New magnetic symmetries in $(d + 2)$ -dimensional QED

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ABSTRACT: Previous analyses of asymptotic symmetries in QED have shown that the subleading soft photon theorem implies a Ward identity corresponding to a charge generating divergent large gauge transformations on the asymptotic states at null infinity. In this work, we demonstrate that the subleading soft photon theorem is equivalent to a more general Ward identity. The charge corresponding to this Ward identity can be decomposed into an electric piece and a magnetic piece. The electric piece generates the Ward identity that was previously studied, but the magnetic piece is novel, and implies the existence of an additional asymptotic “magnetic” symmetry in QED.

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1 Introduction

In recent years, an intricate relationship between soft theorems and asymptotic symmetries in asymptotically flat spacetimes has been discovered and extensively studied (for a detailed review of this subject, see [1]). It began with the discovery that the leading soft theorems in both four dimensional gauge and gravity theories are equivalent to Ward identities associated to charges generating asymptotic symmetries of the theory [2–7]. These results were later extended to all higher dimensions [8–11].

The relationship between asymptotic symmetries and soft theorems became more intriguing when it was observed that there is also a relationship between the subleading soft theorems and asymptotic charges generating divergent large gauge symmetries in all dimensions [12–19]. However, unlike the case involving the leading soft theorems, the subleading soft theorem is oftentimes a *stronger* condition than the associated Ward identity. While the subleading soft theorem implies the Ward identity, the reverse is not necessarily true.

Traditionally, one conjectures a ‘matching condition’ that relates a specific component of a field at past null infinity $\mathcal{I}^-$ to that at future null infinity $\mathcal{I}^+$ (see section 3.1),¹

¹These matching conditions were rigorously studied (and proved) in QED and gravity in [20, 21].

which can then be massaged into a Ward identity (in the semi-classical theory) for the S -matrix. For example, in gauge theories one imposes such a matching condition for the radial electric field E_r , and the corresponding Ward identity is equivalent to the leading soft photon theorem. Similarly, in gravitational theories the matching condition for the electric part of the Weyl tensor is equivalent to the leading soft graviton theorem. Crucially, each independent matching condition leads to an independent Ward identity or soft theorem. This does not imply a contradiction for the d leading soft photon theorems (one for each polarization), which are not all independent. Rather, they satisfy a trivial identity that results in a single independent leading soft theorem [10]; this is the soft theorem that is equivalent to the matching condition described above. A similar argument holds for the leading soft graviton theorem as well.

However, such an identity does not hold for the subleading soft photon theorem, thereby implying there are indeed d independent subleading soft theorems. It is therefore not possible to demonstrate its equivalence with a Ward identity arising from matching the radial electric field. Rather, the matching condition for the radial electric field leads to a particular linear combination of the subleading soft theorem [19], which we shall henceforth call the subleading *electric* Ward identity. The origin (from the perspective of asymptotic symmetries or matching conditions) of the remaining $d - 1$ independent soft theorems is, so far, unknown.

In this paper, we conjecture a matching condition for the d angular components of the magnetic field (a vector matching condition), and then show that it is equivalent to the d independent subleading soft photon theorems. Naturally, our ansatz that there are d matching conditions instead of just one implies that the associated Ward identities must correspond to new symmetries. As it turns out, the subleading electric Ward identity corresponds precisely to one of the d matching conditions. The remaining $d - 1$ matching conditions then give rise to Ward identities corresponding to charges that generate magnetic large gauge transformations, and we shall call these Ward identities the subleading *magnetic* Ward identities. This suggests that even though there are no global magnetic charges in our theory, there exists finite large gauge symmetries that are generated by asymptotic magnetic charges.

This paper is organized as follows. In section 2, we will summarize all the notations and conventions used throughout the paper. We will also derive the asymptotic expansion of the gauge field near $\mathcal{I}^\pm$; because much of the technology used was introduced in [10], we refer the reader there for more details. In section 3, we will conjecture the set of d matching conditions and derive the corresponding Ward identities. Next, in section 4, we prove the equivalence between the subleading soft theorems and the Ward identities. Finally, we explain in section 5 the interpretations of the charges that correspond to these new Ward identities.

2 Asymptotic behavior of gauge field

2.1 Preliminaries

In this section, we introduce the notations employed in this paper (following the conventions of [10]) and review related previous work.

Spacetime coordinates. We work in flat null coordinates $x^\mu = (u, r, x^a)$, $a = 1, \dots, d$, where $d \geq 2$. These are related to Cartesian coordinates by

$$X^A = \frac{r}{2} \left(1 + x^2 + \frac{u}{r}, 2x^a, 1 - x^2 - \frac{u}{r} \right), \quad (2.1)$$

and the standard Minkowski metric in flat null coordinates takes the form

$$ds^2 = -du dr + r^2 \delta_{ab} dx^a dx^b. \quad (2.2)$$

$\mathcal{I}^\pm$ is located at $r \rightarrow \pm\infty$ while keeping (u, x^a) fixed, and these surfaces have the topology $S^d \times \mathbb{R}$. The point coordinatized by x^a on $\mathcal{I}^+$ is antipodal to the point with the same coordinate on $\mathcal{I}^-$. The boundaries of $\mathcal{I}^+$ and $\mathcal{I}^-$ are located at $u = \pm\infty$ and are denoted by $\mathcal{I}_\pm^+$ and $\mathcal{I}_\pm^-$, respectively.

Momenta coordinates. We will focus on the scattering of massless particles, which satisfy $p^A p_A = 0$. We parameterize such momenta using flat null coordinates so that

$$p^A(\omega, x) = \omega \hat{p}^A(x), \quad \hat{p}^A(x) = \frac{1}{2} \left(1 + x^2, 2x^a, 1 - x^2 \right). \quad (2.3)$$

Massless gauge fields transform under the vector representation of the little group $\mathcal{O}(d)$ and have d polarizations. The d polarization vectors $\varepsilon_a^A(x)$ are

$$\varepsilon_a^A(x) = \partial_a \hat{p}^A(x) = (x_a, \delta_a^b, -x_a). \quad (2.4)$$

One particle in- and out-states with momenta $\vec{p}$ are created from the vacuum by *in* ($-$) and *out* ($+$) creation and annihilation operators denoted by $\mathcal{O}_\alpha^{(\pm)\dagger}(p)$ and $\mathcal{O}_\alpha^{(\pm)}(p)$, where α labels the polarization of the particle. They are canonically normalized, i.e.

$$\left\{ \mathcal{O}_\alpha^{(\pm)}(p), \mathcal{O}_\beta^{(\pm)\dagger}(p') \right\} = \delta_{\alpha\beta} (2p^0) (2\pi)^{d+1} \delta^{(d+1)}(\vec{p} - \vec{p}'), \quad (2.5)$$

where $[\cdot, \cdot]$ indicates an anticommutator if both operators are fermionic and a commutator otherwise. Using the parameterization (2.3), this can be written as

$$\left\{ \mathcal{O}_\alpha^{(\pm)}(\omega, x), \mathcal{O}_\beta^{(\pm)\dagger}(\omega', x') \right\} = 2\omega^{1-d} (2\pi)^{d+1} \delta_{\alpha\beta} \delta(\omega - \omega') \delta^{(d)}(x - x'). \quad (2.6)$$

Poincaré algebra. The Poincaré algebra is generated by P_A and M_{AB} and takes the form

$$\begin{aligned} [P_A, P_B] &= 0, & [P_A, M_{BC}] &= -i(\eta_{AB} P_C - \eta_{AC} P_B) \\ [M_{AB}, M_{CD}] &= i(\eta_{AC} M_{BD} + \eta_{BD} M_{AC} - \eta_{AD} M_{BC} - \eta_{BC} M_{AD}). \end{aligned} \quad (2.7)$$

We define

$$P_\pm = -P_0 \mp P_{d+1}, \quad T_a = M_{0a} - M_{(d+1)a}, \quad D = M_{(d+1)0}, \quad K_a = M_{0a} + M_{(d+1)a}, \quad (2.8)$$

so that the nonzero commutators in the Poincaré algebra are given by

$$\begin{aligned} [M_{ab}, M_{cd}] &= i(\delta_{ac} M_{bd} + \delta_{bd} M_{ac} - \delta_{bc} M_{ad} - \delta_{ad} M_{bc}) \\ [M_{ab}, T_c] &= i(\delta_{ac} T_b - \delta_{bc} T_a), & [M_{ab}, K_c] &= i(\delta_{ac} K_b - \delta_{bc} K_a) \\ [T_a, D] &= iT_a, & [K_a, D] &= -iK_a, & [T_a, K_b] &= -2i(\delta_{ab} D + M_{ab}) \\ [M_{ab}, P_c] &= i(\delta_{ac} P_b - \delta_{bc} P_a), & [P_\pm, D] &= \mp iP_\pm, & [P_a, T_b] &= -i\delta_{ab} P_- \\ [P_+, T_a] &= -2iP_a, & [P_a, K_b] &= -i\delta_{ab} P_+, & [P_-, K_a] &= -2iP_a. \end{aligned} \quad (2.9)$$

In addition to the Poincaré transformations, we will also consider scale transformations $X^A \rightarrow \lambda X^A$, which appears as an effective symmetry in the infrared (near the asymptotic regions of spacetime). We denote the generator of scale transformations by S , which satisfies the commutation relations

$$[P_A, S] = iP_A, \quad [M_{AB}, S] = 0. \quad (2.10)$$

Gauge theory. A U(1) gauge theory is described in terms of a 2-form field strength $F_{\mu\nu}$ that satisfies Maxwell's equations, i.e.

$$\nabla^\mu F_{\mu\nu} = e^2 J_\nu, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \quad \nabla^\mu J_\mu = 0, \quad (2.11)$$

where J_μ is the matter current. The theory is invariant under the gauge transformations

$$A_\mu \rightarrow A_\mu + \partial_\mu \varepsilon, \quad \Psi_i \rightarrow e^{iQ_i \varepsilon} \Psi_i, \quad (2.12)$$

where $\varepsilon \sim \varepsilon + 2\pi$ and $Q_i \in \mathbb{Z}$ is the U(1) charge of the matter field Ψ_i . Gauge transformations that vanish at infinity map physically equivalent solutions to each other and are therefore merely redundancies of the theory. We will use this redundancy to impose the gauge condition

$$A_u = 0 \quad (2.13)$$

when carrying out the asymptotic expansion of the radiative field. Note that (2.13) is consistent with the choice of polarization in (2.4).

