

Kibble mechanism for electroweak magnetic monopoles and magnetic fields

Teerthal Patel and Tanmay Vachaspati

*Physics Department, Arizona State University,
Tempe, AZ 85287, U.S.A.*

E-mail: tpatel28@asu.edu, tvchasp@asu.edu

ABSTRACT: The vacuum manifold of the standard electroweak model is a three-sphere when one considers *homogeneous* Higgs field configurations. For inhomogeneous configurations we argue that the vacuum manifold is the Hopf fibered three sphere and that this viewpoint leads to general criteria to detect electroweak monopoles and Z-strings. We extend the Kibble mechanism to study the formation of electroweak monopoles and strings during electroweak symmetry breaking. The distribution of magnetic monopoles produces magnetic fields that have a spectrum $B_\lambda \propto \lambda^{-2}$, where λ is a smearing length scale. Even as the magnetic monopoles annihilate due to the confining Z-strings, the magnetic field evolves with the turbulent plasma and may be relevant for cosmological observations.

KEYWORDS: Solitons Monopoles and Instantons, Cosmology of Theories beyond the SM, Nonperturbative Effects

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1 Introduction

The distribution of topological defects formed after spontaneous symmetry breaking (SSB) is often analyzed by implementing the “Kibble mechanism” [1–3]. During SSB a field takes on a non-trivial vacuum expectation value (VEV) that lies on the “vacuum manifold”. Distant spatial points are randomly selected on the vacuum manifold and if the vacuum manifold has non-trivial topology, the VEV of the field may end up in a non-trivial topological configuration, in which case a topological defect would be formed. Numerical simulations of the Kibble mechanism have been central to our understanding of topological defect formation during spontaneous symmetry breaking. Notably the cosmic string network was shown to be dominated by infinite strings that don’t close on themselves, while the sub-dominant distribution of closed loops was found to be scale invariant [2] (for reviews see [4–6]).

Here we are interested in the implications of the Kibble mechanism when the electroweak Higgs field, denoted Φ , acquires a VEV. The electroweak vacuum manifold is a three-sphere with trivial first and second homotopy groups and there are no topological magnetic monopoles or cosmic strings by these criteria. However, electroweak monopoles and Z-strings that connect the magnetic monopoles do exist in the model [7–9]. We will show that a suitably modified algorithm like that in the case of topological defects can still be used to obtain the distribution of electroweak monopoles and strings. The distribution can be used as an initial condition for further evolution. Since the monopoles and antimonopoles are confined by strings, they will quickly annihilate. Yet the annihilation will leave behind a distribution of magnetic fields [10, 11] that can be of observational interest and may have important ramifications for cosmology [12–15].

In section 2 we describe our viewpoint that the electroweak vacuum manifold is better described as $S^2 \times S^1$, i.e. as a Hopf fibered S^3 , and thus contains electroweak monopoles and strings. We describe the prototype Nambu monopole in section 3 and implement the Kibble mechanism in section 4 to find a distribution of electroweak monopoles and Z-strings. With

evolution, the network of monopoles and strings will leave behind a distribution of magnetic fields that we characterize in section 5. We summarize our conclusions in section 6.

2 Electroweak vacuum manifold

The vacuum manifold of the electroweak manifold is the set of all spatially homogeneous and static Higgs fields for which the energy function vanishes. The Higgs VEV is an SU(2) doublet¹

$$\Phi = \begin{pmatrix} \phi_1 + i\phi_2 \\ \phi_3 + i\phi_4 \end{pmatrix} \tag{2.1}$$

and since the Higgs potential is,

$$V(\Phi) = \lambda(|\Phi|^2 - \eta^2)^2 \tag{2.2}$$

the vacuum manifold is an S^3 given by

$$|\Phi|^2 = \phi_1^2 + \phi_2^2 + \phi_3^2 + \phi_4^2 = \eta^2. \tag{2.3}$$

One issue is that the symmetry of the potential consists of rotations of the four dimensional vector $(\phi_1, \phi_2, \phi_3, \phi_4)$, hence it is O(4), whereas the electroweak symmetry is the smaller $[\text{SU}(2)_L \times \text{U}(1)_Y]/Z_2$. The reduced symmetry is due to the derivative terms in the model and these are completely ignored in discussions that are based solely on the vacuum manifold. Derivative terms vanish for homogeneous Higgs configurations and so the S^3 vacuum manifold is appropriate for such configurations. On the other hand, the Kibble mechanism relies on VEVs that are different in different regions of space. Hence the Higgs configurations are necessarily inhomogeneous. We have learned from semilocal strings that the vacuum manifold does not give the complete picture when one considers inhomogeneous Higgs fields configuration for then the gradient energy terms can also be important.

