta^{\mu\nu} - 2\alpha' q^\mu p^\nu\,, \qquad \zeta\cdot h\cdot\xi \equiv \zeta_\mu h^{\mu\nu}\xi_\nu = \zeta\cdot\xi - 2\alpha'\zeta\cdot q\,\xi\cdot p\,. \tag{3.14}$$

Therefore the worldsheet OPE (3.3) can be written as

$$\zeta\cdot\dot{X}e^{ip\cdot X}(y_1)\xi\cdot\dot{X}e^{iq\cdot X}(y_2) \tag{3.15}$$
$$\sim -2\alpha' y_{12}^{2\alpha' p\cdot q-2}e^{i(p+q)\cdot X}\zeta\cdot h\cdot\xi \tag{3.16}$$
$$-\,2i\,\alpha' y_{12}^{2\alpha' p\cdot q-1}e^{i(p+q)\cdot X}\Big(\zeta\cdot q\,\xi\cdot\varepsilon_c - \xi\cdot p\,\zeta\cdot\varepsilon_c + \zeta\cdot h\cdot\xi\,p\cdot\varepsilon_c\Big)\varepsilon_c\cdot\dot{X} \tag{3.17}$$
$$-\,2i\,\alpha'\frac{\zeta\cdot h\cdot\xi}{\omega_k}\dot{X}\cdot n\,p\cdot q\,y_{12}^{2\alpha' p\cdot q-1}e^{i(p+q)\cdot X} \tag{3.18}$$
$$-\,2\,\alpha'\zeta\cdot h\cdot\xi\,\frac{\omega_p}{\omega_k}y_{12}^{2\alpha' p\cdot q-1}\partial e^{i(p+q)\cdot X} \tag{3.19}$$
$$+\,\mathscr{O}(y_{12}^{2\alpha' p\cdot q}) + \mathscr{O}(2\alpha' p\cdot q)\,, \tag{3.20}$$

where all terms on the right hand side are evaluated at y_2. This is the worldsheet OPE in the collinear limit. To obtain the celestial OPE, we use the following identities to analyze

[16] Note that $\zeta\cdot n = 0$ can be realized by choosing a specific gauge. This is inessential as the amplitude is gauge invariant or BRST invariant.

the dominant contributions. For infinitesimal ϵ and positive $x > 0$, we have[17]

$$\epsilon x^{\epsilon-1} = 2\delta(x)\,, \quad \epsilon x^{\epsilon-1-n} = \frac{2(-1)^n}{n!}\delta^n(x)\,, \quad \epsilon x^{\epsilon+n} = 0\,, \quad n = 0, 1, 2, \cdots\,. \tag{3.21}$$

We first observe that in the worldsheet OPE, if we only look at the X factors, (3.16) and (3.17) are just the integrand in the vertex operators of tachyon and gluon. Furthermore, if we take the limit $2\alpha' p\cdot q \to 1$ and use identities in (3.21), (3.16) gives the factor $y_{12}^{2\alpha' p\cdot q-2} \propto \delta(y_{12})/(2\alpha' p\cdot q - 1)$ and the pole just singles out the tachyon. Therefore in the limit $2\alpha' p\cdot q \to 1$, the term (3.16) dominates the OPE. Doing the y integral, (3.16) indeed gives the tachyon vertex operator, whose mass-shell condition is just $(p+q)^2 = 2p\cdot q = 1/\alpha'$.

The more interesting limit for our purpose is the collinear limit $2\alpha' p\cdot q \to 0$. We will argue that in this limit only (3.17) survives as a singular term in the OPE at leading order, while the rest are sub-dominant.

For (3.18), the factor $p\cdot q\, y_{12}^{2\alpha' p\cdot q-1}$ in the collinear limit $p\cdot q \to 0$ becomes $\delta(y_{12})$ after using (3.21). This is of order one, hence we will ignore it.

For (3.16), the factor $y_{12}^{2\alpha' p\cdot q-2}$ just leads to $\delta'(y_{12})/p\cdot q$ following (3.21). This looks singular in the collinear limit. For vertex operators, we need to perform the integral as in (3.2), which schematically gives

$$\int dy_1 dy_2\, \theta(y_1-y_2)\delta'(y_{12})e^{i(p+q)\cdot X}(y_2) \to \int dy_1 dy_2\, [\delta(y_1-y_2)]^2 e^{i(p+q)\cdot X}(y_2) + \int dy_2\, \partial e^{i(p+q)\cdot X}(y_2)\,. \tag{3.22}$$

The $[\delta(y_1-y_2)]^2$ factor in the first term arises from the collision of y_1, y_2 and it is a contact type term. After integrating over y_1, we end up with $\delta(0)$ multiplied by $\int dy_2\, e^{i(p+q)\cdot X}(y_2)$, which is however not BRST closed as $(p+q)^2 \to 0$. The second type of term with an unrestricted integration range is a total derivative, which normally does not have contributions in amplitudes due to BRST invariance. When we insert it in the correlator, there are also other operator insertions, then the second type of term gives rise to the boundary term when the position of two operators in OPE coincides with the rest of operator insertions. We will refer to these types of terms as *boundary contact terms*. Since these boundary contact terms arise when operator insertions collide and correspond to very singular configurations, we expect that they would finally cancel out and play no role in our OPE analysis. On the other hand, (3.16) corresponds to tachyon whose mass-squared is negative, the boundary contact terms can thus be partially understood as the remnant contribution from fields in string theory with lower masses. Physically, we also expect the collinear limit should kill the contribution from these fields at leading order.

[17] They arise from the generalized function

$$\lim_{\epsilon\to 0} \frac{\epsilon}{2}|x|^{\epsilon-1} = \delta(x)$$

as well as its derivatives. For this representation of delta function, see https://mathworld.wolfram.com/DeltaFunction.html. The equality here should be understood when we integrate them against some test functions with proper fall-off conditions. In the current context of computing string amplitudes, this kind of equality should be understood in the correlator of worldsheet CFT. In unitary CFT, any correlations at two different points are power-law suppressed with distance, which is expected to give some fall-off conditions.

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For (3.19), $\partial e^{i(p+q)\cdot X} = \partial e^{ik\cdot X} + \mathscr{O}(\epsilon) = k \cdot \dot{X} e^{ik\cdot X} + \mathscr{O}(\epsilon)$. Since the polarization is along the momentum direction, this is supposed to a pure gauge or BRST-exact term and hence does not contribute. Nevertheless, we can also analyze this term as before. The factor $y_{12}^{2\alpha' p\cdot q-1}$ becomes $\delta(y_{12})/p\cdot q$ in the collinear limit. And then performing the integral gives

$$\int dy_1 dy_2\, \theta(y_1-y_2)\delta(y_{12})\partial e^{i(p+q)\cdot X}(y_2) \to \int dy_2\, \partial e^{i(p+q)\cdot X}(y_2)\,, \tag{3.23}$$

which takes the same form we encountered in (3.22).

Actually we can combine (3.16) and (3.19) together and get

$$\begin{aligned} &-2\alpha' y_{12}^{2\alpha' p\cdot q-2} e^{i(p+q)\cdot X}\zeta\cdot h\cdot\xi - 2\,\alpha'\zeta\cdot h\cdot\xi\,\frac{\omega_p}{\omega_k} y_{12}^{2\alpha' p\cdot q-1}\partial e^{i(p+q)\cdot X} \\ &= -2\alpha' y_{12}^{2\alpha' p\cdot q-2} e^{i(p+q)\cdot X}\zeta\cdot h\cdot\xi\frac{2\omega_p\alpha' p\cdot q+\omega_k-\omega_p}{\omega_k} - 2\,\alpha'\zeta\cdot h\cdot\xi\,\frac{\omega_p}{\omega_k}\partial\Big[y_{12}^{2\alpha' p\cdot q-1} e^{i(p+q)\cdot X}\Big]\,, \end{aligned} \tag{3.24}$$

where the first term is the same as (3.16) up to a constant, while the second term is a total derivative and again would lead to at most boundary contact terms.

Therefore, up to the subtle boundary contact terms, all terms except for (3.17) in the world-sheet OPE do not give rise to singular terms in the collinear limit $p\cdot q\to 0$. So we just need to focus on (3.17).

To proceed, we choose the polarization vectors as the basis defined in (2.2) $\zeta = \varepsilon_a(p)$, $\xi = \varepsilon_b(q)$, and furthermore use the notation $p = \omega_p\hat{p}(x_1)$, $q = \omega_q\hat{q}(x_2)$, $k = \omega_k\hat{k}(x_3)$ where $x_{1,2,3}$ are the coordinates on the celestial sphere. Then the world-sheet OPE (3.3), (3.17) becomes

$$\begin{aligned} &\varepsilon_a(p)\cdot\dot{X}e^{ip\cdot X}(y_1)\varepsilon_b(q)\cdot\dot{X}e^{iq\cdot X}(y_2) \\ &\sim -2i\,\alpha' y_{12}^{2\alpha' p\cdot q-1} e^{i(p+q)\cdot X}\Big(\varepsilon_a(p)\cdot q\,\varepsilon_b(q)\cdot\varepsilon_c - \varepsilon_b(q)\cdot p\,\varepsilon_a(p)\cdot\varepsilon_c + \varepsilon_a(p)\cdot h\cdot\varepsilon_b(q)\,p\cdot\varepsilon_c\Big)\varepsilon_c\cdot\dot{X}\,. \end{aligned} \tag{3.25}$$

The next step is to rewrite all the terms on the right hand side in terms of celestial variables ω_i, x_i. Using identities in (2.4),

$$\hat{k}(x_i)\cdot\hat{k}(x_j) \equiv \hat{k}_i^\mu\cdot\hat{k}_j^\mu = -\frac{1}{2}(x_{ij})^2\,, \qquad \hat{k}_i\cdot\varepsilon_a(x_j) = x_{ij}^a\,, \qquad \varepsilon_a(x_i)\cdot\varepsilon_b(x_j) = \delta^{ab}\,, \tag{3.26}$$

we find

$$\varepsilon_a(p)\cdot h\cdot\varepsilon_b(q) = \varepsilon_a(p)\cdot\varepsilon_b(q) - 2\alpha'\varepsilon_a(p)\cdot q\,\varepsilon_b(q)\cdot p = \delta^{ab} + 2\alpha'\omega_p\omega_q x_{12}^a x_{12}^b\,, \tag{3.27}$$

and

$$\varepsilon_a(p)\cdot q\,\varepsilon_b(q)\cdot\varepsilon_c = -\omega_q x_{12}^a\delta^{bc}\,, \qquad \varepsilon_b(q)\cdot p\,\varepsilon_a(p)\cdot\varepsilon_c = \omega_p x_{12}^b\delta^{ac}\,. \tag{3.28}$$

To simplify $p\cdot\varepsilon_c$, we need to use the momentum conservation. Using (2.1), the momentum conservation $K = p+q = k+\epsilon v$ can be written as

$$\omega_p\left(\frac{1+(x_1)^2}{2}, x_1^a, \frac{1-(x_1)^2}{2}\right) + \omega_q\left(\frac{1+(x_2)^2}{2}, x_2^a, \frac{1-(x_2)^2}{2}\right) = \omega_k\left(\frac{1+(x_3)^2}{2}, x_3^a, \frac{1-(x_3)^2}{2}\right) + \epsilon v\,, \tag{3.29}$$

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where the infinitesimal quantity ϵ is now given by

$$\epsilon = 2\alpha' p \cdot q = -\alpha' \omega_p \omega_q (x_{12})^2 \,. \tag{3.30}$$

From (3.29), it is easy to see that

$$\omega_k = \omega_p + \omega_q + \mathscr{O}(\epsilon) \,, \qquad \omega_k x_3^a = \omega_p x_1^a + \omega_q x_2^a + \mathscr{O}(\epsilon) \,. \tag{3.31}$$

As a result, we have

$$x_{13}^a = \frac{\omega_q}{\omega_p + \omega_q} x_{12}^a + \mathscr{O}(\epsilon) \,, \qquad x_{32}^a = \frac{\omega_p}{\omega_p + \omega_q} x_{12}^a + \mathscr{O}(\epsilon) \,, \tag{3.32}$$

and furthermore

$$p \cdot \varepsilon_c = \omega_p \hat{p} \cdot \varepsilon_c(k) = \omega_p x_{13}^c = \frac{\omega_p \omega_q}{\omega_p + \omega_q} x_{12}^c + \mathscr{O}(\epsilon) \,. \tag{3.33}$$

With these identities (3.27), (3.28), (3.33), the OPE (3.25) simplifies to

$$\begin{aligned}
&\varepsilon_a(p) \cdot \dot{X} e^{ip \cdot X}(y_1) \varepsilon_b(q) \cdot \dot{X} e^{iq \cdot X}(y_2) \qquad (3.34)\\
&\sim i\, \epsilon y_{12}^{\epsilon - 1} \frac{\omega_p x_{12}^b \delta^{ac} + \omega_q x_{12}^a \delta^{bc} - \frac{\omega_p \omega_q}{\omega_p + \omega_q} x_{12}^c \delta_{ab} - 2\alpha' \frac{\omega_p^2 \omega_q^2}{\omega_p + \omega_q} x_{12}^a x_{12}^b x_{12}^c}{p \cdot q} \varepsilon_c \cdot \dot{X}\, e^{i(p+q) \cdot X} + \mathscr{O}(\epsilon) \\
&\sim -4i\, \delta(y_{12})\, I_c\Big(p, \varepsilon_a(p); q, \varepsilon_b(q); \alpha'\Big)\, \varepsilon_c \cdot \dot{X}\, e^{i(p+q) \cdot X} + \mathscr{O}(\epsilon) \,,
\end{aligned}$$

where used (3.21) and have defined

$$\begin{aligned}
I_c\Big(p, \varepsilon_a(p); q, \varepsilon_b(q); \alpha'\Big) &= \frac{\varepsilon_a(p) \cdot q\, \varepsilon_b(q) \cdot \varepsilon_c \; - \varepsilon_b(q) \cdot p\, \varepsilon_a(p) \cdot \varepsilon_c \; + \varepsilon_a(p) \cdot h \cdot \varepsilon_b(q)\, p \cdot \varepsilon_c}{2p \cdot q} && (3.35)\\
&= \frac{\omega_p x_{12}^b \delta^{ac} + \omega_q x_{12}^a \delta^{bc} - \frac{\omega_p \omega_q}{\omega_p + \omega_q} x_{12}^c \delta^{ab} - 2\alpha' \frac{\omega_p^2 \omega_q^2}{\omega_p + \omega_q} x_{12}^a x_{12}^b x_{12}^c}{\omega_p \omega_q (x_{12})^2} && (3.36)\\
&= -I_c\Big(q, \varepsilon_b(q); p, \varepsilon_a(p); \alpha'\Big) \,, && (3.37)
\end{aligned}$$

which is anti-symmetric in p and q, as shown above.

Doing the integral as in (3.2), we thus get

$$\begin{aligned}
\mathcal{V}_a^A(p) \mathcal{V}_b^B(q) &= \int dy_1 dy_2\, \theta(y_1 - y_2) \Big(- 4i\, \delta(y_{12}) I_c(p, \varepsilon_a(p); q, \varepsilon_b(q); \alpha') \Big) \varepsilon_c \cdot \dot{X}\, e^{i(p+q) \cdot X}(y_2) t^A t^B \\
&\quad + \int dy_1 dy_2\, \theta(y_2 - y_1) \Big(- 4i\, \delta(y_{21}) I_c(q, \varepsilon_b(q); p, \varepsilon_a(p); \alpha') \Big) \varepsilon_c \cdot \dot{X}\, e^{i(p+q) \cdot X}(y_1) t^B t^A \\
&\propto f^{ABC} I_c\Big(p, \varepsilon_a(p); q, \varepsilon_b(q); \alpha'\Big) \times \int dy_2\, \varepsilon_c \cdot \dot{X}\, e^{i(p+q) \cdot X}(y_2) t^C \\
&\propto f^{ABC} I_c\Big(p, \varepsilon_a(p); q, \varepsilon_b(q); \alpha'\Big) \times \mathcal{V}_c^C(p + q) \,,
\end{aligned}$$

where we used the anti-symmetric property of I_c in (3.37) and the notation $\mathcal{V}_a(p) = \mathcal{V}(p, \varepsilon_a(p))$, $\mathcal{V}_b(q) = \mathcal{V}(q, \varepsilon_b(q))$ and so on. We also used the integral

$$\int dy_1 dy_2\, \theta(y_1 - y_2)\delta(y_{12}) f(y_2) = \int dy_2\, \theta(0) f(y_2) = \frac{1}{2}\int dy_2\, f(y_2)\,, \tag{3.38}$$

with $\theta(0) = \frac{1}{2}$.

Finally, we need to perform the Mellin transformation (2.39) in order to go to the conformal basis. Using the following integral,

$$\begin{aligned}&\int_0^\infty d\omega_1\, \omega_1^{\Delta_1 - 1} \int_0^\infty d\omega_2\, \omega_2^{\Delta_2 - 1}\, \omega_1^\alpha \omega_2^\beta (\omega_1 + \omega_2)^\gamma\, F(\omega_1 + \omega_2) \\ &= B\Big(\Delta_1 + \alpha, \Delta_2 + \beta\Big) \int_0^\infty d\omega\, \omega^{\Delta_P - 1}\, F(\omega)\,, \qquad \Delta_P = \Delta_1 + \Delta_2 + \alpha + \beta + \gamma\,,\end{aligned} \tag{3.39}$$

we obtain the final result for celestial gluon OPE

$$\begin{aligned}&\mathcal{V}^A_{\Delta_1,a}(x_1)\mathcal{V}^B_{\Delta_2,b}(x_2) \\ &\sim f^{ABC} \frac{x_{12}^a \delta^{bc} B(\Delta_1 - 1, \Delta_2) + x_{12}^b \delta^{ac} B(\Delta_1, \Delta_2 - 1) - x_{12}^c \delta^{ab} B(\Delta_1, \Delta_2)}{(x_{12})^2} \mathcal{V}^C_{\Delta_1+\Delta_2-1,\, c}(x_2) \\ &\quad -2\alpha' f^{ABC} \frac{x_{12}^a x_{12}^b x_{12}^c B(\Delta_1 + 1, \Delta_2 + 1)}{(x_{12})^2} \mathcal{V}^C_{\Delta_1+\Delta_2+1,\, c}(x_2)\,.\end{aligned} \tag{3.40}$$

3.1.1 General structure of worldsheet OPE

Based on our previous OPE analysis of two gluon vertex operators, we would like now to discuss the general structure of worldsheet OPE. From (3.3) and the derivation above, one expects that in general the world-sheet OPE takes the form

$$\mathscr{V}^{A,\ell_1} e^{ip\cdot X}(y_1) \mathscr{V}^{B,\ell_2} e^{iq\cdot X}(y_2) \sim \sum_{\ell=0}^{\infty} y_{12}^{2\alpha' p\cdot q + \ell - \ell_1 - \ell_2} F_{C,\ell}^{A,\ell_1;B,\ell_2}(p,q) \mathscr{V}^{C,\ell} e^{i(p+q)\cdot X}(y_2)\,, \tag{3.41}$$

where

$$\mathscr{V}^{A,\ell} e^{ik\cdot X} \sim \prod_i \partial^{\ell_i} X^{\mu_i} e^{ik\cdot X}, \qquad \ell = \sum_i \ell_i: \qquad h = \alpha' k^2 + \ell \xrightarrow{\text{on-shell}} \ell - \alpha' M^2 = 1\,. \tag{3.42}$$

Note that in (3.41) we did not impose the on-shell condition on the momenta and it holds even off-shell.

Now we take p, q on-shell, namely setting $p^2 = -m_1^2 = (1 - \ell_1)/\alpha'$, $q^2 = -m_2^2 = (1 - \ell_2)/\alpha'$, then we find

$$2\alpha' p\cdot q + \ell - \ell_1 - \ell_2 = \alpha'(p+q)^2 - 2 + \ell\,. \tag{3.43}$$

For $p+q$, we take it slightly off-shell, namely $(p+q)^2 = -m_3^2 + \epsilon/\alpha' = (1-\ell_3)/\alpha' + \epsilon/\alpha'$, then

$$2\alpha' p\cdot q + \ell - \ell_1 - \ell_2 = (1 - \ell_3) + \epsilon - 2 + \ell = \epsilon + \ell - \ell_3 - 1\,. \tag{3.44}$$

Therefore (3.41) becomes

$$\mathscr{V}^{A,\ell_1}e^{ip\cdot X}\mathscr{V}^{B,\ell_2}e^{iq\cdot X} = \sum_{\ell=0}^{\infty} y_{12}^{\epsilon+\ell-\ell_3-1}F_{C,\ell}^{A,\ell_1;B,\ell_2}(p,q)\mathscr{V}^{C,\ell}e^{i(p+q)\cdot X}(y_2) \tag{3.45}$$

$$\xrightarrow{\epsilon\to 0} \sum_{\ell=0}^{\ell_3}\frac{1}{\epsilon}\delta^{\ell_3-\ell}(y_{12})F_{C,\ell}^{A,\ell_1;B,\ell_2}(p,q)\mathscr{V}^{C,\ell}e^{i(p+q)\cdot X}(y_2)\,. \tag{3.46}$$

where we used (3.21) and ignored some unimportant factors.

After doing the worldsheet integral, we then expect to find

$$(p+q)^2\to(1-\ell_3)/\alpha':\qquad \mathcal{V}_p^{A,\ell_1}\mathcal{V}_q^{B,\ell_2}\sim\frac{1}{(p+q)^2-(1-\ell_3)/\alpha'}F_{C,\ell_3}^{A,\ell_1;B,\ell_2}\mathcal{V}_{p+q}^{C,\ell_3}+\cdots\,, \tag{3.47}$$

where the dots are boundary contact terms of the form (3.22) arising from $\ell=0,\cdots,\ell_3-1$ as well some terms in $\ell=\ell_3$. The prefactor on the right hand side is just the propagator $1/((p+q)^2+m_3^2)$.

3.2 OPE in closed string

Now we switch to the closed string. As before, we need to first compute

$$\mathcal{V}(p,e_1)\mathcal{V}(q,e_2) = \int d^2z_1d^2z_2\; V_{p,e_1}(z_1,\bar z_1)V_{q,e_2}(z_2,\bar z_2) \tag{3.48}$$

$$= \int d^2z_1d^2z_2\; e_{1\mu\bar\mu}\partial X^\mu\bar\partial X^{\bar\mu}e^{ip\cdot X}(z_1,\bar z_1)\; e_{2\nu\bar\nu}\partial X^\nu\bar\partial X^{\bar\nu}e^{iq\cdot X}(z_2,\bar z_2)\,, \tag{3.49}$$

and then perform the Mellin transformation to obtain the celestial OPE.

As the starting point, we need to know the world-sheet OPE of two operators in the integrand of (3.49), which is computed in (A.29):

$$\partial X^\mu\bar\partial X^{\bar\mu}e^{ip\cdot X}(z_1,\bar z_1)\partial X^\mu\bar\partial X^{\bar\nu}e^{ip\cdot X}(z_2,\bar z_2) \tag{3.50}$$

$$\sim\frac{\alpha'^2}{4}|z_{12}|^{\alpha'p\cdot q-4}e^{i(p+q)\cdot X} \tag{3.51}$$

$$\times\left[\left(\eta^{\mu\nu}-\frac{1}{2}\alpha'q^\mu p^\nu\right)+iz_{12}\left(q^\mu\partial X^\nu-p^\nu\partial X^\mu+\left(\eta^{\mu\nu}-\frac{1}{2}\alpha'q^\mu q^\nu\right)p\cdot\partial X\right)+\cdots\right] \tag{3.52}$$

$$\times\left[\left(\eta^{\bar\mu\bar\nu}-\frac{1}{2}\alpha'q^{\bar\mu}p^{\bar\nu}\right)+i\bar z_{12}\left(q^{\bar\mu}\partial X^{\bar\nu}-p^{\bar\nu}\bar\partial X^{\bar\mu}+\left(\eta^{\bar\mu\bar\nu}-\frac{1}{2}\alpha'q^{\bar\mu}q^{\nu}\right)p\cdot\bar\partial X\right)+\cdots\right](z_2,\bar z_2). \tag{3.53}$$

This OPE is essentially the "square" of open string world-sheet OPE (3.3), up to the rescaling of α'. In particular, the terms proportional to $z_{12},\bar z_{12}$ can be simplified as in the open string case; in each bracket, we can get expressions which are similar to (3.15).

Although many cross product terms from the left and right moving sectors in (3.50) would be generated, we only need to consider the diagonal terms, namely those terms which only depend on the modulus $|z_{12}|$. The non-diagonal terms, like those with factor z_{12}, would become zero after integrating along the angular direction of z_{12}.

