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$O(n, n)$ invariance and Wald entropy formula in the Heterotic Superstring effective action at first order in α'

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ABSTRACT: We perform the toroidal compactification of the full Bergshoeff-de Roo version of the Heterotic Superstring effective action to first order in α' . The dimensionally-reduced action is given in a manifestly- $O(n, n)$ -invariant form which we use to derive a manifestly- $O(n, n)$ -invariant Wald entropy formula which we then use to compute the entropy of α' -corrected, 4-dimensional, 4-charge, static, extremal, supersymmetric black holes.

KEYWORDS: Black Holes in String Theory, String Duality

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Introduction. In a recent paper [1] we have performed the dimensional reduction of the Heterotic Superstring effective action in a circle to first-order in α' with two goals in mind:

1. To study T duality in the dimensionally-reduced theory and the effect that the first-order α' corrections have in it. In particular, we wanted to recover the first-order in α' corrections to the Buscher T duality rules [2, 3] found in ref. [4],¹ and show explicitly that the whole action is invariant under them to that order.
2. To derive a T duality-invariant formula for the Wald entropy using the Iyer-Wald prescription developed in refs. [7–9].²

This formula, though, can only be applied to black holes which can be obtained from a solution of the 10-dimensional theory by 1 non-trivial and several trivial dimensional

¹See also [5, 6].

²A discussion of the caveats in the direct use of this prescription in the Heterotic Superstring effective action can be found in the Introduction of ref. [1]. On the other hand, it is clear that the entropy formula derived in that reference by using this prescription gives results which coincide with those obtained by microstate counting [10] and also satisfy the fundamental thermodynamic relation $\frac{\partial S}{\partial M} = \frac{1}{T}$ in black hole with finite temperature [11]. See also the discussion in the Conclusions section.

reductions over circles. This severely limits its applicability to 5-dimensional black holes and certain 4-dimensional ones.

It is natural to try to extend those results to non-trivial toroidal compactifications, testing the $O(n, n)$ invariance of the dimensionally-reduced action³ to first order in α' and obtaining a manifestly $O(n, n)$ -invariant Wald entropy formula that can be applied to more general black-hole solutions such as, for instance, the heterotic version of the 4-dimensional, 4-charge, static, extremal black holes whose microscopic entropy was first computed in refs. [13, 14].

Earlier work on the effect of α' corrections on the T duality invariance of the Heterotic Superstring effective action more or less complete in different forms and schemes [15], including Double Field Theory, can be found in refs. [16–22] some of which we will comment upon in the main body of this paper. Here we will use the Bergshoeff-de Roo action ref. [23] obtained by supersymmetric completion of the Lorentz Chern-Simons terms in the Kalb-Ramond field strength [24].

This paper is organized as follows: in section 1 we introduce the 10-dimensional Heterotic Superstring effective action in the Bergshoeff-de Roo formulation. In section 2, as a warm-up exercise, we review the toroidal dimensional reduction of the zeroth-order action, rewriting it in a manifestly $O(n, n)$ -invariant form. In section 3 we add Yang-Mills fields, and rewrite the dimensionally-reduced action in an apparently manifestly $O(n, n + n_V)$ -invariant form, reproducing, in the Abelian case (when that invariance is real), the results of Maharana and Schwarz [12]. In section 4 we consider the full $\mathcal{O}(\alpha')$ action, which amounts to the addition of the torsionful spin-connection terms. The full action can be regarded, formally, as that of the previous section with more gauge fields and a gauge group which is the direct product of the Yang-Mills gauge group and the 10-dimensional Lorentz group $SO(1, 9)$ [24] and, in a first stage (section 4.1.1), we can simply use the results of the previous section. This cannot be the final result, though, because, as different from the Yang-Mills group, the 10-dimensional Lorentz group is broken into the $(10 - n)$ -dimensional one and $O(n)$. Thus, in a second stage (section 4.1.2), we perform this decomposition leaving the dimensionally-reduced action in a manifestly gauge-, $(10 - n)$ -dimensional Lorentz-, diffeomorphism- and $O(n, n)$ -invariant form. Then, in section 5 we derive from that action a Wald entropy formula that we test on 4-dimensional 4-charge black holes. We present our conclusions in section 6. Appendix A contains relevant formulae concerning the $O(n, n)/(O(n) \times O(n))$ coset space that we use in the manifestly- $O(n, n)$ -invariant action.

1 The Heterotic Superstring effective action to $\mathcal{O}(\alpha')$

Let us first introduce the Heterotic Superstring effective action to $\mathcal{O}(\alpha')$, where α' is the Regge slope parameter,⁴ in the formulation of ref. [23] but using the conventions of ref. [25].⁵

³If the n_V 10-dimensional gauge fields are Abelian, the theory is expected to have a larger duality group: $O(n, n + n_V)$ [12], but this group is obviously broken when they are non-Abelian, since they cannot be rotated into the Abelian Kaluza-Klein and winding vector fields. Here we will focus mostly on the $O(n, n)$ duality group which is expected to always be present.

⁴The Regge slope parameter is related to the string length ℓ_s by $\alpha' = \ell_s^2$.

⁵The relation between the normalizations of the fields in ref. [23] and here can be found in ref. [26].

The torsionful spin connection and Kalb-Ramond field strength, which are two fundamental ingredients of the action, can be constructed recursively order by order in α' . At zeroth-order, the field strength of the Kalb-Ramond 2-form $B_{\mu\nu}$ is defined as

$$H^{(0)}_{\mu\nu\rho} \equiv 3\partial_{[\mu}B_{\nu\rho]}, \quad (1.1)$$

and it is added as torsion to the (torsionless, metric-compatible) Levi-Civita spin connection 1-form $\omega_\mu{}^a{}_b$ as

$$\Omega_{(\pm)\mu}^{(0)}{}^a{}_b = \omega_\mu{}^a{}_b \pm \frac{1}{2}H^{(0)}_{\mu}{}^a{}_b, \quad (1.2)$$

to construct the zeroth-order torsionful spin connections.

The corresponding zeroth-order Lorentz curvature 2-forms and Chern-Simons 3-forms are defined as

$$R_{(\pm)\mu\nu}^{(0)}{}^a{}_b = 2\partial_{[\mu}\Omega_{(\pm)|\nu]}^{(0)}{}^a{}_b - 2\Omega_{(\pm)[\mu}^{(0)}{}^a{}_c\Omega_{(\pm)|\nu]}^{(0)}{}^c{}_b, \quad (1.3)$$

$$\omega_{(\pm)}^{L(0)} = 3R_{(\pm)[\mu\nu]}^{(0)}{}^a{}_b\Omega_{(\pm)|\rho]}^{(0)}{}^b{}_a + 2\Omega_{(\pm)[\mu}^{(0)}{}^a{}_b\Omega_{(\pm)|\nu]}^{(0)}{}^b{}_c\Omega_{(\pm)|\rho]}^{(0)}{}^c{}_a. \quad (1.4)$$

At first order in α' we also have to take into account the Yang-Mills fields. The gauge field is denoted by A^A_μ , where $A, B, C, \dots$ are the adjoint gauge indices of some group that we will not specify. The corresponding gauge field strength and the Chern-Simons 3-forms are defined by

$$F^A_{\mu\nu} = 2\partial_{[\mu}A^A_{\nu]} + f_{BC}{}^A A^B_{[\mu}A^C_{\nu]}, \quad (1.5)$$

$$\omega^{\text{YM}} = 3F_A{}_{[\mu\nu}A^A_{\rho]} - f_{ABC}A^A_{[\mu}A^B_{\nu}A^C_{\rho]}, \quad (1.6)$$

where we have lowered the adjoint group indices using the Killing metric of K_{AB} : $f_{ABC} \equiv f_{AB}{}^D K_{DB}$ and of the gauge fields $F_{A\mu\nu} \equiv K_{AB}F^B_{\mu\nu}$.

Then, the first-order Kalb-Ramond field strength is given by

$$H^{(1)}_{\mu\nu\rho} = 3\partial_{[\mu}B_{\nu\rho]} + \frac{\alpha'}{4}(\omega^{\text{YM}}_{\mu\nu\rho} + \omega_{(-)}^{L(0)}{}_{\mu\nu\rho}), \quad (1.7)$$

and now it is the torsion of the first-order torsionful spin connection

$$\Omega_{(\pm)\mu}^{(1)}{}^a{}_b = \omega_\mu{}^a{}_b \pm \frac{1}{2}H^{(1)}_{\mu}{}^a{}_b, \quad (1.8)$$

whose curvature $R_{(\pm)\mu\nu}^{(1)}{}^a{}_b$ and Chern-Simons form $\omega_{(\pm)}^{L(1)}$ are now used to define the second-order Kalb-Ramond field strength $H^{(2)}$ and so on.

Only $\Omega_{(\pm)\mu}^{(0)}$, $R_{(\pm)\mu\nu}^{(0)}{}^a{}_b$, $\omega_{(\pm)\mu\nu\rho}^{L(0)}$ and $H^{(1)}_{\mu\nu\rho}$ (plus the Yang-Mills fields) occur in the $\mathcal{O}(\alpha')$ action and, in terms of these objects plus the dilaton field ϕ and the Ricci scalar R of the metric $g_{\mu\nu}$, the first-order in α' Heterotic Superstring effective action in the string frame takes the form

$$S = \frac{g_s^2}{16\pi G_N^{(10)}} \int d^{10}x \sqrt{|g|} e^{-2\phi} \left\{ R - 4(\partial\phi)^2 + \frac{1}{12}H^2 - \frac{\alpha'}{8} [F_A \cdot F^A + R_{(-)}{}^a{}_b \cdot R_{(-)}{}^b{}_a] \right\}, \quad (1.9)$$

here g_s is the Heterotic Superstring coupling constant, which is given by the vacuum expectation value of e^ϕ , the dot indicates the contraction of the indices of 2-forms: $F_A \cdot F^A \equiv F_{A\mu\nu} F^{A\mu\nu}$ and the 10-dimensional Newton constant $G_N^{(10)}$ is related to the string length and coupling constants by

$$G_N^{(10)} = 8\pi^6 g_s^2 \ell_s^8. \quad (1.10)$$

2 Dimensional reduction on T^n at zeroth order in α'

As a warm-up exercise (and also because of the recursive definition of the action), we review the well-known dimensional reduction of the action at zeroth order in α' using the Scherk-Schwarz formalism [27]. We add hats to all the 10-dimensional objects (fields, indices, coordinates) and split the 10-dimensional world and tangent-space indices as $(\hat{\mu}) = (\mu, m)$ and $(\hat{a}) = (a, i)$, with $\mu, \nu, \dots$ and $a, b, \dots = 0, 1, \dots, 9 - n$ and $m, n, \dots$ and $i, j, \dots = 1, \dots, n$.

The Zehnbain and inverse-Zehnbain components $\hat{e}_{\hat{\mu}}^{\hat{a}}$ and $\hat{e}_{\hat{a}}^{\hat{\mu}}$ can be put in an upper-triangular form by a local Lorentz transformation and, then, they can be decomposed in terms of the $10 - n$ -dimensional Vielbein and inverse Vielbein components e_μ^a, e_a^μ , Kaluza-Klein (KK) vectors A^m_μ and internal (T^n) metric Vielbeins and inverse Vielbein e_m^i, e_i^m

$$\left(\hat{e}_{\hat{\mu}}^{\hat{a}}\right) = \begin{pmatrix} e_\mu^a & A^m_\mu e_m^i \\ 0 & e_m^i \end{pmatrix}, \quad \left(\hat{e}_{\hat{a}}^{\hat{\mu}}\right) = \begin{pmatrix} e_a^\mu & -A^m_a \\ 0 & e_i^m \end{pmatrix}. \quad (2.1)$$

where $A^m_a = e_a^\mu A^m_\mu$. We will always assume that all the $(10 - n)$ -dimensional fields with Lorentz indices are $(10 - n)$ -dimensional world tensors contracted with the $(10 - n)$ -dimensional Vielbeins. For instance, the field strengths of the KK vector fields F^m_{ab} are

$$F^m_{ab} = e_a^\mu e_b^\nu F^m_{\mu\nu}, \quad F^m_{\mu\nu} \equiv 2\partial_{[\mu} A^m_{\nu]}. \quad (2.2)$$

We denote the internal metric by

$$G_{mn} \equiv e_m^i e_n^j \delta_{ij}. \quad (2.3)$$

The relation between the components of the 10-dimensional metric and $(10 - n)$ -dimensional KK fields is

$$\hat{g}_{\mu\nu} = g_{\mu\nu} - G_{mn} A^m_\mu A^n_\nu, \quad (2.4a)$$

$$\hat{g}_{\mu m} = -G_{mn} A^n_\mu, \quad (2.4b)$$

$$\hat{g}_{mn} = -G_{mn}. \quad (2.4c)$$

The components of the 10-dimensional spin connection $\hat{\omega}_{\hat{a}\hat{b}\hat{c}}$ decompose into those of the $(10 - n)$ -dimensional one ω_{abc} , the KK vector field strengths $e_{im} F^m_{ab}$ and the pullback of the $O(n)$ connection 1-form A^{ij} defined in eq. (A.10), as follows:

$$\begin{aligned} \hat{\omega}_{abc} &= \omega_{abc}, & \hat{\omega}_{abi} &= -\frac{1}{2} e_{im} F^m_{ab}, \\ \hat{\omega}_{ibc} &= -\hat{\omega}_{bci}, & \hat{\omega}_{aij} &= A^{ij}_a, \\ \hat{\omega}_{ibj} &= -\frac{1}{2} e_i^m e_j^n \partial_b G_{mn}, \end{aligned} \quad (2.5)$$

where we have used

$$e_{(i|}{}^m \partial_a e_{|m|j)} = -\frac{1}{2} e_i{}^m e_j{}^n \partial_a G_{mn} . \quad (2.6)$$

Then, using the Palatini identity, it is not difficult to see that the first two terms in the action eq. (1.9) take the following $(10-n)$ -dimensional form (up to a total derivative):

$$\begin{aligned} & \int d^{10} \hat{x} \sqrt{|\hat{g}|} e^{-2\hat{\phi}} \left\{ \hat{R} - 4(\partial\hat{\phi})^2 \right\} \\ &= \int d^n z \int d^{10-n} x \sqrt{|g|} e^{-2\phi} \left\{ R - 4(\partial\phi)^2 - \frac{1}{4} \partial_a G_{mn} \partial^a G^{mn} - \frac{1}{4} G_{mn} F^m \cdot F^n \right\} , \end{aligned} \quad (2.7)$$

where the $(10-n)$ -dimensional dilaton field is related to the 10-dimensional one by

$$\phi \equiv \hat{\phi} - \frac{1}{2} \log |G| , \quad |G| \equiv \det(G_{mn}) . \quad (2.8)$$