Let us clarify this further by discussing the semilocal limit of the electroweak model. Then the $\text{SU}(2)_L$ gauge coupling is set to vanish: $g = 0$. In that case, one can consider Higgs configurations that lie entirely on the vacuum manifold but whose energy cannot vanish. This is because the gauged $\text{U}(1)_Y$ symmetry defines S^1 gauge orbits on the vacuum manifold. Only Higgs gradients along these orbits can be compensated by the gauge field so that the covariant gradient energy vanishes; if the Higgs VEV does not lie on a gauge orbit, the gradient energy cannot vanish.

An alternative “semilocal” limit that has not previously been considered in this context is to take the $\text{U}(1)_Y$ coupling to vanish: $g' = 0$. In that case the gauge orbits on the vacuum manifold are S^2 's. If we restrict attention to asymptotic Higgs fields configurations that have vanishing potential and gradient energy, the Higgs VEV would have to lie on an S^2 and this has the right topology for magnetic monopoles.

The standard electroweak model has $g = 0.65$ and $g' = 0.34$, so neither coupling vanishes, even though the $\text{SU}(2)_L$ coupling is larger. However the fibered structure still exists — the vacuum manifold S^3 has S^2 and S^1 gauge orbits. These gauge orbits are

¹For convenience we will write Φ instead of $\langle \Phi \rangle$ throughout this paper.

precisely defined by the Hopf fibration of S^3 , as was originally pointed out in the $g' = 0$ semilocal limit [16, 17]. The Hopf fibration of S^3 provides a map from S^3 to S^2 with S^1 fibers. The electroweak monopole is due to winding around the S^2 base manifold and the Z-string is due to winding around the S^1 fiber. Because of the non-trivial global structure of the Hopf fibration, the Z-string is attached to the electroweak monopole.

3 Nambu monopole

It is instructive to first consider the explicit configuration for the Nambu monopole [7] for which the asymptotic Higgs VEV is,

$$\Phi_m = \frac{v}{\sqrt{2}} \begin{pmatrix} \cos(\theta/2) \\ \sin(\theta/2)e^{i\phi} \end{pmatrix} \quad (3.1)$$

where θ, ϕ are spherical angles. Note that the configuration is singular at $\theta = \pi$. To see the presence of the monopole in this configuration, construct

$$\hat{n}_m = -\hat{\Phi}_m^\dagger \vec{\sigma} \hat{\Phi}_m = -(\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta) = -\hat{r} \quad (3.2)$$

where $\hat{\Phi} \equiv \Phi/|\Phi|$, σ^a ($a = 1, 2, 3$) are the Pauli spin matrices and the overall sign is chosen so that $\hat{n} = \hat{z}$ when $\Phi^T = v(0, 1)/\sqrt{2}$. Now \hat{n}_m is regular for all θ and ϕ and is in the (inner) radial direction. This is also called the ‘‘hedgehog’’ configuration and immediately implies the presence of a singularity of \hat{n}_m at the origin that corresponds to a magnetic monopole [18, 19]. Going back to Φ_m , the singularity at $\theta = \pi$ signifies the Z-string attached to the monopole.

The above explicit example suggests that to apply the Kibble mechanism to the electroweak model we should start by considering a distribution of the vector field \hat{n} . Since \hat{n} lives on a two-sphere (S^2) that has non-trivial second homotopy, there will be hedgehog configurations of \hat{n} (e.g. $\hat{n} = \hat{r}$). As for ’t Hooft-Polyakov monopoles [18, 19], the topological winding of \hat{n} in a spherical volume of radius R is given by the surface integral

$$n_M = \frac{1}{4\pi} \int_R dS^i \epsilon_{abc} \epsilon_{ijk} \hat{n}^a \partial_j \hat{n}^b \partial_k \hat{n}^c. \quad (3.3)$$

The discrete winding number $n_M \in \mathbb{Z}$ must remain constant as $R \rightarrow 0$. For $n_M \neq 0$ this implies that \hat{n} is singular within the spherical volume.