Before simplifying this OPE, let us first introduce some useful identities. In the open string case, (3.21) plays an important role. Now we would like to find similar identities for the closed string. We first parametrize the complex coordinate as

$$z = u + iv = \varrho e^{i\vartheta} \,, \qquad \bar{z} = u - iv = \varrho e^{-i\vartheta} \,, \qquad |z| = \varrho \,. \tag{3.54}$$

Then we have the following formula[18]

$$\delta^{(2)}(z) = \delta(u)\delta(v) = \frac{\delta(\varrho)}{\pi\varrho} = \frac{1}{2\pi}\epsilon\varrho^{\epsilon-2} = \frac{1}{2\pi}\epsilon|z|^{\epsilon-2} \,, \qquad d^2z = dudv = \varrho d\vartheta d\varrho \,, \tag{3.55}$$

where ϵ is again infinitesimal and we used the identity in (3.21). Further taking derivatives of the delta-function gives

$$\partial\delta^{(2)}(z) = -\frac{1}{2\pi}\epsilon|z|^{\epsilon-2}z^{-1} \,, \quad \bar{\partial}\delta^{(2)}(z) = -\frac{1}{2\pi}\epsilon|z|^{\epsilon-2}\bar{z}^{-1} \,, \quad \partial\bar{\partial}\delta^{(2)}(z) = \frac{1}{2\pi}\epsilon|z|^{\epsilon-4} \,, \quad \cdots \,. \tag{3.56}$$

We are interested in the behavior of OPE (3.50) in the collinear limit $p \cdot q \to 0$. As in the open string case, one can show that most terms in the OPE are not relevant in the collinear limit; they are either regular in the limit $p \cdot q \to 0$, or only contribute boundary contact terms. For example, the tachyon arising from the first term in two square brackets of (3.50) takes the form

$$|z_{12}|^{\alpha' p\cdot q-4} e^{i(p+q)\cdot X} \to \frac{\partial\bar{\partial}\delta^{(2)}(z_{12})}{p\cdot q} e^{i(p+q)\cdot X} \,, \tag{3.57}$$

where we used identity in (3.56). Following (3.49), we need to integrate over z_1, z_2. This type of total derivative can contribute at most boundary terms after doing the integral. Other terms in the OPE (3.50) can be analyzed similarly. In particular, one can rewrite terms in each square bracket in (3.50) and get expressions similar to the open string case (3.15). Then one can show that most terms are not important for our purpose.

After doing simplification, the only relevant terms in OPE (3.50) in the collinear limit $p \cdot q \to 0$ are given by

$$\partial X^\mu \bar{\partial} X^{\bar{\mu}} e^{ip\cdot X}(z_1, \bar{z}_1) \partial X^\nu \bar{\partial} X^{\bar{\nu}} e^{ip\cdot X}(z_2, \bar{z}_2) \tag{3.58}$$

$$\sim -\frac{\alpha'^2}{4}|z_{12}|^{\alpha' p\cdot q-2} e^{i(p+q)\cdot X} \times \left(q^\mu\, \varepsilon_c^\nu - p^\nu\, \varepsilon_c^\mu + \left(\eta^{\mu\nu} - \frac{1}{2}\alpha' q^\mu\, q^\nu\right) p\cdot\varepsilon_c\right) \tag{3.59}$$

$$\times \left(q^{\bar{\mu}}\, \varepsilon_{\tilde{c}}^{\bar{\nu}} - p^{\bar{\nu}}\, \varepsilon_{\tilde{c}}^{\bar{\mu}} + \left(\eta^{\bar{\mu}\bar{\nu}} - \frac{1}{2}\alpha' q^{\bar{\mu}}\, q^{\nu}\right) p\cdot\varepsilon_{\tilde{c}}\right) \times \varepsilon_c \cdot \partial X\; \varepsilon_{\tilde{c}} \cdot \bar{\partial} X \,, \tag{3.60}$$

which is just the "square" of (3.17).

[18] In this convention, we thus have

$$\partial_{\bar{z}}\frac{1}{z} = \pi\delta^{(2)}(z) \,, \qquad \int d^2z\; f(z)\delta^{(2)}(z) = f(0) \,.$$

For the representation of delta function in polar coordinates, see https://mathworld.wolfram.com/DeltaFunction.html.

To simplify further, we choose the polarization tensors as the basis constructed in (2.5):

$$\varepsilon^{\mu\bar{\mu}}_{a\tilde{a}}(p) = \varepsilon^{\mu}_{a}(p)\varepsilon^{\bar{\mu}}_{\tilde{a}}(p)\,, \qquad \varepsilon^{\nu\bar{\nu}}_{b\tilde{b}}(q) = \varepsilon^{\nu}_{b}(q)\varepsilon^{\bar{\nu}}_{\tilde{b}}(q)\,. \tag{3.61}$$

Any polarization for graviton/dilaton/KR field can be expanded in this basis using (2.15), (2.17).

Contracting the OPE (3.58) with (3.61) and taking the limit $p\cdot q \to 0$ gives

$$\varepsilon_{a\tilde{a}\;\mu\bar{\mu}}(p)\partial X^{\mu}\bar{\partial}X^{\bar{\mu}}e^{ip\cdot X}(z_1,\bar{z}_1)\varepsilon_{b\tilde{b}\;\nu\bar{\nu}}(q)\partial X^{\nu}\bar{\partial}X^{\bar{\nu}}e^{ip\cdot X}(z_2,\bar{z}_2) \tag{3.62}$$

$$\sim -\frac{\alpha'^2}{4}2\pi\frac{1}{\alpha' p\cdot q}\delta^2(z_{12})e^{i(p+q)\cdot X}\times\left(q^{\mu}\;\varepsilon^{\nu}_{c} - p^{\nu}\;\varepsilon^{\mu}_{c} + \left(\eta^{\mu\nu}-\frac{1}{2}\alpha' q^{\mu}\;q^{\nu}\right)p\cdot\varepsilon_c\right)\varepsilon^{\mu}_{a}\varepsilon^{\nu}_{b}$$
$$\times\left(q^{\bar{\mu}}\;\varepsilon^{\bar{\nu}}_{\tilde{c}} - p^{\bar{\nu}}\;\varepsilon^{\bar{\mu}}_{\tilde{c}} + \left(\eta^{\bar{\mu}\bar{\nu}}-\frac{1}{2}\alpha' q^{\bar{\mu}}\;q^{\nu}\right)p\cdot\varepsilon_{\tilde{c}}\right)\varepsilon^{\bar{\mu}}_{\tilde{a}}\varepsilon^{\bar{\nu}}_{\tilde{b}}\;\varepsilon_{c}\cdot\partial X\;\varepsilon_{\tilde{c}}\cdot\bar{\partial}X \tag{3.63}$$

$$\sim \pi\alpha'\;\delta^2(z_{12})\,\mathcal{I}_{c\tilde{c}}\left(p,\varepsilon_{a\tilde{a}}(p);q,\varepsilon_{b\tilde{b}}(q);\alpha',\alpha'\right)\;\varepsilon_{c\tilde{c}\;\sigma\bar{\sigma}}\partial X^{\sigma}\bar{\partial}X^{\bar{\sigma}}\,, \tag{3.64}$$

where we used the identity in (3.55) and defined

$$\mathcal{I}_{c\tilde{c}}\Big(p,\varepsilon_{a\tilde{a}}(p);q,\varepsilon_{b\tilde{b}}(q);\alpha,\beta\Big) \tag{3.65}$$

$$=\frac{1}{-2p\cdot q}\left[\varepsilon_a(p)\cdot q\,\varepsilon_b(q)\cdot\varepsilon_c - \varepsilon_b(q)\cdot p\,\varepsilon_a(p)\cdot\varepsilon_c + \left(\varepsilon_a(p)\cdot\varepsilon_b(q) - \frac{\alpha}{2}\varepsilon_a(p)\cdot q\,\varepsilon_b(q)\cdot p\right)p\cdot\varepsilon_c\right]$$
$$\times\left[\varepsilon_{\tilde{a}}(p)\cdot q\,\varepsilon_{\tilde{b}}(q)\cdot\varepsilon_{\tilde{c}} - \varepsilon_{\tilde{b}}(q)\cdot p\,\varepsilon_{\tilde{a}}(p)\cdot\varepsilon_{\tilde{c}} + \left(\varepsilon_{\tilde{a}}(p)\cdot\varepsilon_{\tilde{b}}(q) - \frac{\beta}{2}\varepsilon_{\tilde{a}}(p)\cdot q\,\varepsilon_{\tilde{b}}(q)\cdot p\right)p\cdot\varepsilon_{\tilde{c}}\right]$$
$$=\omega_p\omega_q(x_{12})^2\,I_c\left(p,\varepsilon_a(p);q,\varepsilon_b(q);\frac{1}{4}\alpha\right)\;I_{\tilde{c}}\left(p,\varepsilon_{\tilde{a}}(p);q,\varepsilon_{\tilde{b}}(q);\frac{1}{4}\beta\right)\,. \tag{3.66}$$

The expression for $I_c\Big(p,\varepsilon_a(p);q,\varepsilon_b(q);\alpha'\Big)$ is given in (3.35).

Following (3.49) and using (3.64), we can now compute the OPE between two vertex operators of massless fields in closed string theory:

$$\begin{aligned}\mathcal{V}_{a\tilde{a}}(p)\mathcal{V}_{b\tilde{b}}(q) &= \int d^2z_1d^2z_2\;\varepsilon_{a\tilde{a}\;\mu\bar{\mu}}(p)\partial X^{\mu}\bar{\partial}X^{\bar{\mu}}e^{ip\cdot X}(z_1,\bar{z}_1)\varepsilon_{b\tilde{b}\;\nu\bar{\nu}}(q)\partial X^{\nu}\bar{\partial}X^{\bar{\nu}}e^{ip\cdot X}(z_2,\bar{z}_2)\\ &= \int d^2z_1d^2z_2\;\pi\alpha'\;\delta^2(z_{12})\,\mathcal{I}_{c\tilde{c}}\Big(p,\varepsilon_{a\tilde{a}}(p);q,\varepsilon_{b\tilde{b}}(q);\alpha',\alpha'\Big)\;\varepsilon_{c\tilde{c}\;\sigma\bar{\sigma}}\partial X^{\sigma}\bar{\partial}X^{\bar{\sigma}}\end{aligned} \tag{3.67}$$

$$\propto \mathcal{I}_{c\tilde{c}}\Big(p,\varepsilon_{a\tilde{a}}(p);q,\varepsilon_{b\tilde{b}}(q);\alpha',\alpha'\Big)\;\mathcal{V}_{c\tilde{c}}(p+q)\,, \tag{3.68}$$

where we used the notation $\mathcal{V}_{a\tilde{a}}(p) \equiv \mathcal{V}(p,\varepsilon_{a\tilde{a}}(p))$ and similarly for q, $p+q$.

Finally, we need to go to the conformal basis of the vertex operators (2.39). Using the formula (3.39) and performing the Mellin transformation, we finally obtain the celestial OPE for graviton/dilaton/KR field in closed string theory

$$\mathcal{V}_{\Delta_1,a\tilde{a}}(x_1)\mathcal{V}_{\Delta_2,b\tilde{b}}(x_2) \sim \mathcal{K}\,, \tag{3.69}$$

where[19]

$$
\begin{aligned}
(x_{12})^2\mathcal{K}=\Bigg[\bigg(&\delta^{bc}\delta^{\tilde{b}\tilde{c}}x^a_{12}x^{\tilde{a}}_{12}B(\Delta_1-1,\Delta_2+1)-\delta^{ac}\delta^{\tilde{a}\tilde{b}}x^b_{12}x^{\tilde{c}}_{12}B(\Delta_1+1,\Delta_2)-\delta^{ab}\delta^{\tilde{b}\tilde{c}}x^c_{12}x^{\tilde{a}}_{12}B(\Delta_1,\Delta_2+1)\\
&+\frac{1}{2}\delta^{ab}\delta^{\tilde{a}\tilde{b}}x^c_{12}x^{\tilde{c}}_{12}B(\Delta_1+1,\Delta_2+1)+\delta^{bc}\delta^{\tilde{a}\tilde{c}}x^a_{12}x^{\tilde{b}}_{12}B(\Delta_1,\Delta_2)\bigg)\\
&+\Big(a\leftrightarrow b,\tilde{a}\leftrightarrow\tilde{b},\Delta_1\leftrightarrow\Delta_2\Big)\Bigg]\mathcal{V}_{\Delta_1+\Delta_2,c\tilde{c}}(x_2) \qquad (3.70)\\
&+\frac{\alpha'}{2}\Bigg[\bigg(\frac{1}{2}\delta^{\tilde{a}\tilde{b}}x^a_{12}x^b_{12}x^c_{12}x^{\tilde{c}}_{12}B(\Delta_1+2,\Delta_2+2)-\delta^{ac}x^b_{12}x^{\tilde{b}}_{12}x^{\tilde{a}}_{12}x^{\tilde{c}}_{12}B(\Delta_1+2,\Delta_2+1)\\
&+(a\leftrightarrow\tilde{a},b\leftrightarrow\tilde{b},c\leftrightarrow\tilde{c})\bigg)+\Big(a\leftrightarrow b,\tilde{a}\leftrightarrow\tilde{b},\Delta_1\leftrightarrow\Delta_2\Big)\Bigg]\mathcal{V}_{\Delta_1+\Delta_2+2,c\tilde{c}}(x_2) \qquad (3.71)\\
&+\frac{\alpha'^2}{4}\Big[x^a_{12}x^b_{12}x^c_{12}x^{\tilde{a}}_{12}x^{\tilde{b}}_{12}x^{\tilde{c}}_{12}B(\Delta_1+3,\Delta_2+3)\Big]\mathcal{V}_{\Delta_1+\Delta_2+4,c\tilde{c}}(x_2). \qquad (3.72)
\end{aligned}
$$

4 Celestial OPE from worldsheet OPE in superstring

In this section, we will generalize the previous discussions on OPEs in bosonic string to superstring. The computations are similar, and especially we can also simplify the results in the same way. The main difference is that in superstring, the worldsheet has supersymmetry. As a result, the bosonic and fermionic contributions can cancel each other, which leads to a simpler result. In particular, the α' corrections to the worldsheet OPEs and thus also to the celestial OPEs are absent in the supersymmetric sector. This behavior agrees with the three-point amplitudes in superstring theory where the α' corrections also don't appear in the supersymmetric sector.

We will be mainly focusing on the heterotic string case. The heterotic string is a good playground for our studies of OPEs because we can realize both graviton and gluon easily in terms of closed string. In this setup, we compute all the OPEs involving gluon and graviton/dilaton/KR field, including their mixed OPEs. Up to α' corrections, the OPEs derived in heterotic string agree with those derived in bosonic string. We will also briefly discuss the case of type I and IIA/IIB string, which are very similar to heterotic string.

4.1 Heterotic string

In this subsection, we will first review the vertex operators of gluon and graviton/dilaton/ KR field in heterotic string, and then derive the celestial OPEs from worldsheet OPEs.

4.1.1 Vertex operator in heterotic string

In bosonic closed string, the vertex operators take two forms, either integrated form with worldsheet integration or unintegrated form with $c\tilde{c}$ attachment. In superstring theory, the vertex operators further take a variety of forms, called pictures [33, 34]. In each supersymmetric left or right moving sector, we can label the vertex operator with the ghost charges q or $\tilde{q}$. The vertex operator in the q or $\tilde{q}$ picture contains a factor $e^{q\phi}$ or

[19]Note that the exchange in the parentheses is the simultaneous exchange of all items.

$e^{\tilde{q}\tilde{\phi}}$, where ϕ is the bosonized field of $\beta\gamma$ ghost system and similarly for $\tilde{\phi}$. The vertex operators in different pictures can be related via the picture changing operator. The total ghost charges in each supersymmetric sector is determined by the worldsheet genus. In particular, at tree level of interest in this paper, the total ghost charge should be -2 in order to soak up the fermionic zero modes.

More specifically, in heterotic string, the gluon vertex operator in the picture -1 and 0 are respectively given by [34]

$$V^{(-1)}(z,\bar{z}) = e^{-\tilde{\phi}} J^A \tilde{\psi}^\nu e^{ik\cdot X} = V_L(z) V_R^{(0)}(\bar{z})\,, \tag{4.1}$$

$$V^{(0)}(z,\bar{z}) = J^A \left(i\bar{\partial} X^\nu + \frac{1}{2}\alpha' k\cdot\tilde{\psi}\tilde{\psi}^\nu \right) e^{ik\cdot X} = V_L(z) V_R^{(-1)}(\bar{z})\,, \tag{4.2}$$

where $J^A(z)$ are the Kac-Moody currents in the left-moving sector, while $\tilde{\phi}$ and $\tilde{\psi}$ are the ghost and world-sheet fermion in the right-moving sector, respectively. Similarly, for the massless graviton/dilaton/KR fields in heterotic string, their vertex operators in the -1 and 0 picture are given by

$$W^{(-1)}(z,\bar{z}) = e^{-\tilde{\phi}} \partial X^\mu \tilde{\psi}^\nu e^{ik\cdot X} = W_L(z) W_R^{(-1)}(\bar{z})\,, \tag{4.3}$$

$$W^{(0)}(z,\bar{z}) = \partial X^\mu \left(i\bar{\partial} X^\nu + \frac{1}{2}\alpha' k\cdot\tilde{\psi}\tilde{\psi}^\nu \right) e^{ik\cdot X} = W_L(z) W_R^{(0)}(\bar{z})\,. \tag{4.4}$$

In these vertex operators, we also write them in the product form of left and right moving parts:

$$V_L(z) = J^A(z) e^{ik\cdot X_L}\,, \qquad W_L(z) = \partial X^\mu e^{ik\cdot X_L}\,, \tag{4.5}$$

$$V_R^{(0)}(\bar{z}) = W_R^{(0)}(\bar{z}) = \left(i\bar{\partial} X^\mu + \frac{1}{2}\alpha' k\cdot\tilde{\psi}\tilde{\psi}^\mu \right) e^{ik\cdot X_R}\,, \tag{4.6}$$

$$V_R^{(-1)}(\bar{z}) = W_R^{(-1)}(\bar{z}) = e^{-\tilde{\phi}} \tilde{\psi}^\mu \, e^{ik\cdot X_R}\,, \tag{4.7}$$

where $X(z,\bar{z}) = X_L(z) + X_R(\bar{z})$. This decomposition enables us to compute the OPE in the left and right moving sectors independently. The Kac-Moody currents $J^A(z)$ satisfy the OPE

$$J^A(z_1) J^B(z_2) \sim \frac{\mathsf{k}\delta^{AB}}{z_{12}^2} + \frac{f^{ABC} J^C(z_2)}{z_{12}}\,. \tag{4.8}$$

Without loss of generality, one can set the level k to one, which can always be realized by rescaling the Kac-Moody currents and structure constants.

4.1.2 Celestial OPE from heterotic worldsheet

Now we derive the celestial OPEs in heterotic string from worldsheet.

Gluon-gluon OPE. We want to compute the OPE between gluons. Although vertex operators in different pictures are physically equivalent, it turns out to be simpler to consider the OPE of two vertex operators in the -1 and 0 picture, namely $V^{(-1)}(z_1,\bar{z}_1) V^{(0)}(z_2,\bar{z}_2)$, which gives another vertex operator in -1 picture.

We first consider the left-moving OPE $V_L(z_1)V_L(z_2)$:

$$\begin{aligned}
J^A e^{ip\cdot X_L}(z_1) J^B e^{iq\cdot X_L}(z_2) &= J^A(z_1)J^B(z_2)\times e^{ip\cdot X_L}(z_1)e^{iq\cdot X_L}(z_2)\\
&= \left(\frac{\mathsf{k}\delta^{AB}}{z_{12}^2}+\frac{f^{ABC}J^C(z_2)}{z_{12}}+\cdots\right)\times z_{12}^{\frac{1}{2}\alpha' p\cdot q}\Big(1+i\,z_{12}p\cdot\partial X+\cdots\Big)e^{i(p+q)\cdot X_L}(z_2)\\
&= z_{12}^{\frac{1}{2}\alpha' p\cdot q-1}\left(\frac{\mathsf{k}\delta^{AB}}{z_{12}}+f^{ABC}J^C+i\,\mathsf{k}\delta^{AB}\,p\cdot\partial X+\cdots\right)e^{i(p+q)\cdot X_L}(z_2)\,.
\end{aligned}\tag{4.9}$$

Following the derivation of (3.8), we have

$$p\cdot\partial X = p\cdot\varepsilon_c\,\varepsilon_c\cdot\partial X+\frac{1}{\omega_k}p\cdot q\,\partial X\cdot n+\frac{\omega_p}{\omega_k}k\cdot\partial X+\mathscr{O}(\epsilon)\,.\tag{4.10}$$

Due to the $p\cdot q$ factor, the second term on the right hand side is regular in the collinear limit $p\cdot q\to 0$. For the third term, we can combine it with extra factors in (4.9), leading to

$$z_{12}^{\frac{1}{2}\alpha' p\cdot q-1}\,k\cdot\partial X\,e^{i(p+q)\cdot X_L}=-i\,z_{12}^{\frac{1}{2}\alpha' p\cdot q-1}\,\partial\,e^{ik\cdot X_L}+\mathscr{O}(\epsilon)\,,\tag{4.11}$$

where we used $e^{i(p+q)\cdot X_L}=e^{ik\cdot X_L}+\mathscr{O}(\epsilon)$ as $p+q=k+\mathscr{O}(\epsilon)$ in the collinear limit. Since the polarization is along the momentum direction, the is supposed to be a pure gauge and thus can be ignored. The more subtle issue of boundary contact terms is the same as that in open bosonic string case, see (3.23). Therefore, we can just keep the first term in (4.10).

As a result, in the collinear limit the left-moving OPE simplifies to

$$J^A e^{ip\cdot X_L}(z_1)J^B e^{iq\cdot X_L}(z_2)=z_{12}^{\frac{1}{2}\alpha' p\cdot q-1}\left(\frac{\mathsf{k}\delta^{AB}}{z_{12}}+f^{ABC}J^C+i\,\mathsf{k}\delta^{AB}\,p\cdot\varepsilon_c\,\varepsilon_c\cdot\partial X+\cdots\right)e^{i(p+q)\cdot X_L}(z_2)\,.\tag{4.12}$$

For the right-moving OPE $V_R^{(-1)}(\bar z_1)V_R^{(0)}(\bar z_2)$, it is computed in (A.41)

$$\zeta\cdot\tilde\psi e^{ip\cdot X_R}(\bar z_1)\left(i\xi\cdot\bar\partial X+\frac{1}{2}\alpha' q\cdot\tilde\psi\xi\cdot\tilde\psi\right)e^{iq\cdot X_R}(\bar z_2)\tag{4.13}$$

$$\sim\frac{\alpha'}{2}\bar z_{12}^{\frac{1}{2}\alpha' p\cdot q-1}e^{i(p+q)\cdot X_R}\Big(\zeta\cdot q\,\xi\cdot\tilde\psi-\xi\cdot p\,\zeta\cdot\tilde\psi-\zeta\cdot\xi\,q\cdot\tilde\psi\Big)\,.\tag{4.14}$$

Note that terms in the bracket of (4.14) are very similar to the terms proportional y_{12} in the bracket of (3.3): one just needs to replace $\dot X$ with ψ, exchange p,ζ and q,ξ, and ignore α' corrections. Therefore we can also simplify the formula in the same manner. Following the steps in the derivation of (3.8) and (3.12), we now have[20]

$$q\cdot\tilde\psi=-p\cdot\varepsilon_{\tilde c}\,\varepsilon_{\tilde c}\cdot\tilde\psi+\frac{1}{\omega_k}p\cdot q\,\tilde\psi\cdot n+\frac{\omega_q}{\omega_k}k\cdot\tilde\psi+\mathscr{O}(\epsilon)\,,\tag{4.15}$$

and

$$\zeta\cdot q\,\xi\cdot\tilde\psi-\xi\cdot p\,\zeta\cdot\tilde\psi=\zeta\cdot q\,\xi\cdot\varepsilon_{\tilde c}\,\varepsilon_{\tilde c}\cdot\tilde\psi-\xi\cdot p\,\zeta\cdot\varepsilon_{\tilde c}\,\varepsilon_{\tilde c}\cdot\tilde\psi+\mathscr{O}(\epsilon)\,.\tag{4.16}$$

[20]Note $(p+q)\cdot\varepsilon_{\tilde c}=k\cdot\varepsilon_{\tilde c}+\mathscr{O}(\epsilon)=\mathscr{O}(\epsilon)$ as $k\cdot\varepsilon_{\tilde c}=0$.