In flat null coordinates, Maxwell's equations take the form

$$\begin{aligned} e^2 J_u &= \partial_u \left(2\partial_u A_r - \frac{1}{r^2} \partial^a A_a \right) \\ e^2 J_r &= -\frac{2}{r^d} \partial_r \left(r^d \partial_u A_r \right) - \frac{1}{r^2} \partial_r \partial^a A_a + \frac{1}{r^2} \partial^2 A_r \\ e^2 J_a &= -2\partial_u \partial_r A_a + 2\partial_u \partial_a A_r - \frac{2}{r^{d-2}} \partial_u \partial_r \left(r^{d-2} A_a \right) - \frac{1}{r^2} \partial_a \partial^b A_b + \frac{1}{r^2} \partial^2 A_a, \end{aligned} \quad (2.14)$$

where $\partial^2 \equiv \partial^a \partial_a$. The gauge field can be split into two pieces $A_\mu = A_\mu^{(R)} + A_\mu^{(C)}$. The radiative field $A_\mu^{(R)}$ satisfies the sourceless Maxwell's equations, whereas the Coulombic field $A_\mu^{(C)}$ is the inhomogeneous solution to Maxwell's equations and is uniquely fixed by a choice of Green's function.² We are interested in the asymptotic *in* (−) and *out* (+) solutions, which are respectively obtained by choosing the retarded and advanced Green's functions. The corresponding radiative and Coulombic fields are then denoted by $A_\mu^{(R\pm)}$ and $A_\mu^{(C\pm)}$.

2.2 Radiative field

In this subsection, we study the radiative gauge field near $\mathcal{I}^\pm$ by expanding it in powers of $1/r$. Because we adopt the same strategy and techniques introduced in [10], we refer the reader there for additional details and explanations.

²We remark that in dimensions $d > 2$, the Coulombic field falls off more quickly in powers of $1/r$ than the radiative field.

The radiative gauge field satisfies the sourceless Maxwell's equations and hence admits the mode expansion

$$A_A^{(R\pm)}(X) = e \int \frac{d^{d+1}q}{(2\pi)^{d+1}} \frac{1}{2q^0} \left(\varepsilon_A^a(\vec{q}) \mathcal{O}_a^{(\pm)}(\vec{q}) e^{iq \cdot X} + \varepsilon_A^a(\vec{q})^* \mathcal{O}_a^{(\pm)\dagger}(\vec{q}) e^{-iq \cdot X} \right), \quad (2.15)$$

where $q^0 = |\vec{q}|$ and ε^a is the polarization vector defined in (2.4). Switching to the flat null coordinate parametrization of momenta defined in (2.3), we obtain

$$\begin{aligned} A_r^{(R\pm)}(u, r, x) &= \frac{e}{2(2\pi)^{d+1}r} \int_0^\infty d\omega \omega^{d-2} \int d^d y \left(i\partial^a \mathcal{O}_a^{(\pm)}(\omega, x+y) e^{-\frac{i}{2}\omega u - \frac{i}{2}\omega r y^2} + \text{c.c.} \right) \\ A_a^{(R\pm)}(u, r, x) &= \frac{er}{2(2\pi)^{d+1}} \int_0^\infty d\omega \omega^{d-1} \int d^d y \left(\mathcal{O}_a^{(\pm)}(\omega, x+y) e^{-\frac{i}{2}\omega u - \frac{i}{2}\omega r y^2} + \text{c.c.} \right). \end{aligned} \quad (2.16)$$

To perform the large $|r|$ expansion, we assume that the creation and annihilation operators admit the Fourier expansion

$$\mathcal{O}_a^{(\pm)}(\omega, x) = \int \frac{d^d k}{(2\pi)^d} \mathcal{O}_a^{(\pm)}(\omega, k) e^{ik \cdot x}, \quad (2.17)$$

and that the Fourier coefficients in turn admit a soft expansion, i.e. they could be written as³

$$\mathcal{O}_a^{(\pm)}(\omega, x) = \sum_{n=0}^\infty \omega^{n-1} \mathcal{O}_a^{(\pm, n)}(x). \quad (2.18)$$

Substituting (2.17) and (2.18) into (2.16), and performing the integral over ω , we obtain

$$\begin{aligned} A_r^{(R\pm)}(u, r, x) &= -\frac{e}{(2\pi)^{\frac{d}{2}+1}} \sum_{n=0}^\infty \int \frac{d^d k}{(2\pi)^d} \left[\frac{ie^{ik \cdot x}}{(ir)^{\frac{d}{2}+\nu_n}} k^a \mathcal{O}_a^{(\pm, n)}(k) \frac{k^{\nu_n-1} K_{\nu_n-1}(kz)}{z^{\nu_n-1}} + \text{c.c.} \right] \\ A_a^{(R\pm)}(u, r, x) &= -\frac{e}{(2\pi)^{\frac{d}{2}+1}} \sum_{n=0}^\infty \int \frac{d^d k}{(2\pi)^d} \left[\frac{ie^{ik \cdot x}}{(ir)^{\frac{d}{2}+\nu_n-1}} \mathcal{O}_a^{(\pm, n)}(k) \frac{k^{\nu_n} K_{\nu_n}(kz)}{z^{\nu_n}} + \text{c.c.} \right], \end{aligned} \quad (2.19)$$

where K_ν is the modified Bessel function of the second kind, $k \equiv |\vec{k}|$, $z \equiv \frac{\sqrt{i u}}{\sqrt{i r}}$, and $\nu_n = \frac{d}{2} - 1 + n$. Because large $|r|$ corresponds to small z , we can expand the Bessel function about $z = 0$. This asymptotic expansion for the Bessel function is qualitatively different depending on whether ν_n is an integer (d even) or a half-integer (d odd), so we will consider these cases separately.

The full large $|r|$ expansion of the radiative gauge field components in all dimensions is presented in appendix A for completeness, though for our purposes we only need the large $|r|$ expansion to evaluate $(1 - u\partial_u) F_{ra}^{(R\pm, d)} \Big|_{\mathcal{I}^\pm_\mp}$ (see section 3.1), where we have adopted the notation $f^{(\pm, n)}$ to denote the coefficient of $|r|^{-n}$ near $r = \pm\infty$ after expanding the field f in large powers of $1/|r|$. Thus, we only need to focus on the terms in the expansion that are $O(1/r^d)$ and $O(u^0)$; the $O(u)$ terms are projected out by $1 - u\partial_u$, and $O(1/u)$ terms

³In the soft expansion given, we are ignoring potential $\log \omega$ terms.

vanish at $\mathcal{J}_\mp^\pm$. In even dimensions, these terms are

$$\begin{aligned} A_r^{(R\pm)} &= \dots + \frac{1}{r^d} \left[-\frac{ie(\Theta(u) - \Theta(r))}{2(4\pi)^{\frac{d}{2}} \Gamma\left(\frac{d}{2}\right)} (-\partial^2)^{\frac{d}{2}-1} \partial^a \mathcal{O}_a^{(\pm,1)}(x) \right] + \dots \\ A_a^{(R\pm)} &= \dots + \frac{1}{r^{d-1}} \left[-\frac{ie(\Theta(u) - \Theta(r))}{2d(4\pi)^{\frac{d}{2}} \Gamma\left(\frac{d}{2}\right)} (-\partial^2)^{\frac{d}{2}} \mathcal{O}_a^{(\pm,1)}(x) \right] + \dots, \end{aligned} \quad (2.20)$$

where we have simplified the expression by assuming

$$\mathcal{O}_a^{(\pm,0)}(x) = \mathcal{O}_a^{(\pm,0)\dagger}(x), \quad \mathcal{O}_a^{(\pm,1)}(x) = -\mathcal{O}_a^{(\pm,1)\dagger}(x). \quad (2.21)$$

These assumptions are required to cancel logarithmic divergences in the asymptotic expansion in order to render the charge finite, and are discussed in greater detail in [10, 19]. In odd dimensions, the relevant terms are

$$\begin{aligned} A_r^{(R\pm)} &= \dots + \frac{1}{r^d} \left[\frac{ie\Gamma(d-1)}{4\pi^{d+1}(-1)^{\frac{d-1}{2}}} \int d^d y \frac{\partial^a \mathcal{O}_a^{(\pm,1)}(y)}{[(x-y)^2]^{d-1}} \right] + \dots \\ A_a^{(R\pm)} &= \dots + \frac{1}{r^{d-1}} \left[\frac{ie\Gamma(d-1)}{4d\pi^{d+1}(-1)^{\frac{d-1}{2}}} \int d^d y \frac{(-\partial^2) \mathcal{O}_a^{(\pm,1)}(y)}{[(x-y)^2]^{d-1}} \right] + \dots. \end{aligned} \quad (2.22)$$

It follows from (2.20) that in even dimensions,

$$(1 - u\partial_u) F_{ra}^{(R\pm,d)} \Big|_{\mathcal{J}_\mp^\pm} = \mp \frac{ie}{d(4\pi)^{\frac{d}{2}} \Gamma\left(\frac{d}{2}\right)} (-\partial^2)^{\frac{d}{2}-1} \left(d\partial_a \partial^b - (d-1)\delta_a^b \partial^2 \right) \mathcal{O}_b^{(\pm,1)}(x), \quad (2.23)$$

whereas it follows from (2.22) that in odd dimensions,

$$(1 - u\partial_u) F_{ra}^{(R\pm,d)} \Big|_{\mathcal{J}_\mp^\pm} = \mp \frac{ie\Gamma(d-1)}{4d\pi^{d+1}(-1)^{\frac{d-1}{2}}} \left(d\partial_a \partial^b - (d-1)\delta_a^b \partial^2 \right) \int d^d y \frac{\mathcal{O}_b^{(\pm,1)}(y)}{[(x-y)^2]^{d-1}}. \quad (2.24)$$

2.3 Coulombic field

Following the approach taken in [10], we know that the Coulombic gauge field $A_\mu^{(C)}$ has a large $|r|$ expansion given by

$$A_r^{(C\pm)} = \sum_{n=0}^{\infty} \frac{A_r^{(C\pm,d-1+n)}}{|r|^{d-1+n}}, \quad A_a^{(C\pm)} = \sum_{n=0}^{\infty} \frac{A_a^{(C\pm,d-2+n)}}{|r|^{d-2+n}}. \quad (2.25)$$