Now we consider the Φ field that corresponds to the hedgehog configuration of \hat{n} . The relation between Φ and \hat{n} for $|\Phi| \neq 0$ is,

$$\hat{n} = -\hat{\Phi}^\dagger \vec{\sigma} \hat{\Phi} \quad (3.4)$$

Therefore the singularity in \hat{n} for non-trivial n_M requires that $\Phi = 0$ at the singular point where there is a magnetic monopole.

The Z-string attached to the monopole appears when we try and invert (3.4) to obtain Φ . For \hat{n} with non-trivial winding ($n_M \neq 0$), the reconstruction will necessarily give a singularity in Φ (as in (3.1)). This singularity is the location of the Z-string on the sphere surrounding the monopole. We will describe the explicit algorithm for finding the location of the Z-string in section 4.

4 Kibble mechanism

The Higgs VEV of (2.1) can also be parametrized as,

$$\Phi = \frac{v}{\sqrt{2}} \begin{pmatrix} \cos \alpha e^{i\beta} \\ \sin \alpha e^{i\gamma} \end{pmatrix} \quad (4.1)$$

where $v = 246$ GeV and $\alpha \in [0, \pi/2]$, $\beta \in [0, 2\pi]$, $\gamma \in [0, 2\pi]$ are Hopf angular coordinates on the vacuum manifold: $\Phi^\dagger \Phi = v^2/2$. The volume measure on the vacuum manifold in terms of Hopf coordinates is $(1/2)d(\cos(2\alpha))d\beta d\gamma$. Hence in any given spatial region, the values of $u \equiv \cos(2\alpha)$, β and γ are selected from uniform probability distributions in their respective ranges. In spatial regions that are separated by more than some correlation length, (u, β, γ) can be chosen independently. There is a lot of theoretical and experimental literature (for a review see [20]) on the determination of the correlation length and, more recently, a full quantum calculation for the growth of the correlation length [21, 22]. However, the precise value of the correlation length is not a critical quantity for us since this only sets a length scale for the topological defects and does not affect the scaling laws for their distribution.

In the numerical implementation we calculate the (discretized) topological winding for monopoles given by the surface integral in eq. (3.3) as was done for 't Hooft-Polyakov monopoles [23–25]. The implementation also assumes the “geodesic rule”: a triangular plaquette of the spatial lattice gets mapped to a spherical triangle on the vacuum manifold, but three points on a two-sphere define two complementary spherical triangles and we choose the one with the smaller area [2, 26].

We now turn to the Z-strings that connect the monopoles. First we note that \hat{n} is invariant under $K \equiv [U(1)_L \times U(1)_Y]/Z_2$ transformations, where $U(1)_L \subset SU(2)_L$ consists of rotations about the axis \hat{n} and $U(1)_Y$ are phase rotations of Φ . (The Z_2 consists of the common elements, $\pm \mathbf{1}$, contained in both $U(1)_L$ and $U(1)_Y$.) The group K can also be thought of as $U(1)_Z \times U(1)_Q$ where Q denotes the generator of the electromagnetic group and is given by $Q = (\mathbf{1} + \hat{n} \cdot \vec{\sigma})/2$. The generator of $U(1)_Z$ is

$$T_Z = \frac{\mathbf{1} - \hat{n} \cdot \vec{\sigma}}{2}. \quad (4.2)$$

The VEV of Φ is invariant under the electromagnetic $U(1)_Q$ since $Q\Phi = 0$. Thus, for a fixed \hat{n} , there is an entire circles worth of Φ 's given by rotations by $U(1)_Z$. As we go around a spatial plaquette, rotations of the \hat{n} vectors define “parallel transport” of the Φ fields, which may differ from the actual Φ by an element of $U(1)_Z$, as explained in figure 1. Non-trivial winding of the $U(1)_Z$ phase factor implies the existence of a Z-string passing through the plaquette.