These two formulae enable us simplify (4.14)

$$\zeta \cdot \tilde{\psi} e^{ip\cdot X_R}(\bar{z}_1) \left(i\xi \cdot \bar{\partial} X + \frac{1}{2}\alpha' q \cdot \tilde{\psi} \xi \cdot \tilde{\psi} \right) e^{iq\cdot X_R}(\bar{z}_2) \tag{4.17}$$

$$\sim \frac{\alpha'}{2} \bar{z}_{12}^{\frac{1}{2}\alpha' p\cdot q-1} e^{i(p+q)\cdot X_R} \Big(\zeta \cdot q \, \xi \cdot \varepsilon_{\tilde{c}} - \xi \cdot p \, \zeta \cdot \varepsilon_{\tilde{c}} + \zeta \cdot \xi \, p \cdot \varepsilon_{\tilde{c}} \Big) \, \varepsilon_{\tilde{c}} \cdot \tilde{\psi} \tag{4.18}$$

$$+ \frac{\alpha'}{2} \bar{z}_{12}^{\frac{1}{2}\alpha' p\cdot q-1} e^{i(p+q)\cdot X_R} \left(-\zeta \cdot \xi \, \frac{1}{\omega_k} p \cdot q \, \tilde{\psi} \cdot n - \zeta \cdot \xi \, \frac{\omega_q}{\omega_k} k \cdot \tilde{\psi} \right) . \tag{4.19}$$

In the collinear limit $p \cdot q \to 0$, the first term in (4.19) is regular. For the second term, we can rewrite it as:[21]

$$e^{i(p+q)\cdot X_R} k \cdot \tilde{\psi} = e^{ik\cdot X_R} k \cdot \tilde{\psi} + \mathscr{O}(\epsilon) \, . \tag{4.20}$$

If we combine (4.20) with the factor $z_{12}^{\frac{1}{2}\alpha' p\cdot q-1}$ from the left-moving sector,[22] we get

$$|z_{12}|^{\alpha' p\cdot q-2} e^{ik\cdot X_R} k \cdot \tilde{\psi} \xrightarrow{p\cdot q \to 0} \frac{\delta^2(z_{12})}{p \cdot q} k \cdot \tilde{\psi} e^{ik\cdot X_R} \xrightarrow{\int d^2 z_1 \, d^2 z_2} \frac{1}{p \cdot q} \int d^2 z_2 \; k \cdot \tilde{\psi} e^{ik\cdot X_R} \, . \tag{4.21}$$

Since the polarization is along the momentum direction, this is a pure gauge. More specifically it is a BRST exact term and thus plays no role in string amplitude. Therefore, we can drop all the terms in (4.19).

Combining the left-moving OPE (4.12) and right-moving OPE (4.18), we then find

$$e^{-\tilde{\phi}} J^A \tilde{\psi}^\mu e^{ip\cdot X}(z_1, \bar{z}_1) J^B \left(i\bar{\partial} X^\nu + \frac{1}{2}\alpha' q \cdot \tilde{\psi} \tilde{\psi}^\nu \right) e^{ik\cdot X}(z_2, \bar{z}_2) \tag{4.22}$$

$$\sim \frac{\alpha'}{2} |z_{12}|^{\alpha' p\cdot q-2} e^{-\tilde{\phi}} \left(\frac{\mathsf{k}\delta^{AB}}{z_{12}} + f^{ABC} J^C + i \, \mathsf{k}\delta^{AB} \, p \cdot \varepsilon_c \, \varepsilon_c \cdot \partial X + \cdots \right)$$

$$\times \Big(q^\mu \, \varepsilon_{\tilde{c}}^\nu - p^\nu \, \varepsilon_{\tilde{c}}^\mu + \eta^{\mu\nu} \, p \cdot \varepsilon_{\tilde{c}} \Big) \, \varepsilon_{\tilde{c}} \cdot \tilde{\psi} \, e^{i(p+q)\cdot X} \, . \tag{4.23}$$

First, we have the leading term $\mathsf{k}\delta^{AB}$ coming with a factor $|z_{12}|^{\alpha' p\cdot q-2} z_{12}^{-1}$. This leads to zero after performing the z_1, z_2 integral due to the cancellation from the angular direction of z_{12}. After contracting with the basis of polarization, the rest of terms in (4.23) become

$$e^{-\tilde{\phi}} J^A \varepsilon_{\tilde{a}}(p) \cdot \tilde{\psi} e^{ip\cdot X}(z_1, \bar{z}_1) J^A \left(i\varepsilon_{\tilde{b}}(q) \cdot \bar{\partial} X + \frac{1}{2}\alpha' q \cdot \tilde{\psi} \, \varepsilon_{\tilde{b}}(q) \cdot \tilde{\psi} \right) e^{ik\cdot X}(z_2, \bar{z}_2) \tag{4.24}$$

$$\sim \frac{\alpha'}{2} |z_{12}|^{\alpha' p\cdot q-2} \Big(\varepsilon_{\tilde{a}}(p) \cdot q \, \varepsilon_{\tilde{b}}(q) \cdot \varepsilon_{\tilde{c}} - \varepsilon_{\tilde{b}}(q) \cdot p \, \varepsilon_{\tilde{a}}(p) \cdot \varepsilon_{\tilde{c}} + \varepsilon_{\tilde{a}}(p) \cdot \varepsilon_{\tilde{b}}(q) \, p \cdot \varepsilon_{\tilde{c}} \Big)$$

$$\times \Big[f^{ABC} \, e^{-\tilde{\phi}} J^C \, \varepsilon_{\tilde{c}} \cdot \tilde{\psi} \, e^{i(p+q)\cdot X} + i \, \mathsf{k}\delta^{AB} \, p \cdot \varepsilon_c \, e^{-\tilde{\phi}} \, \varepsilon_c \cdot \partial X \varepsilon_{\tilde{c}} \cdot \tilde{\psi} \, e^{i(p+q)\cdot X} \Big] \tag{4.25}$$

$$\sim 2\pi \, \delta^2(z_{12}) \, I_{\tilde{c}}\Big(p, \varepsilon_{\tilde{a}}(p); q, \varepsilon_{\tilde{b}}(q); 0 \Big)$$

$$\times \Big[f^{ABC} \, e^{-\tilde{\phi}} J^C \, \varepsilon_{\tilde{c}} \cdot \tilde{\psi} \, e^{i(p+q)\cdot X} + i \, \mathsf{k}\delta^{AB} \, p \cdot \varepsilon_c \, e^{-\tilde{\phi}} \, \varepsilon_c \cdot \partial X \varepsilon_{\tilde{c}} \cdot \tilde{\psi} \, e^{i(p+q)\cdot X} \Big] , \tag{4.26}$$

[21] Recall that $p + q = k + \epsilon v$ where k is null while $p + q$ is slightly off-shell.

[22] The left-moving sector may also contribute $z_{12}^{\frac{1}{2}\alpha' p\cdot q-2}$, then in total we have $|z_{12}|^{\alpha' p\cdot q-2} z_{12}^{-1}$. This gives zero after integrating along the angular direction of z_{12}.

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where we used identity (3.55) and the expression for $I_{\tilde{c}}(p, \varepsilon_{\tilde{a}}(p); q, \varepsilon_{\tilde{b}}(q); 0)$ in (3.35) but without α' corrections. Compared with (4.1), (4.3), it is easy to recognize that terms in the square bracket are exactly the vertex operators of gluon and graviton/dilaton/KR field. After performing the integral over z_1, z_2, we get

$$\begin{aligned}\mathcal{V}_{\tilde{a}}^{(-1)A}(p)\mathcal{V}_{\tilde{b}}^{(0)B}(q) \sim f^{ABC}\, I_{\tilde{c}}\Big(p, \varepsilon_{\tilde{a}}(p); q, \varepsilon_{\tilde{b}}(q); 0\Big)\mathcal{V}_{\tilde{c}}^{(-1)C}(p+q) \\ +\delta^{AB}\, p\cdot\varepsilon_c\, I_{\tilde{c}}\Big(p, \varepsilon_{\tilde{a}}(p); q, \varepsilon_{\tilde{b}}(q); 0\Big)\mathcal{W}_{c\tilde{c}}^{(-1)C}(p+q)\,,\end{aligned} \tag{4.27}$$

up to some coefficients in front of gluon and graviton contributions, respectively.

We can further write in terms of celestial variables. In particular,

$$p\cdot\varepsilon_c = \omega_p \hat{p}\cdot\varepsilon_c = \omega_p (x_{13})^c = \frac{\omega_p\omega_q}{\omega_p+\omega_q} x_{12}^c\,, \tag{4.28}$$

where we used (2.4) and (3.32).

Finally performing the Mellin transformation yields the celestial OPE of two gluons

$$\begin{aligned}&\mathcal{V}^A_{\Delta_1,\tilde{a}}(x_1)\mathcal{V}^B_{\Delta_2,\tilde{b}}(x_2) \\ &\sim f^{ABC}\frac{x_{12}^{\tilde{a}}\delta^{\tilde{b}\tilde{c}}B(\Delta_1-1,\Delta_2)+x_{12}^{\tilde{b}}\delta^{\tilde{a}\tilde{c}}B(\Delta_1,\Delta_2-1)-x_{12}^{\tilde{c}}\delta^{\tilde{a}\tilde{b}}B(\Delta_1,\Delta_2)}{(x_{12})^2}\mathcal{V}^C_{\Delta_1+\Delta_2-1,\,\tilde{c}}(x_2) \\ &\quad+\delta^{AB}x_{12}^c\frac{x_{12}^{\tilde{a}}\delta^{\tilde{b}\tilde{c}}B(\Delta_1,\Delta_2+1)+x_{12}^{\tilde{b}}\delta^{\tilde{a}\tilde{c}}B(\Delta_1+1,\Delta_2)-x_{12}^{\tilde{c}}\delta^{\tilde{a}\tilde{b}}B(\Delta_1+1,\Delta_2+1)}{(x_{12})^2}\mathcal{W}_{\Delta_1+\Delta_2,\,c\tilde{c}}(x_2)\,,\end{aligned} \tag{4.29}$$

where the first and second line just correspond to the gluon and graviton, respectively. The OPE of gluons among themselves is the same as that in bosonic string case (3.40), except for the absence of α' corrections.

Gluon-graviton/dilaton/KR OPE. Now we consider the OPE between gluon and graviton/dilaton/KR field, namely $W^{(-1)}(z_1,\bar{z}_1)V^{(0)}(z_2,\bar{z}_2)$. The left-moving OPE $W_L(z_1) \cdot V_L(z_2)$ is simply given by

$$\partial X^\rho e^{ip\cdot X_L}(z_1)J^A e^{iq\cdot X_L}(z_2) = z_{12}^{\frac{1}{2}\alpha' p\cdot q-1}\left(-\frac{i}{2}\alpha' q^\rho+\cdots\right)J^A e^{i(p+q)\cdot X_L}(z_2)\,, \tag{4.30}$$

which follows from the same derivation of (A.14). The right-moving OPE $W_R^{(-1)}(\bar{z}_1)V_R^{(0)}(\bar{z}_2) = V_R^{(-1)}(\bar{z}_1)V_R^{(0)}(\bar{z}_2)$ has been discussed above and the final result is given in (4.18).

Combining the left and right moving sectors, we get

$$e^{-\tilde{\phi}}\varepsilon_a\cdot\partial X\varepsilon_{\tilde{a}}\cdot\tilde{\psi}e^{ip\cdot X}(z_1,\bar{z}_1)\; J^A\left(i\varepsilon_{\tilde{b}}\cdot\bar{\partial}X^\nu+\frac{1}{2}\alpha' p\cdot\tilde{\psi}\;\varepsilon_{\tilde{b}}\cdot\tilde{\psi}^\nu\right)e^{iq\cdot X}(z_2,\bar{z}_2) \tag{4.31}$$

$$\sim -\frac{i}{4}\alpha'^2|z_{12}|^{\alpha' p\cdot q-2}q\cdot\varepsilon_a\Big(\varepsilon_{\tilde{a}}\cdot q\;\varepsilon_{\tilde{b}}\cdot\varepsilon_{\tilde{c}}-\varepsilon_{\tilde{b}}\cdot p\;\varepsilon_{\tilde{a}}\cdot\varepsilon_{\tilde{c}}+\varepsilon_{\tilde{a}}\cdot\varepsilon_{\tilde{b}}\;p\cdot\varepsilon_{\tilde{c}}\Big)\;e^{-\tilde{\phi}}J^A\varepsilon_{\tilde{c}}\cdot\tilde{\psi}\;e^{i(p+q)\cdot X} \tag{4.32}$$

$$\sim -i\,\pi\alpha'\delta^2(z_{12})\;q\cdot\varepsilon_a\,I_{\tilde{c}}\Big(p,\varepsilon_{\tilde{a}}(p);q,\varepsilon_{\tilde{b}}(q);0\Big)\;e^{-\tilde{\phi}}J^A\varepsilon_{\tilde{c}}\cdot\tilde{\psi}\;e^{i(p+q)\cdot X}\,. \tag{4.33}$$

Integrating over z_1, z_2 gives

$$\mathcal{W}_{a\tilde{a}}^{(-1)}(p)\mathcal{V}_{\tilde{b}}^{(0)A}(q) \propto -q\cdot\varepsilon_a\,I_{\tilde{c}}\Big(p,\varepsilon_{\tilde{a}}(p);q,\varepsilon_{\tilde{b}}(q);0\Big)\mathcal{V}_{\tilde{c}}^{(-1)A}(p+q)\,, \tag{4.34}$$

where we ignore the overall constant for simplicity.

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Further writing in terms of celestial variables and performing the Mellin transformation, we get

$$\begin{aligned}&\mathcal{W}_{\Delta_1,a\tilde{a}}(x_1)\mathcal{V}^A_{\Delta_2,\tilde{b}}(x_2) &(4.35)\\ &\sim x^a_{12}\frac{x^{\tilde{a}}_{12}\delta^{\tilde{b}\tilde{c}}B(\Delta_1-1,\Delta_2+1)+x^{\tilde{b}}_{12}\delta^{\tilde{a}\tilde{c}}B(\Delta_1,\Delta_2)-x^{\tilde{c}}_{12}\delta^{\tilde{a}\tilde{b}}B(\Delta_1,\Delta_2+1)}{(x_{12})^2}\mathcal{V}^A_{\Delta_1+\Delta_2,\tilde{c}}(x_2)\,.\end{aligned}$$

Graviton/dilaton/KR OPE. Finally we want to discuss the OPE of graviton/dilaton/KR field, namely $W^{(-1)}(z_1,\bar{z}_1)W^{(0)}(z_2,\bar{z}_2)$. The OPE in the left moving sector $W_L(z_1)W_L(z_2)$ is given in (A.27) and it is the same as the left moving sector of closed bosonic string. Up to the boundary contact terms, we can simplify this sector in the same way as that in the bosonic string.

The right-moving OPE $W^{(-1)}_R(\bar{z}_1)W^{(0)}_R(\bar{z}_2) = V^{(-1)}_R(\bar{z}_1)V^{(0)}_R(\bar{z}_2)$ has been discussed above and the final result is given in (4.18). Combining the left and right moving sectors, we arrive at almost the same OPE as that in the closed bosonic string (3.58) except that we need to remove the α' corrections in the right-moving sector (3.60), which are absent in heterotic string due to supersymmetry (4.18). As a result, in heterotic string the OPE (3.68) becomes

$$\mathcal{W}^{(-1)}_{a\tilde{a}}(p)\mathcal{W}^{(0)}_{b\tilde{b}}(q) \sim \mathcal{I}_{c\tilde{c}}\Big(p,\varepsilon_{a\tilde{a}}(p);q,\varepsilon_{b\tilde{b}}(q);\alpha',0\Big)\,\mathcal{W}^{(-1)}_{c\tilde{c}}(p+q)\,. \qquad (4.36)$$

Then one can derive the celestial OPE in the same way as bosonic string. The final result is given by (3.69): ignoring the α'^2 terms in (3.72), and keeping all the terms in (3.70) as well as the α' terms in (3.71) which are proportional to $x^a_{12}x^b_{12}x^c_{12}$.[23]

4.2 Type I, IIA, IIB string

Now we briefly discuss the OPEs of closed string massless fields in type I and IIA/IIB superstring. The vertex operators for massless NS-NS fields in type I and IIA/IIB string are [34]

$$W^{(-1,-1)}(z,\bar{z}) = e^{-\phi-\tilde{\phi}}\psi^\mu\tilde{\psi}^\nu e^{ik\cdot X}\,, \qquad (4.37)$$

$$W^{(0,0)}(z,\bar{z}) = \left(i\partial X^\mu+\frac{1}{2}\alpha' k\cdot\psi\psi^\mu\right)\left(i\bar{\partial}X^\nu+\frac{1}{2}\alpha' k\cdot\tilde{\psi}\tilde{\psi}^\nu\right)e^{ik\cdot X}\,. \qquad (4.38)$$

Now both the left and right-moving sectors are supersymmetric. Note that for type I superstring, the KR 2-form field is projected out and we are only left with graviton and dilaton.

It is simpler to study OPE of two operators in -1 and 0 picture, namely $W^{(-1,-1)}(z_1,\bar{z}_1)\cdot W^{(0,0)}(z_2,\bar{z}_2)$. The vertex operators (4.37), (4.38) can again be decomposed into the product of left and right moving parts. For both the left and right moving sectors, the OPE

[23] More specifically, the terms in the square bracket of (3.71) is now given by:

$$\left[x^a_{12}x^b_{12}x^c_{12}\Big(B(\Delta_1+2,\Delta_2+2)\delta^{\tilde{a}\tilde{b}}x^{\tilde{c}}_{12}-B(\Delta_1+1,\Delta_2+2)\delta^{\tilde{b}\tilde{c}}x^{\tilde{a}}_{12}-B(\Delta_1+2,\Delta_2+1)\delta^{\tilde{a}\tilde{c}}x^{\tilde{b}}_{12}\Big)\right].$$

is given by (4.18) or its conjugate. Combining the two sectors together, the OPE of $W^{(-1,-1)}(z_1,\bar{z}_1)W^{(0,0)}(z_2,\bar{z}_2)$ is similar to that in closed bosonic string (3.59), (3.60) except that we need to replace $\partial X, \bar{\partial}X \to \psi, \tilde{\psi}$ and remove all the α' corrections, which are absent now due to greater supersymmetry. It is amusing that the tachyon also does not appear in the OPE, although we have not performed the GSO projection. The final celestial OPE is given by (3.69) with all α' terms removed. The absence of α' correction is consistent with the string amplitude, as we will see in the next section.

5 Celestial OPE from collinear factorization

In the previous two sections, we have derived the celestial OPEs from the string worldsheet perspective. In this section, we will compute the celestial OPEs using a different method based on the collinear factorization of scattering amplitudes. It turns out that the two approaches give the same result all the time.

Before going to the computational details, let us first describe the general strategy of deriving celestial OPEs based on the collinear factorization. We are interested in the scattering amplitude of massless fields. In the collinear limit two momenta become parallel $p_1//p_2$, and the total momentum $P \equiv p_1 + p_2$ also becomes almost on-shell, namely $P^2 = 2p_1 \cdot p_2 \to 0$. The amplitude then factorizes in the collinear limit as[24]

$$A_{n+1}(p_1^{s_1}, p_2^{s_2}, \cdots) \xrightarrow{p_1//p_2} \sum_s A_3(p_1^{s_1}, p_2^{s_2}, -P^s)\frac{1}{P^2}A_n(P^s, \cdots)\,, \tag{5.1}$$

where s_i are the extra quantum numbers labelling the particles. The prefactor in front of A_n is essentially the split function characterizing the collinear behavior:

$$\mathsf{Split}(p_1^{s_1} + p_2^{s_2} \to P^s) = \frac{1}{P^2}A_3(p_1^{s_1}, p_2^{s_2}, -P^s) = \frac{1}{2p_1 \cdot p_2}A_3(p_1^{s_1}, p_2^{s_2}, -P^s)\,. \tag{5.2}$$

Therefore, once we know the three-point amplitude, we also know the split function. Performing the Mellin transformation then gives the corresponding celestial OPEs.

5.1 Celestial OPE for gluon

In bosonic open string theory, the (color-ordered) amplitude for three gluons is given by [31][25]

$$A^o_{ggg} = e_{1\mu}e_{2\alpha}e_{3\rho}T^{\mu\alpha\rho}(4\alpha')\,, \tag{5.3}$$

while in heterotic string, the three gluon amplitude is [34]

$$A^H_{ggg} = e_{1\mu}e_{2\alpha}e_{3\rho}T^{\mu\alpha\rho}(0)\,, \tag{5.4}$$

[24]Since the two collinear particles are massless, the collinear limit thus singles out the massless field propagator connecting A_3 and A_n. More generally, to identify the contribution from the field with mass M, one can instead use the propagator $1/(P^2 + M^2)$ in (5.2) and take the limit $P^2 \to -M^2$.

[25]Like footnote 11, we set the overall factors in all the string amplitudes to be one. In the final results of celestial OPEs, we will choose proper overall factors to make the formulae as simple as possible. We also use the subscripts g and G for gluon and graviton/dilaton/KR field, respectively.

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where we introduced the tensor

$$T^{\mu\alpha\rho}(\alpha') = p_{23}^{\mu}\,\eta^{\alpha\rho} + p_{31}^{\alpha}\,\eta^{\rho\mu} + p_{12}^{\rho}\,\eta^{\mu\alpha} + \frac{\alpha'}{8} p_{23}^{\mu}\,p_{31}^{\alpha}\,p_{12}^{\rho}\,, \qquad p_{ij} = p_i - p_j\,. \tag{5.5}$$

So up to α' corrections, the three gluon amplitude is the same in bosonic and heterotic string. This is not surprising as the low energy effective field actions of both theories contain the Yang-Mills theory, which is responsible for the leading non-α' amplitude above. The absence of α' correction in the heterotic string is due to supersymmetry, and it is consistent with the absence of α' corrections in the celestial OPEs in heterotic string that we derived before from worldsheet.

Since the heterotic gluon amplitude can be regarded as the special limit of the bosonic one without α' correction, we will just focus on the bosonic string gluon amplitude (5.3). More explicitly, the three gluon amplitude (5.3) can be written as

$$A_{ggg}^{o} = e_1\cdot p_{23}\; e_2\cdot e_3 + e_2\cdot p_{31}\; e_3\cdot e_1 + e_3\cdot p_{12}\; e_1\cdot e_2 + \frac{\alpha'}{2} e_1\cdot p_{23}\; e_2\cdot p_{31}\; e_3\cdot p_{12} \tag{5.6}$$

$$= 2\Big[e_1\cdot p_2 e_2\cdot e_3 - e_2\cdot p_1 e_1\cdot e_3 - e_3\cdot p_2 e_1\cdot e_2 + 2\alpha' e_1\cdot p_2\; e_2\cdot p_1\; e_3\cdot p_2\Big]\,, \tag{5.7}$$

where in the second equality we used momentum conservation $p_1+p_2+p_3=0$ and $e_i\cdot p_i=0$ to simplify.

We choose p_1, p_2 out-going, namely $\eta_1=\eta_2=1$, then p_3 is incoming $\eta_3=-1$. We also choose the polarization vectors as $e_i=\varepsilon_{a_i}(p_i)$. Using (2.4), we have

$$p_i\cdot e_j \equiv p_i\cdot\varepsilon_{a_j}(x_j) = \eta_i\omega_i\hat{p}_i\cdot\varepsilon_{a_j}(x_j) = \eta_i\,\omega_i\, x_{ij}^{a_j}\,, \qquad \varepsilon_{a_i}(x_i)\cdot\varepsilon_{a_j}(x_j) = \delta^{a_i a_j}\,, \tag{5.8}$$

which enables us to rewrite (5.7) as

$$\frac{1}{2}A_{ggg}^{o}(p_i,\varepsilon_{a_i}) = e_1\cdot p_2\; e_2\cdot e_3 - e_2\cdot p_1\; e_3\cdot e_1 - e_3\cdot p_2\; e_1\cdot e_2 + 2\alpha' e_1\cdot p_2\; e_2\cdot p_1\; e_3\cdot p_2 \tag{5.9}$$

$$= \omega_2 x_{21}^{a_1}\delta^{a_2a_3} - \omega_1 x_{12}^{a_2}\delta^{a_1a_3} - \omega_2 x_{23}^{a_3}\delta^{a_1a_2} + 2\alpha'\omega_1\omega_2^2\, x_{21}^{a_1}x_{12}^{a_2}x_{23}^{a_3}\,. \tag{5.10}$$

To have sensible results for collinear factorization, we take p_1, p_2 on-shell but $P\equiv p_1+p_2$ slightly off-shell. Then the momentum conservation $P=p_1+p_2=-p_3+\epsilon v$ is

$$\omega_1\left(\frac{1+(x_1)^2}{2}, x_1^a, \frac{1-(x_1)^2}{2}\right) + \omega_2\left(\frac{1+(x_2)^2}{2}, x_2^a, \frac{1-(x_2)^2}{2}\right) = \omega_3\left(\frac{1+(x_3)^2}{2}, x_3^a, \frac{1-(x_3)^2}{2}\right) + \epsilon v\,, \tag{5.11}$$

where ϵ characterizes the deviation from the strict collinear limit and v is an order one vector. Note $P^2 = 2p_1\cdot p_2 = -\omega_1\omega_2(x_{12})^2 = -\epsilon p_3\cdot v + \mathscr{O}(\epsilon)$, hence $\epsilon\sim(x_{12})^2$. This also introduces $\mathscr{O}(\epsilon)$ uncertainty in the numerator of (5.2), and thus an order one $(x_{12})^0$ uncertainty in the split function. Nevertheless, it would not affect the singular terms in the split function which we are really interested in.