The conserved current that couples to the gauge field also admits a similar expansion:

$$J_u = \sum_{n=0}^{\infty} \frac{J_u^{(C\pm,d+n)}}{|r|^{d+n}}, \quad J_a = \sum_{n=0}^{\infty} \frac{J_a^{(C\pm,d+n)}}{|r|^{d+n}}, \quad J_r = \sum_{n=0}^{\infty} \frac{J_r^{(C\pm,d+2+n)}}{|r|^{d+2+n}}. \quad (2.26)$$

Substituting these expressions into Maxwell's equations (2.14), we derive various constraint equations order-by-order in large $|r|$. In particular, we have

$$\partial_u A_r^{(C\pm,d-1)} = 0, \quad \partial_u A_a^{(C\pm,d-2)} = 0, \quad (d-2)\partial^a A_a^{(C\pm,d-2)} + \partial^2 A_r^{(C\pm,d-1)} = 0. \quad (2.27)$$

These equations in turn imply

$$\begin{aligned} 2\partial_u^2 A_r^{(C\pm,d)} &= e^2 J_u^{(\pm,d)} \\ \pm 2d\partial_u A_a^{(C\pm,d-1)} &= e^2 J_a^{(\pm,d)} - 2\partial_u \partial_a A_r^{(C\pm,d)} - \left(\delta_a^b \partial^2 - \partial_a \partial^b\right) A_b^{(C\pm,d-2)}. \end{aligned} \quad (2.28)$$

The coefficient of $|r|^{-d}$ in the expansion of $F_{ra}^{(C\pm)}$ is

$$F_{ra}^{(C\pm,d)} = \mp(d-1)A_a^{(C\pm,d-1)} - \partial_a A_r^{(C\pm,d)}. \quad (2.29)$$

Acting on both sides with ∂_u^2 and using (2.27), (2.28), we find

$$\partial_u^2 F_{ra}^{(C\pm,d)} = -\frac{e^2}{2d} \left[(d-1)\partial_u J_a^{(\pm,d)} + \partial_a J_u^{(\pm,d)} \right]. \quad (2.30)$$

3 Ward identity

3.1 Matching condition

In [19], it was shown that the Ward identity corresponding to the insertion of $\partial^a \mathcal{O}_a^{(\pm,1)}$ is associated with the antipodal matching condition⁴

$$(1 - u\partial_u) F_{ur}^{(+,d+1)} \Big|_{\mathcal{I}_-^+} = (1 - u\partial_u) F_{ur}^{(-,d+1)} \Big|_{\mathcal{I}_+^-}. \quad (3.1)$$

However, we now want to derive a set of d independent Ward identities involving the insertion of $\mathcal{O}_a^{(\pm,1)}$, which will ultimately be equivalent to the d subleading soft photon theorems (one for each polarization of the soft photon). To motivate the appropriate matching condition, we begin by noting that Maxwell's equations imply

$$2F_{ur}^{(\pm,d+1)} = \pm e^2 J_r^{(\pm,d+2)} \pm \partial^a F_{ra}^{(\pm,d)}. \quad (3.2)$$

It follows that (3.1) is equivalent to matching $\partial^a F_{ra}^{(\pm,d)}$ across spatial infinity, as the current vanishes on the boundaries of null infinity. Since this matching condition gives rise to a Ward identity corresponding to inserting $\partial^a \mathcal{O}_a^{(\pm,1)}$, it is natural to conjecture that in order to obtain d independent Ward identities corresponding to inserting $\mathcal{O}_a^{(\pm,1)}$, we require the matching condition

$$(1 - u\partial_u) F_{ra}^{(+,d)} \Big|_{\mathcal{I}_-^+} = - (1 - u\partial_u) F_{ra}^{(-,d)} \Big|_{\mathcal{I}_+^-}. \quad (3.3)$$

If we define the charge⁵

$$\mathcal{Q}_Y^\pm \equiv \pm \frac{2}{e^2} \int_{\mathcal{I}_\mp^\pm} d^d x Y^a(x) (1 - u\partial_u) F_{ra}^{(\pm,d)}, \quad (3.4)$$

where $Y^a(x)$ is a vector field on the transverse space $\mathbb{R}^d$, the matching condition (3.3) immediately implies

$$\mathcal{Q}_Y^+ = \mathcal{Q}_Y^-, \quad (3.5)$$

so that the charge is classically conserved.

⁴Coordinate x^a on $\mathcal{I}_+^+$ and $\mathcal{I}_-^-$ correspond to antipodal points on the celestial sphere.

⁵To verify that this is indeed the charge whose Ward identity implies the subleading soft photon theorem, it must generate appropriate divergent large gauge transformations on the in- and out-states. This is verified in the next subsection.

3.2 Soft and hard charges

In the semiclassical picture, (3.5) implies the following Ward identity for the charge $\mathcal{Q}_Y$:

$$\langle \text{out} | (\mathcal{Q}_Y^+ - \mathcal{Q}_Y^-) | \text{in} \rangle = 0. \quad (3.6)$$

Analogous to our decomposition of the gauge field A_μ into a radiative field $A_\mu^{(R)}$ and a Coulombic field $A_\mu^{(C)}$, we may decompose the charge into a soft piece and a hard piece, i.e.

$$\mathcal{Q}_Y^\pm = \mathcal{Q}_Y^{\pm S} + \mathcal{Q}_Y^{\pm H}, \quad (3.7)$$

where

$$\begin{aligned} \mathcal{Q}_Y^{\pm S} &\equiv \pm \frac{2}{e^2} \int_{\mathcal{S}_\mp^\pm} d^d x Y^a(x) (1 - u \partial_u) F_{ra}^{(R\pm, d)} \\ \mathcal{Q}_Y^{\pm H} &\equiv \pm \frac{2}{e^2} \int_{\mathcal{S}_\mp^\pm} d^d x Y^a(x) (1 - u \partial_u) F_{ra}^{(C\pm, d)}. \end{aligned} \quad (3.8)$$

Thus, the Ward identity becomes

$$\langle \text{out} | (\mathcal{Q}_Y^{+S} - \mathcal{Q}_Y^{-S}) | \text{in} \rangle = -\langle \text{out} | (\mathcal{Q}_Y^{+H} - \mathcal{Q}_Y^{-H}) | \text{in} \rangle. \quad (3.9)$$

The form of the soft charge depends on the spacetime dimension. Using (2.23), the soft charge in even dimensions is

$$\mathcal{Q}_Y^{\pm S} = -\frac{i}{e(4\pi)^{\frac{d}{2}} \Gamma\left(\frac{d}{2} + 1\right)} \int d^d y Y^a(y) (-\partial^2)^{\frac{d}{2}-1} \left(d \partial_a \partial^b - (d-1) \delta_a^b \partial^2 \right) \mathcal{O}_b^{(\pm, 1)}(y), \quad (3.10)$$

and using (2.24), the soft charge in odd dimensions is

$$\mathcal{Q}_Y^{\pm S} = \frac{i \Gamma(d-1)}{2d e \pi^{d+1}} (-1)^{\frac{d+1}{2}} \int d^d x Y^a(x) \int d^d y \frac{\left(d \partial_a \partial^b - (d-1) \delta_a^b \partial^2 \right) \mathcal{O}_b^{(\pm, 1)}(y)}{[(x-y)^2]^{d-1}}. \quad (3.11)$$

The form of the hard charge, on the other hand, is independent of dimension, and using (2.30) is given by

$$\mathcal{Q}_Y^{\pm H} = \frac{1}{d} \int_{\mathcal{S}^\pm} du d^d x Y^a(x) \left[(d-1) J_a^{(\pm, d)} - u \partial_a J_u^{(\pm, d)} \right], \quad (3.12)$$

where we have assumed that there are no stable massive particles in the system so that the contribution to $\mathcal{Q}_Y$ from $\mathcal{S}_\pm^\pm$ vanishes. We will demonstrate in the section 3.2.1 below that⁶

$$\begin{aligned} \langle \omega_i, x_i | \mathcal{Q}_Y^{+H} &= \frac{2i(d-1)Q_i}{\omega_i d} \left[Y^a(x_i) \partial_{x_i^a} - \frac{1}{d-1} \partial_a Y^a(x_i) \omega_i \partial_{\omega_i} + i \partial^a Y^b(x_i) \mathcal{S}_{iab} \right] \langle \omega_i, x_i | \\ \mathcal{Q}_Y^{-H} | \omega_i, x_i \rangle &= -\frac{2i(d-1)Q_i}{\omega_i d} \\ &\times \left[Y^a(x_i) \partial_{x_i^a} - \frac{1}{d-1} \partial_a Y^a(x_i) \omega_i \partial_{\omega_i} + i \partial^a Y^b(x_i) \mathcal{S}_{iab} \right] | \omega_i, x_i \rangle. \end{aligned} \quad (3.13)$$

⁶Readers who are mainly interested in the final result should feel free to skip section 3.2.1.

In an S -matrix element, the hard charge acts on multi-particle states as a tensor product of one-particle states. Thus, we can rewrite (3.9) as

$$\begin{aligned} \langle \text{out} | \left(\mathcal{Q}_Y^{+S} - \mathcal{Q}_Y^{-S} \right) | \text{in} \rangle \\ = - \sum_{i=1}^n \frac{2i(d-1)Q_i}{\omega_i d} \left(Y^a(x_i) \partial_{x_i^a} - \frac{1}{d-1} \partial_a Y^a(x_i) \omega_i \partial_{\omega_i} + i \partial^a Y^b(x_i) \mathcal{S}_{iab} \right) \langle \text{out} | \text{in} \rangle. \end{aligned} \quad (3.14)$$

3.2.1 Action of hard charges

In this sub-subsection, we will prove (3.13). For notational simplicity, we do not distinguish between in- and out-states and drop the superscripts ($\pm$) on all operators. We will also drop the subscripts on the annihilation operators and simply denote them as $\mathcal{O}(\omega, x)$.