At zeroth order in α' , the last term that we have to reduce is the kinetic term of the Kalb-Ramond 2-form $\sim (\hat{H}^{(0)})^2$. Following Scherk and Schwarz, we consider the Lorentz components of the 3-form field strength, because they are automatically combinations of gauge-invariant objects. These are given in terms of the world-indices components by

$$\hat{H}_{ijk} = e_i{}^m e_j{}^n e_k{}^p \hat{H}_{mnp} , \quad (2.9a)$$

$$\hat{H}_{aij} = e_i{}^m e_j{}^n e_a{}^\mu \left[\hat{H}_{\mu mn} - A^p{}_\mu \hat{H}_{pmn} \right] , \quad (2.9b)$$

$$\hat{H}_{abi} = e_i{}^m e_a{}^\mu e_b{}^\nu \left[\hat{H}_{\mu\nu m} - 2A^n{}_{[\nu} \hat{H}_{\mu]nm} + A^n{}_{[\mu} A^p{}_{\nu]} \hat{H}_{npm} \right] , \quad (2.9c)$$

$$\hat{H}_{abc} = e_a{}^\mu e_b{}^\nu e_c{}^\rho \left[\hat{H}_{\mu\nu\rho} - 3A^m{}_{[\mu} \hat{H}_{\nu\rho]m} + 3A^m{}_{[\mu} A^n{}_{\nu} \hat{H}_{\rho]mn} - A^m{}_{[\mu} A^n{}_{\nu} A^p{}_{\rho]} \hat{H}_{mnp} \right] , \quad (2.9d)$$

in general. At zeroth order in α' , $\hat{H}^{(0)}{}_{mnp} = 0$ and the above expressions are simplified to

$$\hat{H}^{(0)}{}_{ijk} = 0 , \quad (2.10a)$$

$$\begin{aligned} \hat{H}^{(0)}{}_{aij} &= e_i{}^m e_j{}^n e_a{}^\mu \hat{H}^{(0)}{}_{\mu mn} \\ &= e_i{}^m e_j{}^n \partial_a B^{(0)}{}_{mn} , \end{aligned} \quad (2.10b)$$

$$\begin{aligned} \hat{H}^{(0)}{}_{abi} &= e_i{}^m e_a{}^\mu e_b{}^\nu \left[\hat{H}^{(0)}{}_{\mu\nu m} - 2A^n{}_{[\nu} \hat{H}^{(0)}{}_{\mu]nm} \right] \\ &= e_i{}^m \left[G^{(0)}{}_{mab} - B_{mn} F^n{}_{ab} \right] , \end{aligned} \quad (2.10c)$$

$$\begin{aligned} \hat{H}^{(0)}{}_{abc} &= e_a{}^\mu e_b{}^\nu e_c{}^\rho \left[\hat{H}^{(0)}{}_{\mu\nu\rho} - 3A^m{}_{[\mu} \hat{H}^{(0)}{}_{\nu\rho]m} + 3A^m{}_{[\mu} A^n{}_{\nu} \hat{H}^{(0)}{}_{\rho]mn} \right] \\ &= H^{(0)}{}_{abc} , \end{aligned} \quad (2.10d)$$

where we have defined the potentials

$$B_{mn} \equiv \hat{B}^{(0)}{}_{mn} , \quad (2.11a)$$

$$B^{(0)}{}_{m\mu} \equiv \hat{B}_{\mu m} + \hat{B}_{mn} A^n{}_\mu , \quad (2.11b)$$

$$B^{(0)}{}_{\mu\nu} \equiv \hat{B}_{\mu\nu} + A^m{}_{[\mu} \hat{B}_{\nu]m} , \quad (2.11c)$$

and the field strengths

$$G^{(0)}_{m\mu\nu} \equiv 2\partial_{[\mu}B^{(0)}_{m|\nu]}, \quad (2.12a)$$

$$H^{(0)}_{\mu\nu\rho} \equiv 3\partial_{[\mu}B^{(0)}_{\nu\rho]} - \frac{3}{2}A^m_{[\mu}G^{(0)}_{m|\nu\rho]} - \frac{3}{2}B^{(0)}_{m[\mu}F^m_{\nu\rho]}. \quad (2.12b)$$

Then, the reduction of the Kalb-Ramond kinetic term gives

$$\begin{aligned} \hat{H}^{(0)2} &= H^{(0)2} - 3G^{mn} \left(G^{(0)}_m - B^{(0)}_{mp}F^p \right) \cdot \left(G^{(0)}_n - B^{(0)}_{nq}F^q \right) \\ &\quad + 3G^{mn}G^{pq}\partial_a B^{(0)}_{mp}\partial^a B^{(0)}_{nq}, \end{aligned} \quad (2.13)$$

and, after integrating over the length of the compact coordinates z^m ($2\pi\ell_s$ by convention) it can be checked that the whole $\mathcal{O}(1)$ action can be written in the compact form⁶

$$\begin{aligned} S^{(0)} &= \frac{g_s^{(10-n)2}}{16\pi G_N^{(10-n)}} \int d^{10-n}x \sqrt{|g|} e^{-2\phi} \left\{ R - 4(\partial\phi)^2 - \frac{1}{8} \text{Tr} \left(\partial_a M^{(0)-1} \partial^a M^{(0)} \right) \right. \\ &\quad \left. - \frac{1}{4} \mathcal{F}^{(0)\text{T}} M^{(0)-1} \cdot \mathcal{F}^{(0)} + \frac{1}{12} H^{(0)2} \right\}, \end{aligned} \quad (2.15)$$

where $M^{(0)}$ is the $\text{O}(n, n)$ matrix defined in eq. (A.4) of appendix A with B_{mn} replaced by $B^{(0)}_{mn}$ and where we have defined the $\text{O}(n, n)$ vectors of 1-forms and 2-form field strengths

$$\mathcal{A}^{(0)} \equiv \begin{pmatrix} A^m \\ B^{(0)}_m \end{pmatrix}, \quad \mathcal{F}^{(0)} \equiv \begin{pmatrix} F^m \\ G^{(0)}_m \end{pmatrix}. \quad (2.16)$$

It is easy to show that $M^{(0)}$ is, indeed, a $\text{O}(n, n)$ matrix

$$M^{(0)}\Omega^{(0)}M^{(0)}\Omega^{(0)} = 1, \quad (2.17)$$

and rewrite the Kalb-Ramond field strength in the manifestly $\text{O}(n, n)$ -invariant form

$$H^{(0)}_{\mu\nu\rho} = 3\partial_{[\mu}B^{(0)}_{\nu\rho]} - \frac{3}{2}\mathcal{A}^{(0)\text{T}}_{[\mu}\Omega^{(0)}\mathcal{F}^{(0)}_{\nu\rho]}. \quad (2.18)$$

Actually, as it is well-known, the zeroth-order action $S^{(0)}$ given in eq. (2.15) is manifestly invariant under $\text{O}(n, n)$ transformations which are understood as T-duality transformations from the 10-dimensional point of view.

3 Dimensional reduction on T^n with Yang-Mills fields and Heterotic Supergravity

In this section we are only going to take into account the addition of the Yang-Mills fields which occur at first order in α' , ignoring for the moment the terms that involve

⁶The 10-dimensional string coupling constant g_s and Newton constant $G_N^{(10)}$ and the $(10-n)$ -dimensional ones $g_s^{(10-n)}$ and $G_N^{(d)}$ are related by

$$g_s^2 = V_n / (2\pi\ell_s)^n g_s^{(10-n)2}, \quad (2.14a)$$

$$G_N^{(10)} = G_N^{(10-n)} V_n, \quad (2.14b)$$

where V_n is the volume of the n -dimensional compact space. Then,

$$\frac{g_s^2}{16\pi G_N^{(10)}} \int d^n z = \frac{g_s^2 (2\pi\ell_s)^n}{16\pi G_N^{(10)}} = \frac{g_s^{(10-n)2}}{16\pi G_N^{(10-n)}}.$$

the torsionful spin connection. This truncation, which constitutes an intermediate step towards our final goal, is interesting by itself because it corresponds to the bosonic sector of a theory with exact local supersymmetry:⁷ $\mathcal{N} = 1, d = 10$ supergravity coupled to non-Abelian vector supermultiplets, also known as *Heterotic Supergravity*, constructed in ref. [28]. The action of this theory is

$$S^{(h)} = \frac{g_s^2}{16\pi G_N^{(10)}} \int d^{10}x \sqrt{|g|} e^{-2\phi} \left\{ R - 4(\partial\phi)^2 + \frac{1}{12} H^{(h)2} - \frac{\alpha'}{8} F_A \cdot F^A \right\}, \quad (3.1)$$

where

$$H^{(h)}_{\mu\nu\rho} = 3\partial_{[\mu} B_{\nu\rho]} + \frac{\alpha'}{4} \omega^{\text{YM}}_{\mu\nu\rho}, \quad (3.2)$$

and F^A and ω^{YM} are defined in eqs. (1.5) and (1.6), respectively.

Notice that the $\mathcal{O}(\alpha'^2)$ terms of this action have to be kept in order to have exact local supersymmetry.

The toroidal dimensional reduction of this theory in the case in which the gauge group is Abelian was carried out along the same lines we are going to follow here in refs. [12, 29]. In the second of these references the $O(n, n + n_V)$ global symmetry of the resulting action was related to the T-duality transformations of the Heterotic Superstring. In the non-Abelian case, the gauge fields coming from the 10-dimensional gauge fields cannot be rotated into Kaluza-Klein and winding vector fields coming from the 10-dimensional metric and Kalb-Ramond fields. As a result, $O(n, n + n_V)$ is broken to $O(n, n)$, or $O(n, n + n_A)$ where n_A is the number of Abelian gauge fields.

The reduction of the Einstein-Hilbert term and of the scalar kinetic term are not modified by the inclusion of α' corrections. The definitions of $(10 - n)$ -dimensional metric $g_{\mu\nu}$, dilaton ϕ , KK vectors A^m_μ and scalars G_{mn} in terms of the 10-dimensional fields are not modified by them either and they are still given by eqs. (2.4) and (2.8).

Because of the additional Yang-Mills Chern-Simons term in the Kalb-Ramond field strength, we do expect modifications in the definitions of the definitions of the $(10 - n)$ -fields that originate in the Kalb-Ramond 2-form, namely the $(10 - n)$ -dimensional Kalb-Ramond 2-form $B^{(h)}_{\mu\nu}$, the winding vectors $B^{(h)}_{m\mu}$, with respect to their zeroth-order counterparts defined in eqs. (2.11).⁸

3.1 Reduction of the Yang-Mills fields

It is convenient to start by studying the dimensional reduction of the Yang-Mills fields. The Lorentz-indices decomposition of the gauge field is

$$\hat{A}^A_i = e_i^m \hat{A}^A_m \equiv \varphi^A_i, \quad (3.3a)$$

$$\hat{A}^A_a = \hat{e}_a^\mu \hat{A}^A_\mu = e_a^\mu \left(\hat{A}^A_\mu - \hat{A}^A_m A^m_\mu \right) \equiv e_a^\mu A^A_\mu, \quad (3.3b)$$

⁷That is, the action is exactly invariant, not just up to terms of higher order in α' .

⁸We use the superscript (h) to indicate that these are the fields that arise in the reduction of Heterotic Supergravity and that possible contributions from the torsionful spin connection have not been taken into account.

which leads to the definition of the $(10 - n)$ -dimensional adjoint scalars φ^A_i and gauge vectors

$$\varphi^A_i \equiv e_i^m \hat{A}_m^A, \quad (3.4a)$$

$$A^A_\mu \equiv \hat{A}_\mu^A - \hat{A}_m^A A^m_\mu. \quad (3.4b)$$

The components of 10-dimensional gauge field strength can be decomposed in terms of these fields as follows:

$$\hat{F}^A_{ij} = f^A_{BC} \varphi^B_i \varphi^C_j, \quad (3.5a)$$

$$\hat{F}^A_{ai} = \mathfrak{D}_a \varphi^A_i + \frac{1}{2} \varphi^A_j e^j_m e^i_n \partial_a G^{mn}, \quad (3.5b)$$

$$\hat{F}^A_{ab} = F^A_{ab} + \varphi^A_i e^i_m F^m_{ab}, \quad (3.5c)$$

where $F^A_{\mu\nu}$ is the standard Yang-Mills gauge field strength for the $(10 - n)$ -dimensional gauge fields A^A_μ and $\mathfrak{D}$ is the Yang-Mills and $O(n)$ -covariant derivative

$$\mathfrak{D}_a \varphi^A_i = \partial_a \varphi^A_i + f^A_{BC} A^B_a \varphi^C_i + A^{ij}_a \varphi^A_j, \quad (3.6)$$

where the $SO(n)$ composite connection is given in eq. (A.10).

$$\begin{aligned} \hat{F}_A \cdot \hat{F}^A &= F_A \cdot F^A + 2\varphi^A_i e^i_m F^m \cdot F_A + \varphi^A_i \varphi_{Aj} e^i_m e^j_n F^m \cdot F^n \\ &\quad - \frac{1}{2} \varphi^A_i \varphi_{Aj} e^i_m e^j_n G_{pq} \partial^a G^{mp} \partial_a G^{nq} - 2\mathfrak{D}^a \varphi^A_i \mathfrak{D}_a \varphi_{Ai} \\ &\quad - 2\mathfrak{D}^a \varphi^A_i \varphi_{Aj} e^i_m e^j_n \partial_a G^{mn} + f_{ABC} f^A_{DE} \varphi^B_i \varphi^D_j \varphi^C_k \varphi^E_k. \end{aligned} \quad (3.7)$$

Our next goal is the reduction of the Kalb-Ramond 3-form field strength. It is convenient to start with the reduction of the Yang-Mills Chern-Simons term

$$\hat{\omega}^{YM}_{ijk} = 2f_{ABC} \varphi^A_i \varphi^B_j \varphi^C_k, \quad (3.8a)$$

$$\hat{\omega}^{YM}_{aij} = 2\mathfrak{D}_a \varphi^A_{[i} \varphi_{A]j}, \quad (3.8b)$$

$$\hat{\omega}^{YM}_{abi} = \left(2F^A_{ab} + \varphi^A_j e^j_m F^m_{ab} \right) \varphi_{Ai} - 2e_{[a}^\mu e_{b]}^\nu e_i^m \partial_\mu \left(\hat{A}_\nu^A \hat{A}_{Am} \right), \quad (3.8c)$$

$$\hat{\omega}^{YM}_{abc} = \omega^{YM}_{abc} + \frac{3}{2} \varphi_{Ai} e^i_m A^A_{[a} F^m_{bc]}. \quad (3.8d)$$

3.2 Reduction of the Kalb-Ramond field

Combining the results in the reduction of $\hat{H}^{(0)}$ with the reduction of the Yang-Mills fields, we find

$$\hat{H}^{(h)}_{ijk} = \frac{\alpha'}{2} f_{ABC} \varphi^A_i \varphi^B_j \varphi^C_k, \quad (3.9a)$$

$$\hat{H}^{(h)}_{aij} = e_i^m e_j^n \partial_a B_{mn} + \frac{\alpha'}{2} \mathfrak{D}_a \varphi^A_{[i} \varphi_{A]j}, \quad (3.9b)$$

$$\begin{aligned} \hat{H}^{(h)}_{abi} &= e_i^m e_{[a}^\mu e_{b]}^\nu \left\{ 2\partial_\mu \left[\hat{B}_{\nu m} + B_{mn} A^n_\nu - \frac{\alpha'}{4} A^A_\nu \hat{A}_{Am} \right] \right. \\ &\quad \left. - \left[B_{mn} - \frac{\alpha'}{4} \hat{A}_m^A \hat{A}_{An} \right] F^n_{\mu\nu} + \frac{\alpha'}{2} \hat{A}_{Am} F^A_{\mu\nu} \right\} \end{aligned} \quad (3.9c)$$

$$\begin{aligned}
 \hat{H}^{(h)}{}_{abc} = & e_{[a}{}^\mu e_b{}^\nu e_{c]}{}^\rho \left\{ 3\partial_\mu \left[\hat{B}_{\nu\rho} + A^m{}_\nu \hat{B}_{\rho m} + \frac{\alpha'}{4} \hat{A}_{Am} A^m{}_\nu A^A{}_\rho \right] \right. \\
 & - 3A^m{}_\mu \partial_\nu \left[\hat{B}_{\rho m} + B_{mn} A^n{}_\rho - \frac{\alpha'}{4} A^A{}_\rho \hat{A}_{Am} \right] \\
 & \left. - 3 \left[\hat{B}_{\mu m} + B_{mn} A^n{}_\mu - \frac{\alpha'}{4} A^A{}_\mu \hat{A}_{Am} \right] \partial_\nu A^m{}_\rho + \frac{\alpha'}{4} \omega^{YM}{}_{\mu\nu\rho} \right\}. \quad (3.9d)
 \end{aligned}$$