From (5.11), we have

$$\omega_1 x_1^a + \omega_2 x_2^a = \omega_3 x_3^a + \mathscr{O}(\epsilon)\,, \qquad \omega_1+\omega_2 = \omega_3 + \mathscr{O}(\epsilon)\,, \tag{5.12}$$

and thus

$$x_{13}^{a} = \frac{\omega_2}{\omega_1+\omega_2} x_{12}^{a} + \mathscr{O}(\epsilon)\,, \qquad x_{32}^{a} = \frac{\omega_1}{\omega_1+\omega_2} x_{12}^{a} + \mathscr{O}(\epsilon)\,. \tag{5.13}$$

Then (5.10) can be simplified further as

$$\widehat{A}_3(\omega_i,\varepsilon_{a_i},\alpha') = 2\left[\omega_2 x_{21}^{a_1}\delta^{a_2a_3} - \omega_1 x_{12}^{a_2}\delta^{a_1a_3} + \frac{\omega_1\omega_2}{\omega_1+\omega_2} x_{12}^{a_3}\delta^{a_1a_2} + 2\alpha' \frac{\omega_1^2\omega_2^2}{\omega_1+\omega_2}\, x_{12}^{a_1}x_{12}^{a_2}x_{12}^{a_3}\right]. \tag{5.14}$$

Substituting into (5.2), we get the split function

$$\mathsf{Split}(p_1^{a_1}+p_2^{a_2}\to P^{a_3}) = \frac{\widehat{A}_3(\omega_i,\varepsilon_{a_i},\alpha') + \mathscr{O}(\epsilon)}{2p_1\cdot p_2} \tag{5.15}$$

$$= 2\frac{\omega_2 x_{12}^{a_1}\delta^{a_2a_3} + \omega_1 x_{12}^{a_2}\delta^{a_1a_3} - \frac{\omega_1\omega_2}{\omega_1+\omega_2} x_{12}^{a_3}\delta^{a_1a_2} - 2\alpha'\frac{\omega_1^2\omega_2^2}{\omega_1+\omega_2}\, x_{12}^{a_1}x_{12}^{a_2}x_{12}^{a_3}}{\omega_1\omega_2\,(x_{12})^2} + \mathscr{O}(\epsilon^0)\,. \tag{5.16}$$

Up to an overall factor, this is the same as the I introduced in (3.35).

Further performing the Mellin transformation leads to the following celestial OPE for gluons:

$$\begin{aligned}
&\mathcal{O}^{A}_{\Delta_1,a_1}(x_1)\mathcal{O}^{B}_{\Delta_2,a_2}(x_2) \\
&\sim f^{ABC}\frac{x_{12}^{a_1}\delta^{a_2a_3}B(\Delta_1-1,\Delta_2) + x_{12}^{a_2}\delta^{a_1a_3}B(\Delta_1,\Delta_2-1) - x_{12}^{a_3}\delta^{a_1a_2}B(\Delta_1,\Delta_2)}{(x_{12})^2}\mathcal{O}^{C}_{\Delta_1+\Delta_2-1,\,a_3}(x_2) \\
&- 2\alpha' f^{ABC}\frac{x_{12}^{a_1}x_{12}^{a_2}x_{12}^{a_3}B(\Delta_1+1,\Delta_2+1)}{(x_{12})^2}\mathcal{O}^{C}_{\Delta_1+\Delta_2+1,\,a_3}(x_2)\,,
\end{aligned} \tag{5.17}$$

where we restore the color factor f^{ABC} and choose a proper overall normalization to make the formula as simple as possible. Here $\mathcal{O}^{A}_{\Delta,a}$ denotes the celestial gluon operator with the polarization vector ε_a and color index A. This agrees with (3.40) derived from worldsheet.

5.2 Celestial OPE for graviton/dilaton/KR field

In bosonic closed string theory, the amplitude for the closed string massless fields is given by [31]

$$A^{c}_{GGG} = e_{1\mu\nu}e_{2\alpha\beta}e_{3\rho\sigma}T^{\mu\alpha\rho}(\alpha')T^{\nu\beta\sigma}(\alpha')\,. \tag{5.18}$$

So up to the rescaling of α', the closed string amplitude is essentially the "square" of the open string amplitude (5.3). This is the simplest example of the famous KLT relation [35].

In heterotic string, the amplitude for massless fields is [34]

$$A^{H}_{GGG} = e_{1\mu\nu}e_{2\alpha\beta}e_{3\rho\sigma}T^{\mu\alpha\rho}(\alpha')T^{\nu\beta\sigma}(0)\,, \tag{5.19}$$

where the α' corrections in the right moving sector is absent due to supersymmetry. For type I and IIA/IIB string, they have supersymmetry in both left and right moving sectors, and the corresponding massless NS-NS amplitude has no α' corrections at all:

$$A^{\mathrm{I/II}}_{GGG} = e_{1\mu\nu}e_{2\alpha\beta}e_{3\rho\sigma}T^{\mu\alpha\rho}(0)T^{\nu\beta\sigma}(0)\,. \tag{5.20}$$

The absence of α' in three-point amplitude agrees with the absence of α' correction in celestial OPEs we derived before from worldsheet.

Let us just focus on the amplitude of massless fields in closed bosonic string (5.18). As before, we choose p_1, p_2 out-going and p_3 in-coming, namely $\eta_1 = \eta_2 = -\eta_3 = 1$. We also choose the polarization tensors $e_i^{\mu\nu} = \varepsilon^{\mu}_{a_i}\varepsilon^{\nu}_{\tilde{a}_i}$, which are the basis for polarizations. Then we can simplify the three-point amplitude and write it in terms of celestial variables. The steps are almost identical to the open string case in the previous subsection, except for the doubling. The final result is

$$A^c_{GGG} = \widehat{A}_3\left(\omega_i, \varepsilon_{a_i}, \frac{1}{4}\alpha'\right)\widehat{A}_3\left(\omega_i, \varepsilon_{\tilde{a}_i}, \frac{1}{4}\alpha'\right). \tag{5.21}$$

Substituting into (5.2), we get the split function

$$\mathsf{Split}(p_1^{a_1\tilde{a}_1} + p_2^{a_2\tilde{a}_2} \to P^{a_3\tilde{a}_3}) = \frac{A^c_{GGG}}{2p_1\cdot p_2} = -\frac{\widehat{A}_3\left(\omega_i, \varepsilon_{a_i}, \frac{1}{4}\alpha'\right)\widehat{A}_3\left(\omega_i, \varepsilon_{\tilde{a}_i}, \frac{1}{4}\alpha'\right)}{(x_{12})^2\omega_1\omega_2}. \tag{5.22}$$

Up to an overall factor, the split function here is the same as that in (3.66). We then need to perform Mellin transformation to obtain the celestial OPEs. The steps are identical, and the results are also just given by (3.69)–(3.72) except for the replacement $a \to a_1$, $b \to a_2$, $c \to a_3$, $\mathcal{V} \to \mathcal{O}$.

5.3 Celestial OPE for gluon and graviton/dilaton/KR field

In heterotic string, the three-point amplitude involving two gluons and one graviton/dilaton/KR field is [34]

$$A^H_{ggG} = e_{1\mu\nu}e_{2\rho}e_{3\sigma}p^{\mu}_{23}T^{\nu\rho\sigma}(0)\delta^{AB}. \tag{5.23}$$

Note that there is no α' correction due to supersymmetry. In the case of graviton, the interaction responsible for this amplitude is just the minimal coupling between gravitons and gluons. For bosonic string, one can also compute the two gluons and one graviton amplitude from the open-closed string setup. This will be discussed in appendix B and the amplitude is given by (B.15), which suffers α' corrections as we expected. Here we will just focus on the heterotic case (5.23).

Following the same procedure as before, one can derive the celestial OPEs from this three-point amplitude.

We first derive the celestial OPE between gluon and graviton/dilaton/KR field. So we take p_1, p_2 out-going and also choose the polarizations as $e_1^{\mu\nu} = \varepsilon^{\mu}_{a_1}\varepsilon^{\nu}_{\tilde{a}_1}$, $e_2 = \varepsilon_{\tilde{a}_2}(p_2)$, $e_3 = \varepsilon_{\tilde{a}_3}(p_3)$. Then (5.23) becomes

$$A^H_{ggG} = 2\varepsilon_{a_1}\cdot p_2\, \widehat{A}_3(\omega_i, \varepsilon_{\tilde{a}_i}, 0)\delta^{AB}. \tag{5.24}$$

Substituting into (5.2), and performing the Mellin transformation, we get the celestial OPE between gluon and graviton/dilaton/KR field

$$\begin{aligned}&\mathcal{O}_{\Delta_1,a\tilde{a}}(x_1)\mathcal{O}^A_{\Delta_2,\tilde{b}}(x_2)\\ &\sim x^a_{12}\frac{x^{\tilde{a}}_{12}\delta^{\tilde{b}\tilde{c}}B(\Delta_1-1,\Delta_2+1)+x^{\tilde{b}}_{12}\delta^{\tilde{a}\tilde{c}}B(\Delta_1,\Delta_2)-x^{\tilde{c}}_{12}\delta^{\tilde{a}\tilde{b}}B(\Delta_1,\Delta_2+1)}{(x_{12})^2}\mathcal{O}^A_{\Delta_1+\Delta_2,\tilde{c}}(x_2),\end{aligned} \tag{5.25}$$

which agrees with (4.35).

The celestial OPE corresponding to the fusion of two gluons into one graviton/dilaton/KR field can be similarly derived. To make the formula more standard, we exchange the label of 1 and 3 in (5.23). Repeating the same steps above leads us to the following celestial OPE between two gluons

$$\begin{aligned}&\mathcal{O}^A_{\Delta_1,\tilde{a}}(x_1)\mathcal{O}^B_{\Delta_2,\tilde{b}}(x_2)\\ &\sim \delta^{AB}x^c_{12}\frac{x^{\tilde{a}}_{12}\delta^{\tilde{b}\tilde{c}}B(\Delta_1,\Delta_2+1)+x^{\tilde{b}}_{12}\delta^{\tilde{a}\tilde{c}}B(\Delta_1+1,\Delta_2)-x^{\tilde{c}}_{12}\delta^{\tilde{a}\tilde{b}}B(\Delta_1+1,\Delta_2+1)}{(x_{12})^2}\mathcal{O}_{\Delta_1+\Delta_2,\,c\tilde{c}}(x_2)\,,\end{aligned} \tag{5.26}$$

which agrees with the second line of (4.29).

5.4 Celestial OPE in four dimensions

So far, we have derived the celestial OPEs in two different ways. The final formula is a bit complicated. Now we specialize to 4D as a consistency check of our results. In 4D, it is very convenient to use the helicity basis for gluon and graviton. They are related to the previous polarization basis through some linear combinations.

Following (2.20) and (2.21), we define the celestial gluon and graviton operators in the helicity basis as

$$\mathcal{O}_{\Delta,\pm}(\mathsf{z},\bar{\mathsf{z}}) = \frac{1}{\sqrt{2}}\Big(\mathcal{O}_{\Delta,1}(x)\mp i\mathcal{O}_{\Delta,2}(x)\Big)\,, \tag{5.27}$$

$$\mathcal{O}_{\Delta,\pm 2}(\mathsf{z},\bar{\mathsf{z}}) = \frac{1}{2}\Big(\mathcal{O}_{\Delta,11}(x)-\mathcal{O}_{\Delta,22}(x)\mp i\mathcal{O}_{\Delta,12}(x)\mp i\mathcal{O}_{\Delta,21}(x)\Big)\,, \tag{5.28}$$

where the coordinates $\mathsf{z},\bar{\mathsf{z}}$ and x are related through (2.18).

Gluon OPE. The general form of celestial gluon OPEs is given in (5.17). Specializing to 4D and transforming to the helicity basis (5.27), we find that the first line of (5.17) without α' reduces

$$\mathcal{O}^A_{\Delta_1,+}(\mathsf{z}_1,\bar{\mathsf{z}}_1)\mathcal{O}^B_{\Delta_2,+}(\mathsf{z}_2,\bar{\mathsf{z}}_2) \sim \frac{f^{ABC}}{\mathsf{z}_{12}}B(\Delta_1-1,\Delta_2-1)\mathcal{O}^c_{\Delta_1+\Delta_2-1,+}(\mathsf{z}_2,\bar{\mathsf{z}}_2)\,, \tag{5.29}$$

$$\mathcal{O}^A_{\Delta_1,-}(\mathsf{z}_1,\bar{\mathsf{z}}_1)\mathcal{O}^B_{\Delta_2,-}(\mathsf{z}_2,\bar{\mathsf{z}}_2) \sim \frac{f^{ABC}}{\bar{\mathsf{z}}_{12}}B(\Delta_1-1,\Delta_2-1)\mathcal{O}^c_{\Delta_1+\Delta_2-1,-}(\mathsf{z}_2,\bar{\mathsf{z}}_2)\,, \tag{5.30}$$

and

$$\begin{aligned}O^A_{\Delta_1,+}(\mathsf{z}_1,\bar{\mathsf{z}}_1)O^B_{\Delta_2,-}(\mathsf{z}_2,\bar{\mathsf{z}}_2) \sim{}& \frac{f^{ABC}}{\bar{\mathsf{z}}_{12}}B(\Delta_1+1,\Delta_2-1)O^c_{\Delta_1+\Delta_2-1,+}(\mathsf{z}_2,\bar{\mathsf{z}}_2)\\ &+\frac{f^{ABC}}{\mathsf{z}_{12}}B(\Delta_1-1,\Delta_2+1)O^c_{\Delta_1+\Delta_2-1,-}(\mathsf{z}_2,\bar{\mathsf{z}}_2)\,.\end{aligned} \tag{5.31}$$

These are indeed the celestial OPEs for gluon in Yang-Mills theory [18, 21].

The second line of (5.17) with α' coefficient arises from the higher derivative interactions. We can similarly transform it into the helicity basis. In particular, we find that only the following two OPEs have singular terms

$$\mathcal{O}^A_{\Delta_1,+}(\mathsf{z}_1,\bar{\mathsf{z}}_1)\mathcal{O}^B_{\Delta_2,+}(\mathsf{z}_2,\bar{\mathsf{z}}_2) \sim -\alpha' f^{ABC}\frac{\bar{\mathsf{z}}_{12}^2}{\mathsf{z}_{12}}B(\Delta_1+1,\Delta_2+1)\mathcal{O}^C_{\Delta_1+\Delta_2+1,-}(\mathsf{z}_2,\bar{\mathsf{z}}_2)\,, \tag{5.32}$$

$$\mathcal{O}^A_{\Delta_1,-}(\mathsf{z}_1,\bar{\mathsf{z}}_1)\mathcal{O}^B_{\Delta_2,-}(\mathsf{z}_2,\bar{\mathsf{z}}_2) \sim -\alpha' f^{ABC}\frac{\mathsf{z}_{12}^2}{\bar{\mathsf{z}}_{12}}B(\Delta_1+1,\Delta_2+1)\mathcal{O}^C_{\Delta_1+\Delta_2+1,+}(\mathsf{z}_2,\bar{\mathsf{z}}_2)\,, \tag{5.33}$$

while the rest of OPEs are all regular. The two OPEs above with singular terms have exactly the same structure as that predicted by the general formula of OPE in [15, 22]. The rest of helicity configurations only give rise to regular OPEs, as their corresponding amplitudes vanish on-shell. Indeed in 4D, the three-point on-shell amplitudes are fully determined by the helicities due to little group scaling and locality. Each singular OPE above is in one-to-one correspondence with the on-shell three-point amplitudes of gluons arising from either YM theory or higher derivative interactions.

Graviton OPE. Similarly, one can transform the celestial graviton OPE in (3.70) into the helicity basis in 4D. The final result, up to an overall constant, is given by

$$\mathcal{O}_{\Delta_1,+2}(\mathsf{z}_1,\bar{\mathsf{z}}_1)\mathcal{O}_{\Delta_2,+2}(\mathsf{z}_2,\bar{\mathsf{z}}_2) \sim -\frac{\bar{\mathsf{z}}_{12}}{\mathsf{z}_{12}}B(\Delta_1-1,\Delta_2-1)\mathcal{O}_{\Delta_1+\Delta_2,+2}(\mathsf{z}_2,\bar{\mathsf{z}}_2) \tag{5.34}$$

$$-\frac{\alpha'^2}{16}\frac{\bar{\mathsf{z}}_{12}^5}{\mathsf{z}_{12}}B(\Delta_1+3,\Delta_2+3)\mathcal{O}_{\Delta_1+\Delta_2+4,-2}(\mathsf{z}_2,\bar{\mathsf{z}}_2)\,, \tag{5.35}$$

$$\mathcal{O}_{\Delta_1,-2}(\mathsf{z}_1,\bar{\mathsf{z}}_1)\mathcal{O}_{\Delta_2,-2}(\mathsf{z}_2,\bar{\mathsf{z}}_2) \sim -\frac{\mathsf{z}_{12}}{\bar{\mathsf{z}}_{12}}B(\Delta_1-1,\Delta_2-1)\mathcal{O}_{\Delta_1+\Delta_2,-2}(\mathsf{z}_2,\bar{\mathsf{z}}_2) \tag{5.36}$$

$$-\frac{\alpha'^2}{16}\frac{\mathsf{z}_{12}^5}{\bar{\mathsf{z}}_{12}}B(\Delta_1+3,\Delta_2+3)\mathcal{O}_{\Delta_1+\Delta_2+4,+2}(\mathsf{z}_2,\bar{\mathsf{z}}_2)\,, \tag{5.37}$$

$$\mathcal{O}_{\Delta_1,+2}(\mathsf{z}_1,\bar{\mathsf{z}}_1)\mathcal{O}_{\Delta_2,-2}(\mathsf{z}_2,\bar{\mathsf{z}}_2) \sim -\frac{\bar{\mathsf{z}}_{12}}{\mathsf{z}_{12}}B(\Delta_1-1,\Delta_2+3)\mathcal{O}_{\Delta_1+\Delta_2,-2}(\mathsf{z}_2,\bar{\mathsf{z}}_2) \tag{5.38}$$

$$-\frac{\mathsf{z}_{12}}{\bar{\mathsf{z}}_{12}}B(\Delta_1+3,\Delta_2-1)\mathcal{O}_{\Delta_1+\Delta_2,+2}(\mathsf{z}_2,\bar{\mathsf{z}}_2)\,, \tag{5.39}$$

where we have only shown the singular terms. The structures of OPEs are again in perfect agreement with [15, 21, 22]. Here those terms in OPE without α' correspond to the Einstein gravity, while the terms with α'^2 coefficient come from higher derivative interactions of graviton. Note the α' pieces in (3.70) are absent in the above formulae because they are regular. Each singular term in the OPEs above is again in one-to-one correspondence with the on-shell three-point amplitudes of gravitons.

Mixed OPE for gluon and graviton. Now we consider the mixed OPEs involving both gluons and gravitons. The general formula is given by (5.25) and (5.26). Specializing to 4D and writing in the helicity basis, they become

$$\mathcal{O}_{\Delta_1,\pm2}(\mathsf{z}_1,\bar{\mathsf{z}}_1)\mathcal{O}^A_{\Delta_2,\pm}(\mathsf{z}_2,\bar{\mathsf{z}}_2) \sim -\left(\frac{\bar{\mathsf{z}}_{12}}{\mathsf{z}_{12}}\right)^{\pm1}B(\Delta_1-1,\Delta_2)\mathcal{O}^A_{\Delta_1+\Delta_2,\pm}(\mathsf{z}_2,\bar{\mathsf{z}}_2)\,, \tag{5.40}$$

$$\mathcal{O}_{\Delta_1,\pm2}(\mathsf{z}_1,\bar{\mathsf{z}}_1)\mathcal{O}^A_{\Delta_2,\mp}(\mathsf{z}_2,\bar{\mathsf{z}}_2) \sim -\left(\frac{\bar{\mathsf{z}}_{12}}{\mathsf{z}_{12}}\right)^{\pm1}B(\Delta_1-1,\Delta_2+2)\mathcal{O}^A_{\Delta_1+\Delta_2,\mp}(\mathsf{z}_2,\bar{\mathsf{z}}_2)\,, \tag{5.41}$$

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and

$$\mathcal{O}^A_{\Delta_1,+}(\mathsf{z}_1,\bar{\mathsf{z}}_1)\mathcal{O}^B_{\Delta_2,-}(\mathsf{z}_2,\bar{\mathsf{z}}_2) \sim \delta^{AB}\frac{\bar{\mathsf{z}}_{12}}{\mathsf{z}_{12}}B(\Delta_1,\Delta_2+2)\mathcal{O}_{\Delta_1+\Delta_2,-2} + \delta^{AB}\frac{\mathsf{z}_{12}}{\bar{\mathsf{z}}_{12}}B(\Delta_1+2,\Delta_2)\mathcal{O}_{\Delta_1+\Delta_2,+2}\,, \tag{5.42}$$

which are again consistent with [21] and [15, 22].

6 Celestial OPE in $\mathcal{N}=2$ string and $w_{1+\infty}$ algebra

In this section, we will further generalize previous OPE discussions to $\mathcal{N}=2$ string theory [25–27]. The $\mathcal{N}=2$ string theory has four dimensional target spacetime with $(2,2)$ signature. So we will first discuss the kinematics and celestial variables in $(2,2)$ signature. Then as before, we can compute the worldsheet OPE and celestial OPE. An interesting feature is that we can even compute all the $\overline{\mathrm{SL}(2,\mathbb{R})}$ descendants in the OPE,[26] and this essentially comes from the momentum conservation in $(2,2)$ signature. The soft sector of such OPE with descendants just gives the $w_{1+\infty}$ algebra, after rewriting in terms of chiral modes. Therefore, we give an indirect derivation of $w_{1+\infty}$ from worldsheet in $\mathcal{N}=2$ string theory.

6.1 Kinematics in $(2,2)$ signature

Now we consider the spacetime in $(2,2)$ signature. In this case, it is very convenient to introduce complex coordinates $\mathcal{X}^1=(X^1+iX^2)/\sqrt{2}, \mathcal{X}^2=(X^3+iX^4)/\sqrt{2}$ as well as their complex conjugates. The metric is then

$$ds^2=(dX^1)^2+(dX^2)^2-(dX^3)^2-(dX^4)^2=\eta_{\mu\nu}d\mathcal{X}^\mu d\mathcal{X}^\nu=2d\mathcal{X}^1d\bar{\mathcal{X}}^1-2d\mathcal{X}^2d\bar{\mathcal{X}}^2\,. \tag{6.1}$$

We will write the vector as

$$\boldsymbol{A}=(A^1,A^2,A^{\bar{1}},A^{\bar{2}})\,. \tag{6.2}$$

We will also use the Greek indices $\mu,\nu,\dots=1,\bar{1},2,\bar{2}$, unbarred indices $i,j,\dots=1,2$, and barred indices $\bar{i},\bar{j},\dots=\bar{1},\bar{2}$. Note we will identify $\bar{A}^i\equiv\bar{A}^{\bar{i}}\equiv A^{\bar{i}}$. For real vector, $A^{\bar{i}}$ is just the complex conjugate of A^i.

The metric $\eta_{\mu\nu}$ is given by

$$\eta_{i\bar{j}}=\eta_{\bar{j}i}=\eta^{i\bar{j}}=\eta^{\bar{j}i}=(-1)^{i+1}\delta^{ij}\,,\qquad \eta_{ij}=\eta_{\bar{i}\bar{j}}=\eta^{ij}=\eta^{\bar{i}\bar{j}}=0\,. \tag{6.3}$$

We introduce the following notation

$$A\cdot\bar{B}=\bar{B}\cdot A=A^1\bar{B}^1-A^2\bar{B}^2\,, \tag{6.4}$$

where $A=(A^1,A^2), \bar{B}=(B^{\bar{1}},B^{\bar{2}})$.

[26] This is just half of the Lorentz group. More precisely $\mathrm{SL}(2,\mathbb{R})\times\overline{\mathrm{SL}(2,\mathbb{R})}$ is the double cover of $\mathrm{SO}(2,2)$ which is the Lorentz group in $(2,2)$ signature.

Given two vectors $\boldsymbol{A}, \boldsymbol{B}$, we can define their inner product as

$$\boldsymbol{B} \cdot \boldsymbol{A} = \boldsymbol{A} \cdot \boldsymbol{B} = \eta_{\mu\nu} A^\mu B^\nu = A \cdot \bar{B} + B \cdot \bar{A} = A^1 \bar{B}^1 - A^2 \bar{B}^2 + B^1 \bar{A}^1 - B^2 \bar{A}^2 \,. \tag{6.5}$$

This is real for real vectors A, B.