Begin by defining the light-ray operators (LROs)

$$\mathbb{Q}(x) \equiv \int_{-\infty}^{\infty} du J_u^{(d)}(u, x), \quad \mathbb{J}_a(x) \equiv \int_{-\infty}^{\infty} du \left[(d-1) J_a^{(d)} - u \partial_a J_u^{(d)} \right]. \quad (3.15)$$

Note that $\mathbb{Q}(x)$ is the LRO appearing in the leading hard charge (see [10]), and $\mathbb{J}(x)$ is the LRO appearing in the subleading hard charge:

$$\mathcal{Q}_\varepsilon^H = \int d^d x \varepsilon(x) \mathbb{Q}(x), \quad \mathcal{Q}_Y^H = \frac{1}{d} \int d^d x Y^a(x) \mathbb{J}_a(x). \quad (3.16)$$

Recall that massless one-particle states are created out of the vacuum by creation and annihilation operators. Since the LROs annihilate the vacuum [22], we have

$$\langle \omega, x | \mathbb{Q}(x') = \langle 0 | [\mathcal{O}(\omega, x), \mathbb{Q}(x')], \quad \langle \omega, x | \mathbb{J}_a(x') = \langle 0 | [\mathcal{O}(\omega, x), \mathbb{J}_a(x')]. \quad (3.17)$$

Our objective now is to determine the following commutators

$$[\mathcal{O}(\omega, x), \mathbb{Q}(x')], \quad [\mathcal{O}(\omega, x), \mathbb{J}_a(x')]. \quad (3.18)$$

We will follow the spirit of the procedure outlined in [23], where the authors determined the commutators of LROs by using only a few basic assumptions. Their argument relied on properties of unitary conformal field theories and crucially required invariance under special conformal transformations. In our case, we are interested in QFTs with an S -matrix; hence, we do not generically have access to scale or special conformal invariance. However, in theories with only massless particles, S -matrices have an effective scale symmetry in the low energy limit. We will utilize this effective scale symmetry (along with Poincaré invariance) to completely fix the commutator between the LROs in (3.15) and the annihilation operators $\mathcal{O}(\omega, x)$, and is based on the following assumptions:

1. *Microcausality: spacelike separated LROs commute with each other.*

To utilize this assumption, we note that $\mathcal{O}(\omega, x)$ is itself a LRO. For instance, the scalar annihilation operator can be constructed out of a scalar field as

$$\mathcal{O}(\omega, x) = \frac{(2\pi i)^{\frac{d}{2}}}{\omega^{\frac{d}{2}-1}} \int_{-\infty}^{\infty} du e^{\frac{i}{2}\omega u} \Phi^{(d/2)}(u, x), \quad (3.19)$$

Quantity	(J, τ)
$\mathbb{Q}$	(d, d)
$\mathbb{J}_a$	$(d, d+1)$
$\mathcal{O}_\alpha$	$(0, \frac{d}{2})$
∂_a	$(1, 1)$
ω	$(1, 0)$

Table 1. This table tabulates the boost charges and twists of various relevant operators.

where $\Phi^{(d/2)}(u, x) = \lim_{r \rightarrow \infty} [r^{\frac{d}{2}} \Phi(u, r, x)]$. This implies that $\mathcal{O}(\omega, x)$ is a LRO localized on the null-ray x on $\mathcal{I}^\pm$. This assumption then implies that the commutators (3.18) depend only on the Dirac delta function $\delta^{(d)}(x - x')$ and its derivatives.

2. *Unitarity: all operators transform in representations of the Poincaré and scaling symmetry algebra.*
3. *Ward Identities: the charge $\hat{Q} = \int d^d x \mathbb{Q}(x)$ generates global U(1) transformations on the operators.*

This implies

$$[\mathcal{O}(\omega, x), \hat{Q}] = Q \mathcal{O}(\omega, x). \quad (3.20)$$

4. *Minimal Coupling: the commutator of an annihilation operator with a LRO is also an annihilation operator (or derivatives thereof) of the same particle type.*

This is required in order to fix the commutators uniquely. As was shown in [24], the subleading soft theorem receives corrections from higher-derivative operators and is therefore not universal. In this paper, we will only focus on the universal part of the subleading soft theorem in minimally coupled theories.⁷

Having outlined our assumptions, we now proceed to prove (3.13). Begin by labeling pertinent operators by their boost charge J (eigenvalue under D) and twist $\tau = J - \Delta$ (Δ is the eigenvalue under the scale charge S), which we have for convenience listed in the table below (see appendix B for details). We will use it to fix the commutators $[\mathcal{O}(\omega, x), \mathbb{Q}(x')]$ and $[\mathcal{O}(\omega, x), \mathbb{J}_a(x')]$.

$[\mathcal{O}(\omega, x), \mathbb{Q}(x')]$. The Lorentzian separation between the two operators is $(x - x')^2$. They are therefore spacelike separated as long as $x \neq x'$, which by assumption 1 implies

$$[\mathcal{O}(\omega, x), \mathbb{Q}(x')] = \delta^{(d)}(x - x') L(\omega, x) + \partial_a \delta^{(d)}(x - x') L^a(\omega, x) + \dots, \quad (3.21)$$

where $\dots$ represents terms involving additional derivatives of the delta function. Integrating over the transverse directions x' , we obtain

$$[\mathcal{O}(\omega, x), \hat{Q}] = L(\omega, x). \quad (3.22)$$

Comparing to (3.20), it follows that $L(\omega, x) = Q \mathcal{O}(\omega, x)$.

⁷The non-universal part of the subleading soft theorem also has an interpretation as a Ward identity [17].

Next, consider $L^a(\omega, x)$. Utilizing table 1, we observe that the twist of the left-hand-side of (3.21) is $\frac{d}{2} + d$, and the twist of $\partial_A \delta^{(d)}(x - x')$ is $d + 1$. This implies that the twist of L^a is $\frac{d}{2} - 1$. On the other hand, by Assumption 4, L^a is locally constructed from $\mathcal{O}$ and must therefore have the general form⁸

$$L^a(\omega, x) = \sum_{n,q=0}^{\infty} \sum_{p=-\infty}^{\infty} c_{p,q}^{aa_1 \dots a_n} \omega^p (\omega \partial_\omega)^q \partial_{a_1} \dots \partial_{a_n} \mathcal{O}(\omega, x). \quad (3.23)$$

All the operators on the right-hand-side have twist $\frac{d}{2} + n$. Since $n \geq 0$, there are no operators with twist $\frac{d}{2} - 1$,⁹ thereby implying that $L^a(\omega, x) = 0$. Likewise, terms with additional derivatives of the delta function in (3.21) are excluded by the same argument, and so

$$[\mathcal{O}(\omega, x), \mathbb{Q}(x')] = Q \delta^{(d)}(x - x') \mathcal{O}(\omega, x). \quad (3.24)$$

$[\mathcal{O}(\omega, x), \mathbb{J}_a(x')]$. By Assumption 1, the commutator takes the form

$$[\mathcal{O}(\omega, x), \mathbb{J}_a(x')] = \delta^{(d)}(x - x') K_a^{(0)}(\omega, x) + \partial^b \delta^{(d)}(x - x') K_{ab}^{(1)}(\omega, x), \quad (3.25)$$

where we have excluded terms involving higher derivatives on the delta function using the same twist argument as above. By Assumption 4, $K^{(0)}$ and $K^{(1)}$ have the general form

$$\begin{aligned} K_a^{(0)}(\omega, x) &= \sum_{n,q=0}^{\infty} \sum_{p=-\infty}^{\infty} (c_{p,q}^{(0)})_a{}^{a_1 \dots a_n} \omega^p (\omega \partial_\omega)^q \partial_{a_1} \dots \partial_{a_n} \mathcal{O}(\omega, x) \\ K_{ab}^{(1)}(\omega, x) &= \sum_{n,q=0}^{\infty} \sum_{p=-\infty}^{\infty} (c_{p,q}^{(1)})_{ab}{}^{a_1 \dots a_n} \omega^p (\omega \partial_\omega)^q \partial_{a_1} \dots \partial_{a_n} \mathcal{O}(\omega, x). \end{aligned} \quad (3.26)$$

From (3.25), we can deduce that the twists of $K^{(0)}$ and $K^{(1)}$ are $\frac{d}{2} + 1$ and $\frac{d}{2}$, respectively, and table 1 further implies that the boost charges of $K^{(0)}$ and $K^{(1)}$ are 0 and -1 , respectively. Matching the twists and boost charges on both sides of (3.26), we find that $p = -1$ for both $K^{(0)}$ and $K^{(1)}$, while $n = 1$ for $K^{(0)}$ and $n = 0$ for $K^{(1)}$. There are no constraints on q , so

$$K_a^{(0)}(\omega, x) = \frac{1}{\omega} \mathcal{R}_{ab}(\omega \partial_\omega) \partial^b \mathcal{O}(\omega, x), \quad K_{ab}^{(1)}(\omega, x) = \frac{1}{\omega} \mathcal{B}_{ab}(\omega \partial_\omega) \mathcal{O}(\omega, x), \quad (3.27)$$

where

$$\mathcal{R}_{ab}(\omega \partial_\omega) = \sum_{q=0}^{\infty} (c_q^{(0)})_{ab} (\omega \partial_\omega)^q, \quad \mathcal{B}_{ab}(\omega \partial_\omega) = \sum_{q=0}^{\infty} (c_q^{(1)})_{ab} (\omega \partial_\omega)^q. \quad (3.28)$$

As we shall next show, $\mathcal{R}_{ab}$ and $\mathcal{B}_{ab}$ can be fixed by checking consistency with the Jacobi identity

$$[[\mathcal{O}(\omega, x), \mathbb{J}_a(x')], X] = [\mathcal{O}(\omega, x), [\mathbb{J}_a(x'), X]] + [[\mathcal{O}(\omega, x), X], \mathbb{J}_a(x')], \quad (3.29)$$

where X is a generator of the Poincaré algebra. In particular, choosing X to be P_- , P_a , K_a , and M_{ab} suffices to fix $\mathcal{R}_{ab}$ and $\mathcal{B}_{ab}$. Readers interested in detailed derivations of the commutators below should refer to appendix B.

⁸Lorentz invariance forbids explicit factors of x^a .

⁹The assumption of minimal coupling implies that the Poincaré invariant constants $c_{p,q}^{a_1 \dots a_n}$ are dimensionless and therefore have vanishing twists.