Consider one leg of a triangular plaquette as shown in figure 1. The vector \hat{n}_1 is rotated into \hat{n}_2 , i.e. $\hat{n}_2 = R_{21}\hat{n}_1$, by an $SO(3)$ rotation about the axis \hat{a}_{21} and by angle θ_{21} ,

$$\hat{a}_{21} = \frac{\hat{n}_1 \times \hat{n}_2}{|\hat{n}_1 \times \hat{n}_2|}, \quad \theta_{21} = \cos^{-1}(\hat{n}_1 \cdot \hat{n}_2) \quad (4.3)$$

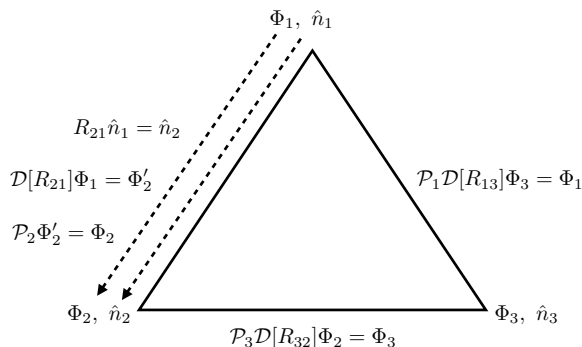


Figure 1. A triangular plaquette is assigned values of Φ at its vertices, from which we determine corresponding values of \hat{n} using (3.4). We find the rotation R_{21} that takes \hat{n}^1 to \hat{n}^2 . This rotation in $SO(3)$ also defines a rotation, $\mathcal{D}[R_{21}]$ in $SU(2)_L$ that acts on Φ_1 to give Φ'_2 which in general differs from Φ_2 by rotation by an element $\mathcal{P}_2 \in U(1)_Z$, at vertex 2. Similarly we can obtain the rotations that take Φ_2 to Φ_3 , and Φ_3 to Φ_1 . The total rotation in going from vertex 1 around the triangle and back to vertex 1 is: $\mathcal{P}_1\mathcal{D}[R_{13}]\mathcal{P}_3\mathcal{D}[R_{32}]\mathcal{P}_2\mathcal{D}[R_{21}]$, and this rotation acts on Φ_1 to give back Φ_1 . If the net Z -phase rotation in going around the plaquette is $\pm 2\pi$, there is a Z -string (or anti-string) passing through the plaquette.

and we take $0 \leq \theta_{12} \leq \pi$. A corresponding $SU(2)_L$ rotation is²

$$\mathcal{D}[R_{21}] = \exp\left(-i\hat{a}_{21} \cdot \vec{\sigma} \frac{\theta_{21}}{2}\right) \tag{4.4}$$

and rotates Φ_1 to,

$$\Phi'_2 = \mathcal{D}[R_{21}]\Phi_1 \tag{4.5}$$

In general, $\Phi'_2 \neq \Phi_2$ and an additional $U(1)_Z$ rotation, \mathcal{P}_2 , may be necessary to rotate Φ_1 to Φ_2 ,

$$\Phi_2 = \mathcal{P}_2 \Phi'_2 = \mathcal{P}_2 \mathcal{D}[R_{21}]\Phi_1 \tag{4.6}$$

where $\mathcal{P}_2 = e^{iT_{Z2}\delta_2}$. T_{Z2} is as defined in (4.2) with $\hat{n} = \hat{n}_2$, and δ_2 is a phase angle. To determine δ_2 we use,

$$e^{i\delta_2} = \Phi_2^\dagger \Phi'_2 \tag{4.7}$$

which can be derived using (4.4). We will choose δ_2 with the smallest value of $|\delta_2|$ in accordance with the geodesic rule [2, 26]. Note that $\mathcal{P}_2 = e^{iT_{Z2}\delta_2}$ acts on Φ'_2 to simply give a phase factor $\exp(i\delta_2)$,

$$e^{iT_{Z2}\delta_2}\Phi'_2 = e^{i\delta_2}\Phi'_2 \tag{4.8}$$

because $\hat{n}_2 = -\hat{\Phi}_2^\dagger \vec{\sigma} \hat{\Phi}_2 = -\hat{\Phi}_2'^\dagger \vec{\sigma} \hat{\Phi}_2'$.

In this way we can go around all the sides of the triangular plaquette and obtain

$$\Phi_1 = \mathcal{P}_1 \mathcal{D}[R_{13}]\mathcal{P}_3 \mathcal{D}[R_{32}]\mathcal{P}_2 \mathcal{D}[R_{21}]\Phi_1 \equiv \mathbf{R}\Phi_1 \tag{4.9}$$

²There are two elements of $SU(2)_L$, namely $\pm\mathcal{D}[R_{21}]$, that correspond to the $SO(3)$ rotation R_{21} . This ambiguity will be absorbed in \mathcal{P}_2 defined in (4.6) as \mathcal{P}_2 also gives a phase factor in its action on Φ'_2 as shown in (4.8).