This result suggests the following definitions of $(10 - n)$ -dimensional fields:

$$B^{(h)}{}_{mn} \equiv \hat{B}_{mn} - \frac{\alpha'}{4} \hat{A}^A{}_m \hat{A}_{An}, \quad (3.10a)$$

$$\begin{aligned}
 B^{(h)}{}_{m\mu} & \equiv \hat{B}_{\mu m} + B_{mn} A^n{}_\mu - \frac{\alpha'}{4} A^A{}_\mu \hat{A}_{Am} \\
 & = \hat{B}_{\mu m} + \left(\hat{B}_{mn} - \frac{\alpha'}{4} \hat{A}^A{}_m \hat{A}_{An} \right) \hat{g}^{np} \hat{g}_{p\mu} - \frac{\alpha'}{4} \hat{A}_{Am} \hat{A}^A{}_\mu, \quad (3.10b)
 \end{aligned}$$

$$\begin{aligned}
 B^{(h)}{}_{\mu\nu} & \equiv \hat{B}_{\mu\nu} + A^m{}_{[\mu} \hat{B}_{\nu]m} + \frac{\alpha'}{4} \hat{A}_{Am} A^m{}_{[\mu} A^A{}_{\nu]} \\
 & = \hat{B}_{\mu\nu} + \hat{g}^{mn} \hat{g}_{m[\mu} \hat{B}_{\nu]n} + \frac{\alpha'}{4} \hat{A}_{Am} \hat{g}^{mn} \hat{g}_{n[\mu} \hat{A}^A{}_{\nu]}, \quad (3.10c)
 \end{aligned}$$

and $(10 - n)$ -dimensional field strengths

$$G^{(h)}{}_{m\mu\nu} \equiv 2\partial_{[\mu} B^{(h)}{}_{m|\nu]}, \quad (3.11a)$$

$$H^{(h)}{}_{\mu\nu\rho} \equiv 3\partial_{[\mu} B^{(h)}{}_{\nu\rho]} - \frac{3}{2} A^m{}_{[\mu} G^{(h)}{}_{m|\nu\rho]} - \frac{3}{2} B^{(h)}{}_{m[\mu} F^m{}_{\nu\rho]} + \frac{\alpha'}{4} \omega^{YM}{}_{\mu\nu\rho}, \quad (3.11b)$$

This allows us to rewrite the components of the Kalb-Ramond field strength in the form

$$\hat{H}^{(h)}{}_{ijk} = \frac{\alpha'}{2} f_{ABC} \varphi^A{}_i \varphi^B{}_j \varphi^C{}_k, \quad (3.12a)$$

$$\hat{H}^{(h)}{}_{aij} = e_i{}^m e_j{}^n \partial_a B^{(h)}{}_{mn} + \frac{\alpha'}{2} \mathfrak{D}_a \varphi^A{}_i \varphi_{Aj}, \quad (3.12b)$$

$$\hat{H}^{(h)}{}_{abi} = e_i{}^m \left(G^{(h)}{}_{mab} - B^{(h)}{}_{mn} F^n{}_{ab} \right) + \frac{\alpha'}{2} F^A{}_{ab} \varphi_{Ai}, \quad (3.12c)$$

$$\hat{H}^{(h)}{}_{abc} = H^{(h)}{}_{abc}, \quad (3.12d)$$

so that the reduction of the kinetic term is

$$\begin{aligned}
 \hat{H}^{(h)2} = & H^{(h)2} - 3G^{mn} \left(G^{(h)}{}_m - B^{(h)}{}_{mp} F^p \right) \cdot \left(G^{(h)}{}_n - B^{(h)}{}_{nq} F^q \right) \\
 & - \frac{3\alpha'^2}{4} \varphi^A{}_i \varphi^B{}_i F^A \cdot F^B - 3\alpha' e_i{}^m \varphi^A{}_i \left(G^{(h)}{}_m - B^{(h)}{}_{mn} F^n \right) \cdot F^A \\
 & + 3G^{mn} G^{pq} \partial^a B^{(h)}{}_{mp} \partial_a B^{(h)}{}_{nq} + \frac{3\alpha'^2}{4} \varphi^A{}_j \varphi^B{}_j \mathfrak{D}^a \varphi_{Ai} \mathfrak{D}_a \varphi_{Bi} \\
 & + 3\alpha' \mathfrak{D}^a \varphi_{Ai} \varphi^A{}_j e_i{}^m e_j{}^n \partial_a B^{(h)}{}_{mn} - \frac{\alpha'^2}{4} f_{ABC} f_{A'B'C'} \varphi^A{}_i \varphi^{A'}{}_i \varphi^B{}_j \varphi^{B'}{}_j \varphi^C{}_k \varphi^{C'}{}_k. \quad (3.13)
 \end{aligned}$$

Collecting all the terms (that is: eqs. (2.7), (3.7) and (3.13)), we get

$$\begin{aligned}
 S^{(h)} = & \frac{g_s^{(10-n)2}}{16\pi G_N^{(10-n)}} \int d^{10-n} x \sqrt{|g|} e^{-2\phi} \left\{ R - 4(\partial\phi)^2 \right. \\
 & - \frac{1}{4} \partial_a G_{mn} \partial^a G^{mn} + \frac{\alpha'}{16} \varphi^A{}_i \varphi_{Aj} e^i{}_m e^j{}_n G_{pq} \partial^a G^{mp} \partial_a G^{nq} + \frac{1}{4} G^{mn} G^{pq} \partial^a B^{(h)}{}_{mp} \partial_a B^{(h)}{}_{nq} \\
 & + \frac{\alpha'}{4} \mathfrak{D}^a \varphi^A{}_i \mathfrak{D}_a \varphi_{Ai} + \frac{\alpha'^2}{16} \varphi^A{}_j \varphi^B{}_j \mathfrak{D}^a \varphi_{Ai} \mathfrak{D}_a \varphi_{Bi} \\
 & + \frac{\alpha'}{4} \mathfrak{D}^a \varphi_{Ai} \varphi^A{}_j e^i{}_m e^j{}_n \partial_a B^{(h)}{}_{mn} + \frac{\alpha'}{4} \mathfrak{D}^a \varphi^A{}_i \varphi_{Aj} e^i{}_m e^j{}_n \partial_a G^{mn} \\
 & - \frac{1}{4} \left(G_{mn} - B^{(h)}{}_{mp} G^{pq} B^{(h)}{}_{qn} + \frac{\alpha'}{2} \varphi^A{}_i \varphi_{Aj} e^i{}_m e^j{}_n \right) F^m \cdot F^n \\
 & - \frac{1}{4} G^{mn} G^{(h)}{}_m \cdot G^{(h)}{}_n + \frac{1}{2} G^{mp} B^{(h)}{}_{pn} G^{(h)}{}_m \cdot F^n \\
 & - \frac{\alpha'}{8} \left(K_{AB} + \frac{\alpha'}{2} \varphi_{Ai} \varphi_{Bi} \right) F^A \cdot F^B - \frac{\alpha'}{4} \varphi_{Ai} e^i{}_m G^{mn} G^{(h)}{}_n \cdot F^A \\
 & \left. - \frac{\alpha'}{4} \varphi_{Ai} e^i{}_m \left(\delta^m{}_n - G^{mp} B^{(h)}{}_{pn} \right) F^n \cdot F^A + \frac{1}{12} H^{(h)} \cdot H^{(h)} - V(\varphi) \right\}, \tag{3.14}
 \end{aligned}$$

where K_{AB} is the Killing metric of the gauge group and we have defined the scalar potential

$$V(\varphi) \equiv \frac{\alpha'}{8} f_{ABC} f^A{}_{DE} \varphi^B{}_i \varphi^D{}_i \varphi^C{}_j \varphi^E{}_j + \frac{\alpha'^2}{48} f_{ABC} f^A{}_{B'C'} \varphi^A{}_i \varphi^{A'}{}_i \varphi^B{}_j \varphi^{B'}{}_j \varphi^C{}_k \varphi^{C'}{}_k. \tag{3.15}$$

Defining the scalar matrices

$$G \equiv (G_{mn}), \quad B^{(h)} \equiv (B^{(h)}{}_{mn}), \quad \varphi \equiv (\varphi^A{}_i e^i{}_m), \quad K \equiv \frac{\alpha'}{2} (K_{AB}), \tag{3.16}$$

and the $O(n, n + n_V)$ vector of 2-form field strengths

$$\mathcal{F}^{(h)}{}_{\mu\nu} \equiv \begin{pmatrix} F^m{}_{\mu\nu} \\ G^{(h)}{}_{m\mu\nu} \\ F^A{}_{\mu\nu} \end{pmatrix}, \tag{3.17}$$

we can rewrite their kinetic terms in the form

$$- \frac{1}{4} \mathcal{F}^{(h)\text{T}} M^{(h)} \cdot \mathcal{F}^{(h)}, \tag{3.18}$$

where $M^{(h)}$ is the symmetric matrix

$$M^{(h)-1} \equiv \begin{pmatrix} G + B^{(h)\text{T}} G^{-1} B^{(h)} + \varphi^{\text{T}} K \varphi & -B^{(h)\text{T}} G^{-1} & \left(\mathbb{1}_{n \times n} - B^{(h)\text{T}} G^{-1} \right) \varphi^{\text{T}} K \\ -G^{-1} B^{(h)} & G^{-1} & G^{-1} \varphi^{\text{T}} K \\ K \varphi \left(\mathbb{1}_{n \times n} - G^{-1} B^{(h)} \right) & K \varphi G^{-1} & K + K \varphi G^{-1} \varphi^{\text{T}} K \end{pmatrix}. \tag{3.19}$$

This is an $O(n, n + n_V)$ matrix (for n_V gauge fields) because it satisfies

$$M^{(h)}\Omega^{(h)}M^{(h)}\Omega^{(h)} = 1, \quad \text{with} \quad \Omega^{(h)} \equiv \begin{pmatrix} 0 & \mathbb{1}_{n \times n} & 0 \\ \mathbb{1}_{n \times n} & 0 & 0 \\ 0 & 0 & -K \end{pmatrix}, \quad (3.20)$$

in a basis in which the Killing metric is not simply the identity.

The kinetic terms of the scalar fields can be written in the form

$$-\frac{1}{8}\mathfrak{D}^a M^{(h)}\mathfrak{D}_a M^{(h)-1}, \quad (3.21)$$

where the covariant derivative only acts on the gauge group.⁹ The total action takes a form very similar to the zeroth-order one eq. (2.15):

$$\begin{aligned} S^{(h)} = & \frac{g_s^{(10-n)2}}{16\pi G_N^{(10-n)}} \int d^{10-n}x \sqrt{|g|} e^{-2\phi} \left\{ R - 4(\partial\phi)^2 - \frac{1}{8}\text{Tr} \left(\mathfrak{D}^a M^{(h)}\mathfrak{D}_a M^{(h)-1} \right) \right. \\ & \left. - \frac{1}{4}\mathcal{F}^{(h)\text{T}} M^{(h)-1} \cdot \mathcal{F}^{(h)} + \frac{1}{12}H^{(h)2} - V^{(h)}(\varphi) \right\}. \end{aligned} \quad (3.22)$$

At first sight, this action is formally $O(n, n + n_V)$ -invariant except for the scalar potential. However, we cannot transform Abelian into non-Abelian fields and *vice-versa* and the $O(n, n + n_V)$ invariance is broken in the Chern-Simons term of $H^{(h)}$, in the kinetic term of the vector fields and also in the kinetic terms of the scalars and therefore, generically, the invariance is broken to just $O(n, n)$. If the gauge group is Abelian, the scalar potential disappears, the covariant derivatives of the scalars can be rewritten entirely in terms of partial derivatives and $H^{(h)}$ takes the manifestly $O(n, n + n_V)$ -invariant form

$$H^{(h)}_{\mu\nu\rho} = 3\partial_{[\mu}B^{(h)}_{\nu\rho]} - \frac{3}{2}\mathcal{A}^{(h)\text{T}}_{[\mu}\Omega^{(h)}_{\nu\rho]}\mathcal{F}^{(h)}_{\nu\rho]}. \quad (3.23)$$

4 Complete dimensional reduction on T^n to $\mathcal{O}(\alpha')$

4.1 Torsionful spin connection

The reduction of the terms involving the torsionful spin connection $\hat{\Omega}_{(-)\hat{\mu}}^{(0)\hat{a}\hat{b}}$ can be carried out in two steps: first we just treat it as just another Yang-Mills field but with the particular gauge group $SO(1, 9)$. Then, we decompose the gauge group indices into $SO(1, 9 - n) \times SO(n)$ indices. As a matter of fact, we can just take the results of the previous section and assume that the gauge group has been extended to include $SO(1, 9)$.

Let us carry out the first step.

4.1.1 First step

As in the general Yang-Mills case, the reduction of the torsionful spin connection gives two fields

$$\varphi^{\hat{a}\hat{b}}_i \equiv e_i^m \hat{\Omega}_{(-)m}^{(0)\hat{a}\hat{b}}, \quad (4.1a)$$

$$A^{\hat{a}\hat{b}}_\mu \equiv \hat{\Omega}_{(-)\mu}^{(0)\hat{a}\hat{b}} - \hat{\Omega}_{(-)m}^{(0)\hat{a}\hat{b}} A^m_\mu, \quad (4.1b)$$

⁹There are no free $SO(n)$ indices in $M^{(h)}$.

and the different components of its curvature give

$$\hat{R}_{(-)}^{(0)\hat{a}\hat{b}}{}_{ij} = -2\varphi^{\hat{a}}{}_{\hat{c}}[\varphi^{\hat{c}\hat{b}}{}_{j}], \quad (4.2a)$$

$$\hat{R}_{(-)}^{(0)\hat{a}\hat{b}}{}_{ci} = \tilde{\mathfrak{D}}_c\varphi^{\hat{a}\hat{b}}{}_i + \frac{1}{2}\varphi^{\hat{a}\hat{b}}{}_je^j{}_me^i{}_n\partial_c G^{mn}, \quad (4.2b)$$

$$\hat{R}_{(-)}^{(0)\hat{a}\hat{b}}{}_{cd} = F^{\hat{a}\hat{b}}{}_{cd} + \varphi^{\hat{a}\hat{b}}{}_ie^i{}_m F^m{}_{cd}. \quad (4.2c)$$

It is worth stressing that $F^{\hat{a}\hat{b}}{}_{cd}$ is the $(10-n)$ -dimensional curvature of the $(10-n)$ -dimensional $\text{SO}(1,9)$ gauge field $A^{\hat{a}\hat{b}}{}_{\mu}$

$$F^{\hat{a}\hat{b}}{}_{\mu\nu} \equiv 2\partial_{[\mu}A^{\hat{a}\hat{b}}{}_{\nu]} - 2A^{\hat{a}}{}_{\hat{c}}[\mu A^{\hat{c}\hat{b}}{}_{\nu]}, \quad (4.3)$$

and $\tilde{\mathfrak{D}}_c$ is a $\text{SO}(1,9) \times \text{SO}(n)$ covariant derivative with the same connection plus the composite $\text{SO}(n)$ connection in eq. (A.10):

$$\tilde{\mathfrak{D}}_c\varphi^{\hat{a}\hat{b}}{}_i = \partial_c\varphi^{\hat{a}\hat{b}}{}_i - 2A^{[\hat{a}}{}_{\hat{d}c}\varphi^{\hat{d}\hat{b}}{}_{i]} + A^{ij}{}_c\varphi^{\hat{a}\hat{b}}{}_j. \quad (4.4)$$