- P_- : we note

$$[\mathcal{O}(\omega, x), P_-] = -\omega \mathcal{O}(\omega, x), \quad [\mathbb{J}_a(x), P_-] = -2i\partial_a \mathbb{Q}(x). \quad (3.30)$$

Eq. (3.29) then implies

$$[\mathcal{R}_{ab}(\omega \partial_\omega), \omega] = 0, \quad [\mathcal{B}_{ab}(\omega \partial_\omega), \omega] + 2iQ\delta_{ab}\omega = 0, \quad (3.31)$$

which means

$$\mathcal{R}_{ab}(\omega \partial_\omega) = \mathcal{R}'_{ab}, \quad \mathcal{B}_{ab}(\omega \partial_\omega) = -2iQ\delta_{ab}\omega \partial_\omega + \mathcal{B}'_{ab}, \quad (3.32)$$

where $\mathcal{R}'_{ab}$ and $\mathcal{B}'_{ab}$ are operators independent of $\omega \partial_\omega$.

- P_a : we note

$$[\mathcal{O}(\omega, x), P_a] = -\omega x_a \mathcal{O}(\omega, x), \quad [\mathbb{J}_a(x), P_b] = -2ix_b \partial_a \mathbb{Q}(x) - 2idd_{ab} \mathbb{Q}(x). \quad (3.33)$$

Using this, (3.29) and (3.32) imply

$$\mathcal{R}_{ab}(\omega \partial_\omega) = \mathcal{R}'_{ab} = 2iQ(d-1)\delta_{ab}. \quad (3.34)$$

- M_{ab} : we note

$$\begin{aligned} [\mathcal{O}(\omega, x), M_{ab}] &= -i(x_a \partial_b - x_b \partial_a) \mathcal{O}(\omega, x) + \mathcal{S}_{ab} \mathcal{O}(\omega, x) \\ [\mathbb{J}_a(x), M_{bc}] &= -i(x_b \partial_c - x_c \partial_b) \mathbb{J}_a(x) - i\delta_{ab} \mathbb{J}_c(x) + i\delta_{ac} \mathbb{J}_b(x). \end{aligned} \quad (3.35)$$

Using this, the Jacobi identity implies

$$[\mathcal{S}_{ab}, \mathcal{B}'_{cd}] = i(\delta_{ac} \mathcal{B}'_{bd} + \delta_{bd} \mathcal{B}'_{ac} - \delta_{bc} \mathcal{B}'_{ad} - \delta_{ad} \mathcal{B}'_{bc}), \quad (3.36)$$

which is simply the statement that $\mathcal{B}'_{ab}$ is an $\mathcal{O}(d)$ covariant matrix.

- K_a : we note

$$\begin{aligned} [\mathcal{O}(\omega, x), K_a] &= i(x^2 \partial_a - 2x_a x^b \partial_b + 2x_a \omega \partial_\omega) \mathcal{O}(\omega, x) + 2x^b \mathcal{S}_{ab} \mathcal{O}(\omega, x) \\ [\mathbb{J}_a(x), K_b] &= i[x^2 \partial_b - 2x_b(x^c \partial_c + d)] \mathbb{J}_a(x) + 2ix_a \mathbb{J}_b(x) - 2i\delta_{ab} x^c \mathbb{J}_c(x). \end{aligned} \quad (3.37)$$

Eq. (3.29) along with (3.34) and (3.36) then implies

$$2Q(d-1)\mathcal{S}_{ab} = \delta_{ab} \delta^{cd} \mathcal{B}'_{cd} - \mathcal{B}'_{ba}. \quad (3.38)$$

Symmetrizing and antisymmetrizing the indices ab on both sides yields

$$\mathcal{B}'_{(ab)} = \delta_{ab} \delta^{cd} \mathcal{B}'_{cd}, \quad \mathcal{B}'_{[ab]} = 2Q(d-1)\mathcal{S}_{ab}. \quad (3.39)$$

Tracing (3.38) over ab , we immediately find $\delta^{ab} \mathcal{B}'_{ab} = 0 \implies \mathcal{B}'_{(ab)} = 0$. Hence, it follows by (3.32) that

$$\mathcal{B}_{ab}(\omega \partial_\omega) = -2iQ\delta_{ab}\omega \partial_\omega + 2Q(d-1)\mathcal{S}_{ab}. \quad (3.40)$$

Having determined both $\mathcal{R}_{ab}$ in (3.34) and $\mathcal{B}_{ab}$ in (3.40), we substitute everything back into (3.25) to get

$$\begin{aligned} [\mathcal{O}(\omega, x), \mathbb{J}_a(x')] &= \frac{2iQ}{\omega} \left[(d-1)\delta^{(d)}(x-x')\partial_a - \partial_a \delta^{(d)}(x-x')\omega \partial_\omega \right. \\ &\quad \left. - i(d-1)\partial^b \delta^{(d)}(x-x')\mathcal{S}_{ab} \right] \mathcal{O}(\omega, x). \end{aligned} \quad (3.41)$$

Substituting this into (3.17) and noting (3.16), we obtain (3.13), as promised.

4 Connection to the subleading soft theorem

We begin by recalling the subleading soft photon theorem in its standard momentum coordinates. Let $\mathcal{A}_n(p_1, \dots, p_n)$ be a scattering amplitude involving n particles with momenta $p_1, \dots, p_n$, and let $\mathcal{A}_{n+1}^{\text{out}}(\vec{p}_\gamma, \varepsilon_a; p_1, \dots, p_n)$ be the same amplitude with an additional outgoing photon with momentum p_γ and polarization ε_a . In a minimally coupled theory, the soft limit ($E_\gamma \rightarrow 0$) of the amplitude has the form

$$\mathcal{A}_{n+1}^{\text{out}}(p_\gamma, \varepsilon_a; p_1, \dots, p_n) = O\left(\frac{1}{E_\gamma}\right) + S_a^{(1)} \mathcal{A}_n(p_1, \dots, p_n) + O(E_\gamma), \quad (4.1)$$

where the $\frac{1}{E_\gamma}$ pole is related to the leading soft photon theorem [25, 26], and

$$S_a^{(1)} = -ie \sum_{i=1}^n Q_i \frac{p_\gamma^A \varepsilon_a^B(\vec{p}_\gamma)}{p_i \cdot p_\gamma} \mathcal{J}_{iAB} \quad (4.2)$$

is the subleading soft factor. Here, $\mathcal{J}_{iAB}$ is the angular momentum operator, which is the sum of the orbital and spin angular momenta:

$$\mathcal{J}_{iAB} = \mathcal{L}_{iAB} + \mathcal{S}_{iAB} = -i \left(p_{iA} \frac{\partial}{\partial p_i^B} - p_{iB} \frac{\partial}{\partial p_i^A} \right) + \mathcal{S}_{iAB}. \quad (4.3)$$

To write the subleading soft factor in flat null coordinates, we parametrize p_i^A and p_γ^A via (2.3) as (ω_i, x_i^a) and (ω, x^a) , respectively. Using (2.4), we have

$$S_a^{(1)} = e \sum_{i=1}^n \frac{Q_i}{\omega_i} \left[\partial^b \log[(x - x_i)^2] (\delta_{ab} \omega_i \partial_{\omega_i} - i \mathcal{S}_{iab}) - \mathcal{I}_a^b (x - x_i) \partial_{x_i^b} \right], \quad (4.4)$$

where

$$\mathcal{I}_{ab}(x) = \frac{x^2}{2} \partial_a \partial_b \log x^2 = \delta_{ab} - \frac{2x_a x_b}{x^2}. \quad (4.5)$$

By the LSZ reduction formula, the left-hand-side of (4.1) corresponds to the insertion of the operator $\mathcal{O}_a^{(+,1)}(x) - \mathcal{O}_a^{(-,1)}(x)$ in the S -matrix.¹⁰ One can then rewrite the subleading soft photon theorem as

$$\begin{aligned} & \langle \text{out} | \left(\mathcal{O}_b^{(+,1)}(x) - \mathcal{O}_b^{(-,1)}(x) \right) | \text{in} \rangle \\ &= e \sum_{i=1}^n \frac{Q_i}{\omega_i} \left[\partial^c \log[(x - x_i)^2] (\delta_{bc} \omega_i \partial_{\omega_i} - i \mathcal{S}_{ibc}) - \mathcal{I}_b^c (x - x_i) \partial_{x_i^c} \right] \langle \text{out} | \text{in} \rangle. \end{aligned} \quad (4.6)$$

4.1 Soft theorem $\implies$ Ward identity

In this subsection, we want to use the subleading soft photon theorem (4.6) to derive the Ward identity (3.14). For the even dimensional case, we act on both sides of (4.6) with the operator

$$- \frac{i}{e(4\pi)^{\frac{d}{2}} \Gamma\left(\frac{d}{2} + 1\right)} \int d^d x Y^a(x) (-\partial^2)^{\frac{d}{2}-1} \left(d \partial_a \partial^b - (d-1) \delta_a^b \partial^2 \right). \quad (4.7)$$

¹⁰See appendix C of [10] for details.

Applying (3.10) and using the fact that in even dimensions

$$(-\partial^2)^{\frac{d}{2}} \log [(x - x_i)^2] = -(4\pi)^{\frac{d}{2}} \Gamma\left(\frac{d}{2}\right) \delta^{(d)}(x - x_i), \quad (4.8)$$

we obtain

$$\begin{aligned} \langle \text{out} | (\mathcal{Q}_Y^{+S} - \mathcal{Q}_Y^{-S}) | \text{in} \rangle &= \\ &= - \sum_{i=1}^n \frac{2i(d-1)Q_i}{\omega_i d} \left(Y^a(x_i) \partial_{x_i^a} - \frac{1}{d-1} \partial_a Y^a(x_i) \omega_i \partial_{\omega_i} + i \partial^a Y^b(x_i) \mathcal{S}_{iab} \right) \langle \text{out} | \text{in} \rangle, \end{aligned} \quad (4.9)$$

which is precisely (3.14).