Given the vector in (6.2), we also define its dual $\boldsymbol{A}^\vee$ as

$$\boldsymbol{A} = (A^1, A^2, A^{\bar{1}}, A^{\bar{2}}) \,, \qquad \boldsymbol{A}^\vee = (A^1, A^2, -A^{\bar{1}}, -A^{\bar{2}}) \,. \tag{6.6}$$

Note that $\boldsymbol{A}^\vee$ is an imaginary vector as $(A^\vee)^{\bar{i}} = -((A^\vee)^i)^*$, if $\boldsymbol{A}$ is a real vector. And we have the inner product

$$\boldsymbol{A}^\vee \cdot \boldsymbol{B} = A \cdot \bar{B} - B \cdot \bar{A} = A^1 \bar{B}^1 - A^2 \bar{B}^2 - B^1 \bar{A}^1 + B^2 \bar{A}^2 \,, \tag{6.7}$$

which is purely imaginary for real vectors A, B. They satisfy $\boldsymbol{A} \cdot \boldsymbol{B} = -\boldsymbol{A}^\vee \cdot \boldsymbol{B}^\vee$, $\boldsymbol{A}^\vee \cdot \boldsymbol{B} = -\boldsymbol{B}^\vee \cdot \boldsymbol{A}$.

We are particularly interested in the null momentum satisfying $\boldsymbol{k} \cdot \boldsymbol{k} = 0$. It can then be parametrized as

$$\boldsymbol{k} = \omega_k \, \hat{\boldsymbol{k}} \,, \qquad \omega_k \in \mathbb{R} \,. \tag{6.8}$$

Here the null vector is

$$\hat{\boldsymbol{k}}(\mathsf{z}, \bar{\mathsf{z}}) = \Big(1 + \mathsf{z}\bar{\mathsf{z}} + i(\mathsf{z} - \bar{\mathsf{z}}),\; 1 - \mathsf{z}\bar{\mathsf{z}} + i(\mathsf{z} + \bar{\mathsf{z}}),\; 1 + \mathsf{z}\bar{\mathsf{z}} - i(\mathsf{z} - \bar{\mathsf{z}}),\; 1 - \mathsf{z}\bar{\mathsf{z}} - i(\mathsf{z} + \bar{\mathsf{z}})\Big) \,, \tag{6.9}$$

where $\mathsf{z}, \bar{\mathsf{z}}$ are the coordinates on the celestial torus [36], instead of the celestial sphere. It is worth emphasizing that the two variables $\mathsf{z}, \bar{\mathsf{z}}$ are real and independent, instead of the complex conjugate of each other.

Then the polarizations are given by

$$\begin{aligned} \boldsymbol{\varepsilon}_+ &= -\frac{i}{2} \partial_{\mathsf{z}} \hat{\boldsymbol{k}} = \frac{1}{2}(1 - i\bar{\mathsf{z}}, 1 + i\bar{\mathsf{z}}, -1 - i\bar{\mathsf{z}}, -1 + i\bar{\mathsf{z}}) \,, &(6.10)\\ \boldsymbol{\varepsilon}_- &= -\frac{i}{2} \partial_{\bar{\mathsf{z}}} \hat{\boldsymbol{k}} = \frac{1}{2}(-1 - i\mathsf{z}, 1 + i\mathsf{z}, 1 - i\mathsf{z}, -1 + i\mathsf{z}) \,. &(6.11)\end{aligned}$$

They satisfy

$$\boldsymbol{\varepsilon}_+ \cdot \boldsymbol{\varepsilon}_+ = \boldsymbol{\varepsilon}_- \cdot \boldsymbol{\varepsilon}_- = \hat{\boldsymbol{k}} \cdot \hat{\boldsymbol{k}} = \boldsymbol{\varepsilon}_+ \cdot \hat{\boldsymbol{k}} = \boldsymbol{\varepsilon}_- \cdot \hat{\boldsymbol{k}} = 0 \,, \qquad \boldsymbol{\varepsilon}_+ \cdot \boldsymbol{\varepsilon}_- = 1 \,. \tag{6.12}$$

For different momenta with $\hat{\boldsymbol{k}}_i = \hat{\boldsymbol{k}}(\mathsf{z}_i, \bar{\mathsf{z}}_i)$, we have the identities

$$\hat{\boldsymbol{k}}_i \cdot \hat{\boldsymbol{k}}_j = 4\, \mathsf{z}_{ij}\, \bar{\mathsf{z}}_{ij} \,, \qquad \hat{\boldsymbol{k}}_i \cdot \boldsymbol{\varepsilon}_{+j} = 2i\, \bar{\mathsf{z}}_{ij} \,, \qquad \hat{\boldsymbol{k}}_i \cdot \boldsymbol{\varepsilon}_{-j} = 2i\, \mathsf{z}_{ij} \,, \tag{6.13}$$

and

$$\boldsymbol{\varepsilon}_{\pm i} \cdot \boldsymbol{\varepsilon}_{\pm j} = 0 \,, \qquad \boldsymbol{\varepsilon}_{\pm i} \cdot \boldsymbol{\varepsilon}_{\mp j} = 1 \,. \tag{6.14}$$

For null momentum in (6.8), (6.9), its dual satisfies

$$\boldsymbol{k}^\vee = \omega_k \hat{\boldsymbol{k}}^\vee = \omega_k \Big(2(1 + \mathsf{z}^2) \boldsymbol{\varepsilon}_+ + i\, \mathsf{z}\, \hat{\boldsymbol{k}}\Big) \,, \tag{6.15}$$

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so up to a gauge transformation and an overall factor, $\boldsymbol{k}^\vee$ is essentially the positive helicity polarization vector.

We also have[27]

$$\hat{\boldsymbol{k}}_i \cdot \hat{\boldsymbol{k}}_j = -\hat{\boldsymbol{k}}_i^\vee \cdot \hat{\boldsymbol{k}}_j^\vee = 4\mathsf{z}_{ij}\bar{\mathsf{z}}_{ij}\,, \qquad \hat{\boldsymbol{k}}_i^\vee \cdot \hat{\boldsymbol{k}}_j = -\hat{\boldsymbol{k}}_j^\vee \cdot \hat{\boldsymbol{k}}_i = -4i(1+\mathsf{z}_i\,\mathsf{z}_j)\bar{\mathsf{z}}_{ij}\,. \tag{6.16}$$

6.2 Vertex operator in $\mathcal{N}=2$ string theory

The $\mathcal{N}=2$ string theory is constructed from the $\mathcal{N}=2$ non-linear σ-model [25]

$$S = \frac{1}{\pi}\int d^2z\, d^2\theta\, d^2\bar\theta\; K(\mathscr{X},\bar{\mathscr{X}})\,, \tag{6.17}$$

where $\mathscr{X}^i$ are $\mathcal{N}=2$ chiral superfields

$$\mathscr{X}^i(Z,\bar Z;\theta^-,\bar\theta^-) = \mathcal{X}^i(Z,\bar Z) + \psi_L^i(Z,\bar Z)\theta^- + \psi_R^i(Z,\bar Z)\bar\theta^- + F^i(Z,\bar Z)\bar\theta^-\theta^-\,, \quad Z = z-\theta^+\theta^-\,, \tag{6.18}$$

and $K(\mathscr{X},\bar{\mathscr{X}}) = \mathscr{X}\cdot\bar{\mathscr{X}} = \mathscr{X}^1\bar{\mathscr{X}}^{\bar 1} - \mathscr{X}^2\bar{\mathscr{X}}^{\bar 2}$ is the Kähler potential for flat metric. More explicitly in terms of component fields, the action reads

$$S = \frac{1}{\pi}\int d^2z\, \Big(\partial\mathcal{X}\cdot\bar\partial\bar{\mathcal{X}} + \bar\partial\mathcal{X}\cdot\partial\bar{\mathcal{X}} + \bar\psi_L\cdot\bar\partial\psi_L + \bar\psi_R\cdot\partial\psi_R + \bar F\cdot F\Big)\,. \tag{6.19}$$

Thus we have $F=0$ and the following OPEs[28]

$$\mathcal{X}^i(z_1,\bar z_1)\bar{\mathcal{X}}^{\bar j}(z_2,\bar z_2) \sim -\eta^{i\bar j}\ln|z_{12}|\,, \qquad \psi_L^i(z_1)\bar\psi_L^{\bar j}(z_2) \sim \frac{\eta^{i\bar j}}{z_{12}}\,, \qquad \psi_R^i(\bar z_1)\bar\psi_R^{\bar j}(\bar z_2) \sim \frac{\eta^{i\bar j}}{\bar z_{12}}\,. \tag{6.20}$$

The critical dimension of $\mathcal{N}=2$ string is four [25, 37]. In order to have $\mathcal{N}=2$ supersymmetry on the world-sheet, the target spacetime should be endowed with a complex structure, implying that the signature of spacetime can be either $(4,0)$ or $(2,2)$. In the former $(4,0)$ case, $\mathcal{N}=2$ string only has ground state as the physical state in the first-quantized string, and is thus not interesting. For the latter $(2,2)$ case and in the simplest version of $\mathcal{N}=2$ string, there is a massless field in the spectrum, and it obeys a non-linear differential equation. The actual Lorentz group of $\mathcal{N}=2$ string is $\mathrm{U}(1,1)\simeq\mathrm{U}(1)\times\mathrm{SU}(1,1)$, instead of $\mathrm{SO}(2,2)$ [25]. Note that $\mathrm{SU}(1,1)$ is also isomorphic to $\mathrm{SL}(2,\mathbb{R})$.

The vertex operator for the massless field in $\mathcal{N}=2$ string theory is given by [25]

$$\widehat V(k) = \int d^2\theta\, d^2\bar\theta\; e^{i(k\cdot\bar{\mathscr{X}}+\bar k\cdot\mathscr{X})} \tag{6.21}$$

$$= \Big(ik\cdot\partial\bar{\mathcal{X}} - i\bar k\cdot\partial\mathcal{X} - k\cdot\bar\psi_L\,\bar k\cdot\psi_L\Big)\Big(ik\cdot\bar\partial\bar{\mathcal{X}} - i\bar k\cdot\bar\partial\mathcal{X} - k\cdot\bar\psi_R\,\bar k\cdot\psi_R\Big)e^{i(k\cdot\bar{\mathcal{X}}+\bar k\cdot\mathcal{X})}\,. \tag{6.22}$$

[27]It is worth mentioning that although $\hat{\boldsymbol{k}}^\vee$ is essentially the polarization vector, their inner products are non-vanishing, in contrast to the vanishing inner product of two polarizations with the same helicity in (6.14). The non-vanishing $\hat{\boldsymbol{k}}_i^\vee\cdot\hat{\boldsymbol{k}}_j^\vee$ is just due to the gauge transformation (6.15) and (6.13).

[28]Note we set $\alpha'=1$ here.

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This looks similar to the vertex operator (4.38) in type II string, up to the choice of polarization. Indeed we can write (6.22) as

$$\widehat{V}(\boldsymbol{k}) = \left(i\boldsymbol{k}^\vee \cdot \partial\boldsymbol{\mathcal{X}} - \frac{1}{2}(\boldsymbol{k}^\vee \cdot \boldsymbol{\psi}_L)(\boldsymbol{k}\cdot\boldsymbol{\psi}_L)\right)\left(i\boldsymbol{k}^\vee \cdot \bar{\partial}\boldsymbol{\mathcal{X}} - \frac{1}{2}(\boldsymbol{k}^\vee \cdot \boldsymbol{\psi}_R)(\boldsymbol{k}\cdot\boldsymbol{\psi}_R)\right) e^{i\boldsymbol{k}\cdot\boldsymbol{\mathcal{X}}}, \quad (6.23)$$

where $\boldsymbol{k}^\vee$ is the dual of $\boldsymbol{k}$ defined in (6.6). As shown in (6.15), the vector $\boldsymbol{k}^\vee$ is essentially the positive helicity polarization vector, up to a gauge transformation and an overall rescaling.

Using this vertex operator, the three-point amplitude was computed, while higher point amplitudes vanish [25]. See the review [27] for other aspects of $\mathcal{N}=2$ string theory.

6.3 OPE in $\mathcal{N}=2$ string theory

We would like to compute the OPE of vertex operators (6.22), or equivalently (6.23). The steps are similar to the previous computations, and the final result is given in (A.68):

$$\begin{aligned}\widehat{V}(\boldsymbol{k}_1)\widehat{V}(\boldsymbol{k}_2) \sim \frac{1}{4}\left(\boldsymbol{k}_1^\vee\cdot\boldsymbol{k}_2\right)^2 |z_{12}|^{\boldsymbol{k}_1\cdot\boldsymbol{k}_2-2} e^{i\boldsymbol{K}_3\cdot\boldsymbol{\mathcal{X}}} \qquad (6.24)\\ \times\left[i\boldsymbol{K}_3^\vee\cdot\partial\boldsymbol{\mathcal{X}} - \frac{1}{2}(\boldsymbol{K}_3^\vee\cdot\boldsymbol{\psi}_L\ \boldsymbol{K}_3\cdot\boldsymbol{\psi}_L) + \mathscr{O}(\boldsymbol{k}_1\cdot\boldsymbol{k}_2) + \mathscr{O}(\boldsymbol{k}_1^\vee\cdot\boldsymbol{k}_2^\vee) + \mathscr{O}(z_{12})\right]\\ \times\left[i\boldsymbol{K}_3^\vee\cdot\bar{\partial}\boldsymbol{\mathcal{X}} - \frac{1}{2}(\boldsymbol{K}_3^\vee\cdot\boldsymbol{\psi}_R\ \boldsymbol{K}_3\cdot\boldsymbol{\psi}_R) + \mathscr{O}(\boldsymbol{k}_1\cdot\boldsymbol{k}_2) + \mathscr{O}(\boldsymbol{k}_1^\vee\cdot\boldsymbol{k}_2^\vee) + \mathscr{O}(\bar{z}_{12})\right],\end{aligned}$$

where $\boldsymbol{K}_3 = \boldsymbol{k}_1 + \boldsymbol{k}_2$. It is easy to recognize that we just get $\widehat{V}(\boldsymbol{K}_3)$ on the right-hand side. Taking the collinear limit $\boldsymbol{k}_1\cdot\boldsymbol{k}_2 \to 0$ and using the identity (3.55), (6.16), we arrive at

$$\begin{aligned}\widehat{V}(\boldsymbol{k}_1)\widehat{V}(\boldsymbol{k}_2) &\sim \frac{\pi}{2}\delta^2(z_{12}) e^{i\boldsymbol{k}_3\cdot\boldsymbol{\mathcal{X}}_L}\frac{\left(\boldsymbol{k}_1^\vee\cdot\boldsymbol{k}_2\right)^2}{\boldsymbol{k}_1\cdot\boldsymbol{k}_2}\left[\widehat{V}(\boldsymbol{K}_3) + \mathscr{O}(\boldsymbol{k}_1\cdot\boldsymbol{k}_2) + \mathscr{O}(z_{12},\bar{z}_{12})\right] \qquad (6.25)\\ &\sim -2\pi\delta^2(z_{12})\omega_1\omega_2\frac{(1+\mathsf{z}_1\mathsf{z}_2)^2(\bar{\mathsf{z}}_{12})^2}{\mathsf{z}_{12}\bar{\mathsf{z}}_{12}}\left[\widehat{V}(\boldsymbol{K}_3) + \mathscr{O}(z_{12},\bar{z}_{12}) + \mathscr{O}((\mathsf{z}_{12}\bar{\mathsf{z}}_{12})^1)\right], \qquad (6.26)\end{aligned}$$

where $\mathscr{O}(\boldsymbol{k}_1\cdot\boldsymbol{k}_2) = \mathscr{O}((\mathsf{z}_{12}\bar{\mathsf{z}}_{12})^1)$ with z_{12} and $\bar{\mathsf{z}}_{12}$ coming together in a product form.

Performing the integration over z_1, z_2 on the worldsheet, we obtain

$$\widehat{\mathcal{V}}(\omega_1,\mathsf{z}_1,\bar{\mathsf{z}}_1)\widehat{\mathcal{V}}(\omega_2,\mathsf{z}_2,\bar{\mathsf{z}}_2) \sim -2\pi\omega_1\omega_2(1+\mathsf{z}_1\mathsf{z}_2)^2\,\frac{\bar{\mathsf{z}}_{12}}{\mathsf{z}_{12}}\left[\widehat{\mathcal{V}}(\omega_3,\mathsf{z}_3,\bar{\mathsf{z}}_3) + \mathscr{O}((\mathsf{z}_{12}\bar{\mathsf{z}}_{12})^1)\right], \quad (6.27)$$

where we used $\boldsymbol{k}_3 = \omega_3\hat{\boldsymbol{k}}(\mathsf{z}_3,\bar{\mathsf{z}}_3)$ is approximately the same as $\boldsymbol{K}_3 = \boldsymbol{k}_1 + \boldsymbol{k}_2$ in the collinear limit, up to order $\mathscr{O}((\mathsf{z}_{12}\bar{\mathsf{z}}_{12})^1)$ corrections. The coefficient here is a little awkward. This is because we were using the vertex operators where the polarization has an extra factor (6.15): $\boldsymbol{k}^\vee \simeq 2\omega_k(1+\mathsf{z}^2)\varepsilon_+$, up to gauge transformation. To make the equation nicer and make contact with the standard convention, we redefine the vertex operators as follows

$$\widehat{\mathcal{V}}(\omega,\mathsf{z},\bar{\mathsf{z}}) = 2\pi\,\omega^2(1+\mathsf{z}^2)^2\,\mathcal{V}(\omega,\mathsf{z},\bar{\mathsf{z}}). \quad (6.28)$$

Then (6.27) takes a nicer form

$$\mathcal{V}(\omega_1,\mathsf{z}_1,\bar{\mathsf{z}}_1)\mathcal{V}(\omega_2,\mathsf{z}_2,\bar{\mathsf{z}}_2) \sim -\frac{(\omega_1+\omega_2)^2}{\omega_1\omega_2}\frac{\bar{\mathsf{z}}_{12}}{\mathsf{z}_{12}}\left[\mathcal{V}(\omega_3,\mathsf{z}_3,\bar{\mathsf{z}}_3) + \mathscr{O}((\mathsf{z}_{12}\bar{\mathsf{z}}_{12})^1) + \mathscr{O}(\mathsf{z}_{12})\right]. \quad (6.29)$$

In the collinear limit, we can just set $\mathsf{z}_3, \bar{\mathsf{z}}_3 \to \mathsf{z}_2, \bar{\mathsf{z}}_2, \omega_3 \to \omega_1 + \omega_2$ as they are close on the celestial sphere and the energy is additive. Performing the Mellin transformation, we get the celestial OPE:

$$\mathcal{V}_{\Delta_1}(\mathsf{z}_1, \bar{\mathsf{z}}_1)\mathcal{V}_{\Delta_2}(\mathsf{z}_2, \bar{\mathsf{z}}_2) \sim -\frac{\bar{\mathsf{z}}_{12}}{\mathsf{z}_{12}} B(\Delta_1 - 1, \Delta_2 - 1)\, \mathcal{V}_{\Delta_1+\Delta_2}(\mathsf{z}_2, \bar{\mathsf{z}}_2)\,. \tag{6.30}$$

This just unsurprisingly reproduces the celestial OPE of two gravitons both with positive helicity (5.34).

Actually, in the current situation, we can go further. In the previous sections, we were considering the spacetime in Minkowski signature and correspondingly the celestial sphere is in Euclidean signature. This is different from the current $(2, 2)$ split signature in a subtle but interesting way. In 4D Minkowski spacetime with $(1, 3)$ signature, three particles satisfying momentum conservation can not become on-shell simultaneously, unless all momenta are strictly parallel and point along the same direction. This constraint is relaxed in $(2, 2)$ signature. Given two null momenta $\boldsymbol{k}_1, \boldsymbol{k}_2$ and $\boldsymbol{K}_3 = \boldsymbol{k}_1 + \boldsymbol{k}_2$, the equation (6.13) tells us $\boldsymbol{K}_3^2 = 2\boldsymbol{k}_1 \cdot \boldsymbol{k}_2 \propto \mathsf{z}_{12}\, \bar{\mathsf{z}}_{12}$. So we can make $\boldsymbol{K}_3$ null by just setting $\mathsf{z}_{12} = 0$ or $\bar{\mathsf{z}}_{12} = 0$, without forcing the strict alignment of $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$. In particular, in $(2, 2)$ signature, z_{12} and $\bar{\mathsf{z}}_{12}$ are independent. We will approach the collinear limit $\boldsymbol{k}_1 \cdot \boldsymbol{k}_2 \to 0$ by setting $\mathsf{z}_1 \to \mathsf{z}_2$, while keeping $\bar{\mathsf{z}}_{12}$ arbitrary. In particular, using (6.9), the momentum conservation $\boldsymbol{k}_3 = \boldsymbol{k}_1 + \boldsymbol{k}_2 + \mathscr{O}(\mathsf{z}_{12})$ implies

$$\omega_3 = \omega_1 + \omega_2 + \mathscr{O}(\mathsf{z}_{12})\,, \quad \mathsf{z}_3 = \mathsf{z}_1 + \mathscr{O}(\mathsf{z}_{12}) = \mathsf{z}_2 + \mathscr{O}(\mathsf{z}_{12})\,, \quad \bar{\mathsf{z}}_3 = \bar{\mathsf{z}}_2 + \frac{\omega_1}{\omega_1 + \omega_2}\bar{\mathsf{z}}_{12} + \mathscr{O}(\mathsf{z}_{12})\,. \tag{6.31}$$

Now let us reconsider (6.29). Since we are only interested in terms at leading order in z_{12}, this means we just need to consider z_{12}^0 terms inside the square bracket of (6.29). Therefore we can just set $\mathsf{z}_3 = \mathsf{z}_2$ in $\mathcal{V}(\omega_3, \mathsf{z}_3, \bar{\mathsf{z}}_3)$ and furthermore use the relation in (6.31). As a result, (6.29) becomes

$$\mathcal{V}(\omega_1, \mathsf{z}_1, \bar{\mathsf{z}}_1)\mathcal{V}(\omega_2, \mathsf{z}_2, \bar{\mathsf{z}}_2) \sim -\frac{(\omega_1 + \omega_2)^2}{\omega_1\omega_2}\frac{\bar{\mathsf{z}}_{12}}{\mathsf{z}_{12}}\left[\mathcal{V}\left(\omega_1 + \omega_2, \mathsf{z}_2, \bar{\mathsf{z}}_2 + \frac{\omega_1}{\omega_1 + \omega_2}\bar{\mathsf{z}}_{12}\right) + \mathscr{O}(\mathsf{z}_{12})\right] \tag{6.32}$$

$$\sim -\sum_{n=0}^{\infty}\frac{1}{n!}\omega_1^{-1+n}\omega_2^{-1}(\omega_1 + \omega_2)^{2-n}\frac{\bar{\mathsf{z}}_{12}^{n+1}}{\mathsf{z}_{12}}\bar{\partial}^n\mathcal{V}(\omega_3, \mathsf{z}_2, \bar{\mathsf{z}}_2) + \mathscr{O}((\mathsf{z}_{12})^0)\,, \tag{6.33}$$

where we essentially performed a Taylor expansion in the second line. Performing the Mellin transformation gives

$$\mathcal{V}_{\Delta_1}(\mathsf{z}_1, \bar{\mathsf{z}}_1)\mathcal{V}_{\Delta_2}(\mathsf{z}_2, \bar{\mathsf{z}}_2) \sim -\sum_{n=0}^{\infty}\frac{1}{n!}\frac{\bar{\mathsf{z}}_{12}^{n+1}}{\mathsf{z}_{12}}\bar{\partial}^n\mathcal{V}_{\Delta_1+\Delta_2}(\mathsf{z}_2, \bar{\mathsf{z}}_2)B(\Delta_1 + n - 1, \Delta_2 - 1)\,, \tag{6.34}$$

which is exact to all orders in $\bar{\mathsf{z}}_{12}$ but to leading order in z_{12}. This coincides with the celestial OPE of two positive helicity gravitons including all the $\overline{\mathrm{SL}(2, \mathbb{R})}$ descendant contributions [13, 15].

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6.4 Descendant in OPE from momentum conservation

In the previous discussions, we derive the celestial OPE for the massless field in $\mathcal{N}=2$ string with all $\overline{\mathrm{SL}(2,\mathbb{R})}$ descendant contributions included. In this subsection, we want to show that this is a general feature and can be easily generalized to all celestial OPEs in $(2,2)$ signature.