At this stage we can use the results of the previous section (the Heterotic Supergravity case) to write the $(10-n)$ -dimensional action that one gets after completing the first step, because it has exactly the same form as that of the Heterotic Supergravity case eq. (3.22) if we define a new gauge index X that includes the 10-dimensional adjoint gauge group index A and the adjoint 10-dimensional Lorentz index $[\hat{a}\hat{b}]$: $X = A, [\hat{a}\hat{b}]$. We can write, directly and formally

$$\begin{aligned} S^{(1)} = & \frac{g_s^{(10-n)^2}}{16\pi G_N^{(10-n)}} \int d^{10-n}x \sqrt{|g|} e^{-2\phi} \left\{ R - 4(\partial\phi)^2 - \frac{1}{8}\text{Tr} \left(\tilde{\mathfrak{D}}^a \tilde{M}^{(1)} \tilde{\mathfrak{D}}_a \tilde{M}^{(1)-1} \right) \right. \\ & \left. - \frac{1}{4} \tilde{\mathcal{F}}^{(1)\text{T}} \tilde{M}^{(1)-1} \cdot \tilde{\mathcal{F}}^{(1)} + \frac{1}{12} \tilde{H}^{(1)2} - V^{(1)}(\varphi) \right\}, \end{aligned} \quad (4.5)$$

where the covariant derivative $\tilde{\mathfrak{D}}$ only acts on X indices,¹⁰ $\tilde{\mathcal{F}}^{(1)}$ is the vector of $(10-n)$ -dimensional 2-form field strengths

$$\tilde{\mathcal{F}}^{(1)} \equiv \begin{pmatrix} F^m \\ G^{(1)}{}_m \\ F^X \end{pmatrix}, \quad \text{with} \quad (F^X) \equiv \begin{pmatrix} F^A \\ F^{\hat{a}\hat{b}} \end{pmatrix}, \quad (4.6)$$

where F^m has been defined in eq. (2.2), F^A is the field strength of $(10-n)$ -dimensional Yang-Mills field, $F^{\hat{a}\hat{b}}$ is the field strengths of the $(10-n)$ -dimensional $\text{SO}(1,9)$ gauge field defined in eq. (4.3) and

$$G^{(1)}{}_{m\mu\nu} \equiv 2\partial_{[\mu}B^{(1)}{}_{m|\nu]}. \quad (4.7)$$

$M^{(1)-1}$ is the matrix

$$\tilde{M}^{(1)-1} \equiv \begin{pmatrix} G + B^{(1)\text{T}}G^{-1}B^{(1)} + \varphi^{\text{T}}K\varphi & -B^{(1)\text{T}}G^{-1} & (\mathbf{1}_{n \times n} - B^{(1)\text{T}}G^{-1})\varphi^{\text{T}}K \\ -G^{-1}B^{(1)} & G^{-1} & G^{-1}\varphi^{\text{T}}K \\ K\varphi(\mathbf{1}_{n \times n} - G^{-1}B^{(1)}) & K\varphi G^{-1} & K + K\varphi G^{-1}\varphi^{\text{T}}K \end{pmatrix}, \quad (4.8)$$

¹⁰With the connection A^A on YM indices and with the connection $A^{\hat{a}\hat{b}}$ on $\text{SO}(1,9)$ indices.

where

$$G \equiv (G_{mn}), \quad B^{(1)} \equiv (B^{(1)}_{mn}), \quad \varphi \equiv (\varphi^X_i e^i_m) = (\varphi^A_i e^i_m, \varphi^{\hat{a}\hat{b}}_i e^i_m),$$

$$K \equiv \frac{\alpha'}{2} (K_{XY}) = \frac{\alpha'}{2} \begin{pmatrix} K_{AB} & 0 \\ 0 & -\hat{\eta}^{\hat{a}\hat{b}}_{\hat{c}\hat{d}} \end{pmatrix}. \quad (4.9)$$

Finally, the $(10 - n)$ -dimensional Kalb-Ramond field strength $\tilde{H}^{(1)}$ and the “scalar potential”¹¹ $V^{(1)}$ are given by¹²

$$\begin{aligned} \tilde{H}^{(1)}_{\mu\nu\rho} &\equiv 3\partial_{[\mu} B^{(1)}_{\nu\rho]} - \frac{3}{2} A^m_{[\mu} G^{(1)}_{m|\nu\rho]} - \frac{3}{2} B^{(1)}_{m[\mu} F^m_{\nu\rho]} \\ &\quad + \frac{\alpha'}{4} (\omega^{YM}_{\mu\nu\rho} + \tilde{\omega}^{L(0)}_{(-)\mu\nu\rho}), \end{aligned} \quad (4.10)$$

$$\begin{aligned} V^{(1)}(\varphi) &\equiv \frac{\alpha'}{8} \left(2\varphi^{\hat{a}}_{\hat{b}i} \varphi^{\hat{b}}_{\hat{c}j} \varphi^{\hat{c}}_{\hat{d}i} \varphi^{\hat{d}}_{\hat{a}j} - 2\varphi^{\hat{a}}_{\hat{b}i} \varphi^{\hat{b}}_{\hat{c}i} \varphi^{\hat{c}}_{\hat{d}j} \varphi^{\hat{d}}_{\hat{a}j} \right. \\ &\quad \left. + f_{ABC} f^A_{DE} \varphi^B_i \varphi^D_i \varphi^C_j \varphi^E_j \right). \end{aligned} \quad (4.11)$$

In the above expressions we have used the $(10 - n)$ -dimensional fields $B^{(1)}_{mn}$, $B^{(1)}_{m\mu}$ and $B^{(1)}_{\mu\nu}$. They are defined in terms of the 10-dimensional ones by

$$B^{(1)}_{mn} \equiv \hat{B}_{mn} - \frac{\alpha'}{4} \left(\hat{A}^A_m \hat{A}_{An} + \hat{\Omega}^{(0)}_{(-)m}{}^{\hat{a}}{}_{\hat{b}} \hat{\Omega}^{(0)}_{(-)n}{}^{\hat{b}}{}_{\hat{a}} \right), \quad (4.12a)$$

$$\begin{aligned} B^{(1)}_{m\mu} &\equiv \hat{B}_{m\mu} + \left[\hat{B}_{mn} - \frac{\alpha'}{4} \left(\hat{A}^A_m \hat{A}_{An} + \hat{\Omega}^{(0)}_{(-)m}{}^{\hat{a}}{}_{\hat{b}} \hat{\Omega}^{(0)}_{(-)n}{}^{\hat{b}}{}_{\hat{a}} \right) \right] \hat{g}^{np} \hat{g}_{p\mu} \\ &\quad - \frac{\alpha'}{4} \left(\hat{A}_{Am} \hat{A}^A_{\mu} + \hat{\Omega}^{(0)}_{(-)m}{}^{\hat{a}}{}_{\hat{b}} \hat{\Omega}^{(0)}_{(-)\mu}{}^{\hat{b}}{}_{\hat{a}} \right), \end{aligned} \quad (4.12b)$$

$$B^{(1)}_{\mu\nu} \equiv \hat{B}_{\mu\nu} + \hat{g}^{mn} \hat{g}_{m[\mu} \hat{B}_{\nu]n} - \frac{\alpha'}{4} \left(\hat{A}_{Am} \hat{A}^A_{[\mu} + \hat{\Omega}^{(0)}_{(-)m}{}^{\hat{a}}{}_{\hat{b}} \hat{\Omega}^{(0)}_{(-)[\mu}{}^{\hat{b}}{}_{\hat{a}} \right] \hat{g}^{mn} \hat{g}_{\nu]n}. \quad (4.12c)$$

The action eq. (4.5) contains implicitly $\mathcal{O}(\alpha'^2)$ terms such as $H^{(1)2}$, as the original action eq. (1.9), but it is convenient to keep them in order to have more compact and gauge-invariant expressions. Eliminating all the $\mathcal{O}(\alpha'^2)$ terms we would get an action which is gauge invariant only to $\mathcal{O}(\alpha')$, but this gauge invariance and possible duality invariance (which we are going to discuss next) would not be manifest.

Although the action eq. (4.5) is apparently manifestly $O(n, n)$ -invariant, this is not so clear because the statement assumes that all the terms not directly affected by the linear $O(n, n)$ transformations remain invariant. However, some of those terms, such as $F^{\hat{a}}_{\hat{b}} \cdot F^{\hat{a}}_{\hat{b}}$, for instance, depend on the internal Vielbein e^i_m and/or the KK vectors A^m_μ which are not invariant. We have to move to the second phase and expand the terms that depend on the fields with $SO(1, 9)$ indices in terms of $(10 - n)$ -dimensional fields.

¹¹The variables φ^A_i become true $(10 - n)$ -dimensional scalars with $O(n)$ and adjoint gauge indices, but the variables $\varphi^{\hat{a}}_{\hat{b}i}$ become both $(10 - n)$ -dimensional tensors and scalars, see eqs. (4.13).

¹²We have neglected the $\mathcal{O}(\alpha'^2)$ terms in $V^{(1)}$.

4.1.2 Second step

The fields with $\text{SO}(1, 9)$ indices $\varphi^{\hat{a}\hat{b}}_i$ and $A^{\hat{a}\hat{b}}_\mu$ are further reduced as follows:

$$\varphi^{\hat{a}\hat{b}}_i \longrightarrow \begin{cases} \varphi^{ab}_i = -\frac{1}{2}K_{(+)}^{(0) i ab}, \\ \varphi^{ai}_j = -P_{(-)}^{(0) a}{}_{ij}, \\ \varphi^{ij}_k = 0, \end{cases} \quad (4.13)$$

and

$$A^{\hat{a}\hat{b}}_\mu \longrightarrow \begin{cases} A^{ab}_\mu = \Omega_{(-)\mu}^{(0) ab}, \\ A^{ai}_\mu = -\frac{1}{2}K_{(-)\mu}^{(0) i a}, \\ A^{ij}_\mu = A_{(-)\mu}^{(0) ij}, \end{cases} \quad (4.14)$$

where the 2-form $K_{(-)\mu\nu}^{(0) i}$, the $\text{O}(n)$ connection 1-form $A_{(-)}^{(0) ij}$ and the Vielbein $P_{(-)ij}^{(0)}$ we have been defined in eq. (A.20) with B_{mn} replaced by $B_{mn}^{(0)}$.

Taking into account these expressions, the components of the $(10 - n)$ -dimensional $\text{SO}(1, 9)$ field strength $F^{\hat{a}\hat{b}}_{cd}$ that occurs in the reduction of the curvature of the torsionful spin connection eq. (4.2) are decomposed as follows:

$$F^{ab}_{\mu\nu} = R_{(-)\mu\nu}^{(0) ab} - \frac{1}{2}K_{(-)[\mu}^{(0) i}{}^a K_{(-)\nu]}^{(0) i}{}^b, \quad (4.15a)$$

$$F^{ai}_{\mu\nu} = -\mathfrak{D}_{(-)[\mu} K_{(-)\nu]}^{(0) i}{}^a, \quad (4.15b)$$

$$F^{ij}_{\mu\nu} = F_{(-)\mu\nu}^{(0) ij} + \frac{1}{2}K_{(-)[\mu}^{(0) i}{}^a K_{(-)\nu]}^{(0) j}{}^a, \quad (4.15c)$$

where $\mathfrak{D}_{(-)}$ is the $\text{SO}(1, 9 - n) \times \text{O}(n)$ covariant derivative with the “(0)(-)” connections, that is

$$\mathfrak{D}_{(-)[\mu} K_{(-)\nu]}^{(0) i}{}^a = \partial_{[\mu} K_{(-)\nu]}^{(0) i}{}^a + A_{(-)}^{(0) ij}{}_{[\mu} K_{(-)\nu]}^{(0) j}{}^a - \Omega_{(-)[\mu}{}^a{}_b K_{(-)\nu]}^{(0) i}{}^b. \quad (4.16)$$

Now we use all these decompositions into $\tilde{\omega}_{(-)abc}^{L(0)}$, the $(10 - n)$ -dimensional Lorentz Chern-Simons 3-form of the $\text{SO}(1, 9)$ connection, obtaining

$$\tilde{\omega}_{(-)\mu\nu\rho}^{L(0)} = \omega_{(-)\mu\nu\rho}^{L(0)} - \omega_{(-)\mu\nu\rho}^{O(n)} + 3\mathfrak{D}_{(-)[\mu} K_{(-)\nu]}^{(0) i}{}^a K_{(-)\rho]}^{(0) i}{}^a, \quad (4.17a)$$

$$\omega_{(-)\mu\nu\rho}^{O(n)} \equiv 3F_{(-)[\mu\nu]}^{(0) ij} A_{(-)\rho]}^{(0) ij} + 2A_{(-)[\mu]}^{(0) ij} A_{(-)\nu]}^{(0) jk} A_{(-)\rho]}^{(0) ki}. \quad (4.17b)$$

According to the discussion in appendix A the 3-form $\mathfrak{D}_{(-)[\mu} K_{(-)\nu]}^{(-) i}{}^a K_{(-)\rho]}^{(-) i}{}^a$ is $\text{O}(n, n)$ -invariant. Since it is also gauge and Lorentz-invariant it is natural to eliminate it from the definition of the $(10 - n)$ -dimensional Kalb-Ramond field strength $H^{(1)}$:

$$\tilde{H}^{(1)}_{\mu\nu\rho} = H^{(1)}_{\mu\nu\rho} + \frac{3\alpha'}{4}\mathfrak{D}_{(-)[\mu} K_{(-)\nu]}^{(0) i}{}^a K_{(-)\rho]}^{(0) i}{}^a, \quad (4.18a)$$

$$\begin{aligned} H^{(1)}_{\mu\nu\rho} &\equiv 3\partial_{[\mu} B^{(1)}_{\nu\rho]} - \frac{3}{2}A^m_{[\mu} G^{(1)}_{m|\nu\rho]} - \frac{3}{2}B^{(1)}_m{}_{[\mu} F^m_{\nu\rho]} \\ &\quad + \frac{\alpha'}{4} \left(\omega^{YM}_{\mu\nu\rho} + \omega_{(-)\mu\nu\rho}^{L(0)} - \omega_{(-)\mu\nu\rho}^{O(n)} \right). \end{aligned} \quad (4.18b)$$

Observe that the Chern-Simons 3-form of the composite $O(n)_{(-)}$ connection $A_{(-)}^{(0)ij}$ is not $O(n, n)$ -invariant, because it is not invariant under the compensating local $O(n)_{(-)}$ transformation (it is invariant up to a total derivative, as any Chern-Simons 3-form).¹³ It can be compensated with a standard Nicolai-Townsend transformation of the $(10 - n)$ -dimensional 2-form $B^{(1)}$,¹⁴ though. Thus, as different from the $n = 1$ case, this field is now not T-duality invariant to first order in α' . Observe that these transformations do not affect $B^{(0)}$, which is the field that occurs in $\Omega_{(-)\mu}^{(0)a_b}$ and $A_{(-)}^{(0)ij}$.

Next, let us consider the decomposition of the “scalar potential” $V^{(1)}(\varphi)$. Using eqs. (4.13), we get

$$V^{(1)}(\varphi) = \frac{\alpha'}{8} \left\{ \frac{1}{8} K_{(+)}^{(0)i}{}_{ab} K_{(+)}^{(0)i}{}_{cd} K_{(+)}^{(0)j}{}_{ac} K_{(+)}^{(0)j}{}_{bd} + \frac{1}{2} K_{(+)}^{(0)i}{}_{ab} K_{(+)}^{(0)j}{}_{cd} K_{(+)}^{(0)i}{}_{ac} K_{(+)}^{(0)j}{}_{bd} \right. \\ \left. + F_{(-)}^{(0)ij} \cdot F_{(-)}^{(0)ij} + f_{ABC} f^A{}_{DE} \varphi^B{}_i \varphi^D{}_j \varphi^C{}_j \varphi^E{}_j \right\}, \quad (4.19)$$

where we have used the identity eq. (A.15). All terms are manifestly $O(n, n)$ -invariant.