For the odd dimensional case, we act on both sides of (4.6) with the operator

$$\frac{i\Gamma(d-1)}{2de\pi^{d+1}} (-1)^{\frac{d+1}{2}} \int d^d y Y^a(y) \int d^d x \frac{d\partial_a \partial^b - (d-1)\delta_a^b \partial^2}{[(x-y)^2]^{d-1}}. \quad (4.10)$$

Applying (3.11) and using the fact that in odd dimensions

$$\int d^d x \frac{\partial^2 \log [(x - x_i)^2]}{[(x - y)^2]^{d-1}} = \frac{4(-1)^{\frac{d-1}{2}} \pi^{d+1}}{\Gamma(d-1)} \delta^{(d)}(y - x_i), \quad (4.11)$$

we again obtain (3.14).

4.2 Ward identity $\implies$ soft theorem

Now, we want to prove the converse, that the Ward identity (3.14) implies the subleading soft theorem (4.6), thereby proving that the two are completely equivalent. First, recall (4.5) and choose $Y^a(z) = \mathcal{Y}^a(z) \equiv \mathcal{I}^a_b(x-z)\zeta^b$ in (3.14) for a constant ζ , so that

$$[d\partial^a \partial_b - (d-1)\delta_b^a \partial^2] \mathcal{Y}^b(z) = (d-1)\zeta^a (-\partial^2) \log [(x-z)^2], \quad (4.12)$$

where the derivatives are with respect to z . Substituting this choice of Y^a into (3.10) and (3.11), and using (4.8) and (4.11), we determine in both even and odd dimensions that the soft charge takes the form¹¹

$$\mathcal{Q}_Y^{\pm S} = \frac{2i}{de} (d-1) \zeta^a \mathcal{O}_a^{(\pm,1)}(x). \quad (4.13)$$

Substituting this into (3.14) with $Y^a(z) = \mathcal{Y}^a(z)$ and simplifying yields

$$\begin{aligned} \zeta^a \langle \text{out} | (\mathcal{O}_a^{(+,1)}(x) - \mathcal{O}_a^{(-,1)}(x)) | \text{in} \rangle &= \\ &= e \sum_{i=1}^n \frac{Q_i}{\omega_i} \zeta^a \left[\partial^c \log [(x - x_i)^2] (\delta_{ac} \omega_i \partial_{\omega_i} - i \mathcal{S}_{iac}) - \mathcal{I}_a^c(x - x_i) \partial_{x_i^c} \right] \langle \text{out} | \text{in} \rangle. \end{aligned} \quad (4.14)$$

Choosing $\zeta^a = \delta_b^a$ yields

$$\begin{aligned} \langle \text{out} | (\mathcal{O}_b^{(+,1)}(x) - \mathcal{O}_b^{(-,1)}(x)) | \text{in} \rangle &= \\ &= e \sum_{i=1}^n \frac{Q_i}{\omega_i} \left[\partial^c \log [(x - x_i)^2] (\delta_{bc} \omega_i \partial_{\omega_i} - i \mathcal{S}_{ibc}) - \mathcal{I}_b^c(x - x_i) \partial_{x_i^c} \right] \langle \text{out} | \text{in} \rangle, \end{aligned} \quad (4.15)$$

which is precisely the subleading soft photon theorem given in (4.6).

¹¹In proving this, we have integrated by parts in (3.11) and neglected potential boundary terms. To justify this, we can take $Y^a = \mathcal{Y}^a$ in a region encompassing all the points x_i in the amplitude and zero outside, in which case all boundary terms are trivially vanishing.

5 Electric and magnetic large gauge transformations

In previous sections, we have established the equivalence between the subleading soft theorem and a Ward identity for the charge

$$\mathcal{Q}_Y^\pm \equiv \pm \frac{2}{e^2} \int_{\mathcal{S}_\mp^\pm} d^d x Y^a (1 - u \partial_u) F_{ra}^{(\pm, d)}. \quad (5.1)$$

In this section, we will show that this charge generates divergent electric *and* magnetic large gauge transformations. We begin by using Hodge decomposition to decompose the vector field as

$$Y^a(x) = \partial^a \lambda(x) + \partial_b K^{ab}(x), \quad K^{ab}(x) = -K^{ba}(x). \quad (5.2)$$

We can then write the charge as¹²

$$\mathcal{Q}_Y^\pm \equiv \mp \frac{2}{e^2} \int_{\mathcal{S}_\mp^\pm} d^d x \lambda (1 - u \partial_u) \partial^a F_{ra}^{(\pm, d)} \mp \frac{4}{e^2} \int_{\mathcal{S}_\mp^\pm} d^d x K^{ab} (1 - u \partial_u) \partial_{[a} F_{b]r}^{(\pm, d)}. \quad (5.3)$$

Using (3.2) and the fact that the current vanishes at $\mathcal{S}_\pm^\pm$, we can simplify the first term as

$$\mp \frac{2}{e^2} \int_{\mathcal{S}_\mp^\pm} d^d x \lambda (1 - u \partial_u) \partial^a F_{ra}^{(\pm, d)} = -\frac{4}{e^2} \int_{\mathcal{S}_\mp^\pm} d^d x \lambda (1 - u \partial_u) F_{ur}^{(\pm, d+1)}. \quad (5.4)$$

This is precisely the charge studied in [19], where it was shown to generate divergent *electric* large gauge transformations.

To simplify the second term in (5.3), we use the Bianchi identity

$$\partial_r F_{ab} + 2\partial_{[a} F_{b]r} = 0 \quad \implies \quad 2\partial_{[a} F_{b]r}^{(\pm, d)} = \pm(d-1)F_{ab}^{(\pm, d-1)}. \quad (5.5)$$

It then follows that the second term in (5.3) reduces to

$$\mp \frac{4}{e^2} \int_{\mathcal{S}_\mp^\pm} d^d x K^{ab} (1 - u \partial_u) \partial_{[a} F_{b]r}^{(\pm, d)} = -\frac{2}{e^2} (d-1) \int_{\mathcal{S}_\mp^\pm} d^d x K^{ab} (1 - u \partial_u) F_{ab}^{(\pm, d-1)}, \quad (5.6)$$

which, as we shall now show, generates divergent *magnetic* large gauge transformations.

In form notation, the charge that generates magnetic gauge transformations on a hypersurface Σ is¹³

$$\tilde{Q}_\mathcal{K}^\Sigma = \frac{1}{2\pi} \int_{\partial\Sigma} \mathcal{K} \wedge F, \quad (5.7)$$

where $\mathcal{K}$ is a $(d-2)$ -form. This acts on the dual gauge field via

$$\tilde{A} \rightarrow \tilde{A} + d\mathcal{K}, \quad (5.8)$$

where $d\tilde{A} = *dA$. To study this charge with $\Sigma = \mathcal{S}^\pm$, we note

$$(\mathcal{K} \wedge F)_{a_1 \dots a_d} = \frac{d(d-1)}{2} \mathcal{K}_{[a_1 \dots a_{d-2}} F_{a_{d-1} a_d]} = \frac{1}{2} \epsilon_{a_1 \dots a_d} (*_d \mathcal{K})^{ab} F_{ab}, \quad (5.9)$$

¹²We assume λ and K^{ab} fall off sufficiently quickly in $|x|$ so we can integrate by parts and neglect boundary terms.

¹³In form notation, the charge that generates electric gauge transformations is $Q_\epsilon^\Sigma = \frac{1}{e^2} \int_\Sigma \epsilon * F$.

where $*_d$ is the Hodge dual on $\mathbb{R}^d$. Substituting this into (5.7) with $\Sigma = \mathcal{J}^\pm$ yields

$$\tilde{Q}_{\mathcal{K}}^{\pm} = \pm \frac{1}{4\pi} \int_{\mathcal{J}_{\mp}^{\pm}} d^d x \lim_{r \rightarrow \pm\infty} \left[|r|^d (*_d \mathcal{K})^{ab} F_{ab} \right]. \quad (5.10)$$

To match the charges, we take

$$(*_d \mathcal{K})^{ab} = -\frac{8\pi}{re^2} (d-1) K^{ab}. \quad (5.11)$$

Recalling that the field strength near $\mathcal{J}_{\mp}^{\pm}$ admits the expansion

$$F_{ab}(u, r, x) = \frac{F_{ab}^{(\pm, d-2)}(u, x)}{|r|^{d-2}} + \frac{F_{ab}^{(\pm, d-1)}(u, x)}{|r|^{d-1}} + \dots, \quad (5.12)$$

the charge simplifies to

$$\tilde{Q}_{\mathcal{K}}^{\pm} = -\frac{2}{e^2} (d-1) \int_{\mathcal{J}_{\mp}^{\pm}} d^d x K^{ab} \left(|r| F_{ab}^{(\pm, d-2)} + F_{ab}^{(\pm, d-1)} \right). \quad (5.13)$$

This charge is formally divergent, and it is in this sense that the symmetry generated by this charge is a divergent magnetic gauge symmetry. However, when this divergent charge is inserted into an S -matrix element, the divergent contribution (the terms that are $O(|r|)$, which are shown explicitly above, as well as terms that are $O(u)$, which are implicitly present in $F_{ab}^{(\pm, d-1)}$) vanishes due to the constraint equation

$$\langle \text{out} | \left(\partial_a \mathcal{O}_b^{(\pm, 0)}(x) - \partial_b \mathcal{O}_a^{(\pm, 0)}(x) \right) | \text{in} \rangle = 0, \quad (5.14)$$

thereby giving rise to a finite Ward identity. Comparing (5.13) with (5.6), we see that the finite part of the charge is exactly (5.6), the second term of (5.3). This concludes our demonstration that the asymptotic symmetry dual to the subleading soft photon theorem is a divergent electric and magnetic large gauge symmetry.