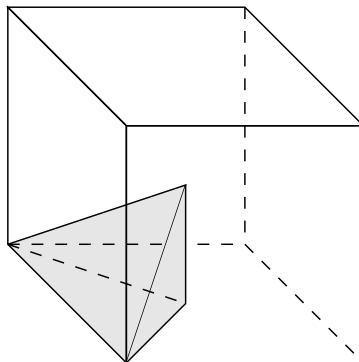


Figure 2. The cubic lattice is divided into tetrahedral cells in our simulations.

The right-most rotation, $\mathcal{D}[R_{21}]\Phi_1$, yields Φ'_2 and, as in (4.8), the action of \mathcal{P}_2 acting on Φ'_2 simply gives a phase factor that commutes with all other rotations in (4.9). Hence the action of \mathcal{P}_2 is to give an overall factor of $e^{i\delta_2}$. Similar arguments apply to the action of \mathcal{P}_1 and \mathcal{P}_3 . Then the action of \mathbf{R} on Φ_1 is equivalent to multiplication by,

$$\mathbf{R} = e^{i(\delta_1 + \delta_2 + \delta_3 + h_{123})} \quad (4.10)$$

where h_{123} denotes the phase angle due to the rotation $\mathcal{D}[R_{13}]\mathcal{D}[R_{32}]\mathcal{D}[R_{21}]$. This rotation implements the parallel transport of Φ_1 all the way around the triangular plaquette and gives the holonomy angle, h_{123} , in this process. To determine h_{123} we use

$$e^{ih_{123}} = \Phi_1^\dagger \mathcal{D}[R_{13}]\mathcal{D}[R_{32}]\mathcal{D}[R_{21}]\Phi_1 \quad (4.11)$$

From (4.9) we must have

$$\delta_1 + \delta_2 + \delta_3 + h_{123} = 0, \pm 2\pi \quad (4.12)$$

and a value of $\pm 2\pi$ signals that a Z-string/anti-string passes through the plaquette.

We have numerically implemented this algorithm to study the distribution of monopoles and strings on a discrete tetrahedral lattice. Each cell of a cubic lattice is divided into 24 tetrahedra [3] as shown in figure 2. At every lattice point, we assign random values of α , β and γ , from which we construct Φ and \hat{n} . We find the monopoles on the lattice by evaluating the monopole winding in (3.3) for every tetrahedral cell, and the strings are found by evaluating the winding in (4.12) for every triangular plaquette. A sample of the monopole distribution with strings is shown in figure 3.

As in earlier simulations of monopole formation [23–25], \hat{n} is uniformly distributed on an S^2 and the magnetic charge within a volume, $\sim L^3$, is given by a surface integral due to Gauss' law, with $N \sim (L/\xi)^2$ independent domains of size ξ on the surface. Hence the root-mean-square magnetic charge within the volume goes as $\sqrt{N} \sim L/\xi$. We have confirmed this scaling in our simulations.

We also evaluate the length distribution of open string segments, i.e. the number density of strings of length between l and $l + dl$, denoted $dn_{\text{open}}(l)$. The dependence of $dn_{\text{open}}(l)$

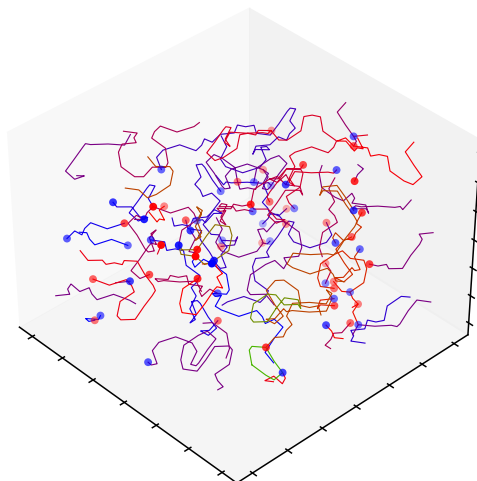


Figure 3. Sample monopole distribution with strings connecting them. Some of the strings are in the form of closed loops.