Next, we have to decompose the kinetic term for the vector field strengths into its $O(n, n)$ invariant form and the rest:

$$\begin{aligned} \widetilde{\mathcal{F}}^{(1)\text{T}} \widetilde{M}^{(1)-1} \cdot \widetilde{\mathcal{F}}^{(1)} &= \mathcal{F}^{(1)\text{T}} M^{(1)-1} \cdot \mathcal{F}^{(1)} + \frac{\alpha'}{2} \left[F_A \cdot F^A + R_{(-)}^{(0)a}{}_b \cdot R_{(-)}^{(0)b}{}_a \right. \\ &+ R_{(-)}^{(0)abcd} \left(K_{(-)}^{(0)i}{}_{ac} K_{(-)}^{(0)i}{}_{bd} + K_{(+)}^{(0)ia}{}_b K_{(+)}^{(0)ic}{}_d \right) \\ &+ \frac{1}{4} K_{(-)}^{(0)i}{}_{ab} K_{(-)}^{(0)i}{}_{cd} K_{(-)}^{(0)j}{}_{ac} K_{(-)}^{(0)j}{}_{bd} - \frac{1}{8} \left(K_{(-)}^{(0)i} \cdot K_{(-)}^{(0)j} \right) \left(K_{(-)}^{(0)i} \cdot K_{(-)}^{(0)j} \right) \\ &- \frac{1}{8} K_{(-)}^{(0)i}{}_{ab} K_{(-)}^{(0)j}{}_{cd} K_{(-)}^{(0)i}{}_{ac} K_{(-)}^{(0)j}{}_{bd} - \frac{1}{2} K_{(+)}^{(0)i}{}_{ab} K_{(+)}^{(0)i}{}_{cd} K_{(-)}^{(0)j}{}_{ac} K_{(-)}^{(0)j}{}_{bd} \\ &+ 2\mathfrak{D}_{(-)}^{(0)[\mu} K_{(-)}^{(0)i}{}_{\nu]}{}^a \mathfrak{D}_{(-)}^{(0)[\mu} K_{(-)}^{(0)i}{}_{\nu]}{}_a - F_{(-)}^{(0)ij} \cdot F_{(-)}^{(0)ij} - F_{(-)}^{(0)ij\mu\nu} K_{(-)}^{(0)i}{}_{\mu}{}^a K_{(-)}^{(0)j}{}_{\nu a} \\ &\left. + 2\varphi_A{}^i F^A \cdot K_{(+)}^{(0)i} + 4P_{(-)}^{(0)a}{}_{ji} \mathfrak{D}_{(-)}^{(0)}{}_{\mu} K_{(-)}^{(0)j}{}_{\nu a} K_{(+)}^{(0)i\mu\nu} \right], \end{aligned} \quad (4.20)$$

where the covariant derivative $\mathfrak{D}_{(-)}^{(0)}$ contains the $O(n)_{(-)}$ connection $A_{(-)}^{(0)ij}$ and the torsionful Lorentz connection $\Omega_{(-)\mu}^{(0)ab}$, and where, now, the matrix $M^{(1)-1}$ is given by the upper-left corner 2×2 part of $\widetilde{M}^{(1)-1}$ in eq. (4.8), namely

$$M^{(1)-1} \equiv \begin{pmatrix} G + B^{(1)\text{T}} G^{-1} B^{(1)} + \varphi^{\text{T}} K \varphi & -B^{(1)\text{T}} G^{-1} \\ -G^{-1} B^{(1)} & G^{-1} \end{pmatrix}, \quad (4.21)$$

and

$$\mathcal{F}^{(1)} \equiv \begin{pmatrix} F^m \\ G^{(1)}{}_m \end{pmatrix}. \quad (4.22)$$

¹³The presence of this term has been observed in ref. [21].

¹⁴The Nicolai-Townsend transformations were first found in ref. [30] in a different context. They were shown to be necessary in the coupling of vector multiplets to $\mathcal{N} = 1, d = 10$ supergravity in [31].

Finally, let us consider the scalar's kinetic term. Expanding this term, reducing and keeping only terms of first order in α' we arrive at

$$\begin{aligned} \text{Tr} \left(\widetilde{\mathfrak{D}}^a \widetilde{M}^{(1)} \widetilde{\mathfrak{D}}_a \widetilde{M}^{(1)-1} \right) &= \text{Tr} \left(\partial^a M^{(1)} \partial_a M^{(1)-1} \right) - \alpha' \mathfrak{D}_{(-)}^{(0)c} \left(\mathcal{U}_{(+)}^i \varphi_{Ai} \right)^T \Omega^{(1)} \mathfrak{D}_{(-)c}^{(0)} \left(\mathcal{U}_{(+)}^j \varphi^A_j \right) \\ &\quad - \frac{\alpha'}{4} \mathfrak{D}_{(-)}^{(0)c} \left(\mathcal{U}_{(+)}^i K_{(+i}^{(0)a}{}_{b)} \right)^T \Omega^{(1)} \mathfrak{D}_{(-)c}^{(0)} \left(\mathcal{U}_{(+)}^j K_{(+j}^{(0)b}{}_{a)} \right) \\ &\quad - 2\alpha' \mathfrak{D}_{(-)}^{(0)c} \left(\mathcal{U}_{(+)}^i P_{(+i}^{(0)a}{}_{kj)} \right)^T \Omega^{(1)} \mathfrak{D}_{(-)c}^{(0)} \left(\mathcal{U}_{(+)}^k P_{(+k}^{(0)a}{}_{ij)} \right). \end{aligned} \quad (4.23)$$

Finally, combining all the partial results we get the final, manifestly- $O(n, n)$ -invariant form of the $(10 - n)$ -dimensional action

$$\begin{aligned} S^{(1)} &= \frac{g_s^{(10-n)2}}{16\pi G_N^{(10-n)}} \int d^{10-n} x \sqrt{|g|} e^{-2\phi} \left\{ R - 4(\partial\phi)^2 - \frac{1}{8} \text{Tr} \left(\partial^a M^{(1)} \partial_a M^{(1)-1} \right) \right. \\ &\quad - \frac{1}{4} \mathcal{F}^{(1)T} M^{(1)-1} \cdot \mathcal{F}^{(1)} + \frac{1}{12} \widetilde{H}^{(1)2} - \frac{\alpha'}{8} \left[F_A \cdot F^A + R_{(-)}^{(0)a}{}_{b} \cdot R_{(-)}^{(0)b}{}_{a} \right. \\ &\quad + R_{(-)}^{(0)abcd} \left(K_{(-)}^{(0)iac} K_{(-)}^{(0)ibd} + K_{(+)}^{(0)iab} K_{(+)}^{(0)icd} \right) \\ &\quad + \frac{1}{4} K_{(-)ab}^{(0)i} K_{(-)cd}^{(0)i} K_{(-)}^{(0)jac} K_{(-)}^{(0)jbd} - \frac{1}{8} \left(K_{(-)}^{(0)i} \cdot K_{(-)}^{(0)j} \right) \left(K_{(-)}^{(0)i} \cdot K_{(-)}^{(0)j} \right) \\ &\quad - \frac{1}{8} K_{(-)ab}^{(0)i} K_{(-)cd}^{(0)j} K_{(-)}^{(0)iac} K_{(-)}^{(0)jbd} - \frac{1}{2} K_{(+ab)}^{(0)i} K_{(+cd)}^{(0)i} K_{(-)}^{(0)jac} K_{(-)}^{(0)jbd} \\ &\quad + \frac{1}{8} K_{(+ab)}^{(0)i} K_{(+cd)}^{(0)i} K_{(+)}^{(0)jac} K_{(+)}^{(0)jbd} + \frac{1}{8} K_{(+ab)}^{(0)i} K_{(+cd)}^{(0)j} K_{(+)}^{(0)iac} K_{(+)}^{(0)jbd} \\ &\quad + 2\mathfrak{D}_{(-)[\mu}^{(0)} K_{(-)\nu]}^{(0)i}{}^a \mathfrak{D}_{(-)}^{(0)[\mu]} K_{(-)}^{(0)i}{}_{\nu]}{}^a - F_{(-)}^{(0)ij\mu\nu} K_{(-)\mu}^{(0)i}{}^a K_{(-)\nu a}^{(0)j} \\ &\quad + 2\varphi_{Ai} F^A \cdot K_{(+)}^{(0)i} + 4P_{(-)ji}^{(0)a} \mathfrak{D}_{(-)\mu}^{(0)} K_{(-)\nu a}^{(0)j} K_{(+)}^{(0)i\mu\nu} \\ &\quad - \mathfrak{D}_{(-)}^{(0)c} \left(\mathcal{U}_{(+)}^i \varphi_{Ai} \right)^T \Omega^{(1)} \mathfrak{D}_{(-)c}^{(0)} \left(\mathcal{U}_{(+)}^j \varphi^A_j \right) \\ &\quad - 2\mathfrak{D}_{(-)}^{(0)c} \left(\mathcal{U}_{(+)}^i P_{(+i}^{(0)a}{}_{kj)} \right)^T \Omega^{(1)} \mathfrak{D}_{(-)c}^{(0)} \left(\mathcal{U}_{(+)}^k P_{(+k}^{(0)a}{}_{ij)} \right) \\ &\quad - \frac{1}{4} \mathfrak{D}_{(-)}^{(0)c} \left(\mathcal{U}_{(+)}^i K_{(+i}^{(0)a}{}_{b)} \right)^T \Omega^{(1)} \mathfrak{D}_{(-)c}^{(0)} \left(\mathcal{U}_{(+)}^j K_{(+j}^{(0)b}{}_{a)} \right) \Big] \\ &\quad \left. + f_{ABC} f^A{}_{DE} \varphi^B{}_i \varphi^D{}_i \varphi^C{}_j \varphi^E{}_j \right\}. \end{aligned} \quad (4.24)$$

This dimensionally-reduced action has two main immediate uses: the derivation of equations of motion and the derivation of an entropy formula. The equations of motion are very complicated and involve, in principle, derivatives higher than 2. According to the lemma proven in ref. [23], the terms of higher-order in derivatives (possibly in combination with some terms of lower order) are, actually, proportional to α' and combinations of the zeroth-order equations of motion and can, in be disregarded in practice. However, in order to find all the terms which can be ignored one has to use the lower-dimensional version of the lemma, which requires the identification of all the fields which originate, purely, on

the 10-dimensional torsionful spin connection. Therefore, it is far easier to deal with the 10-dimensional equations of motion and perform the dimensional reduction of the solution using the rules derived in this paper.

The entropy formula, though, can be readily obtained and used in $(10 - n)$ dimensions, as we are going to show in the next section.

5 Entropy formula

If we change the index 10 by D , the action eq. (1.9) is identical to the action one would obtain by trivial dimensional reduction of the 10-dimensional theory on a $m \equiv (10 - D)$ -dimensional torus. Then, with the same change, eq. (4.24) is the action obtained by a fully non-trivial dimensional reduction on T^n , from D to $d = D - n = 10 - n - m$ dimensions. The solutions of the equations of motion of this d -dimensional theory are solutions which, uplifted to $d + n = D$ dimensions with the rules derived in this paper, are, then, solutions of the equations of motion of the original action eq. (1.9) with 10 replaced by D and which can be trivially uplifted again to 10 dimensions.

An important example of solutions of this type are the heterotic version of the 4-dimensional, 4-charge extremal black holes whose microscopic entropy was originally computed in refs. [13, 14] (see also ref. [32])¹⁵ and whose first-order α' corrections were recently computed in refs. [38, 39].¹⁶

If these 4-dimensional solutions are uplifted to 6 dimensions ($n = 2$), they are solutions of our original action eq. (1.9) with $D = 6$ ($m = 4$). Thus, they are solutions of the equations of motion of the action eq. (4.24) with 10 replaced by 6 and $n = 2$ and we can use that action to compute their Wald entropy using the Iyer-Wald prescription [8, 9].

The direct application of this prescription to eq. (4.24) with 10 replaced by a general dimension D yields the following string-frame entropy formula for $d = (D - n)$ -dimensional black holes:

$$S = -2\pi \int_{\Sigma} d^{d-2}x \sqrt{|h|} \frac{\partial \mathcal{L}}{\partial R_{abcd}} \epsilon_{ab} \epsilon_{cd}, \quad (5.1a)$$

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial R_{abcd}} = & \frac{e^{-2(\phi - \phi_{\infty})}}{16\pi G_N^{(d)}} \left\{ g^{ab, cd} - \frac{\alpha'}{8} \left[H^{(0)abg} \left(\omega_g^{cd} - H_g^{(0)cd} \right) \right. \right. \\ & \left. \left. - 2R_{(-)}^{(0)abcd} + K^{(-)i[a|c} K^{(-)i|b]d} + K^{(+)j ab} K^{(+)j cd} \right] \right\}, \quad (5.1b) \end{aligned}$$

¹⁵These 4-dimensional solutions can be embedded either in the Heterotic or in type II theories. They were first found in ref. [33] and further discussed in refs. [34, 35]. They were rediscovered in ref. [36] in the context of the so-called STU model [37] in a form that made it easier to identify harmonic functions and charges to 10-dimensional string theory extended objects.

¹⁶In these references the first-order in α' corrections to the complete geometry were computed. Earlier work in which only the corrections to the near-horizon geometry were computed and then, used to compute the corrections to the entropy can be found in refs. [40–45]. Some of the drawbacks of these methods, such as the problem of identification of asymptotic charges and the possible incompleteness of the higher-order terms considered have been discussed in refs. [39, 47].

where $|h|$ is the absolute value of the determinant of the metric induced over the event horizon, $g^{ab,cd} = \frac{1}{2}(g^{ac}g^{bd} - g^{ad}g^{bc})$, ϵ^{ab} is the event horizon's binormal normalized so that $\epsilon_{ab}\epsilon^{ab} = -2$ and R_{abcd} is the Riemann tensor.

This manifestly $O(n, n)$ -invariant entropy formula reduces, for $n = 1$ to the formula found in ref. [1] and which has been used to compute the Wald entropy of the α' -corrected Reissner-Nordström black hole of ref. [11] and of the α' -corrected heterotic version of the Strominger-Vafa black hole of ref. [46].