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A Asymptotic expansions

In this appendix, we list the full large $|r|$ expansion of the radiative gauge field components A_r and A_a near $\mathcal{J}^\pm$. Recall that (2.19) states

$$\begin{aligned} A_r^{(R\pm)}(u, r, x) &= -\frac{e}{(2\pi)^{\frac{d}{2}+1}} \sum_{n=0}^{\infty} \int \frac{d^d k}{(2\pi)^d} \left[\frac{ie^{ik \cdot x}}{(ir)^{\frac{d}{2}+\nu_n}} k^a \mathcal{O}_a^{(\pm, n)}(k) \frac{k^{\nu_n-1} K_{\nu_n-1}(kz)}{z^{\nu_n-1}} + \text{c.c.} \right] \\ A_a^{(R\pm)}(u, r, x) &= -\frac{e}{(2\pi)^{\frac{d}{2}+1}} \sum_{n=0}^{\infty} \int \frac{d^d k}{(2\pi)^d} \left[\frac{ie^{ik \cdot x}}{(ir)^{\frac{d}{2}+\nu_n-1}} \mathcal{O}_a^{(\pm, n)}(k) \frac{k^{\nu_n} K_{\nu_n}(kz)}{z^{\nu_n}} + \text{c.c.} \right], \quad (\text{A.1}) \end{aligned}$$

where K_ν is the modified Bessel function of the second kind, $k \equiv |\vec{k}|$, $z \equiv \frac{\sqrt{iu}}{\sqrt{ir}}$, $\nu_n = \frac{d}{2} - 1 + n$, and $\mathcal{O}_a^{(\pm)}$ are the Fourier coefficients of the annihilation operator $\mathcal{O}_a^{(\pm)}$ (see (2.17) and (2.18)). Because large $|r|$ is equivalent to small z , we can perform a large $|r|$ expansion of these components by utilizing the known expansions about $z = 0$ for the Bessel functions and then performing an inverse Fourier transform. For further details, we refer the reader to [10], where this procedure was introduced and more carefully explained.

In even dimensions $D = d+2 > 4$, we only need the $z = 0$ expansion of Bessel functions with nonnegative integer orders. Carrying out the procedure outlined above, we obtain

$$\begin{aligned}
 A_r^{(R\pm)} &= -\frac{e}{16\pi^{\frac{d}{2}+1}} \sum_{n=0}^{\infty} \sum_{s=0}^{\nu_n-2} \frac{(-1)^s \Gamma(\nu_n - s - 1)}{2^{2s-n} \Gamma(s+1)} \frac{(iu)^{s+1-\nu_n}}{(ir)^{\frac{d}{2}+s+1}} (-\partial^2)^s \partial^a \mathcal{O}_a^{(\pm,n)}(x) \\
 &\quad - \frac{8e}{(4\pi)^{\frac{d}{2}+1}} \sum_{n=0}^{\infty} \sum_{s=0}^{\infty} \frac{(-1)^{\nu_n} \left(\log \frac{\sqrt{iu}}{\sqrt{ir}} + c_{s,\nu_n-1} \right)}{2^{2s+n} \Gamma(s+1) \Gamma(s+\nu_n)} \frac{(iu)^s (-\partial^2)^{s+\nu_n-1} \partial^a \mathcal{O}_a^{(\pm,n)}(x)}{(ir)^{d-1+n+s}} \\
 &\quad - \frac{2^d e}{16\pi} \sum_{n=0}^{\infty} \sum_{s=0}^{\infty} \frac{\Gamma(d-2+n+s)}{2^{-n} (-1)^s \Gamma(s+1)} \frac{(iu)^s}{(ir)^{d-1+n+s}} \int d^d y \frac{\partial^a \mathcal{O}_a^{(\pm,n)}(y)}{[(x-y)^2]^{d-2+n+s}} + \text{c.c.} \\
 A_a^{(R\pm)} &= -\frac{e}{8\pi^{\frac{d}{2}+1}} \sum_{n=0}^{\infty} \sum_{s=0}^{\nu_n-1} \frac{(-1)^s \Gamma(\nu_n - s)}{2^{2s-n} \Gamma(s+1)} \frac{i(iu)^{s-\nu_n}}{(ir)^{\frac{d}{2}+s-1}} (-\partial^2)^s \mathcal{O}_a^{(\pm,n)}(x) \\
 &\quad - \frac{e}{2d\pi^{\frac{d}{2}+1}} \sum_{n=0}^{\infty} \sum_{s=0}^{\infty} \frac{(-1)^{\nu_n-1} \left(\log \frac{\sqrt{iu}}{\sqrt{ir}} + c_{s,\nu_n} \right)}{2^{2s+n} \Gamma(s+1) \Gamma(s+\nu_n+1)} \frac{i(iu)^s (-\partial^2)^{s+\nu_n} \mathcal{O}_a^{(\pm,n)}(x)}{(ir)^{d-2+n+s}} \\
 &\quad - \frac{2^d e}{8\pi} \sum_{n=0}^{\infty} \sum_{s=0}^{\infty} \frac{\Gamma(d-1+n+s)}{2^{-n} (-1)^s \Gamma(s+1)} \frac{i(iu)^s}{(ir)^{d-2+n+s}} \int d^d y \frac{\mathcal{O}_a^{(\pm,n)}(y)}{[(x-y)^2]^{d-1+n+s}} + \text{c.c.},
 \end{aligned} \tag{A.2}$$

where $c_{s,\nu} \equiv \gamma_E - \log 2 - \frac{1}{2}(H_s + H_{s+\nu})$, γ_E the Euler-Mascheroni constant, and $H_n = \sum_{k=1}^n \frac{1}{k}$.

For $D = 4$, in addition to needing the $z = 0$ expansion of Bessel functions with nonnegative integer orders, we also need the $z = 0$ expansion of $K_{-1}(kz)$. $A_a^{(R\pm)}$ stays the same as (A.2) with $d = 2$, but the final expression for $A_r^{(R\pm)}$ is

$$\begin{aligned}
 A_r^{(R\pm)} &= \frac{e}{8\pi^3} \frac{1}{ir} \int d^2 y \frac{(x-y)^a}{(x-y)^2} \mathcal{O}_a^{(\pm,0)}(y) + \frac{e}{4\pi} \sum_{s=0}^{\infty} \frac{(-1)^s \Gamma(s+1)}{\Gamma(s+2)} \frac{(iu)^{s+1}}{(ir)^{s+2}} \int d^2 y \frac{\partial^a \mathcal{O}_a^{(\pm,0)}(y)}{[(x-y)^2]^{s+1}} \\
 &\quad - \frac{e}{8\pi^2} \sum_{s=0}^{\infty} \frac{\log \frac{\sqrt{iu}}{\sqrt{ir}} + c_{s,1}}{2^{2s} \Gamma(s+1) \Gamma(s+2)} \frac{(iu)^{s+1}}{(ir)^{s+2}} (-\partial^2)^s \partial^a \mathcal{O}_a^{(\pm,0)}(x) \\
 &\quad - \frac{e}{16\pi^2} \sum_{n=1}^{\infty} \sum_{s=0}^{n-2} \frac{(-1)^s \Gamma(n-s-1)}{2^{2s-n} \Gamma(s+1)} \frac{(iu)^{s-n+1}}{(ir)^{s+2}} (-\partial^2)^s \partial^a \mathcal{O}_a^{(\pm,n)}(x) \\
 &\quad - \frac{e}{2\pi^2} \sum_{n=1}^{\infty} \sum_{s=0}^{\infty} \frac{(-1)^n \left(\log \frac{\sqrt{iu}}{\sqrt{ir}} + c_{s,n-1} \right)}{2^{2s+n} \Gamma(s+1) \Gamma(n+s)} \frac{(iu)^s}{(ir)^{n+s+1}} (-\partial^2)^{n+s-1} \partial^a \mathcal{O}_a^{(\pm,n)}(x) \\
 &\quad - \frac{e}{4\pi} \sum_{n=1}^{\infty} \sum_{s=0}^{\infty} \frac{2^n \Gamma(n+s)}{(-1)^s \Gamma(s+1)} \frac{(iu)^s}{(ir)^{n+s+1}} \int d^2 y \frac{\partial^a \mathcal{O}_a^{(\pm,n)}(y)}{[(x-y)^2]^{n+s}} + \text{c.c.}
 \end{aligned} \tag{A.3}$$

Finally, in odd dimensions $D = d + 2 > 4$, we only need the $z = 0$ expansion of Bessel functions with nonnegative half-integer orders. Repeating the above procedure yields

$$\begin{aligned}
 A_r^{(R\pm)} &= -\frac{e}{16\pi^{\frac{d}{2}}} \sum_{n=0}^{\infty} \sum_{s=0}^{\infty} \frac{2^{n-2s}(-1)^{\frac{d-1}{2}+n}}{\Gamma(s+1)\Gamma(s-\nu_n+2)} \frac{(iu)^{s-\nu_n+1}}{(ir)^{\frac{d}{2}+1+s}} (-\partial^2)^s \partial^a \mathcal{O}_a^{(\pm,n)}(x) \\
 &\quad - \frac{e}{16\pi^{d+1}} \sum_{n=0}^{\infty} \sum_{s=0}^{\infty} \frac{\Gamma(d-2+n+s)}{2^{-n}(-1)^s \Gamma(s+1)} \frac{(iu)^s}{(ir)^{d-1+n+s}} \int d^d y \frac{\partial^a \mathcal{O}_a^{(\pm,n)}(y)}{[(x-y)^2]^{d-2+n+s}} + \text{c.c.} \\
 A_a^{(R\pm)} &= \frac{e}{8\pi^{\frac{d}{2}}} \sum_{n=0}^{\infty} \sum_{s=0}^{\infty} \frac{2^{n-2s}(-1)^{\frac{d-1}{2}+n}}{\Gamma(s+1)\Gamma(1+s-\nu_n)} \frac{i(iu)^{s-\nu_n}}{(ir)^{\frac{d}{2}-1+s}} [-\partial^2]^s \mathcal{O}_a^{(\pm,n)}(x) \\
 &\quad - \frac{e}{8\pi^{d+1}} \sum_{n=0}^{\infty} \sum_{s=0}^{\infty} \frac{\Gamma(d-1+n+s)}{2^{-n}(-1)^s \Gamma(s+1)} \frac{i(iu)^s}{(ir)^{d-2+n+s}} \int d^d y \frac{\mathcal{O}_a^{(\pm,n)}(y)}{[(x-y)^2]^{d-1+n+s}} + \text{c.c.}
 \end{aligned} \tag{A.4}$$

B Poincaré and scale transformations

In this appendix, we determine the Poincaré and scale transformations of the annihilation operators and the LROs (3.15). The goal is to derive table 1 and the commutators used when verifying the Jacobi identity (3.29). For simplicity, both subscripts labeling the annihilation operators and superscripts $\pm$ labeling the currents and LROs will be kept implicit.