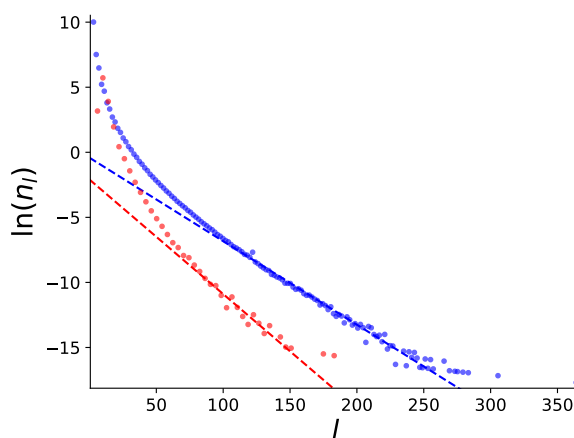


Figure 4. Log-linear plot of number density of open strings (blue) and closed strings (red) vs. length l . The parameters of the dashed fitting curves are given in (4.13) and (4.14).

on l is shown in figure 4 and is fit by a decaying exponential,

$$dn_{\text{open}}(l) = A_o e^{-l/l_o} dl, \quad A_o = 0.12 \pm 0.06, \quad l_o = 6.68 \pm 0.28 \quad (4.13)$$

where the length is measured in units of the step length in going from one tetrahedral cell to its neighboring cell. The number density of closed loops also follows an exponential with,

$$A_c = 0.66 \pm 0.07, \quad l_c = 7.79 \pm 0.08. \quad (4.14)$$

5 Magnetic field

As in the case of topological defects, the Kibble mechanism only provides initial conditions for the evolution of the system. In the case of cosmic strings, small loops formed during the

symmetry breaking will quickly collapse and dissipate, while longer loops and infinite strings will persist and eventually reach a scaling solution. In the electroweak case, monopoles and anti-monopoles will be brought together by the confining strings and rapidly annihilate [27]. However their annihilation will leave behind a magnetic field. Since Maxwell equations hold after electroweak symmetry breaking, the magnetic field can then be evolved with the usual Maxwellian magneto-hydrodynamical (MHD) equations [28]. We now turn to a characterization of the initial magnetic field.

The electromagnetic field strength is defined as

$$A_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i \frac{2 \sin \theta_w}{g} \left(\partial_\mu \hat{\Phi}^\dagger \partial_\nu \hat{\Phi} - \partial_\nu \hat{\Phi}^\dagger \partial_\mu \hat{\Phi} \right) \quad (5.1)$$

where $A_\mu \equiv \sin \theta_w \hat{n}^a W_\mu^a + \cos \theta_w Y_\mu$ and the last term in (5.1) is required for a suitable gauge invariant definition of $A_{\mu\nu}$ [10, 18]. The definition breaks down at points where $|\Phi| = 0$, i.e. in the symmetry restored phase, because \hat{n} and $\hat{\Phi}$ are not well-defined.

The magnetic field of the monopole is

$$\mathbf{B} = \nabla \times \mathbf{A} - i \frac{2 \sin \theta_w}{g} \nabla \hat{\Phi}^\dagger \times \nabla \hat{\Phi} \quad (5.2)$$

With $\Phi = \Phi_m$ of eq. (3.1) and $\mathbf{A} = 0$ we find the monopole magnetic field outside the core of the monopole, $\mathbf{B}_m = \sin \theta_w \hat{r} / (gr^2)$ where r is the radial coordinate. Around the Z-string at $\theta = \pi$ we find $\hat{\Phi}_m \rightarrow e^{i\phi}(0, 1)^T$. Using this form in (5.2) we see that there is no electromagnetic field associated with the Z-string at locations where $\Phi \neq 0$. We can extend the formula (5.2) to the point where $\Phi = 0$ in the Z-string by using continuity, and then the magnetic field vanishes everywhere for the Z-string.