5.1 The Wald entropy of the α' -corrected 4d 4-charge black holes

The α' -corrected 4d 4-charge black-hole solutions correspond to the following 10-dimensional solutions of the action eq. (1.9):

$$ds^2 = \frac{2}{\mathcal{Z}_-} du \left[dv - \frac{1}{2} \mathcal{Z}_+ du \right] - \mathcal{Z}_0 d\sigma^2 - dy^i dy^i, \quad i, j = 1, 2, 3, 4, \quad (5.2a)$$

$$H = d\mathcal{Z}_-^{-1} \wedge du \wedge dv + \star_{(4)} d\mathcal{Z}_0, \quad (5.2b)$$

$$e^{-2\phi} = e^{-2\phi_\infty} \frac{\mathcal{Z}_-}{\mathcal{Z}_0}, \quad (5.2c)$$

where $d\sigma^2$ is the Gibbons-Hawking metric

$$d\sigma^2 = \mathcal{H}^{-1} (d\eta + \chi)^2 + \mathcal{H} dx^x dx^x, \quad x, y, z = 1, 2, 3, \quad d\mathcal{H} = \star_{(3)} d\chi, \quad (5.3)$$

where $\star_{(3)}$ denotes the Hodge dual in $\mathbb{E}^3$. This last equation implies that $\mathcal{H}$ is harmonic in $\mathbb{E}^3$. An appropriate choice of harmonic function $\mathcal{H}$ for single, spherically-symmetric, asymptotically flat black holes is

$$\mathcal{H} = 1 + \frac{\tilde{q}}{r}, \quad (5.4)$$

and, then, the Gibbons-Hawking metric is that of a Kaluza-Klein monopole, which, in spherical coordinates, takes the local form

$$d\sigma^2 = \mathcal{H}^{-1} (d\eta + \tilde{q} \cos \theta d\phi)^2 + \mathcal{H} \left(dr^2 + r^2 d\Omega_{(2)}^2 \right), \quad (5.5)$$

where

$$d\Omega_{(2)}^2 = d\theta^2 + \sin^2 \theta d\phi^2, \quad (5.6)$$

and where η parametrizes a circle of radius R . The Kaluza-Klein charge $\tilde{q}$ has to be quantized according to

$$\tilde{q} = \frac{WR}{2}, \quad W = 1, 2, \dots \quad (5.7)$$

in order to avoid Dirac-Misner strings.

For this choice of $\mathcal{H}$ and to describe single, spherically-symmetric, asymptotically-flat black holes, the functions $\mathcal{Z}_+$, $\mathcal{Z}_-$, $\mathcal{Z}_0$, $\mathcal{H}$ must take the explicit form

$$\mathcal{Z}_+ = 1 + \frac{\tilde{q}_+}{r} + \frac{\alpha' \tilde{q}_+}{2\tilde{q}\tilde{q}_0} \frac{r^2 + r(\tilde{q}_0 + \tilde{q}_- + \tilde{q}) + \tilde{q}\tilde{q}_0 + \tilde{q}\tilde{q}_- + \tilde{q}_0\tilde{q}_-}{(r - \tilde{q})(r + \tilde{q}_0)(r + \tilde{q}_-)} + \mathcal{O}(\alpha'^2), \quad (5.8a)$$

$$\mathcal{Z}_- = 1 + \frac{\tilde{q}_-}{r} + \mathcal{O}(\alpha'^2), \quad (5.8b)$$

$$\mathcal{Z}_0 = 1 + \frac{\tilde{q}_0}{r} - \alpha' \left[\frac{(r + \tilde{q})(r + 2\tilde{q}_0) + \tilde{q}_0^2}{4\tilde{q}(r + \tilde{q})(r + \tilde{q}_0)^2} + \frac{(r + \tilde{q})(r + 2\tilde{q}) + \tilde{q}^2}{4\tilde{q}(r + \tilde{q})^3} \right] + \mathcal{O}(\alpha'^2). \quad (5.8c)$$

As we discussed at the beginning of this section, we can compactify trivially this solution in the T^4 parametrized by the coordinates $y^1, \dots, y^4$. Relabeling $u = k_\infty z$, $\eta = \ell_\infty w$, with $\ell_\infty \equiv R/\ell_s$,¹⁷ and $v = t$, the 6-dimensional solution can be conveniently written in the following form:

$$d\hat{s}^2 = \frac{1}{\mathcal{Z}_+ \mathcal{Z}_-} dt^2 - \mathcal{Z}_0 \mathcal{H} \left(dr^2 + r^2 d\Omega_{(2)}^2 \right) - k_\infty^2 \frac{\mathcal{Z}_+}{\mathcal{Z}_-} \left(dz - \frac{1}{k_\infty \mathcal{Z}_+} dt \right)^2 - \ell_\infty^2 \frac{\mathcal{Z}_0}{\mathcal{H}} (dw + \tilde{q}/\ell_\infty \cos \theta d\phi)^2, \quad (5.9a)$$

$$\hat{H} = d \left(-\frac{k_\infty}{\mathcal{Z}_-} dt \right) \wedge dz + \ell_\infty r^2 \mathcal{Z}_0' d \cos \theta \wedge d\phi \wedge dw, \quad (5.9b)$$

$$e^{-2(\hat{\phi} - \hat{\phi}_\infty)} = \frac{\mathcal{Z}_-}{\mathcal{Z}_0}, \quad (5.9c)$$

where a prime indicates a derivative with respect to the radial coordinate r . In this form we can immediately identify the KK scalars and vector fields

$$G = \begin{pmatrix} k_\infty^2 \mathcal{Z}_+ / \mathcal{Z}_- & 0 \\ 0 & \ell_\infty^2 \mathcal{Z}_0 / \mathcal{H} \end{pmatrix}, \quad (A^m{}_\mu dx^\mu) = \begin{pmatrix} -dt / (k_\infty \mathcal{Z}_+) \\ \tilde{q} / \ell_\infty \cos \theta d\phi \end{pmatrix}, \quad (5.10)$$

and the 4-dimensional string-frame metric

$$ds^2 = \frac{1}{\mathcal{Z}_+ \mathcal{Z}_-} dt^2 - \mathcal{Z}_0 \mathcal{H} \left(dr^2 + r^2 d\Omega_{(2)}^2 \right). \quad (5.11)$$

The 4-dimensional dilaton field is

$$e^{-2(\phi - \phi_\infty)} = \sqrt{\frac{\mathcal{Z}_+ \mathcal{Z}_-}{\mathcal{Z}_0 \mathcal{H}}}, \quad (5.12)$$

where

$$e^{-2\phi_\infty} = e^{-2\hat{\phi}_\infty} k_\infty \ell_\infty, \quad (5.13)$$

and the (modified) Einstein-frame metric is given by

$$ds_E^2 = e^{2(\phi - \phi_\infty)} ds^2 = e^{-2U} dt^2 - e^{2U} d\vec{x}^2, \quad (5.14)$$

$$e^{2U} = (\mathcal{Z}_+ \mathcal{Z}_- \mathcal{Z}_0 \mathcal{H})^{1/2}.$$

Let us now consider the Kalb-Ramond field strength and its decomposition. First, we choose the following Vielbein basis

$$\begin{aligned} \hat{e}^0 &= \frac{1}{\sqrt{\mathcal{Z}_+ \mathcal{Z}_-}} dt, & \hat{e}^1 &= \sqrt{\mathcal{Z}_0 \mathcal{H}} dr, \\ \hat{e}^2 &= \sqrt{\mathcal{Z}_0 \mathcal{H}} r d\theta, & \hat{e}^3 &= \sqrt{\mathcal{Z}_0 \mathcal{H}} r \sin \theta d\phi, \\ \hat{e}^4 &= k_\infty \sqrt{\mathcal{Z}_+ / \mathcal{Z}_-} \left(dz - \frac{1}{k_\infty \mathcal{Z}_+} dt \right), & \hat{e}^5 &= \ell_\infty \sqrt{\mathcal{Z}_0 / \mathcal{H}} (dw + \tilde{q} / \ell_\infty \cos \theta d\phi), \end{aligned} \quad (5.15)$$

¹⁷We normalize the periods of the compact coordinates that parametrize the internal circles, z and w , to $2\pi\ell_s$. The information about the size of these circles is carried by the internal metric and the corresponding 4-dimensional moduli, which are dimensionless. Thus $G_{zz\infty} = k_\infty^2 = (R_z/\ell_s)^2$ and $G_{ww\infty} = \ell_\infty^2 = (R/\ell_s)^2$.

in terms of which, the non-vanishing components of $\hat{H}$ are

$$\hat{H}_{104} = \frac{1}{\sqrt{\mathcal{Z}_0 \mathcal{H}}} (\log \mathcal{Z}_-)' , \quad \hat{H}_{235} = \frac{1}{\sqrt{\mathcal{Z}_0 \mathcal{H}}} (\log \mathcal{Z}_0)' . \quad (5.16)$$

This implies that the 4-dimensional Kalb-Ramond field strength vanishes identically and there are two non-vanishing 4-dimensional winding vector field strengths plus, perhaps, scalars. Computing the 4-dimensional Kalb-Ramond 2-form $B^{(1)}_{\mu\nu}$, the winding vectors $B^{(1)}_{m\mu}$ and the scalars $B^{(1)}_{mn}$ is very complicated because of the α' corrections in their definitions. Fortunately for us, only their zeroth-order values contribute to the entropy formula. At this order

$$\hat{H}^{(0)} = d \left(-\frac{k_\infty}{\mathcal{Z}_-} dt \right) \wedge dz + (-\ell_\infty \tilde{q}_0 \cos \theta d\phi) \wedge dw , \quad (5.17)$$

from which we read the only non-vanishing fields descending from the Kalb-Ramond field:

$$\left(B^{(0)}_{m\mu} dx^\mu \right) = \begin{pmatrix} -k_\infty / \mathcal{Z}_- dt \\ -\ell_\infty \tilde{q}_0 \cos \theta d\phi \end{pmatrix} . \quad (5.18)$$

This allows us to compute the nonvanishing components of the 2-forms $K^{(0)i}_{(\pm)}$:

$$\begin{aligned} K^{(0)401}_{(\pm)} &= \frac{1}{\sqrt{\mathcal{Z}_0 \mathcal{H}}} \left\{ (\log \mathcal{Z}_+)' \pm (\log \mathcal{Z}_-)' \right\} , \\ K^{(0)523}_{(\pm)} &= \frac{1}{r^2 (\mathcal{Z}_0 \mathcal{H})^{3/2}} (\tilde{q} \mathcal{Z}_0 \mp \tilde{q}_0 \mathcal{H}) . \end{aligned} \quad (5.19)$$

Taking into account the vanishing of the 4-dimensional Kalb-Ramond field strength, the entropy formula takes the simple form

$$\begin{aligned} S &= -\frac{1}{8G_N^{(4)}} \int_\Sigma d^2x \sqrt{|h|} e^{-2(\phi-\phi_\infty)} \epsilon_{ab} \epsilon_{cd} \left\{ g^{ab,cd} \right. \\ &\quad \left. - \frac{\alpha'}{8} \left[-2R^{(0)abcd} + K^{(-)iac} K^{(-)ibd} + K^{(+)jab} K^{(+)jcd} \right] \right\} \\ &= \frac{1}{4G_N^{(4)}} \int_\Sigma d^2x \sqrt{|h|} e^{-2(\phi-\phi_\infty)} \left\{ 1 + \frac{\alpha'}{4} \left[-2R^{(0)0101} + \left(K^{(-)401} \right)^2 + \left(K^{(+)401} \right)^2 \right] \right\} \\ &= \frac{1}{4G_N^{(4)}} \left\{ A_{\mathcal{H}} + 2\pi\alpha' \lim_{r \rightarrow 0} e^{2U} r^2 \left[-\sqrt{\frac{\mathcal{Z}_+ \mathcal{Z}_-}{\mathcal{Z}_0 \mathcal{H}}} \left[\frac{1}{\sqrt{\mathcal{Z}_0 \mathcal{H}}} \left(\frac{1}{\sqrt{\mathcal{Z}_+ \mathcal{Z}_-}} \right)' \right]' \right. \right. \\ &\quad \left. \left. + \frac{1}{\mathcal{Z}_0 \mathcal{H}} \left[\left(\frac{\mathcal{Z}'_+}{\mathcal{Z}_+} \right)^2 + \left(\frac{\mathcal{Z}'_-}{\mathcal{Z}_-} \right)^2 \right] \right] \right\} \\ &= \frac{A_{\mathcal{H}}}{4G_N^{(4)}} \left\{ 1 + \frac{\alpha'}{2\tilde{q}_0 \tilde{q}} \right\} \end{aligned} \quad (5.20)$$

where we have assumed that all the charges $\tilde{q}_i$ are different from zero¹⁸ and where

$$A_{\mathcal{H}} = 4\pi\sqrt{\tilde{q}_+\tilde{q}_-\tilde{q}_0\tilde{q}}, \quad (5.21)$$

is the area of the horizon, which does not receive any corrections to first order in α' .

The charges $\tilde{q}_i$ are related to the numbers of solitonic 5-branes N , units of momentum n , winding number w and KK charge W by [39]¹⁹

$$\tilde{q}_0 = \frac{\alpha'}{2R}N, \quad \tilde{q}_+ = \frac{\alpha'^2 g_s^2}{2RR_z^2}n, \quad \tilde{q}_- = \frac{\alpha' g_s^2}{2R}w, \quad q = \frac{WR}{2}, \quad (5.22)$$

and, using the value of the 10-dimensional Newton constant in eq. (1.10) and its relation with the 4-dimensional one (eq. (2.14b) with $V_6 = (2\pi)^6\alpha'^2 RR_z$) we get

$$S = 2\pi\sqrt{NnWw} \left\{ 1 + \frac{2}{NW} \right\}. \quad (5.23)$$

Finally, using the relation between the numbers of objects and the asymptotic charges found in ref. [47]

$$\mathcal{Q}_+ = n \left(1 + \frac{2}{NW} \right), \quad (5.24a)$$

$$\mathcal{Q}_- = w, \quad (5.24b)$$

$$\mathcal{Q}_0 = N \left(1 - \frac{2}{NW} \right), \quad (5.24c)$$

$$\mathcal{Q} = W, \quad (5.24d)$$

we find that

$$S = 2\pi\sqrt{\mathcal{Q}_+\mathcal{Q}_-(\mathcal{Q}_0\mathcal{Q} + 4)}. \quad (5.25)$$

This is the entropy obtained by microstate counting in ref. [49].

The entropy formula we have derived can, evidently, be applied to other, more general, solutions, such as 4-charge extremal non-supersymmetric and non-extremal black holes. The non-supersymmetric extremal ones can be easily obtained from the supersymmetric ones by flipping the sign of a charge [50], but a computation of the α' corrections to their entropy (which at lowest order in α' is identical to that of the supersymmetric ones) requires a computation of the α' corrections to the solutions themselves. This is, nevertheless, an interesting problem whose result should be compared with the result obtained in ref. [43] by a different method.

¹⁸We have assumed the positivity of all the charge parameters because, as a general rule, if these parameters have negative values, there are naked singularities. However, the α' corrections can sometimes eliminate these singularities, as shown in ref. [48]. On the other hand, the signs of the charges are not completely determined by those parameters, but by additional coefficients (± 1) that appear multiplying them in the 4-dimensional gauge fields. These coefficients have the values corresponding to supersymmetric black holes in the example that we are considering.

¹⁹Here we are using the notation of ref. [47].

To conclude this section it is, perhaps, worth stressing that the result we have obtained differs from the result obtained in ref. [39] and it is also worth explaining why.

There is a potential problem with all the results in this field of work: the dangers of confirmation bias. It is easy to conclude that, if the microscopic entropy is reproduced in a certain macroscopic calculation, then this calculation is correct. This temptation is strengthened by the fact that it is very difficult to make additional checks on the calculation, specially when one considers extremal black holes with zero temperature only.

The result of ref. [39], obtained using the Iyer-Wald prescription directly in 10 dimensions agrees with the microscopic calculation if it is interpreted as an approximate result containing the zeroth- and first-order terms in an expansion in α' of the microscopic result. However, further research carried out in ref. [47] showed that the procedure followed in ref. [39] had several problems. In particular, the dependence of the action on the curvature in 10 or in 10-d dimensions is different and, applying the Iyer-Wald formula to it gives different results. The dimensional reduction of the action to $d = 10 - n$ dimensions carried out in this paper was needed to obtain the correct dependence on the curvature. The $n = 1$ case was dealt with in ref. [1] for the same reasons but at least $n = 2$ was needed to consider 4-charge black holes.