We first derive the necessary commutators involving annihilation operators, which transform as

$$\begin{aligned}
 [\mathcal{O}(p), P_A] &= -p_A \mathcal{O}(p) \\
 [\mathcal{O}(p), M_{AB}] &= -i \left(p_A \partial_{p^B} - p_B \partial_{p^A} \right) \mathcal{O}(p) + \mathcal{S}_{AB} \mathcal{O}(p) \\
 [\mathcal{O}(p), S] &= -i \left(p^A \partial_{p^A} + \frac{d}{2} \right) \mathcal{O}(p).
 \end{aligned} \tag{B.1}$$

Parametrizing the momenta in flat null coordinates, i.e.

$$p^A(\omega, x) = \omega \hat{p}^A(x), \quad \hat{p}^A(x) = \frac{1}{2} \left(1 + x^2, 2x^a, 1 - x^2 \right), \tag{B.2}$$

and recalling that we defined in (2.8)

$$P_{\pm} = -P_0 \mp P_{d+1}, \quad T_a = M_{0a} - M_{(d+1)a}, \quad D = M_{(d+1)0}, \quad K_a = M_{0a} + M_{(d+1)a}, \tag{B.3}$$

we find

$$\begin{aligned}
 [\mathcal{O}(\omega, x), P_-] &= -\omega \mathcal{O}(\omega, x), & [\mathcal{O}(\omega, x), P_+] &= -\omega x^2 \mathcal{O}(\omega, x) \\
 [\mathcal{O}(\omega, x), P_a] &= -\omega x_a \mathcal{O}(\omega, x), & [\mathcal{O}(\omega, x), T_a] &= i \partial_a \mathcal{O}(\omega, x) \\
 [\mathcal{O}(\omega, x), M_{ab}] &= -i(x_a \partial_b - x_b \partial_a) \mathcal{O}(\omega, x) + \mathcal{S}_{ab} \mathcal{O}(\omega, x) \\
 [\mathcal{O}(\omega, x), D] &= i(x^a \partial_a - \omega \partial_\omega) \mathcal{O}(\omega, x) \\
 [\mathcal{O}(\omega, x), K_a] &= i \left(x^2 \partial_a - 2x_a x^b \partial_b + 2x_a \omega \partial_\omega \right) \mathcal{O}(\omega, x) + 2x^b \mathcal{S}_{ab} \mathcal{O}(\omega, x) \\
 [\mathcal{O}(\omega, x), S] &= i \left(-\omega \partial_\omega - \frac{d}{2} \right) \mathcal{O}(\omega, x).
 \end{aligned} \tag{B.4}$$

Next, we want to determine the Poincaré and scale transformations of the LROs defined in (3.15), which we repeat here for convenience:

$$\mathbb{Q}(x) \equiv \int_{-\infty}^{\infty} du J_u^{(d)}(u, x), \quad \mathbb{J}_a(x) \equiv \int_{-\infty}^{\infty} du \left[(d-1)J_a^{(d)} - u\partial_a J_u^{(d)} \right]. \quad (\text{B.5})$$

Observing that a conserved current obeys the commutators

$$\begin{aligned} [J_A(X), P_B] &= i\partial_B J_A(X) \\ [J_A(X), M_{BC}] &= -i(X_B\partial_C - X_C\partial_B)J_A(X) - i\eta_{AB}J_C(X) + i\eta_{AC}J_B(X) \\ [J_A(X), S] &= i(X^B\partial_B + d+1)J_A(X), \end{aligned} \quad (\text{B.6})$$

we have in flat null coordinates the J_u commutators

$$\begin{aligned} [J_u, P_-] &= -2i\partial_u J_u, & [J_u, P_+] &= -2i\left(x^2\partial_u + \partial_r - \frac{1}{r}x^a\partial_a\right)J_u \\ [J_u, P_a] &= -2i\left(x_a\partial_u - \frac{1}{2r}\partial_a\right)J_u, & [J_u, T_a] &= i\partial_a J_u \\ [J_u, M_{ab}] &= -i(x_a\partial_b - x_b\partial_a)J_u \\ [J_u, D] &= i(x^a\partial_a + u\partial_u - r\partial_r + 1)J_u \\ [J_u, K_a] &= i\left[\left(x^2 + \frac{u}{r}\right)\partial_a - 2x_a\left(x^b\partial_b + u\partial_u - r\partial_r + 1\right)\right]J_u + \frac{i}{r}J_a \\ [J_u, S] &= i(u\partial_u + r\partial_r + d+1)J_u, \end{aligned} \quad (\text{B.7})$$

and the J_a commutators

$$\begin{aligned} [J_a, P_-] &= -2i\partial_u J_a, & [J_a, P_+] &= -2i\left[x^2\partial_u + \partial_r - \frac{1}{r}(x^b\partial_b + 1)\right]J_a - 4ix_a J_u \\ [J_a, P_b] &= -2i\left(x_b\partial_u - \frac{1}{2r}\partial_b\right)J_a - 2i\delta_{ab}J_u, & [J_a, T_b] &= i\partial_b J_a \\ [J_a, M_{bc}] &= -i(x_b\partial_c - x_c\partial_b)J_a - i\delta_{ab}J_c + i\delta_{ac}J_b \\ [J_a, D] &= i\left(x^b\partial_b + u\partial_u - r\partial_r + 1\right)J_a \\ [J_a, K_b] &= i\left[\left(x^2 + \frac{u}{r}\right)\partial_b - 2x_b(x^c\partial_c + u\partial_u - r\partial_r + 1)\right]J_a \\ &\quad + 2ix_a J_b - 2i\delta_{ab}(x^c J_c + uJ_u - rJ_r) \\ [J_a, S] &= i(u\partial_u + r\partial_r + d)J_a. \end{aligned} \quad (\text{B.8})$$

Extracting the $1/|r|^d$ coefficient from J_u and J_r , and then taking the limit $r \rightarrow \pm\infty$, we obtain

$$\begin{aligned} [J_u^{(d)}, P_-] &= -2i\partial_u J_u^{(d)}, & [J_u^{(d)}, P_+] &= -2ix^2\partial_u J_u^{(d)} \\ [J_u^{(d)}, P_a] &= -2ix_a\partial_u J_u^{(d)}, & [J_u^{(d)}, T_a] &= i\partial_a J_u^{(d)} \\ [J_u^{(d)}, M_{ab}] &= -i(x_a\partial_b - x_b\partial_a)J_u^{(d)} \\ [J_u^{(d)}, D] &= i(x^a\partial_a + u\partial_u + d+1)J_u^{(d)} \\ [J_u^{(d)}, K_a] &= i\left[x^2\partial_a - 2x_a\left(x^b\partial_b + u\partial_u + d+1\right)\right]J_u^{(d)} \\ [J_u^{(d)}, S] &= i[u\partial_u + 1]J_u^{(d)} \end{aligned} \quad (\text{B.9})$$

and

$$\begin{aligned}
[J_a^{(d)}, P_-] &= -2i\partial_u J_a^{(d)}, & [J_a^{(d)}, P_+] &= -2ix^2\partial_u J_a^{(d)} - 4ix_a J_u^{(d)} \\
[J_a^{(d)}, P_b] &= -2ix_b\partial_u J_a^{(d)} - 2i\delta_{ab}J_u^{(d)}, & [J_a^{(d)}, T_b] &= i\partial_b J_a^{(d)} \\
[J_a^{(d)}, M_{bc}] &= -i(x_b\partial_c - x_c\partial_b)J_a^{(d)} - i\delta_{ab}J_c^{(d)} + i\delta_{ac}J_b^{(d)} \\
[J_a^{(d)}, D] &= i\left(x^b\partial_b + u\partial_u + d + 1\right)J_a^{(d)} \\
[J_a^{(d)}, K_b] &= i\left[x^2\partial_b - 2x_b(x^c\partial_c + u\partial_u + d + 1)\right]J_a^{(d)} \\
&\quad + 2ix_a J_b^{(d)} - 2i\delta_{ab}\left(x^c J_c^{(d)} + uJ_u^{(d)}\right) \\
[J_a^{(d)}, S] &= iu\partial_u J_a^{(d)}.
\end{aligned} \tag{B.10}$$

Finally, we integrate over u to determine the commutators involving the LROs given in (B.5). For $\mathbb{Q}(x)$, we have

$$\begin{aligned}
[\mathbb{Q}(x), P_-] &= 0, & [\mathbb{Q}(x), P_+] &= 0 \\
[\mathbb{Q}(x), P_a] &= 0, & [\mathbb{Q}(x), T_a] &= i\partial_a \mathbb{Q}(x) \\
[\mathbb{Q}(x), M_{ab}] &= -i(x_a\partial_b - x_b\partial_a)\mathbb{Q}(x) \\
[\mathbb{Q}(x), D] &= i(x^a\partial_a + d)\mathbb{Q}(x) \\
[\mathbb{Q}(x), K_a] &= i\left[x^2\partial_a - 2x_a(x^b\partial_b + d)\right]\mathbb{Q}(x) \\
[\mathbb{Q}(x), S] &= 0,
\end{aligned} \tag{B.11}$$

and for $\mathbb{J}_a(x)$, we have

$$\begin{aligned}
[\mathbb{J}_a(x), P_-] &= -2i\partial_a \mathbb{Q}(x), & [\mathbb{J}_a(x), P_+] &= -2ix^2\partial_a \mathbb{Q}(x) - 4id x_a \mathbb{Q}(x) \\
[\mathbb{J}_a(x), P_b] &= -2ix_b\partial_a \mathbb{Q}(x) - 2id\delta_{ab}\mathbb{Q}(x), & [\mathbb{J}_a(x), T_b] &= i\partial_b \mathbb{J}_a(x) \\
[\mathbb{J}_a(x), M_{bc}] &= -i(x_b\partial_c - x_c\partial_b)\mathbb{J}_a(x) - i\delta_{ab}\mathbb{J}_c(x) + i\delta_{ac}\mathbb{J}_b(x) \\
[\mathbb{J}_a(x), D] &= i\left(x^b\partial_b + d\right)\mathbb{J}_a(x) \\
[\mathbb{J}_a(x), K_b] &= i\left[x^2\partial_b - 2x_b(x^c\partial_c + d)\right]\mathbb{J}_a(x) + 2ix_a \mathbb{J}_b(x) - 2i\delta_{ab}x^c \mathbb{J}_c(x) \\
[\mathbb{J}_a(x), S] &= -i\mathbb{J}_a(x).
\end{aligned} \tag{B.12}$$

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