The usual characterization of stochastic isotropic magnetic fields is in terms of the two point correlators,

$$\langle B_i(\mathbf{x} + \mathbf{r}) B_j(\mathbf{x}) \rangle = M_N(r) (\delta_{ij} - \hat{r}_i \hat{r}_j) + M_L(r) \hat{r}_i \hat{r}_j + \epsilon_{ijk} r_k M_H(r) \quad (5.3)$$

In Maxwell theory, the correlation functions M_N and M_L are related by the condition that the magnetic field is divergence free,

$$\frac{1}{2r} \frac{d}{dr} \left(r^2 M_L(r) \right) = M_N(r). \quad (5.4)$$

In our case, however, the magnetic field is not divergence-free and M_N and M_L are independent functions. The helical correlator, M_H , vanishes for us since we have not included any source of parity violation in the system.

We have evaluated the magnetic field correlator numerically and find

$$\langle B_i(\mathbf{x} + \mathbf{r}) B_j(\mathbf{x}) \rangle = f(r) \delta_{ij} \quad (5.5)$$

with $f(r)$ exhibiting anti-correlations at small scales. This makes physical sense since it is known that defects are preferentially surrounded by anti-defects [24].

Once the monopoles and antimonopoles have annihilated, the correlator in (5.5) should revert to the form in (5.3) with the standard divergence free condition. We have not yet

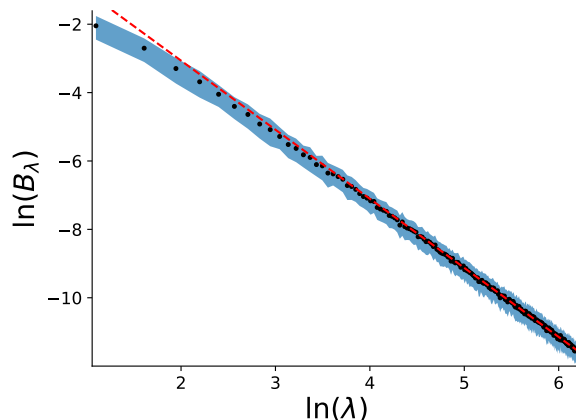


Figure 5. Log-log plot of the smeared magnetic field strength, B_λ , vs. λ . The blue band shows the $1\text{-}\sigma$ spread of the individual Monte Carlo results. The dashed line shows the fit $\ln(B_\lambda) = (-2.02 \pm 0.02)\ln(\lambda) + (0.98 \pm 0.09)$.

studied this evolution. Instead we use a “smearing procedure” to estimate the volume averaged magnetic field due to monopoles,

$$\langle \mathbf{B} \rangle_V = \frac{1}{V} \int_V d^3x \mathbf{B} = -i \frac{2 \sin \theta_w}{gV} \int_{\partial V} d\mathbf{S} \times (\hat{\Phi}^\dagger \nabla \hat{\Phi}) \quad (5.6)$$

where the last expression for the surface integral follows from using (5.2) together with an integration by parts. Note that (5.2) assumes $|\Phi| \neq 0$ and hence is not valid in the interior of the integration volume V in the presence of monopoles. The volume integral in (5.6) is ambiguous because of the divergent magnetic field at the locations of the monopoles. However the surface integral given in (5.6) still applies as the surface of integration does not intersect any monopole cores. The surface may intersect Z-strings but the formula in (5.2) holds by continuity as discussed below (5.2).

For the integration in (5.6) we will consider cubical volumes with side λ . If ξ denotes the size of domains in which the random variable $\hat{\Phi}^\dagger \nabla \hat{\Phi}$ is tightly correlated, the discretized surface integral in (5.6) consists of a sum of $(\lambda/\xi)^2$ independent random terms and the sum itself will go like the square root of this number. Therefore we expect the magnitude $B_\lambda \equiv |\langle \mathbf{B} \rangle_V|$ to grow as $B_\lambda \propto \lambda/V \propto 1/\lambda^2$. We have numerically evaluated B_λ and the result is plotted in figure 5. The fit shows indeed shows that $B_\lambda \propto 1/\lambda^2$.

As a final comment, note that the numerical calculation of the magnetic field does not directly use the network of monopoles and strings discussed in the previous sections. All that is needed is to evaluate the final term of (5.2) from the random distribution of the Higgs VEV.