The difference between the result of ref. [39] and the result obtained applying the Iyer-Wald prescription to the dimensionally-reduced action is very small: the coefficient of one term changes from 1 to 2. The effects of this change are, however, important:

1. The entropy formula obtained in ref. [1] with this coefficient gives an entropy which satisfies the first law of thermodynamics in the case of a non-extremal, α' -corrected Reissner-Nordström black hole. This provides the additional check one needs to make sure the result is correct, avoiding any confirmation bias. There is no microscopic entropy calculation for this particular black hole, but that is a completely different problem.
2. The entropy formula obtained in ref. [1], when applied to the α' -corrected Strominger-Vafa black hole, gives a result for the entropy different from the one obtained in our paper [46], which is analogous to ref. [39] but deals with 3-charge black holes. However, this entropy now coincides identically (not as an approximation) with the microscopic entropy.
3. Now the same happens in the case that we have considered in this paper: the entropy is different from the one obtained in ref. [39], which was interpreted as an approximation, but now it gives the exact result.

Using the dimensionally-reduced action has solved one of the problems that arise in the application of Iyer-Wald prescription to (heterotic) stringy black holes. The rest of the problems discussed in the introduction to ref. [1] still remain, as we are going to discuss in the last section.

6 Conclusions

In this paper we have shown that the bosonic sector of the complete Heterotic Superstring effective action compactified on T^n is $O(n, n)$ invariant to first-order in α' .²⁰ Local supersymmetry (which is preserved by the α' corrections) ensures the $O(n, n)$ invariance of the whole theory at first order in α' . The $(10 - n)$ -dimensional action is not really suitable for the derivation of the $(10 - n)$ -dimensional equations of motion because it is not easy to apply the Bergshoeff-de Roo lemma [23] to it. Nevertheless, one can always work in 10 dimensions, where it is easy to apply it, reducing afterwards the solutions to $(10 - n)$ using the formulae obtained in this paper.²¹ Furthermore, the action can be used to obtain a Wald entropy formula which we have used to compute the entropy of the most basic stringy 4-dimensional, 4-charge black holes. In order to apply it to more general black holes, non-extremal, non-supersymmetric, with more vector and scalar fields or with angular momentum one first needs to find their α' corrections, which can be a non-trivial, albeit highly interesting, task.

Our results leave, however, some important questions unanswered:

- If the vector fields are Abelian, does one recover $O(n, n + n_V)$ invariance in the presence of all the α' corrections (that is: adding the torsionful spin connection terms)? At first sight the answer should be yes: the situation is not qualitatively different from having a number of gauge fields n_A of which are Abelian and n_N of which are non-Abelian, where one should have $O(n, n + n_A)$ invariance. Nevertheless, it would be convenient to rewrite this case in a manifestly $O(n, n + n_V)$ -invariant form.
- It is tempting to conjecture that this invariance will be maintained to all orders in α' . However, since this invariance depends on the conspiracy between quite different terms, and we have not found the systematics behind the invariance, more work would be necessary in order to justify such a conjecture.
- In the 4-dimensional case, is S duality preserved (as expected) [52]? How is it realized? Work in these directions is already in progress [53].
- Why is the Iyer-Wald prescription so successful in this setting? As we have explained in the introduction to ref. [1], the Iyer-Wald prescription is based on two conditions: the invariance of the action under diffeomorphisms and the tensorial character of all the fields in the theory. While the first condition is met by the 10- and $(10 - n)$ -dimensional actions eqs. (1.9) and (4.24) to the order in α' we are considering here,²²

²⁰A similar result using the effective action of ref. [15] and without Yang-Mills fields has been recently published in ref. [22]. Furthermore, it has been argued in ref. [51] that $O(n, n)$ is present at all orders in α' .

²¹One can replace 10 by D in this discussion, as explained at the beginning of section 5.

²²There seems to be some confusion in the literature concerning this point. Sometimes the presence of Lorentz-Chern-Simons terms in the Kalb-Ramond field strength is associated to lack of invariance of the action under diffeomorphisms. However, the action eqs. (1.9) is, by construction, invariant under Yang-Mills, local-Lorentz, general coordinate and local-supersymmetry transformations order by order in α' [23]. If the Kalb-Ramond 2-form is dualized into a 6-form, the Chern-Simons 3-forms disappear from the field strength but reappear in a Chern-Simons 10-form [59]. That dual action is then invariant up to a total derivative which can be dealt with by the procedure considered in ref. [60]. This reference does not deal

the second is not because Yang-Mills fields and Vielbeins have gauge freedoms, a fact recognized and partially dealt with in refs. [54–58]. The case of a Kalb-Ramond field with Nicolai-Townsend gauge transformations (which is the case that arises in this formulation of the Heterotic superstring effective action) has not yet been fully studied.

This problem with the applicability of the Iyer-Wald prescription to the Heterotic Superstring effective action results, as a matter of fact, in entropy formulae which are manifestly not invariant under local Lorentz transformations.²³ And, yet, the results we have obtained by using this formula here and in refs. [1, 11] seem to be quite satisfactory, which we interpret as an indication that this formula captures the essential terms that contribute to the entropy.

It is clear that much more work is necessary to find an unambiguous, Lorentz-invariant, entropy formula for the Heterotic Superstring effective action. Hopefully, such a formula must reduce under certain general conditions to the entropy formula we have obtained here, but only when this formula is found it will be possible to make a fully trustable macroscopic calculation of the black hole entropy that we can compare with the microscopic one.

Work in this direction is also well under way [62].

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A $O(n, n)/(O(n) \times O(n))$ coset space

Since this coset space occurs repeatedly in the main body of this paper, we describe it here in some detail. First, we define the $n \times n$ matrices $E \equiv (e^i_m)$, $G \equiv (G_{mn})$ and $B \equiv (B_{mn})$. Evidently $E^{-1} = (e^m_i)$ and $E^T E = G$. With them we construct the $2n \times 2n$ Vielbein $\mathcal{U}$

with the gauge freedoms of the fields of the theory, though. In any case, one should not confuse the actions containing Chern-Simons 10- and the 3-forms, because the latter (ours) is exactly invariant under diffeomorphisms, which is the reason why we do not need the results of ref. [60].

²³A more rigorous derivation, performed in ref. [61] which takes into account these transformations shows the presence of an additional term not captured by the Iyer-Wald prescription that may restore the (expected) local Lorentz invariance of the Wald entropy formula. This term may not contribute in many relevant cases but it remains to be explained why and when. On the other hand, a proof of the first law of black hole mechanics for the theory at hands may have to be found in order to identify unambiguously the entropy.

and its inverse $\mathcal{U}^{-1}$

$$\mathcal{U} = -\frac{1}{\sqrt{2}} \begin{pmatrix} E^{-1} & E^{-1} \\ E^T + BE^{-1} & -E^T + BE^{-1} \end{pmatrix} = -\frac{1}{\sqrt{2}} \begin{pmatrix} e^m{}_i & -e^{mi} \\ -e_{mi} + B_{mn}e^n{}_i & -e_m{}^i - B_{mn}e^{ni} \end{pmatrix}, \quad (\text{A.1a})$$

$$\mathcal{U}^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} -E + E^{-1T}B & -E^{-1T} \\ -E - E^{-1T}B & E^{-1T} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} -e^i{}_m - e^{in}B_{nm} & e^{im} \\ e_{im} - e_i{}^n B_{nm} & e_i{}^m \end{pmatrix} \quad (\text{A.1b})$$

where we have used the metric $-\delta_{ij}$ to raise and lower $\text{SO}(n)$ indices i, j , consistently with our mostly minus convention for the 10-dimensional spacetime metric.

In terms of the non-diagonal $\text{O}(n, n)$ metric $\Omega = \Omega^{-1} = \begin{pmatrix} 0 & \mathbf{1}_{n \times n} \\ \mathbf{1}_{n \times n} & 0 \end{pmatrix}$ we can define $\text{O}(n, n)$ transformations Λ as those satisfying

$$\Lambda^T \Omega \Lambda = \Omega. \quad (\text{A.2})$$

Under a $\text{O}(n, n)$ transformation Λ acting from the left, the Vielbein $\mathcal{U}$ transforms as

$$\Lambda \mathcal{U} = \mathcal{U}' R, \quad R = \begin{pmatrix} R_{(+)} & 0 \\ 0 & R_{(-)} \end{pmatrix} \in \text{O}(n)_{(+)} \times \text{O}(n)_{(-)}. \quad (\text{A.3})$$

Thus, using $\mathcal{U}$, we can construct a symmetric matrix M

$$M \equiv \mathcal{U} \mathcal{U}^T = \begin{pmatrix} G^{-1} & -G^{-1}B \\ -B^T G^{-1} & G + B^T G^{-1}B \end{pmatrix}, \quad (\text{A.4})$$

which transforms under $\text{O}(n, n)$ as

$$M' = \Lambda M \Lambda^T. \quad (\text{A.5})$$

The inverse of M is given by

$$M^{-1} = \mathcal{U}^{-1T} \mathcal{U}^{-1} = \begin{pmatrix} G + B^T G^{-1}B & -B^T G^{-1} \\ -G^{-1}B^T & G^{-1} \end{pmatrix}, \quad (\text{A.6})$$

and it is not difficult to check that

$$M^{-1} = \Omega M \Omega, \quad (\text{A.7})$$

which implies that M is a $\text{O}(n, n)$ matrix itself.

The left-invariant Maurer-Cartan 1-form is

$$-\mathcal{U}^{-1} d\mathcal{U} = \begin{pmatrix} A_{(+)}{}^i{}_j & P_{(+)}{}^{ij} \\ P_{(-)}{}_{ij} & A_{(-)}{}^j{}_i \end{pmatrix}, \quad (\text{A.8})$$

where the $\text{O}(n)_{(+)}$ connection $A_{(+)}{}^i{}_j$, the $\text{O}(n)_{(-)}$ connection $A_{(-)}{}^j{}_i$ and the Vielbein $P_{(+)}{}_{ij} = P_{(-)}{}^{ji}$ (n^2 degrees of freedom) are given by²⁴

$$A_{(\pm)}{}^{ij} \equiv A^{ij} \mp \frac{1}{2} e^{im} e^{jn} dB_{mn}, \quad (\text{A.9a})$$

$$P_{(\pm)}{}^{ij} \equiv \frac{1}{2} e^{im} e^{jn} d(G_{mn} \mp B_{mn}), \quad (\text{A.9b})$$

²⁴We have made use of eq. (2.6).

where, in its turn

$$A^{ij} \equiv -e^{[i|m} de_m{}^{j]} . \quad (\text{A.10})$$

Observe that the Vielbein transforms under both $O(n)$ groups:

$$P'_{(\pm)} = R_{\pm} P_{(\pm)} R_{\mp} . \quad (\text{A.11})$$

The Maurer-Cartan equations, obtained by taking the exterior derivative of the left-invariant Maurer-Cartan 1-forms lead to the following identities:

$$F_{(+)}^{ij} = P_{(+)}^{ik} \wedge P_{(-)}^k{}_j , \quad (\text{A.12a})$$

$$F_{(-)}^{ij} = P_{(-)}^{ik} \wedge P_{(+)}^k{}_j , \quad (\text{A.12b})$$

$$\mathfrak{D}_{(\pm\mp)} P_{(\pm)}^{ij} = 0 , \quad (\text{A.12c})$$

where

$$F_{(\pm)}^{ij} = dA_{(\pm)}^{ij} + A_{(\pm)}^{ik} \wedge A_{(\pm)}^k{}_j , \quad (\text{A.13})$$

are the curvatures of the two $O(n)$ connections and where the covariant derivative $\mathfrak{D}_{(\mp\pm)}$ uses the $(\mp)$ connection on the first $O(n)$ index and the $(\pm)$ connection on the second of $P_{(\pm)}^{ij}$, that is:

$$\mathfrak{D}_{(\pm\mp)} P_{(\pm)}^{ij} = dP_{(\pm)}^{ij} + A_{(\pm)}^{ik} \wedge P_{(\pm)}^k{}_j + \wedge P_{(\pm)}^{ik} \wedge A_{(\mp)}^k{}_j . \quad (\text{A.14})$$

From the Maurer-Cartan eqs. (A.12a) and (A.12b) we find

$$F_{(\pm)}^{ij} \cdot F_{(\pm)}^{ij} = 2P_{(\pm)}^{(0)a}{}_{ki} P_{(\pm)}^{(0)a}{}_{li} P_{(\pm)}^{(0)b}{}_{ki} P_{(\pm)}^{(0)b}{}_{li} - 2P_{(\pm)}^{(0)a}{}_{ki} P_{(\pm)}^{(0)a}{}_{lj} P_{(\pm)}^{(0)b}{}_{li} P_{(\pm)}^{(0)b}{}_{kj} , \quad (\text{A.15})$$

and the relation $P_{(\pm)}^{ij} = P_{(\mp)}^{ji}$ implies that

$$F_{(+)}^{ij} \cdot F_{(+)}^{ij} = F_{(-)}^{ij} \cdot F_{(-)}^{ij} . \quad (\text{A.16})$$

The scalar kinetic term is given by

$$\begin{aligned} P_{(+)}^{\mu}{}_{ij} P_{(-)}^{\mu}{}_{ji} &= \frac{1}{4} G^{mn} G^{pq} \partial_{\mu} (G_{mp} + B_{mp}) \partial^{\mu} (G_{nq} - B_{nq}) \\ &= -\frac{1}{4} \partial_{\mu} G_{mn} \partial^{\mu} G^{mn} - \frac{1}{4} G^{mn} G^{pq} \partial_{\mu} B_{mp} \partial^{\mu} B_{nq} , \end{aligned} \quad (\text{A.17})$$

or by the equivalent expression

$$-\frac{1}{2} \text{Tr} \left(\partial_{\mu} M \partial^{\mu} M^{-1} \right) . \quad (\text{A.18})$$

Using the $O(n, n)$ vector

$$\mathcal{F} \equiv \begin{pmatrix} F^m \\ G_m \end{pmatrix} , \quad (\text{A.19})$$

one can construct the following combinations that occur naturally in some of the expressions in the main paper:

$$\mathcal{U}^{-1} \mathcal{F} = \frac{1}{\sqrt{2}} \begin{pmatrix} e^i{}_m \left(F^m + G^{mn} G^{(0)}{}_n - G^{mn} B_{np} F^p \right) \\ e^i{}_m \left(F^m - G^{mn} G^{(0)}{}_n + G^{mn} B_{np} F^p \right) \end{pmatrix} \equiv -\frac{1}{\sqrt{2}} \begin{pmatrix} K_{(+)}^i \\ K_{(-)}^i \end{pmatrix} . \quad (\text{A.20})$$

Observe that $K_{(\pm)}^i$ only transform under $O(n)_\pm$ rotations, respectively. We can construct $O(n, n)$ invariants by building $O(n)_\pm$ invariants. For instance:

$$K_{(\pm)i} K_{(\pm)i} = \mathcal{F}^T M^{-1} \mathcal{F} \pm 2 \mathcal{F}^T \Omega \mathcal{F}, \quad (\text{A.21})$$

which is clearly consistent with

$$(\mathcal{U}^{-1} \mathcal{F})^T \mathcal{U}^{-1} \mathcal{F} = \mathcal{F}^T M^{-1} \mathcal{F}. \quad (\text{A.22})$$

Other examples of $O(n, n)$ invariants built in the same way that arise in the main text are

$$\mathfrak{D}_{(-)\mu} K_{(-)\nu\rho}^i K_{(-)\lambda\sigma}^i, \quad \text{and} \quad P_{(-)ji}^a \mathfrak{D}_{(-)\mu} K_{(-)\nu\rho}^j K_{(+)\lambda\sigma}^i. \quad (\text{A.23})$$

On the other hand, if we have a $O(n)_\pm$ vector ξ^i , we can construct a $O(n, n)$ vector using the Vielbein

$$\mathcal{U}_i^\pm \xi^i = \frac{1}{\sqrt{2}} \begin{pmatrix} -e^m{}_i \xi^i \\ \pm e_{mi} \xi^i - B_{mn} e^n{}_i \xi^i \end{pmatrix}, \quad (\text{A.24})$$

and, then, the $O(n, n)$ invariants

$$\left(\mathcal{U}_i^\pm \xi^i \right)^T \Omega \mathcal{U}_j^\pm \xi^j = \pm \xi^i \xi_i, \quad (\text{A.25a})$$

$$\mathfrak{D} \left(\mathcal{U}_i^\pm \xi^i \right)^T \Omega \mathfrak{D} \mathcal{U}_j^\pm \xi^j = -\mathfrak{D} \left(e^m{}_i \xi^i \right) \mathfrak{D} \left(\pm e_{mj} \xi^j - B_{mn} e^n{}_j \xi^j \right), \quad (\text{A.25b})$$

the first of which is trivial. The second occurs in the main text with $\xi_i = \varphi^A_i$ transforming under $O(n)_{(+)}$. With this assignment,

$$\varphi^A_i K_{(+)}^i, \quad (\text{A.26})$$

is also $O(n, n)$ -invariant.