As explained, in $(2,2)$ signature we can have three momenta conserved and null without fully pointing along the same direction. In particular we can vary $\mathsf{z}_{12}, \bar{\mathsf{z}}_{12}$ independently. In the collinear limit, the OPE of two operators has the following general structure

$$O(\omega_1,\mathsf{z}_1,\bar{\mathsf{z}}_1)O(\omega_2,\mathsf{z}_2,\bar{\mathsf{z}}_2)\sim S(\omega_1,\omega_2,\mathsf{z}_{12},\bar{\mathsf{z}}_{12})\Big[O(\omega_3,\mathsf{z}_3,\bar{\mathsf{z}}_3)+\mathscr{O}(\mathsf{z}_{12}\bar{\mathsf{z}}_{12})\Big]\,, \tag{6.35}$$

where $\omega_3, \mathsf{z}_3, \bar{\mathsf{z}}_3$ parametrize the null momentum $\boldsymbol{k}_3$ which is approximately the total momentum $\boldsymbol{k}_1+\boldsymbol{k}_2$, The difference between $\boldsymbol{k}_1+\boldsymbol{k}_2$ and $\boldsymbol{k}_3$ vanishes in the strict collinear limit and accounts for the $\mathscr{O}(\mathsf{z}_{12}\bar{\mathsf{z}}_{12})$ uncertainty in the bracket. We would like to realize the collinear limit such that $\mathsf{z}_{12}\to 0$ while keeping $\bar{\mathsf{z}}_{12}$ arbitrary. And in the bracket of (6.35), we only keep the $(\mathsf{z}_{12})^0$ term. This enables us to set $\mathsf{z}_3\to\mathsf{z}_2$ at this order. The momentum conservation in the collinear limit (6.31) further enables us to write $\omega_3, \bar{\mathsf{z}}_3$ in terms of $\omega_1,\omega_2,\bar{\mathsf{z}}_1,\bar{\mathsf{z}}_2$ exactly in $\bar{\mathsf{z}}$ direction but at leading order $\mathscr{O}((\mathsf{z}_{12})^0)$ in z direction. As a consequence, (6.35) finally reduces to

$$O(\omega_1,\mathsf{z}_1,\bar{\mathsf{z}}_1)O(\omega_2,\mathsf{z}_2,\bar{\mathsf{z}}_2)\sim S(\omega_1,\omega_2,\mathsf{z}_{12},\bar{\mathsf{z}}_{12})\left[O\left(\omega_1+\omega_2,\mathsf{z}_2,\bar{\mathsf{z}}_2+\frac{\omega_1}{\omega_1+\omega_2}\bar{\mathsf{z}}_{12}\right)+\mathscr{O}(\mathsf{z}_{12})\right]\,, \tag{6.36}$$

where S only depends on $\mathsf{z}_{12}, \bar{\mathsf{z}}_{12}$ through their differences due to the translational invariance on the celestial sphere/torus. In general we can assume that $S(\omega_1,\omega_2,\mathsf{z}_{12},\bar{\mathsf{z}}_{12})$ has power law dependence on energy

$$S(\omega_1,\omega_2,\mathsf{z}_{12},\bar{\mathsf{z}}_{12})=\omega_1^{\alpha}\omega_2^{\beta}(\omega_1+\omega_2)^{\gamma}T(\mathsf{z}_{12},\bar{\mathsf{z}}_{12})\,, \tag{6.37}$$

and thus

$$S(t\omega,(1-t)\omega,\mathsf{z}_{12},\bar{\mathsf{z}}_{12})=t^{\alpha}(1-t)^{\beta}\,\omega^{\alpha+\beta+\gamma}T(\mathsf{z}_{12},\bar{\mathsf{z}}_{12})\,. \tag{6.38}$$

Performing the Mellin transformation and using the following identity

$$\int d\omega_1\,d\omega_2\,\omega_1^{\Delta_1-1}\omega_2^{\Delta_2-1}=\int_0^1 dt\int d\omega\,\omega^{\Delta_1+\Delta_2-1}t^{\Delta_1-1}(1-t)^{\Delta_2-1}\,,\quad t=\frac{\omega_1}{\omega_1+\omega_2}\,,\quad \omega=\omega_1+\omega_2\,, \tag{6.39}$$

respectively on two sides of (6.36), we obtain

$$\begin{aligned}\mathcal{O}_{\Delta_1}(\mathsf{z}_1,\bar{\mathsf{z}}_1)\mathcal{O}_{\Delta_2}(\mathsf{z}_2,\bar{\mathsf{z}}_2)&\sim\int_0^1 dt\int d\omega\,t^{\Delta_1+\alpha-1}(1-t)^{\Delta_2+\beta-1}\,\omega^{\Delta_1+\Delta_2+\alpha+\beta+\gamma-1}T(\mathsf{z}_{12},\bar{\mathsf{z}}_{12})\mathcal{O}(\omega,\mathsf{z}_2,\bar{\mathsf{z}}_2+t\bar{\mathsf{z}}_{12})\\&\sim T(\mathsf{z}_{12},\bar{\mathsf{z}}_{12})\int_0^1 dt\,\mathcal{O}_{\Delta_1+\Delta_2+\alpha+\beta+\gamma}(\mathsf{z}_2,\bar{\mathsf{z}}_2+t\bar{\mathsf{z}}_{12})t^{\Delta_1+\alpha-1}(1-t)^{\Delta_2+\beta-1}\\&\sim T(\mathsf{z}_{12},\bar{\mathsf{z}}_{12})\sum_{n=0}^{\infty}\frac{(\bar{\mathsf{z}}_{12})^n}{n!}\bar{\partial}^n\mathcal{O}_{\Delta_1+\Delta_2+\alpha+\beta+\gamma}(\mathsf{z}_2,\bar{\mathsf{z}}_2)B(\Delta_1+\alpha+n,\Delta_2+\beta)\,.\end{aligned} \tag{6.40}$$

So once we work out the primary operators in the OPEs, namely the leading $n = 0$ term in the above expansion, then we may read off α, β, γ. The formula above enables us to include all $n > 0$ contributions, and thus compute all the $\overline{\mathrm{SL}(2,\mathbb{R})}$ descendants.

Our derivation above essentially relies on the momentum conservation in $(2,2)$ signature.[29] The same type of formula was derived before from the conformal symmetry in celestial CFT. There the formula for the OPE with $\overline{\mathrm{SL}(2,\mathbb{R})}$ descendants is given by [13, 15]

$$\begin{aligned}&\mathcal{O}_{\Delta_1,J_1}(\mathsf{z}_1,\bar{\mathsf{z}}_1)\mathcal{O}_{\Delta_2,J_2}(\mathsf{z}_2,\bar{\mathsf{z}}_2)\\ &\sim \mathcal{N}_{\mathcal{O}_1\mathcal{O}_2}^{\mathcal{O}_3}\frac{\bar{\mathsf{z}}_{12}^{N-M}}{\mathsf{z}_{12}^{M+N}}\int_0^1 dt\, \mathcal{O}_{\Delta_3,J_3}(\mathsf{z}_2,\bar{\mathsf{z}}_2+t\bar{\mathsf{z}}_{12})\, t^{\Delta_1-J_1-M+N-1}(1-t)^{\Delta_2-J_2-M+N-1}\,,\end{aligned} \tag{6.41}$$

where

$$M = \frac{\Delta_1+\Delta_2-\Delta_3}{2}\,, \qquad N = \frac{J_1+J_2-J_3}{2}\,. \tag{6.42}$$

One easily sees that they are very similar. It is also easy to verify that the descendant contributions in (6.40) are consistent with the general OPE formula with descendants in [15].

6.5 $w_{1+\infty}$ algebra from OPE

We have now derived the celestial OPE (6.34) including all the $\overline{\mathrm{SL}(2,\mathbb{R})}$ descendants in $\mathcal{N} = 2$ string theory. Now we focus on the soft sector, namely the set of operators with special integral dimensions. More specifically, we define the soft current as [13–15]

$$H^l(\mathsf{z},\bar{\mathsf{z}}) = \lim_{\Delta\to l}(\Delta - l)\mathcal{O}_\Delta(\mathsf{z},\bar{\mathsf{z}})\,, \qquad l = 2,1,0,\cdots\,. \tag{6.43}$$

The soft currents can be further decomposed into chiral currents [13–15]

$$H^l(\mathsf{z},\bar{\mathsf{z}}) = \sum_{n=1-i}^{i-1}\bar{\mathsf{z}}^{i-n-1}\frac{\mathcal{H}^i_n(\mathsf{z})}{(i-n-1)!(i+n-1)!}\,, \qquad l = 4-2i\,, \tag{6.44}$$

where the range of indices are

$$i = 1,\ \frac{3}{2},\ 2,\ \cdots, \qquad n = 1-i,\ 2-i,\ \cdots,\ i-1\,. \tag{6.45}$$

There are thus $2i-1$ chiral currents $\mathcal{H}^i_n(\mathsf{z})$ which transform under the $(2i-1)$-dimensional representation of $\overline{\mathrm{SL}(2,\mathbb{R})}$. After doing some algebraic manipulations, one can show that (6.34) gives rise to the following chiral OPEs [14, 15]

$$\mathcal{H}^i_n(\mathsf{z})\,\mathcal{H}^j_m(0) \sim -\frac{2}{\mathsf{z}}\Big(m(i-1)-n(j-1)\Big)\mathcal{H}^{i+j-2}_{n+m}(0)\,. \tag{6.46}$$

In terms of commutators, they are

$$[\mathcal{H}^i_n,\ \mathcal{H}^j_m] = -2\Big(m(i-1)-n(j-1)\Big)\mathcal{H}^{i+j-2}_{n+m}\,. \tag{6.47}$$

[29] Translational symmetries were also used in [28] and [22] to constrain the descendants in OPE purely from the CCFT perspective.

This just gives the $w_{1+\infty}$ algebra, or more precisely, the loop algebra of the wedge algebra of $w_{1+\infty}$ algebra [14].

Therefore, we have derived the $w_{1+\infty}$ algebra in $\mathcal{N}=2$ string theory. Note that the appearance of $w_{1+\infty}$ in $\mathcal{N}=2$ string is not accidental. At classical level, $w_{1+\infty}$ appears as the symmetry group of self-dual gravity in $(2,2)$ signature [29], where the only degree of freedom is Kahler potential. The quantization of this theory is just given by the $\mathcal{N}=2$ string [25].

Here we obtain the $w_{1+\infty}$ algebra by first deriving the celestial OPE (6.34) with all $\overline{\mathrm{SL}(2,\mathbb{R})}$ descendants from worldsheet OPE in $\mathcal{N}=2$ string theory, and then performing the mode expansion into chiral currents. However, our construction is indirect. It would be desirable to construct directly the generators $\mathcal{H}^i_n$ from the worldsheet, and then show that they satisfy the $w_{1+\infty}$ algebra (6.47). Let us add some comments here. One can indeed perform the Mellin transformation (2.39) directly on the vertex operator (6.23) and obtain the conformal vertex operator. It contains several terms each with factor $\Gamma(\Delta+s_i)(-i\hat{\boldsymbol{k}}\cdot\boldsymbol{\mathcal{X}})^{-\Delta-s_i}$, $s_i\in\mathbb{Z}$. Since $\Gamma(\Delta+s_i)$ has a pole when $\Delta+s_i\in\mathbb{Z}_{\leq 0}$, the definition of soft current in (6.43) indeed gives meaningful result when $l=-s_i,-s_i-1,\cdots$. The soft currents are then essentially some polynomials of $(-i\hat{\boldsymbol{k}}\cdot\boldsymbol{\mathcal{X}})$. However, the range of index l seems to not work exactly as we expected. Moreover, it is not clear how to decompose the resulting soft currents into chiral currents (6.44).[30] A detailed and better understanding is needed to solve these confusions, and we leave the direct construction to the future.

7 Conclusion and outlook

To summarize, in this paper we provide an approach to deriving celestial OPEs from the worldsheet in string theory. Our results are corroborated by the collinear factorization of string amplitudes, and are applicable to general dimensions, corresponding to Einstein-Yang-Mills theory with possible higher derivative corrections. For $\mathcal{N}=2$ string theory, we also obtain the descendant contributions in the celestial OPE, whose soft sector leads to the $w_{1+\infty}$ symmetry. The connection between celestial sphere and string worldsheet initiated in this paper may finally help us to find the microscopic celestial CFT dual for string theory. Besides this ambitious goal, various questions about celestial OPEs remain to be further studied.

First of all, our results of celestial OPEs include the α' corrections but not the quantum correction as we were only focusing on the tree level amplitude. It would be interesting to derive the string loop corrections to the celestial OPEs. The derivation from worldsheet at loop level seems to be much more complicated; in particular, one needs to take into account the integration over the moduli space of Riemann surface. Nevertheless, a simpler approach may be considering the factorization of string amplitudes at loop level and then performing Mellin transformation.

[30] Due to the relation (6.9), (6.10), the soft currents are also polynomials in both z and $\bar{\mathsf{z}}$. However, (6.44) shows that the soft currents are supposed to be Laurent series in z.

Even at tree level, it is still not clear how to derive the celestial OPE corresponding to the fusion of two gluon operators from the open string into a graviton operator from the closed string. Although we sidestepped this question by going to heterotic string and obtained the desired celestial OPEs, it is still conceptually very important to derive such an OPE from the open-closed setup.

Moreover, our derivation in this paper is mostly about the primary operators in the OPEs, although we discussed the descendants in $\mathcal{N}=2$ string theory. It would be very interesting to understand how to systematically incorporate the descendant contribution in the OPEs. Since the descendants are fully determined by symmetry, the more basic question may be how to implement various symmetries on the celestial sphere through some worldsheet generators.

Furthermore, it would be important to study the vertex operators in the conformal basis directly. In our current derivation, we first derive the OPEs of worldsheet vertex operators in momentum space, and then Mellin transform to the conformal basis. Although the momentum space offers a bridge, it makes the connection between celestial sphere and worldsheet less transparent. So understanding the vertex operators and their OPEs directly in conformal basis would be very useful.

Last but not least, the $w_{1+\infty}$ symmetry seems to be a very interesting outcome in the study of celestial holography. Although the $w_{1+\infty}$ symmetry is a nice feature of generic gravitational theories, it may be also useful to have a more direct and deeper understanding of such an infinite dimensional symmetry in $\mathcal{N}=2$ string theory. In this paper we derived the $w_{1+\infty}$ symmetry based on the celestial OPEs with descendants purely from string theory, but the origin of $w_{1+\infty}$ symmetry is not clear. A direct construction of $w_{1+\infty}$ generators from the worldsheet would make the symmetry more transparent and also enables us to discover the implications of such an infinite dimensional symmetry algebra. The study of $\mathcal{N}=2$ string theory and self-dual gravity is particularly interesting, as the self-duality of these gravitational theories suggests the chirality of the dual celestial conformal field theory. The chiral nature and the infinite dimensional symmetry bring lots of simplifications, and may finally enable us to find the CCFT dual of $\mathcal{N}=2$ string theory or self-dual gravity.

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A OPE on the string worldsheet

In this appendix, we give all the details of computing the worldsheet OPEs in various string theories.

A.1 OPE in open bosonic string

Let us first consider the OPEs in open bosonic string theory. We will assume that all X^μ satisfy the Neumann boundary conditions and the basic OPE is given by[31]

$$X^\mu(y_1)X^\nu(y_2) \sim -2\alpha'\eta^{\mu\nu}\ln|y_1-y_2|\,. \tag{A.1}$$

From now on, we will assume $y_1 > y_2$ and thus the absolute value symbol in (A.1) can be removed.

Taking derivatives of (A.1), we get

$$\dot{X}^\mu(y_1)X^\nu(y_2) \sim -2\alpha'\eta^{\mu\nu}\frac{1}{y_{12}}\,, \qquad \dot{X}^\mu(y_1)\dot{X}^\nu(y_2) \sim -2\alpha'\eta^{\mu\nu}\frac{1}{y_{12}^2}\,, \tag{A.2}$$

where $\dot{X}(y) = \partial_y X(y)$.

Now, we can compute the OPEs of various composite operators made out of X. The most important formula in our OPE computations is the following identity in free field theory

$$:e^A::e^B:=e^{\langle AB\rangle}:e^{A+B}:, \tag{A.3}$$

where A, B are collections of annihilation and creations operators of free fields, and $:\cdots:$ denotes the normal order product.

OPE without derivative. In particular, we can take A, B in (A.3) as the free fields themselves and then obtain

$$:e^{ip\cdot X(y_1)}::e^{iq\cdot X(y_2)}:=y_{12}^{2\alpha' p\cdot q}:e^{ip\cdot X(y_1)+iq\cdot X(y_2)}:, \tag{A.4}$$

where

$$:e^{ip\cdot X(y_1)+iq\cdot X(y_2)}:=:e^{ip\cdot X(y_1)}e^{iq\cdot X(y_2)}:=\sum_{n=0}^{\infty}\frac{y_{12}^n}{n!}:[\partial^n e^{ip\cdot X}]e^{iq\cdot X}(y_2): \tag{A.5}$$

$$=\sum_{n=0}^{\infty}\frac{y_{12}^n}{n!}:[e^{-ip\cdot X}\partial^n e^{ip\cdot X}]e^{i(p+q)\cdot X}(y_2): \tag{A.6}$$

$$=:\left(1+y_{12}\,ip\cdot\dot{X}+\frac{y_{12}^2}{2}\left(-(p\cdot\dot{X})^2+ip\cdot\ddot{X}\right)+\cdots\right)e^{i(p+q)\cdot X}(y_2):\,. \tag{A.7}$$

As a result, we have

$$e^{ip\cdot X(y_1)}e^{iq\cdot X(y_2)}=y_{12}^{2\alpha' p\cdot q}\left[1+y_{12}\,ip\cdot\dot{X}+\frac{y_{12}^2}{2}\Big(-(p\cdot\dot{X})^2+ip\cdot\ddot{X}\Big)+\cdots\right]e^{i(p+q)\cdot X}(y_2)\,, \tag{A.8}$$

where we have removed the normal ordering symbol for simplicity of notation.[32]

[31] Let us also recall the weights of various operators $h(\partial_y^\ell X(y)) = \ell$, $h(e^{ik\cdot X}) = \alpha' k^2$. Although being non-primary, X formally has weight 0.

[32] Rigorously speaking, different orders of operators inside the normal order product usually give different results. But in our free field OPEs and for the vertex operators under consideration, the difference does not matter. In particular, $:e^{ik\cdot X}\dot{X}:$ and $:\dot{X}e^{ik\cdot X}:$ only differ by $\partial e^{ik\cdot X}$, which is a total derivative and is thus inessential after integration. For this reason, we will not be rigorous about the ordering of operators.

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By further taking derivatives in (A.4), we similarly get the following OPE

$$\partial_{y_1} e^{ip\cdot X(y_1)} \partial_{y_2} e^{iq\cdot X(y_2)} \sim -y_{12}^{2\alpha' p\cdot q-2} e^{i(p+q)\cdot X} \times \Big[2\alpha' p\cdot q(2\alpha' p\cdot q-1) + 2i\alpha' y_{12}\, p\cdot q\Big(-q\cdot \dot{X} + 2\alpha' p\cdot q\, p\cdot \dot{X}\Big) + \cdots\Big](y_2)\,. \tag{A.9}$$

OPE with one derivative. We are also interested in the OPEs of operators involving derivatives $\dot{X}$. This can be done by using the following trick. We can regard $\dot{X}$ as arising from the Taylor expansion of the exponentiation of free fields, namely

$$\dot{X}^\mu e^{ip\cdot X} = -i\frac{\partial}{\partial \zeta_\mu} e^{ip\cdot X + i\zeta\cdot \dot{X}}\Big|_{\zeta=0}\,. \tag{A.10}$$

Since the OPEs of free field exponential operators can be computed using (A.3), we can thus also easily obtain OPEs of composite operators involving $\dot{X}$. This can also be easily generalized to composite operators with multiplet $\dot{X}$ or even higher derivatives $\ddot{X}$, etc.

Using this trick, we can compute the OPE of vertex operators for tachyon and gluon as follows:

$$e^{ip\cdot X}(y_1)\dot{X}^\nu e^{iq\cdot X}(y_2) = -i\frac{\partial}{\partial \xi_\nu} e^{ip\cdot X}(y_1) e^{iq\cdot X + i\xi\cdot \dot{X}}(y_2)\Big|_{\xi=0}\,. \tag{A.11}$$

The OPE on the right hand side can be evaluated using (A.3)

$$e^{ip\cdot X}(y_1) e^{iq\cdot X + i\xi\cdot\dot{X}}(y_2) \tag{A.12}$$
$$= e^{2\alpha' p\cdot q \ln y_{12} - 2\alpha' \xi\cdot p/y_{12}} : e^{ip\cdot X(y_1) + iq\cdot X(y_2)} e^{i\xi\cdot \dot{X}(y_2)} : \tag{A.13}$$
$$\begin{aligned}
&= e^{2\alpha' p\cdot q \ln y_{12} - 2\alpha' \xi\cdot p/y_{12}} : \Big(1 + i\xi\cdot \dot{X}(y_2) + \cdots\Big) e^{ip\cdot X(y_1) + iq\cdot X(y_2)} : \\
&= y_{12}^{2\alpha' p\cdot q}\Big(1 - 2\alpha' \xi\cdot p/y_{12} + \cdots\Big) : \Big(1 + i\xi\cdot \dot{X}(y_2) + \cdots\Big)\Big(1 + y_{12}\, ip\cdot \dot{X}(y_2) + \cdots\Big) e^{i(p+q)\cdot X(y_2)} : \\
&= y_{12}^{2\alpha' p\cdot q-1}\Big(-2\alpha' \xi\cdot p + iy_{12}\xi\cdot \dot{X} + \cdots\Big)\Big(1 + y_{12}\, ip\cdot \dot{X} + \cdots\Big) e^{i(p+q)\cdot X}(y_2) \\
&= y_{12}^{2\alpha' p\cdot q-1}\Big(-2\alpha' \xi\cdot p(1 + y_{12}\, ip\cdot \dot{X}) + iy_{12}\xi\cdot \dot{X}(y_2) + \cdots\Big) e^{i(p+q)\cdot X}(y_2)
\end{aligned}$$

Then the OPE in (A.11) is given by

$$e^{ip\cdot X}(y_1)\dot{X}^\nu e^{iq\cdot X}(y_2) \sim y_{12}^{2\alpha' p\cdot q-1}\Big(2i\,\alpha' p^\nu + y_{12}(\dot{X}^\nu - 2\alpha' p^\nu p\cdot \dot{X}) + \cdots\Big) e^{i(p+q)\cdot X(y_2)}\,. \tag{A.14}$$

OPE with two derivatives. Using the same trick, we can compute the vertex operators of two gluons as follows

$$\dot{X}^\mu e^{ip\cdot X}(y_1)\dot{X}^\nu e^{iq\cdot X}(y_2) = -\frac{\partial}{\partial \zeta_\mu}\frac{\partial}{\partial \xi_\nu} e^{ip\cdot X + i\zeta\cdot\dot{X}}(y_1) e^{iq\cdot X + i\xi\cdot\dot{X}}(y_2)\Big|_{\zeta=\xi=0}\,. \tag{A.15}$$

Using the identity (A.3), the OPE on the right-hand-side can be calculated:

$$e^{ip\cdot X+i\zeta\cdot\dot{X}}(y_1)e^{iq\cdot X+i\xi\cdot\dot{X}}(y_2) \tag{A.16}$$

$$=e^{2\alpha' p\cdot q\ln y_{12}+2\alpha'(\zeta\cdot q-\xi\cdot p)/y_{12}+2\alpha'\zeta\cdot\xi/y_{12}^2}:e^{ip\cdot X(y_1)+iq\cdot X(y_2)}e^{i\zeta\cdot\dot{X}(y_1)}e^{i\xi\cdot\dot{X}(y_2)}: \tag{A.17}$$

$$=e^{2\alpha' p\cdot q\ln y_{12}+2\alpha'(\zeta\cdot q-\xi\cdot p)/y_{12}+2\alpha'\zeta\cdot\xi/y_{12}^2}:\Big(1+i\zeta\cdot\dot{X}(y_1)+\cdots\Big)\Big(1+i\xi\cdot\dot{X}(y_2)+\cdots\Big)e^{ip\cdot X(y_1)+iq\cdot X(y_2)}:$$

$$=y_{12}^{2\alpha' p\cdot q}\Big(1+2\alpha'(\zeta\cdot q-\xi\cdot p)/y_{12}+2\alpha'\zeta\cdot\xi/y_{12}^2+2\alpha'^2(\zeta\cdot q-\xi\cdot p)^2/y_{12}^2+\cdots\Big)$$
$$\times:\Big(1+i\zeta\cdot\dot{X}(y_2)+y_{12}i\zeta\cdot\ddot{X}(y_2)+\cdots\Big)\Big(1+i\xi\cdot\dot{X}(y_2)+\cdots\Big)e^{ip\cdot X(y_1)+iq\cdot X(y_2)}:$$

$$=y_{12}^{2\alpha' p\cdot q}:\left[\frac{1}{y_{12}^2}\Big(2\alpha'\zeta\cdot\xi-4\alpha'^2\zeta\cdot q\xi\cdot p\Big)+\frac{2i\alpha'}{y_{12}}\Big(\zeta\cdot q\xi\cdot\dot{X}(y_2)-\xi\cdot p\zeta\cdot\dot{X}(y_2)\Big)+\cdots\right]$$
$$\times\Big(1+iy_{12}p\cdot\dot{X}(y_2)+\cdots\Big)e^{i(p+q)\cdot X}:(y_2)$$

$$=y_{12}^{2\alpha' p\cdot q-2}e^{i(p+q)\cdot X}\left[\Big(2\alpha'\zeta\cdot\xi-4\alpha'^2\zeta\cdot q\xi\cdot p\Big)\Big(1+iy_{12}p\cdot\dot{X}\Big)+2i\alpha' y_{12}\Big(\zeta\cdot q\xi\cdot\dot{X}-\xi\cdot p\zeta\cdot\dot{X}\Big)+\cdots\right](y_2).$$

where the dots represent terms which are quadratic or higher order in ξ, ζ or in y_{12}.