6 Conclusions

Vacuum configurations of a field theory should include all configurations with minimum energy. Conventional considerations focus on homogeneous fields and then the vacuum manifold is given by the minima of the potential. However, in gauge theories, inhomogeneous

configurations can also have minimum energy provided they lie on gauge orbits on the vacuum manifold. Thus the vacuum manifold has additional structure. In particular, by minimizing the potential of the electroweak model the vacuum manifold is seen to be an S^3 . However the gauge orbits map the S^3 to S^2 with S^1 fibers, i.e. the vacuum manifold is a Hopf fibered S^3 . The topology of $S^2 \times S^1$ leads to electroweak magnetic monopoles that are confined by Z-strings whose distribution we have determined by an extension of the Kibble mechanism. Since the electroweak monopoles are confined by Z-strings, they will annihilate rapidly even as they are formed, leaving behind a cosmological magnetic field whose spectrum falls off slowly with increasing wavelength: $B_k \propto k^2$.

An alternative approach to deriving the properties of the magnetic field is to directly simulate the electroweak symmetry breaking, as has been done in several works [29–33]. These field theory simulations are much more computationally intensive than the present approach and are limited by computer resources. On the flip side, an advantage is that they more completely account for the dynamical evolution during the symmetry breaking, including magnetic fields that may be generated independently of the monopoles (the A_μ terms in (5.1)).

The MHD evolution of magnetic fields depends significantly on the helicity of the field, described by the parity odd M_H correlator in (5.3). There is, however, no source of parity violation in the formulation of the Kibble mechanism, and indeed in the bosonic sector of the electroweak model. Hence the magnetic field will be (globally) non-helical. (The process of monopole annihilation can induce local helicity because, in general, the monopole and antimonopole will be relatively twisted [34].) It is an interesting open question if parity violation from the fermionic sector or extensions of the standard model can be incorporated in the Kibble mechanism, that can then be used to study the generation of helical magnetic fields. Parity violating effects are also necessary for generating cosmic matter-antimatter asymmetry and the connection with magnetic helicity has already been noted [35–41].

The evolution of the magnetic field from the electroweak epoch to the present epoch is affected by several factors: turbulence, cosmic expansion, dissipation, and perhaps novel chiral effects. Magneto-hydrodynamical evolution does not apply initially because the magnetic field is not divergence-free. From general arguments that are supported by numerical simulations, a few percent of the electroweak false vacuum energy goes into magnetic fields during spontaneous symmetry breaking [14]. The coherence scale of the magnetic field at the electroweak epoch, $\xi(t_{EW})$, will depend on the dynamics during electroweak symmetry breaking. To obtain estimates we use an upper bound on the coherence and take it to be the horizon size at the electroweak scale: $\xi(t_{EW}) \sim t_{EW} \sim 1$ cm. Then the magnetic field on length scale λ at the present epoch is given by

$$B_\lambda(t_0) \sim \sqrt{\rho_\gamma(t_0)} \left(\frac{t_{EW}}{\lambda(t_{EW})} \right)^2 \tag{6.1}$$

where $\lambda(t_{EW}) = \lambda(t_0)T_0/T_{EW}$ and T denotes the cosmic temperature. With $T_0 \sim 10^{-4}$ eV, $T_{EW} \sim 10^{11}$ eV, $\sqrt{\rho_\gamma(t_0)} \sim 10^{-6}$ G, we get

$$B_{1 \text{ kpc}}(t_0) \sim 10^{-18} \text{ G}, \tag{6.2}$$

and on Mpc scales the magnetic field is $\sim 10^{-24}$ G. This estimate is much smaller than the blazar lower bounds in the literature: $B_{1\text{Mpc}} \gtrsim 10^{-16} - 10^{-19}$ G [42–45]. Hence the monopoles by themselves cannot provide magnetic fields of the observed strength. Additional ingredients are necessary if the magnetic fields generated during electroweak symmetry breaking are to explain observations. In particular, magnetic helicity can be this necessary ingredient as it can stretch the coherence scale of the magnetic field by a large factor $\sim 10^7$ and increase the field strength estimate to $\sim 10^{-11}$ G on 10 kpc scales [14].

In summary, we have extended the Kibble mechanism and applied it to the electroweak model. Then topological considerations lead to a distribution of magnetic monopoles and Z-strings that we can characterize. The distribution of magnetic monopoles immediately implies the presence of magnetic fields. We have derived the (smeared) magnetic field distribution as a function of the smearing length scale, λ , and find $B_\lambda \propto \lambda^{-2}$. The role of early universe magnetic fields for cosmological observations has been recently reviewed in refs. [12–15].

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