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References

- [1] Z. Elgood and T. Ortín, *T duality and Wald entropy formula in the Heterotic Superstring effective action at first-order in α'* , *JHEP* **10** (2020) 097 [[arXiv:2005.11272](https://arxiv.org/abs/2005.11272)] [[INSPIRE](#)].
- [2] T.H. Buscher, *A Symmetry of the String Background Field Equations*, *Phys. Lett. B* **194** (1987) 59 [[INSPIRE](#)].
- [3] T.H. Buscher, *Path Integral Derivation of Quantum Duality in Nonlinear Sigma Models*, *Phys. Lett. B* **201** (1988) 466 [[INSPIRE](#)].
- [4] E. Bergshoeff, B. Janssen and T. Ortín, *Solution generating transformations and the string effective action*, *Class. Quant. Grav.* **13** (1996) 321 [[hep-th/9506156](https://arxiv.org/abs/hep-th/9506156)] [[INSPIRE](#)].
- [5] M. Serone and M. Trappetti, *A Note on T-duality in heterotic string theory*, *Phys. Lett. B* **637** (2006) 331 [[hep-th/0512272](https://arxiv.org/abs/hep-th/0512272)] [[INSPIRE](#)].

- [6] O.A. Bedoya, D. Marqués and C. Núñez, *Heterotic α' -corrections in Double Field Theory*, *JHEP* **12** (2014) 074 [[arXiv:1407.0365](#)] [[INSPIRE](#)].
- [7] J. Lee and R.M. Wald, *Local symmetries and constraints*, *J. Math. Phys.* **31** (1990) 725 [[INSPIRE](#)].
- [8] R.M. Wald, *Black hole entropy is the Noether charge*, *Phys. Rev. D* **48** (1993) R3427(R) [[gr-qc/9307038](#)] [[INSPIRE](#)].
- [9] V. Iyer and R.M. Wald, *Some properties of Noether charge and a proposal for dynamical black hole entropy*, *Phys. Rev. D* **50** (1994) 846 [[gr-qc/9403028](#)] [[INSPIRE](#)].
- [10] P. Kraus and F. Larsen, *Holographic gravitational anomalies*, *JHEP* **01** (2006) 022 [[hep-th/0508218](#)] [[INSPIRE](#)].
- [11] P.A. Cano, S. Chimento, R. Linares, T. Ortín and P.F. Ramírez, *α' corrections of Reissner-Nordström black holes*, *JHEP* **02** (2020) 031 [[arXiv:1910.14324](#)] [[INSPIRE](#)].
- [12] J. Maharana and J.H. Schwarz, *Noncompact symmetries in string theory*, *Nucl. Phys. B* **390** (1993) 3 [[hep-th/9207016](#)] [[INSPIRE](#)].
- [13] J.M. Maldacena and A. Strominger, *Statistical entropy of four-dimensional extremal black holes*, *Phys. Rev. Lett.* **77** (1996) 428 [[hep-th/9603060](#)] [[INSPIRE](#)].
- [14] C.V. Johnson, R.R. Khuri and R.C. Myers, *Entropy of 4D extremal black holes*, *Phys. Lett. B* **378** (1996) 78 [[hep-th/9603061](#)] [[INSPIRE](#)].
- [15] R.R. Metsaev and A.A. Tseytlin, *Order α' (Two Loop) Equivalence of the String Equations of Motion and the σ -Model Weyl Invariance Conditions: Dependence on the Dilaton and the Antisymmetric Tensor*, *Nucl. Phys. B* **293** (1987) 385 [[INSPIRE](#)].
- [16] K.A. Meissner, *Symmetries of higher order string gravity actions*, *Phys. Lett. B* **392** (1997) 298 [[hep-th/9610131](#)] [[INSPIRE](#)].
- [17] N. Kaloper and K.A. Meissner, *Duality beyond the first loop*, *Phys. Rev. D* **56** (1997) 7940 [[hep-th/9705193](#)] [[INSPIRE](#)].
- [18] O. Hohm and B. Zwiebach, *Green-Schwarz mechanism and α' -deformed Courant brackets*, *JHEP* **01** (2015) 012 [[arXiv:1407.0708](#)] [[INSPIRE](#)].
- [19] D. Marqués and C.A. Núñez, *T-duality and α' -corrections*, *JHEP* **10** (2015) 084 [[arXiv:1507.00652](#)] [[INSPIRE](#)].
- [20] W.H. Baron, J.J. Fernández-Melgarejo, D. Marqués and C. Núñez, *The Odd story of α' -corrections*, *JHEP* **04** (2017) 078 [[arXiv:1702.05489](#)] [[INSPIRE](#)].
- [21] C. Eloy, O. Hohm and H. Samtleben, *Green-Schwarz Mechanism for String Dualities*, *Phys. Rev. Lett.* **124** (2020) 091601 [[arXiv:1912.01700](#)] [[INSPIRE](#)].
- [22] C. Eloy, O. Hohm and H. Samtleben, *Duality Invariance and Higher Derivatives*, *Phys. Rev. D* **101** (2020) 126018 [[arXiv:2004.13140](#)] [[INSPIRE](#)].
- [23] E.A. Bergshoeff and M. de Roo, *The Quartic Effective Action of the Heterotic String and Supersymmetry*, *Nucl. Phys. B* **328** (1989) 439 [[INSPIRE](#)].
- [24] E. Bergshoeff and M. de Roo, *Supersymmetric Chern-Simons Terms in Ten-dimensions*, *Phys. Lett. B* **218** (1989) 210 [[INSPIRE](#)].
- [25] T. Ortín, *Gravity and Strings*, 2nd edition, in *Cambridge Monographs on Mathematical Physics*, Cambridge University Press, Cambridge U.K. (2015) [[INSPIRE](#)].

- [26] A. Fontanella and T. Ortín, *On the supersymmetric solutions of the Heterotic Superstring effective action*, *JHEP* **06** (2020) 106 [[arXiv:1910.08496](#)] [[INSPIRE](#)].
- [27] J. Scherk and J.H. Schwarz, *How to Get Masses from Extra Dimensions*, *Nucl. Phys. B* **153** (1979) 61 [[INSPIRE](#)].
- [28] G.F. Chapline and N.S. Manton, *Unification of Yang-Mills Theory and Supergravity in Ten-Dimensions*, *Phys. Lett. B* **120** (1983) 105 [[INSPIRE](#)].
- [29] A.H. Chamseddine, *$N = 4$ Supergravity Coupled to $N = 4$ Matter*, *Nucl. Phys. B* **185** (1981) 403 [[INSPIRE](#)].
- [30] H. Nicolai and P.K. Townsend, *$N = 3$ Supersymmetry Multiplets with Vanishing Trace Anomaly: Building Blocks of the $N > 3$ Supergravities*, *Phys. Lett. B* **98** (1981) 257 [[INSPIRE](#)].
- [31] E. Bergshoeff, M. de Roo, B. de Wit and P. van Nieuwenhuizen, *Ten-Dimensional Maxwell-Einstein Supergravity, Its Currents, and the Issue of Its Auxiliary Fields*, *Nucl. Phys. B* **195** (1982) 97 [[INSPIRE](#)].
- [32] D.M. Kaplan, D.A. Lowe, J.M. Maldacena and A. Strominger, *Microscopic entropy of $N = 2$ extremal black holes*, *Phys. Rev. D* **55** (1997) 4898 [[hep-th/9609204](#)] [[INSPIRE](#)].
- [33] M. Cvetič and D. Youm, *Dyonic BPS saturated black holes of heterotic string on a six torus*, *Phys. Rev. D* **53** (1996) 584 [[hep-th/9507090](#)] [[INSPIRE](#)].
- [34] M. Cvetič and A.A. Tseytlin, *General class of BPS saturated dyonic black holes as exact superstring solutions*, *Phys. Lett. B* **366** (1996) 95 [[hep-th/9510097](#)] [[INSPIRE](#)].
- [35] M. Cvetič and A.A. Tseytlin, *Solitonic strings and BPS saturated dyonic black holes*, *Phys. Rev. D* **53** (1996) 5619 [Erratum *ibid.* **55** (1997) 3907] [[hep-th/9512031](#)] [[INSPIRE](#)].
- [36] J. Rahmfeld, *Extremal black holes as bound states*, *Phys. Lett. B* **372** (1996) 198 [[hep-th/9512089](#)] [[INSPIRE](#)].
- [37] M.J. Duff, J.T. Liu and J. Rahmfeld, *Four-dimensional string-string-string triality*, *Nucl. Phys. B* **459** (1996) 125 [[hep-th/9508094](#)] [[INSPIRE](#)].
- [38] S. Chimento, P. Meessen, T. Ortín, P.F. Ramírez and A. Ruipérez, *On a family of α' -corrected solutions of the Heterotic Superstring effective action*, *JHEP* **07** (2018) 080 [[arXiv:1803.04463](#)] [[INSPIRE](#)].
- [39] P.A. Cano, S. Chimento, P. Meessen, T. Ortín, P.F. Ramírez and A. Ruipérez, *Beyond the near-horizon limit: Stringy corrections to Heterotic Black Holes*, *JHEP* **02** (2019) 192 [[arXiv:1808.03651](#)] [[INSPIRE](#)].
- [40] K. Behrndt, G. Lopes Cardoso, B. de Wit, D. Lüst, T. Mohaupt and W.A. Sabra, *Higher order black hole solutions in $N = 2$ supergravity and Calabi-Yau string backgrounds*, *Phys. Lett. B* **429** (1998) 289 [[hep-th/9801081](#)] [[INSPIRE](#)].
- [41] G. Lopes Cardoso, B. de Wit and T. Mohaupt, *Corrections to macroscopic supersymmetric black hole entropy*, *Phys. Lett. B* **451** (1999) 309 [[hep-th/9812082](#)] [[INSPIRE](#)].
- [42] G. Lopes Cardoso, B. de Wit and T. Mohaupt, *Deviations from the area law for supersymmetric black holes*, *Fortsch. Phys.* **48** (2000) 49 [[hep-th/9904005](#)] [[INSPIRE](#)].
- [43] G. Lopes Cardoso, B. de Wit and T. Mohaupt, *Macroscopic entropy formulae and nonholomorphic corrections for supersymmetric black holes*, *Nucl. Phys. B* **567** (2000) 87 [[hep-th/9906094](#)] [[INSPIRE](#)].

- [44] B. Sahoo and A. Sen, α' corrections to extremal dyonic black holes in heterotic string theory, *JHEP* **01** (2007) 010 [[hep-th/0608182](#)] [[INSPIRE](#)].
- [45] P. Dominis Prester and T. Terzic, α' -exact entropies for BPS and non-BPS extremal dyonic black holes in heterotic string theory from ten-dimensional supersymmetry, *JHEP* **12** (2008) 088 [[arXiv:0809.4954](#)] [[INSPIRE](#)].
- [46] P.A. Cano, P. Meessen, T. Ortín and P.F. Ramírez, α' -corrected black holes in String Theory, *JHEP* **05** (2018) 110 [[arXiv:1803.01919](#)] [[INSPIRE](#)].
- [47] F. Faedo and P.F. Ramírez, Exact charges from heterotic black holes, *JHEP* **10** (2019) 033 [[arXiv:1906.12287](#)] [[INSPIRE](#)].
- [48] P.A. Cano, S. Chimento, T. Ortín and A. Ruipérez, Regular Stringy Black Holes?, *Phys. Rev. D* **99** (2019) 046014 [[arXiv:1806.08377](#)] [[INSPIRE](#)].
- [49] D. Kutasov, F. Larsen and R.G. Leigh, String theory in magnetic monopole backgrounds, *Nucl. Phys. B* **550** (1999) 183 [[hep-th/9812027](#)] [[INSPIRE](#)].
- [50] T. Ortín, Extremality versus supersymmetry in stringy black holes, *Phys. Lett. B* **422** (1998) 93 [[hep-th/9612142](#)] [[INSPIRE](#)].
- [51] O. Hohm, A. Sen and B. Zwiebach, Heterotic Effective Action and Duality Symmetries Revisited, *JHEP* **02** (2015) 079 [[arXiv:1411.5696](#)] [[INSPIRE](#)].
- [52] A. Font, L.E. Ibáñez, D. Lüst and F. Quevedo, Strong-weak coupling duality and non-perturbative effects in string theory, *Phys. Lett. B* **249** (1990) 35 [[INSPIRE](#)].
- [53] Z. Elgood, P. Meessen, Á. Murcia and T. Ortín, work in progress.
- [54] G. Compère, Note on the First Law with p -form potentials, *Phys. Rev. D* **75** (2007) 124020 [[hep-th/0703004](#)] [[INSPIRE](#)].
- [55] T. Jacobson and A. Mohd, Black hole entropy and Lorentz-diffeomorphism Noether charge, *Phys. Rev. D* **92** (2015) 124010 [[arXiv:1507.01054](#)] [[INSPIRE](#)].
- [56] K. Prabhu, The First Law of Black Hole Mechanics for Fields with Internal Gauge Freedom, *Class. Quant. Grav.* **34** (2017) 035011 [[arXiv:1511.00388](#)] [[INSPIRE](#)].
- [57] Z. Elgood, P. Meessen and T. Ortín, The first law of black hole mechanics in the Einstein-Maxwell theory revisited, *JHEP* **09** (2020) 026 [[arXiv:2006.02792](#)] [[INSPIRE](#)].
- [58] P.B. Aneesh, S. Chakraborty, S.J. Hoque and A. Virmani, First law of black hole mechanics with fermions, *Class. Quant. Grav.* **37** (2020) 205014 [[arXiv:2004.10215](#)] [[INSPIRE](#)].
- [59] E.A. Bergshoeff and M. de Roo, The string effective action in the dual formulation of $D = 10$ supergravity, *Phys. Lett. B* **247** (1990) 530 [[INSPIRE](#)].
- [60] Y. Tachikawa, Black hole entropy in the presence of Chern-Simons terms, *Class. Quant. Grav.* **24** (2007) 737 [[hep-th/0611141](#)] [[INSPIRE](#)].
- [61] J.D. Edelstein, K. Sfetsos, J.A. Sierra-García and A. Vilar López, T -duality and high-derivative gravity theories: the BTZ black hole/string paradigm, *JHEP* **06** (2018) 142 [[arXiv:1803.04517](#)] [[INSPIRE](#)].
- [62] Z. Elgood, T. Ortín and D. Pereñíguez, The first law and Wald entropy formula of heterotic stringy black holes at first order in α' , [arXiv:2012.14892](#) [[INSPIRE](#)].