Combining this OPE and (A.15), we then obtain

$$\zeta\cdot\dot{X}e^{ip\cdot X}(y_1)\xi\cdot\dot{X}e^{iq\cdot X}(y_2)\sim -y_{12}^{2\alpha' p\cdot q-2}e^{i(p+q)\cdot X}\Big[2\alpha'\Big(\zeta\cdot\xi-2\alpha'\zeta\cdot q\,\xi\cdot p\Big)$$
$$+2i\alpha'\,y_{12}\Big(\zeta\cdot q\,\xi\cdot\dot{X}-\xi\cdot p\,\zeta\cdot\dot{X}+(\zeta\cdot\xi-2\alpha'\zeta\cdot q\,\xi\cdot p)p\cdot\dot{X}\Big)+\cdots\Big](y_2), \tag{A.18}$$

or equivalently

$$\dot{X}^\mu e^{ip\cdot X}(y_1)\dot{X}^\nu e^{iq\cdot X}(y_2)\sim -y_{12}^{2\alpha' p\cdot q-2}e^{i(p+q)\cdot X}\Big[2\alpha'\Big(\eta^{\mu\nu}-2\alpha' q^\mu\,p^\nu\Big) \tag{A.19}$$
$$+2i\alpha'\,y_{12}\Big(q^\mu\,\dot{X}^\nu-p^\nu\,\dot{X}^\mu+(\eta^{\mu\nu}-2\alpha' q^\mu\,q^\nu)p\cdot\dot{X}\Big)+\cdots\Big](y_2).$$

As a consistency check, we can set $\zeta=ip,\xi=iq$, then (A.18) reduces to (A.9) as expected.

A.2 OPE in closed bosonic string

Now we switch to the closed bosonic string. It has the basic OPE

$$X^\mu(z_1,\bar{z}_1)X^\nu(z_2,\bar{z}_2)\sim -\frac{\alpha'}{2}\eta^{\mu\nu}\ln|z_{12}|^2\,. \tag{A.20}$$

It is more convenient to separate X^μ into the left and right movers:

$$X^\mu(z,\bar{z})=X_L^\mu(z)+X_R^\mu(\bar{z})\,. \tag{A.21}$$

The two sectors are independent and have the following OPEs

$$X_L^\mu(z_1)X_L^\nu(z_2)\sim -\frac{\alpha'}{2}\eta^{\mu\nu}\ln z_{12}\,,\quad X_R^\mu(\bar{z}_1)X_R^\nu(\bar{z}_2)\sim -\frac{\alpha'}{2}\eta^{\mu\nu}\ln\bar{z}_{12}\,,\quad X_L^\mu(z_1)X_R^\nu(\bar{z}_2)\sim 0\,. \tag{A.22}$$

In general any vertex operator in closed string theory can be decomposed into the product of the left and right moving pieces. So to compute the OPE of two vertex operators in closed string theory, one just needs to compute the OPEs in the left and right moving sectors independently, and then take their product. In both the left and right moving sectors, the free OPE (A.22) is almost identical to the open string case (A.1), except for the reduction of α' by a factor of 4.[33]

We are particularly interested in massless fields in closed string, whose vertex operators are given by

$$V^{\mu\bar{\mu}}(z,\bar{z}) = \partial X^{\mu}\bar{\partial}X^{\bar{\mu}}e^{ip\cdot X}(z,\bar{z})\,. \tag{A.23}$$

We can decompose it into the left-moving and right-moving parts:

$$V^{\mu\bar{\mu}}(z,\bar{z}) = V_L^{\mu}(z)V_R^{\bar{\mu}}(z), \quad V_L^{\mu}(z) = \partial X_L e^{ip\cdot X_L} = \partial X e^{ip\cdot X_L}, \quad V_R^{\bar{\mu}}(z) = \bar{\partial}X_R e^{ip\cdot X_R} = \bar{\partial}X e^{ip\cdot X_R}\,. \tag{A.24}$$

We would like to compute the OPE of two such vertex operators, namely

$$\partial X^{\mu}\bar{\partial}X^{\bar{\mu}}e^{ip\cdot X}(z_1,\bar{z}_1)\partial X^{\nu}\bar{\partial}X^{\bar{\nu}}e^{ip\cdot X}(z_1,\bar{z}_1) = V^{\mu\bar{\mu}}(z_1,\bar{z}_1)V^{\nu\bar{\nu}}(z_2,\bar{z}_2) \tag{A.25}$$
$$= V_L^{\mu}(z_1)V_L^{\nu}(z_2)\times V_R^{\bar{\mu}}(\bar{z}_1)V_R^{\bar{\nu}}(\bar{z}_2)\,. \tag{A.26}$$

As described before, we can compute the OPEs in the left and right moving sectors independently. The computation of OPE in the left moving sector is exactly the same as that in the open string case (A.19), and the final result is given by

$$V_L^{\mu}(z_1)V_L^{\nu}(z_2) = \partial X_L^{\mu}e^{ip\cdot X_L}(z_1)\partial X_L^{\nu}e^{iq\cdot X_L}(z_2) \tag{A.27}$$
$$\sim -\frac{\alpha'}{2}z_{12}^{\frac{1}{2}\alpha' p\cdot q-2}e^{i(p+q)\cdot X_L}\left[\left(\eta^{\mu\nu}-\frac{1}{2}\alpha' q^{\mu}\,p^{\nu}\right)\right.$$
$$\left.+i\,z_{12}\left(q^{\mu}\,\partial X^{\nu}-p^{\nu}\,\partial X^{\mu}+\left(\eta^{\mu\nu}-\frac{1}{2}\alpha' q^{\mu}\,q^{\nu}\right)p\cdot\partial X\right)+\cdots\right](z_2)\,. \tag{A.28}$$

The right moving OPE is similar. After combining the left and right moving OPEs together, we get the world-sheet OPE of two vertex operators for massless fields in closed string theory:

$$\partial X^{\mu}\bar{\partial}X^{\bar{\mu}}e^{ip\cdot X}(z_1,\bar{z}_1)\partial X^{\nu}\bar{\partial}X^{\bar{\nu}}e^{ip\cdot X}(z_2,\bar{z}_2) \tag{A.29}$$
$$\sim \frac{\alpha'^2}{4}|z_{12}|^{\alpha' p\cdot q-4}e^{i(p+q)\cdot X}$$
$$\times\left[\left(\eta^{\mu\nu}-\frac{1}{2}\alpha' q^{\mu}\,p^{\nu}\right)+i\,z_{12}\left(q^{\mu}\,\partial X^{\nu}-p^{\nu}\,\partial X^{\mu}+\left(\eta^{\mu\nu}-\frac{1}{2}\alpha' q^{\mu}\,q^{\nu}\right)p\cdot\partial X\right)+\cdots\right]$$
$$\times\left[\left(\eta^{\bar{\mu}\bar{\nu}}-\frac{1}{2}\alpha' q^{\bar{\mu}}\,p^{\bar{\nu}}\right)+i\,\bar{z}_{12}\left(q^{\bar{\mu}}\,\partial X^{\bar{\nu}}-p^{\bar{\nu}}\,\bar{\partial}X^{\bar{\mu}}+\left(\eta^{\bar{\mu}\bar{\nu}}-\frac{1}{2}\alpha' q^{\bar{\mu}}\,q^{\nu}\right)p\cdot\bar{\partial}X\right)+\cdots\right](z_2,\bar{z}_2)\,.$$

[33]In particular, the weights of $e^{ik\cdot X}$ are also reduced by 4, namely $h(e^{ik\cdot X}) = \bar{h}(e^{ik\cdot X}) = \frac{\alpha'}{4}k^2$.

A.3 OPE in heterotic string

Now we consider the OPE in heterotic string. The bosonic fields are the same as that in (A.21) and (A.22). The new ingredient is the right moving fermions on the worldsheet which have the following OPE

$$\tilde{\psi}^{\mu}(\bar{z}_1)\tilde{\psi}^{\nu}(\bar{z}_2) \sim \frac{\eta^{\mu\nu}}{\bar{z}_{12}}\,. \tag{A.30}$$

We would like to compute the following right moving OPE:

$$V_R^{(-1)}(\bar{z}_1)V_R^{(0)}(\bar{z}_2) = e^{-\tilde{\phi}}\tilde{\psi}^{\mu}e^{ip\cdot X_R}(\bar{z}_1)\left(i\bar{\partial}X^{\nu}+\frac{1}{2}\alpha' k\cdot\tilde{\psi}\tilde{\psi}^{\nu}\right)e^{iq\cdot X_R}(\bar{z}_2)\,. \tag{A.31}$$

For this purpose, let us first present the following two OPEs

$$\tilde{\psi}^{\mu}(\bar{z}_1)k\cdot\tilde{\psi}\tilde{\psi}^{\nu}(\bar{z}_2) \sim \frac{k^{\mu}\tilde{\psi}^{\nu}(\bar{z}_2)-\eta^{\mu\nu}k\cdot\tilde{\psi}(\bar{z}_2)}{\bar{z}_{12}}\,, \tag{A.32}$$

and

$$e^{ip\cdot X_R}(\bar{z}_1)\bar{\partial}X^{\nu}e^{iq\cdot X_R}(\bar{z}_2) \sim \bar{z}_{12}^{\frac{1}{2}\alpha' p\cdot q-1}\left(\frac{i}{2}\alpha' p^{\nu}-\frac{1}{2}\alpha'\bar{z}_{12}p^{\nu}p\cdot\bar{\partial}X+\bar{z}_{12}\bar{\partial}X^{\nu}+\cdots\right)e^{i(p+q)\cdot X}(\bar{z}_2)\,, \tag{A.33}$$

which can be obtained as in the open string case (A.14).

Then the OPE in (A.31) can be evaluated straightforwardly

$$\tilde{\psi}^{\mu}e^{ip\cdot X_R}(\bar{z}_1)\left(i\bar{\partial}X^{\nu}+\frac{1}{2}\alpha' q\cdot\tilde{\psi}\tilde{\psi}^{\nu}\right)e^{iq\cdot X_R}(\bar{z}_2) \tag{A.34}$$

$$= i\tilde{\psi}^{\mu}e^{ip\cdot X_R}(\bar{z}_1)\bar{\partial}X^{\nu}e^{iq\cdot X_R}(\bar{z}_2)+\frac{1}{2}\alpha'\tilde{\psi}^{\mu}(\bar{z}_1)q\cdot\tilde{\psi}\tilde{\psi}^{\nu}(\bar{z}_2)\times e^{ip\cdot X_R}(\bar{z}_1)e^{iq\cdot X_R}(\bar{z}_2) \tag{A.35}$$

$$= i\tilde{\psi}^{\mu}\bar{z}_{12}^{\frac{1}{2}\alpha' p\cdot q-1}\left(\frac{i}{2}\alpha' p^{\nu}-\frac{1}{2}\alpha'\bar{z}_{12}p^{\nu}p\cdot\bar{\partial}X+\bar{z}_{12}\bar{\partial}X^{\nu}+\cdots\right)e^{i(p+q)\cdot X_R(\bar{z}_2)} \tag{A.36}$$

$$+\frac{1}{2}\alpha'\left(\frac{q^{\mu}\tilde{\psi}^{\nu}(\bar{z}_2)-\eta^{\mu\nu}q\cdot\tilde{\psi}(\bar{z}_2)}{\bar{z}_{12}}+\cdots\right)\bar{z}_{12}^{\frac{1}{2}\alpha' p\cdot q}\left(1+\bar{z}_{12}\,ip\cdot\bar{\partial}X+\cdots\right)e^{i(p+q)\cdot X(\bar{z}_2)} \tag{A.37}$$

$$= \bar{z}_{12}^{\frac{1}{2}\alpha' p\cdot q-1}e^{i(p+q)\cdot X_R}\left(\frac{i}{2}\alpha' i\tilde{\psi}^{\mu}p^{\nu}-\frac{1}{2}\alpha'\bar{z}_{12}i\tilde{\psi}^{\mu}p^{\nu}p\cdot\dot{X}+\bar{z}_{12}i\tilde{\psi}^{\mu}\dot{X}^{\nu}+\cdots\right. \tag{A.38}$$

$$\left.+\frac{1}{2}\alpha' q^{\mu}\tilde{\psi}^{\nu}-\frac{1}{2}\alpha'\eta^{\mu\nu}q\cdot\tilde{\psi}+\cdots\right)(\bar{z}_2) \tag{A.39}$$

$$= \bar{z}_{12}^{\frac{1}{2}\alpha' p\cdot q-1}e^{i(p+q)\cdot X_R}\left(\frac{1}{2}\alpha'\left[q^{\mu}\tilde{\psi}^{\nu}-p^{\nu}\tilde{\psi}^{\mu}-\eta^{\mu\nu}q\cdot\tilde{\psi}\right]+\cdots\right)(\bar{z}_2)\,. \tag{A.40}$$

So the final result for the OPE in (A.31) after contracting with polarization vectors is

$$\zeta\cdot\tilde{\psi}e^{ip\cdot X_R}(\bar{z}_1)\left(i\xi\cdot\bar{\partial}X+\frac{1}{2}\alpha' q\cdot\tilde{\psi}\xi\cdot\tilde{\psi}\right)e^{iq\cdot X_R}(\bar{z}_2) \tag{A.41}$$

$$\sim\frac{\alpha'}{2}\bar{z}_{12}^{\frac{1}{2}\alpha' p\cdot q-1}e^{i(p+q)\cdot X_R}\Big(\zeta\cdot q\,\xi\cdot\tilde{\psi}-\xi\cdot p\,\zeta\cdot\tilde{\psi}-\zeta\cdot\xi\,q\cdot\tilde{\psi}\Big)(\bar{z}_2)\,.$$

A.4 OPE in $\mathcal{N}=2$ string

For $\mathcal{N}=2$ string, we have bosonic fields $\mathcal{X}^\mu$ and fermionic fields ψ^μ_L, ψ^μ_R. We can agin decompose $\mathcal{X}^\mu$ into left and right movers

$$\mathcal{X}^\mu(z,\bar{z}) = \mathcal{X}^\mu_L(z) + \mathcal{X}^\mu_R(\bar{z})\,, \tag{A.42}$$

which obey the OPEs [25]

$$\mathcal{X}^i_L(z_1)\bar{\mathcal{X}}^{\bar{j}}_L(z_2) \sim -\frac{\eta^{i\bar{j}}}{2}\ln(z_1-z_2)\,, \qquad \mathcal{X}^i_R(\bar{z}_1)\bar{\mathcal{X}}^{\bar{j}}_R(\bar{z}_2) \sim -\frac{\eta^{i\bar{j}}}{2}\ln(\bar{z}_1-\bar{z}_2)\,. \tag{A.43}$$

The OPEs for fermions are [25]

$$\psi^i_L(z_1)\bar{\psi}^{\bar{j}}_L(z_2) \sim \frac{\eta^{i\bar{j}}}{z_1-z_2}\,, \qquad \psi^i_R(\bar{z}_1)\bar{\psi}^{\bar{j}}_R(\bar{z}_2) \sim \frac{\eta^{i\bar{j}}}{\bar{z}_1-\bar{z}_2}\,. \tag{A.44}$$

It turns out to be more convenient to consider the four component fields, namely $\mathcal{X}^\mu$ and ψ^μ_L, ψ^μ_R where $\mu = i, \bar{i} = 1, 2, \bar{1}, \bar{2}$. Then we have the OPEs

$$\mathcal{X}^\mu_L(z_1)\mathcal{X}^\nu_L(z_2) \sim -\frac{\eta^{\mu\nu}}{2}\ln(z_{12})\,, \qquad \mathcal{X}^\mu_R(\bar{z}_1)\mathcal{X}^\nu_R(\bar{z}_2) \sim -\frac{\eta^{\mu\nu}}{2}\ln(\bar{z}_{12})\,, \tag{A.45}$$

and

$$\psi^\mu_L(z_1)\psi^\nu_L(z_2) \sim \frac{\eta^{\mu\nu}}{z_{12}}\,, \qquad \psi^\mu_R(\bar{z}_1)\psi^\nu_R(\bar{z}_2) \sim \frac{\eta^{\mu\nu}}{\bar{z}_{12}}\,, \tag{A.46}$$

where $\eta^{\mu\nu}$ is given in (6.3).

We would like to compute the OPE of two vertex operators given in (6.23). As before, we can decompose the vertex operators into the independent left- and right-moving sectors:

$$\widehat{V}(\boldsymbol{k},z,\bar{z}) = \widehat{V}_L(\boldsymbol{k},z)\widehat{V}_R(\boldsymbol{k},\bar{z})\,, \tag{A.47}$$

$$\widehat{V}_L(\boldsymbol{k},z) = \left(i\boldsymbol{k}^\vee\cdot\partial\boldsymbol{\mathcal{X}}_L - \frac{1}{2}(\boldsymbol{k}^\vee\cdot\boldsymbol{\psi}_L)(\boldsymbol{k}\cdot\boldsymbol{\psi}_L)\right)e^{i\boldsymbol{k}\cdot\boldsymbol{\mathcal{X}}_L}\,, \tag{A.48}$$

$$\widehat{V}_R(\boldsymbol{k},\bar{z}) = \left(i\boldsymbol{k}^\vee\cdot\bar{\partial}\boldsymbol{\mathcal{X}}_R - \frac{1}{2}(\boldsymbol{k}^\vee\cdot\boldsymbol{\psi}_R)(\boldsymbol{k}\cdot\boldsymbol{\psi}_R)\right)e^{i\boldsymbol{k}\cdot\boldsymbol{\mathcal{X}}_R}\,. \tag{A.49}$$

They can be further written as

$$\widehat{V}_L(\boldsymbol{k},z) = \exp\left[i\boldsymbol{k}^\vee\cdot\partial\boldsymbol{\mathcal{X}}_L + i\boldsymbol{k}\cdot\boldsymbol{\mathcal{X}}_L\right]\left(1-\frac{1}{2}(\boldsymbol{k}^\vee\cdot\boldsymbol{\psi}_L)(\boldsymbol{k}\cdot\boldsymbol{\psi}_L)\right)\Big|_{\text{linear term in }\boldsymbol{k}^\vee}\,, \tag{A.50}$$

and similarly for V_R, where we only keep terms which are linear in the $\boldsymbol{k}^\vee$.

Then bosonic and fermionic parts in (A.50) can be considered separately. The bosonic contribution in the left moving OPE can be derived as in the open sting case (A.18):

$$\begin{aligned}
&e^{i\boldsymbol{k}_1\cdot\boldsymbol{\mathcal{X}}_L+i\boldsymbol{k}_1^\vee\cdot\partial\boldsymbol{\mathcal{X}}_L}(z_1)e^{i\boldsymbol{k}_2\cdot\boldsymbol{\mathcal{X}}_L+i\boldsymbol{k}_2^\vee\cdot\partial\boldsymbol{\mathcal{X}}_L}(z_2)\\
&= z_{12}^{\frac{1}{2}\boldsymbol{k}_1\cdot\boldsymbol{k}_2-2} : \Bigg[\left(\frac{1}{2}\boldsymbol{k}_1^\vee\cdot\boldsymbol{k}_2^\vee - \frac{1}{4}\boldsymbol{k}_1^\vee\cdot\boldsymbol{k}_2\,\boldsymbol{k}_2^\vee\cdot\boldsymbol{k}_1\right)\\
&\qquad +\frac{i}{2}z_{12}\left(\boldsymbol{k}_1^\vee\cdot\boldsymbol{k}_2\,\boldsymbol{k}_2^\vee\cdot\partial\boldsymbol{\mathcal{X}}(z_2) - \boldsymbol{k}_2^\vee\cdot\boldsymbol{k}_1\,\boldsymbol{k}_1^\vee\cdot\partial\boldsymbol{\mathcal{X}}(z_1) - i\boldsymbol{k}_1^\vee\cdot\boldsymbol{k}_2 + i\boldsymbol{k}_2^\vee\cdot\boldsymbol{k}_1\right)\\
&\qquad - z_{12}^2\left(\boldsymbol{k}_1^\vee\cdot\partial\boldsymbol{\mathcal{X}}(z_1)-i\right)\left(\boldsymbol{k}_2^\vee\cdot\partial\boldsymbol{\mathcal{X}}(z_2)-i\right)\\
&\qquad + \mathscr{O}(z_{12}^3)\cdots\Bigg] e^{i\boldsymbol{k}_1\cdot\boldsymbol{\mathcal{X}}_L(z_1)+i\boldsymbol{k}_2\cdot\boldsymbol{\mathcal{X}}_L(z_2)} :,
\end{aligned} \tag{A.51}$$

where the normal ordering of two operators at different points should be understood in the same way as in (A.5).

The fermionic contribution in the left moving OPE is

$$(\boldsymbol{k}_1^\vee \cdot \boldsymbol{\psi}_L)(\boldsymbol{k}_1 \cdot \boldsymbol{\psi}_L)(z_1)(\boldsymbol{k}_2^\vee \cdot \boldsymbol{\psi}_L)(\boldsymbol{k}_2 \cdot \boldsymbol{\psi}_L)(z_2) \tag{A.52}$$

$$= \frac{\boldsymbol{k}_1^\vee \cdot \boldsymbol{k}_2\, \boldsymbol{k}_1 \cdot \boldsymbol{k}_2^\vee - \boldsymbol{k}_1^\vee \cdot \boldsymbol{k}_2^\vee\, \boldsymbol{k}_1 \cdot \boldsymbol{k}_2}{z_{12}^2} \tag{A.53}$$

$$+ \frac{\boldsymbol{k}_1^\vee \cdot \boldsymbol{k}_2}{z_{12}} : \boldsymbol{k}_1 \cdot \boldsymbol{\psi}(z_1) \boldsymbol{k}_2^\vee \cdot \boldsymbol{\psi}(z_2) : + \frac{\boldsymbol{k}_1 \cdot \boldsymbol{k}_2^\vee}{z_{12}} : \boldsymbol{k}_1^\vee \cdot \boldsymbol{\psi}(z_1) \boldsymbol{k}_2 \cdot \boldsymbol{\psi}(z_2) : \tag{A.54}$$

$$- \frac{\boldsymbol{k}_1 \cdot \boldsymbol{k}_2}{z_{12}} : \boldsymbol{k}_1^\vee \cdot \boldsymbol{\psi}(z_1) \boldsymbol{k}_2^\vee \cdot \boldsymbol{\psi}(z_2) : - \frac{\boldsymbol{k}_1^\vee \cdot \boldsymbol{k}_2^\vee}{z_{12}} : \boldsymbol{k}_1 \cdot \boldsymbol{\psi}(z_1) \boldsymbol{k}_2 \cdot \boldsymbol{\psi}(z_2) : \tag{A.55}$$

$$+ : \boldsymbol{k}_1^\vee \cdot \boldsymbol{\psi}_L\, \boldsymbol{k}_1 \cdot \boldsymbol{\psi}_L(z_1) \boldsymbol{k}_2^\vee \cdot \boldsymbol{\psi}_L\, \boldsymbol{k}_2 \cdot \boldsymbol{\psi}_L(z_2) :, \tag{A.56}$$

where we used

$$\boldsymbol{p} \cdot \boldsymbol{\psi}_L(z_1) \boldsymbol{q} \cdot \boldsymbol{\psi}_L(z_2) = \frac{\boldsymbol{p} \cdot \boldsymbol{q}}{z_{12}} + : \boldsymbol{p} \cdot \boldsymbol{\psi}_L(z_1) \boldsymbol{q} \cdot \boldsymbol{\psi}_L(z_2) : . \tag{A.57}$$

Then the OPE of two operators of the form (A.50) is given by

$$\widehat{V}_L(\boldsymbol{k}_1, z_1) \widehat{V}_L(\boldsymbol{k}_2, z_2) \Big|_{\text{linear in } \boldsymbol{k}_1^\vee \text{ and } \boldsymbol{k}_2^\vee} \tag{A.58}$$

$$\sim z_{12}^{\frac{1}{2}\boldsymbol{k}_1 \cdot \boldsymbol{k}_2 - 1} : \left[\frac{1}{2 z_{12}} \boldsymbol{k}_1^\vee \cdot \boldsymbol{k}_2^\vee \left(1 - \frac{1}{2} \boldsymbol{k}_1 \cdot \boldsymbol{k}_2 \right) \right. \tag{A.59}$$

$$+ \frac{i}{2} \boldsymbol{k}_1^\vee \cdot \boldsymbol{k}_2 \left(\boldsymbol{k}_1^\vee \cdot \partial \boldsymbol{\mathcal{X}}(z_1) + \boldsymbol{k}_2^\vee \cdot \partial \boldsymbol{\mathcal{X}}(z_2) \right) \tag{A.60}$$

$$+ \frac{1}{4} \boldsymbol{k}_1^\vee \cdot \boldsymbol{k}_2 \Big(- (\boldsymbol{k}_2^\vee \cdot \boldsymbol{\psi}_L\, \boldsymbol{k}_2 \cdot \boldsymbol{\psi}_L)(z_2) - (\boldsymbol{k}_1^\vee \cdot \boldsymbol{\psi}_L\, \boldsymbol{k}_1 \cdot \boldsymbol{\psi}_L)(z_1) \tag{A.61}$$

$$- : \boldsymbol{k}_2^\vee \cdot \boldsymbol{\psi}(z_2) \boldsymbol{k}_1 \cdot \boldsymbol{\psi}(z_1) : - : \boldsymbol{k}_1^\vee \cdot \boldsymbol{\psi}(z_1) \boldsymbol{k}_2 \cdot \boldsymbol{\psi}(z_2) : \Big) \tag{A.62}$$

$$- \frac{1}{4} \boldsymbol{k}_1 \cdot \boldsymbol{k}_2 : \boldsymbol{k}_1^\vee \cdot \boldsymbol{\psi}(z_1) \boldsymbol{k}_2^\vee \cdot \boldsymbol{\psi}(z_2) : - \frac{1}{4} \boldsymbol{k}_1^\vee \cdot \boldsymbol{k}_2^\vee : \boldsymbol{k}_1 \cdot \boldsymbol{\psi}(z_1) \boldsymbol{k}_2 \cdot \boldsymbol{\psi}(z_2) : \tag{A.63}$$

$$\left. + \mathscr{O}(z_{12}) \right] e^{i \boldsymbol{k}_1 \cdot \boldsymbol{\mathcal{X}}_L(z_1) + i \boldsymbol{k}_2 \cdot \boldsymbol{\mathcal{X}}_L(z_2)} :, \tag{A.64}$$

where we used $\boldsymbol{k}_1^\vee \cdot \boldsymbol{k}_2 = -\boldsymbol{k}_2^\vee \cdot \boldsymbol{k}_1$.

We are interested in the collinear limit $\boldsymbol{k}_1 \cdot \boldsymbol{k}_2 \to 0$. In this limit, we also have $\boldsymbol{k}_1^\vee \cdot \boldsymbol{k}_2^\vee = -\boldsymbol{k}_1 \cdot \boldsymbol{k}_2 \to 0$ following (6.16). Therefore the $1/z_{12}$ term (A.59) and fermion bilinear term (A.63) can be dropped in the collinear limit.[34] The remaining terms can be

[34]Actually, the $1/z_{12}$ term is BRST exact and can be dropped even not in the collinear limit. Recall that $\boldsymbol{k}^\vee$ is just the positive helicity polarization vector $\boldsymbol{\varepsilon}_+$ (6.15), up to an overall rescaling and a gauge transformation. For $\boldsymbol{\varepsilon}_+$, we have the property that $\boldsymbol{\varepsilon}_+(\mathsf{z}_i, \bar{\mathsf{z}}_i) \cdot \boldsymbol{\varepsilon}_+(\mathsf{z}_j, \bar{\mathsf{z}}_j) = 0$ (6.14). So if we replace $\boldsymbol{k}^\vee$ with $\boldsymbol{\varepsilon}_+$, the term $\boldsymbol{k}_1^\vee \cdot \boldsymbol{k}_2^\vee$ should be absent completely. The appearance of the term $\boldsymbol{k}_1^\vee \cdot \boldsymbol{k}_2^\vee$ is due to the gauge transformation, and thus is BRST exact.

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combined into a very simple form:

$$\widehat{V}_L(\boldsymbol{k}_1, z_1)\widehat{V}_L(\boldsymbol{k}_2, z_2)\Big|_{\text{linear in } \boldsymbol{k}_1^\vee \text{ and } \boldsymbol{k}_2^\vee} \tag{A.65}$$

$$\sim \frac{1}{2}\boldsymbol{k}_1^\vee \cdot \boldsymbol{k}_2 \, z_{12}^{\frac{1}{2}\boldsymbol{k}_1\cdot\boldsymbol{k}_2-1} e^{i(\boldsymbol{k}_1+\boldsymbol{k}_2)\cdot\boldsymbol{\mathcal{X}}_L} \tag{A.66}$$

$$\times \left[i(\boldsymbol{k}_1^\vee + \boldsymbol{k}_2^\vee)\cdot\partial\boldsymbol{\mathcal{X}} - \frac{1}{2}(\boldsymbol{k}_1^\vee + \boldsymbol{k}_2^\vee)\cdot\boldsymbol{\psi}_L \, (\boldsymbol{k}_1+\boldsymbol{k}_2)\cdot\boldsymbol{\psi}_L + \mathscr{O}(\boldsymbol{k}_1\cdot\boldsymbol{k}_2) + \mathscr{O}(z_{12})\cdots \right](z_2)\,. \tag{A.67}$$

The computation of $\widehat{V}_R(\boldsymbol{k}_1, \bar{z}_1)\widehat{V}_R(\boldsymbol{k}_2, \bar{z}_2)$ is identical and the final result is just given by (A.65) except for the replacement of z with $\bar{z}$. Combining the left and right moving OPEs together, we get the final OPE

$$\widehat{V}(\boldsymbol{k}_1)\widehat{V}(\boldsymbol{k}_2) \sim \frac{1}{4}\left(\boldsymbol{k}_1^\vee \cdot \boldsymbol{k}_2\right)^2 |z_{12}|^{\boldsymbol{k}_1\cdot\boldsymbol{k}_2-2} e^{i\boldsymbol{K}_3\cdot\boldsymbol{\mathcal{X}}} \tag{A.68}$$

$$\times \left[i\boldsymbol{K}_3^\vee \cdot \partial\boldsymbol{\mathcal{X}} - \frac{1}{2}(\boldsymbol{K}_3^\vee\cdot\boldsymbol{\psi}_L \, \boldsymbol{K}_3\cdot\boldsymbol{\psi}_L) + \mathscr{O}(\boldsymbol{k}_1\cdot\boldsymbol{k}_2) + \mathscr{O}(z_{12}) \right] \tag{A.69}$$

$$\times \left[i\boldsymbol{K}_3^\vee \cdot \bar{\partial}\boldsymbol{\mathcal{X}} - \frac{1}{2}(\boldsymbol{K}_3^\vee\cdot\boldsymbol{\psi}_R \, \boldsymbol{K}_3\cdot\boldsymbol{\psi}_R) + \mathscr{O}(\boldsymbol{k}_1\cdot\boldsymbol{k}_2) + \mathscr{O}(\bar{z}_{12}) \right], \tag{A.70}$$

where $\boldsymbol{K}_3 = \boldsymbol{k}_1 + \boldsymbol{k}_2$.

B OPE and amplitude in open-closed string theory

In this section of appendix, we will discuss the celestial OPEs involving both gluons and gravitons from the open-closed string setup.

Basic OPE in open-closed string. In the open string case with spacetime filling D-brane, the fields X^μ satisfy the Neumann boundary condition on the boundary, namely

$$\partial X^\mu(z,\bar{z}) = \bar{\partial} X^\mu(z,\bar{z}) \qquad \text{for} \qquad z = \bar{z} = y \in \mathbb{R}\,. \tag{B.1}$$

This can be realized by adding a mirror image contribution to the OPE in the closed string (A.20), namely we now have

$$X^\mu(z_1,\bar{z}_1)X^\nu(z_2,\bar{z}_2) \sim -\frac{\alpha'}{2}\eta^{\mu\nu}\ln|z_1-z_2|^2 - \frac{\alpha'}{2}\eta^{\mu\nu}\ln|z_1-\bar{z}_2|^2\,. \tag{B.2}$$

Then it is easy to check that this OPE indeed satisfies the boundary condition in (B.1). On the boundary of the disk, we have $X(y) \equiv X(z=y, \bar{z}=y)$ with $y \in \mathbb{R}$, and we also recover the OPE (A.1) in the open string case.

As before, we can decompose $X(z,\bar{z}) = X_L(z) + X_R(\bar{z})$. Then (B.2) becomes

$$X_L^\mu(z_1)X_L^\nu(z_2) \sim -\frac{\alpha'}{2}\eta^{\mu\nu}\ln z_{12}\,, \qquad X_R^\mu(\bar{z}_1)X_R^\nu(\bar{z}_2) \sim -\frac{\alpha'}{2}\eta^{\mu\nu}\ln \bar{z}_{12}\,, \tag{B.3}$$

and furthermore

$$X_L^\mu(z_1)X_R^\nu(\bar{z}_2) \sim -\frac{\alpha'}{2}\eta^{\mu\nu}\ln(z_1-\bar{z}_2)\,. \tag{B.4}$$

As a result, we have $X(y) = X_L(z = y) + X_R(\bar{z} = y)$ and the following OPEs [38]

$$X^\mu(y)X_L^\nu(z) \sim -\alpha'\eta^{\mu\nu}\ln(y - z)\,, \quad X^\mu(y)X_R^\nu(\bar{z}) \sim -\alpha'\eta^{\mu\nu}\ln(y - \bar{z})\,, \tag{B.5}$$

$$X(y)X(z, \bar{z}) \sim -\alpha'\eta^{\mu\nu}\ln|y - z|^2\,. \tag{B.6}$$

In particular, we also have $\dot{X}(y) = 2\partial X_L(z)|_{z=y} = 2\bar{\partial}X_R(\bar{z})|_{\bar{z}=y}$.

Gluon and graviton mixed amplitude from open-closed string. We want to compute the three-point amplitude of two gluons and one graviton/dilaton/KR field. They are labelled by their momenta p_1, p_2, p_3 and polarizations $\zeta_{1\mu}, \zeta_{2\mu}, e_{\mu\nu}$. They satisfy the momentum conservation and the polarization transversality conditions

$$p_1 + p_2 + p_3 = 0\,, \qquad \zeta_1 \cdot p_1 = \zeta_2 \cdot p_2 = e_{\mu\nu}p_3^\mu = e_{\mu\nu}p_3^\nu = 0\,, \tag{B.7}$$

as well as the on-shell conditions

$$p_1^2 = p_2^2 = p_3^2 = 0\,, \qquad \Rightarrow \qquad p_1 \cdot p_2 = p_2 \cdot p_3 = p_3 \cdot p_1 = 0\,. \tag{B.8}$$

We further assume that the closed string polarization tensor is traceless $e_{\mu\nu}\eta^{\mu\nu} = 0$, so we do not consider dilaton.[35]

We would like to compute three-point amplitude from the open-closed string setup. At tree level, this is given by the correlator on the disk, where we insert the open string vertex operators on the boundary of the disk and closed vertex operators in the interior of the disk. In particular, our three-point amplitude involves two open and one closed string fields, and can be computed by

$$A_3^{ooc} = \int \frac{dy_1\, dy_2\, d^2z}{V_{\text{CKG}}} \langle \zeta_1 \cdot \dot{X}e^{ip_1\cdot X}(y_1)\zeta_2 \cdot \dot{X}e^{ip_2\cdot X}(y_2)e_{\mu\nu}\partial X^\mu\bar{\partial}X^\nu e^{ip_3\cdot X}(z, \bar{z})\rangle\, \text{Tr}(t^A t^B)\,, \tag{B.9}$$

where we need to divide the volume of the conformal Killing group, which is $\text{PSL}(2, \mathbb{R})$. In practice we need to fix the $\text{PSL}(2, \mathbb{R})$ invariance. Following the prescription in [38], we can set

$$y_1 = -y_2 = -y\,, \qquad z = i\,, \qquad \bar{z} = -i\,, \tag{B.10}$$

but we also need to take into account the non-trivial Jacobian of the transformation from the fixed coordinates to the parameters of $\text{PSL}(2, \mathbb{R})$. Using the coordinate transformation $y_1 = \tilde{y} - y, y_2 = \tilde{y} + y, z = u + iv, \bar{z} = u - iv$, the integration measure becomes $dy_1\, dy_2\, d^2z = 2dyd\tilde{y}dudv$. An infinitesimal $\text{PSL}(2, \mathbb{R})$ transformation acts as $\delta z = \alpha + \beta z + \gamma z^2$ where α, β, γ are real. The Jacobian between them is [38]

$$\left|\frac{\partial(u, v, \tilde{y})}{\partial(\alpha, \beta, \gamma)}\right| = 1 + y^2\,, \qquad \text{at} \qquad \tilde{y} = u = 0\,, \quad v = 1\,. \tag{B.11}$$

[35] This is to avoid the contraction between the left and right moving pieces of closed string vertex operators, which turns out to bring divergence and needs to be treated properly. We thank Rodolfo Russo for discussions on this point.

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Then (B.9) becomes

$$A_3^{ooc} = \int dy\, 2(1+y^2)\, \langle \zeta_1 \cdot \dot{X} e^{ip_1\cdot X}(-y)\zeta_2 \cdot \dot{X} e^{ip_2\cdot X}(y) e_{\mu\nu}\partial X^\mu \bar{\partial} X^\nu e^{ip_3\cdot X}(i,-i)\rangle\, \mathrm{Tr}(t^A t^B)\,. \tag{B.12}$$

To compute the correlator in the integrand, we rewrite it as

$$\begin{aligned}&\langle \zeta_1 \cdot \dot{X} e^{ip_1\cdot X}(y_1)\zeta_2 \cdot \dot{X} e^{ip_2\cdot X}(y_2) e_{\mu\nu}\partial X^\mu \bar{\partial} X^\nu e^{ip_3\cdot X}(z,\bar{z})\rangle \\ &= e_{\mu\nu}\frac{\partial^2}{\partial \xi_\mu \partial \bar{\xi}_\nu}\langle e^{ip_1\cdot X + i\zeta_1\cdot\dot{X}}(y_1) e^{ip_2\cdot X + i\zeta_2\cdot\dot{X}}(y_2) e^{ip_3\cdot X_L + i\xi\cdot\partial X_L}(z) e^{ip_3\cdot X_R + i\bar{\xi}\cdot\bar{\partial} X_R}(\bar{z})\rangle\Big|_{\text{linear terms in }\zeta_1,\zeta_2,e}\,,\end{aligned} \tag{B.13}$$

where we only keep the terms which are linear in ζ_1, ζ_2 and e, or equivalently terms linear in ζ_1, ζ_2 and $\xi, \bar{\xi}$ in the correlator. The correlator can be evaluated using the formula (A.3) and OPE (B.5). The computation is straightforward and can be simplified by using (B.7), (B.8). After further performing the y integration, we obtain the final result. Up to an overall constant, the three-point amplitude is given by

$$A_3^{ooc} = \delta^{AB}\Big(\zeta_1\cdot p_2\ \zeta_2\cdot e\cdot p_1 - \zeta_2\cdot p_1\ \zeta_1\cdot e\cdot p_1 + \zeta_1\cdot\zeta_2\ p_1\cdot e\cdot p_1 - \alpha'\zeta_1\cdot p_2\ \zeta_2\cdot p_1\ p_1\cdot e\cdot p_1\Big)\,, \tag{B.14}$$

where $A\cdot e\cdot B \equiv \frac{1}{2}(A^\mu B^\nu e_{\mu\nu} + A^\mu B^\nu e_{\nu\mu})$. Using (B.7), (B.8), (B.14) can be further rewritten as

$$A_3^{ooc} = \frac{1}{8}\, e_{\mu\nu}\zeta_{1\rho}\zeta_{2\sigma}\Big[p_{12}^\nu S^{\rho\sigma\mu}(\alpha') + p_{12}^\mu S^{\rho\sigma\nu}(\alpha')\Big]\delta^{AB}\,, \tag{B.15}$$

where

$$S^{\rho\sigma\mu}(\alpha') = p_{12}^\mu\,\eta^{\rho\sigma} + p_{23}^\rho\,\eta^{\sigma\mu} + p_{31}^\sigma\,\eta^{\mu\rho} + \frac{\alpha'}{4}p_{12}^\mu\,p_{23}^\rho\,p_{31}^\sigma\,. \tag{B.16}$$

The result is similar to the amplitude for two gluon and one graviton/dilaton/KR field in the heterotic string case (5.23), except for the symmetrization of closed string polarization tensor and non-trivial α' corrections which are absent in the heterotic string due to supersymmetry. The tensor $S(\alpha')$ here is essentially $T(2\alpha')$ in (5.5), up to the relabeling. Note that in the purely open and closed string case, we have $T(4\alpha')$ and $T(\alpha')$ in (5.3) and (5.18), respectively.

Since the closed string polarization tensor is always symmetrized, the massless fields from closed string can not be Kalb-Ramond 2-form field in order to have a non-vanishing amplitude.

It is worth mentioning that the result in (B.14) comes from the α'^3 and α'^4 order terms in the correlator (B.13). But actually the leading terms in the correlator (B.13) is of order α'^2, which is non-vanishing and takes the form

$$\alpha'^2\left[\frac{\zeta_1\cdot\xi\ \zeta_2\cdot\bar{\xi}}{(y+i)^4} + \frac{\zeta_2\cdot\xi\ \zeta_1\cdot\bar{\xi}}{(y-i)^4}\right]. \tag{B.17}$$

However, after performing the integration over y, this term vanishes. As a result, the leading interaction between gluon and graviton is indeed the minimal coupling between them, as it should be.

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Celestial OPE of gluon and graviton. Now we want to compute the worldsheet OPE between closed string and open string vertex operators. As before, we can decompose the closed string vertex operator into the left and right moving part, and then use the formula (A.3) and (B.5). The steps are similar to the previous cases, so we only write down the final result:[36]

$$
\begin{aligned}
&e_{\mu\nu}\partial X^{\mu}\bar{\partial}X^{\nu}e^{ip\cdot X}(z,\bar{z})\zeta\cdot Xe^{iq\cdot X}(y) && \text{(B.18)}\\
&\sim i\alpha'^2|z-y|^{2\alpha' p\cdot q-2}e^{i(p+q)\cdot X} && \text{(B.19)}\\
&\quad\times\Bigg[\frac{2i(\alpha' p\cdot\zeta\, q\cdot e\cdot q-q\cdot e\cdot\zeta)}{z-y} && \text{(B.20)}\\
&\qquad+q\cdot e\cdot q\,\zeta\cdot\dot{X}+q\cdot e\cdot\zeta\, p\cdot\dot{X}-p\cdot\zeta\, q\cdot e\cdot\dot{X}-\alpha' p\cdot\zeta\, q\cdot e\cdot q\, p\cdot\dot{X} && \text{(B.21)}\\
&\qquad+\mathscr{O}(z-\bar{z})+\mathscr{O}(z-y)+\mathscr{O}(\bar{z}-y)\Bigg](y)\,, && \text{(B.22)}
\end{aligned}
$$

where we use the relation $\dot{X}=2\partial X_L=2\partial X_R$ on the boundary. Note that only the symmetric part of the closed string polarization tensor contributes. We are interested in the collinear limit $p\cdot q\to 0$. Then the first term (B.20) is supposed to contribute only the boundary contact terms, so we will ignore it. The second line (B.21) is the one relevant here.

To proceed, we choose polarizations as the bases $\zeta=\varepsilon_{\tilde{b}}(q), e^{\mu\nu}=\varepsilon^{\mu}_{a}(p)\varepsilon^{\nu}_{\tilde{a}}(p)$. Then (B.21) can be rewritten as

$$
\varepsilon_a\cdot q\Big[q\cdot\varepsilon_{\tilde{a}}\,\varepsilon_{\tilde{b}}\cdot\dot{X}-p\cdot\varepsilon_{\tilde{b}}\,\varepsilon_{\tilde{a}}\cdot\dot{X}+\Big(\varepsilon_{\tilde{a}}\cdot\varepsilon_{\tilde{b}}-\alpha' p\cdot\varepsilon_{\tilde{b}}\,\varepsilon_{\tilde{a}}\cdot q\Big)\,p\cdot\dot{X}\Big]+\Big(a\leftrightarrow\tilde{a}\Big)\,. \tag{B.23}
$$

The terms in the bracket are very similar to the terms proportional to y_{12} in (3.3) once replacing $\zeta\to\varepsilon_{\tilde{a}},\xi\to\varepsilon_{\tilde{b}}$, and $\alpha'\to\frac{1}{2}\alpha'$. Therefore, the simplification of this formula is also similar to the open string case there. Following the same derivation leading to (3.17), up to the boundary contact terms, the final result here is given by

$$
\varepsilon_a\cdot q\Big[q\cdot\varepsilon_{\tilde{a}}\,\varepsilon_{\tilde{b}}\cdot\varepsilon_{\tilde{c}}-p\cdot\varepsilon_{\tilde{b}}\,\varepsilon_{\tilde{a}}\cdot\varepsilon_{\tilde{c}}+\Big(\varepsilon_{\tilde{a}}\cdot\varepsilon_{\tilde{b}}-\alpha' p\cdot\varepsilon_{\tilde{b}}\,\varepsilon_{\tilde{a}}\cdot q\Big)\,p\cdot\varepsilon_{\tilde{c}}\Big]\varepsilon_{\tilde{c}}\cdot\dot{X}+\Big(a\leftrightarrow\tilde{a}\Big)\,. \tag{B.24}
$$

Up to symmetrization and α' correction, this is similar to the heterotic case (4.31).

In the collinear limit, the factor $|z-y|^{2\alpha' p\cdot q-2}$ localizes to a delta-function $\delta^2(y-z)/p\cdot q$, as one can see from (3.55). Integrating over y and z, we get the OPE

$$
\begin{aligned}
\mathcal{V}_{a\tilde{a}}(p)\mathcal{V}^{A}_{\tilde{b}}(q)&\sim\frac{i\pi\alpha'}{4}\frac{1}{p\cdot q}\varepsilon_a\cdot q\Big[q\cdot\varepsilon_{\tilde{a}}\,\varepsilon_{\tilde{b}}\cdot\varepsilon_{\tilde{c}}-p\cdot\varepsilon_{\tilde{b}}\,\varepsilon_{\tilde{a}}\cdot\varepsilon_{\tilde{c}}+\Big(\varepsilon_{\tilde{a}}\cdot\varepsilon_{\tilde{b}}-\alpha' p\cdot\varepsilon_{\tilde{b}}\,\varepsilon_{\tilde{a}}\cdot q\Big)\,p\cdot\varepsilon_{\tilde{c}}\Big]\mathcal{V}^{A}_{\tilde{c}}(p+q)+\Big(a\leftrightarrow\tilde{a}\Big)\\
&\sim\frac{i\pi\alpha'}{2}\varepsilon_a\cdot q\, I_{\tilde{c}}\Big(p,\varepsilon_{\tilde{a}}(p);q,\varepsilon_{\tilde{b}}(q);\frac{1}{2}\alpha'\Big)+\Big(a\leftrightarrow\tilde{a}\Big)\,,
\end{aligned}
\tag{B.25}
$$

where I is given in (3.35).

[36] Note that inside the square bracket, there are also order one terms which are proportional to $(z-y)/(\bar{z}-y)$ or its inverse; these terms vanish after doing z integral on the upper half plane. This is supposed to be the origin of (B.17), and both of them disappear only after integration.

Further performing the Mellin transformation, we finally get the celestial OPE between open and closed string massless fields, up to an overall constant:

$$\mathcal{V}_{\Delta_1,a\tilde{a}}(x_1)\mathcal{V}^A_{\Delta_2,\tilde{b}}(x_2) \tag{B.26}$$

$$\sim x^a_{12}\frac{x^{\tilde{a}}_{12}\delta^{\tilde{b}\tilde{c}}B(\Delta_1-1,\Delta_2+1)+x^{\tilde{b}}_{12}\delta^{\tilde{a}\tilde{c}}B(\Delta_1,\Delta_2)-x^{\tilde{c}}_{12}\delta^{\tilde{a}\tilde{b}}B(\Delta_1,\Delta_2+1)}{(x_{12})^2}\mathcal{V}^A_{\Delta_1+\Delta_2,\tilde{c}}(x_2)+\Big(a\leftrightarrow\tilde{a}\Big)$$

$$-2\alpha'\frac{x^a_{12}x^{\tilde{a}}_{12}x^{\tilde{b}}_{12}x^{\tilde{c}}_{12}}{(x_{12})^2}B(\Delta_1+1,\Delta_2+2)\mathcal{V}^A_{\Delta_1+\Delta_2+2,\tilde{c}}(x_2)\,. \tag{B.27}$$

Since we have assumed that the polarization tensor is traceless and furthermore only its symmetric part contributes, the closed string massless field under our consideration can only be graviton. We thus need to subtract the trace part in (B.26) following (2.7). This then gives the gluon and graviton OPE from the open-closed string setup. One can check that the same result can be obtained from the collinear factorization (5.2) and the three-point amplitude (B.14) derived before. In particular, the relative coefficients of α' correction are also the same. The perfect agreement thus justifies our amplitude and OPE calculations. Therefore, the celestial OPE can also be obtained from the open-closed string setup. However, the emergence of closed string field from open string field, namely the fusion of two gluons into one graviton, is not clear from the OPE perspective.

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