Global symmetries of Quaternion-Kähler $\mathcal{N} = 4$ supersymmetric mechanics

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ABSTRACT: We analyze the global symmetries of $\mathcal{N} = 4$ supersymmetric mechanics involving $4n$ -dimensional Quaternion-Kähler (QK) 1D sigma models on projective spaces $\mathbb{H}H^n$ and $\mathbb{H}P^n$ as the bosonic core. All Noether charges associated with global worldline symmetries are shown to vanish as a result of equations of motion, which implies that we deal with a severely constrained hamiltonian system. The complete hamiltonian analysis of the bosonic sector is performed.

KEYWORDS: Field Theories in Lower Dimensions, Extended Supersymmetry, Sigma Models

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1 Introduction

After appearance of the first versions of supersymmetric quantum mechanics (SQM) based on $\mathcal{N} = 2$ one-dimensional supersymmetry [1, 2], this class of theories was intensively and extensively studied in numerous articles and reviews (see, e.g., [4] – [7]). The SQM models reveal interesting quantum and geometric properties, some of which cannot be reproduced in the framework of the standard dimensional reduction from the higher-dimensional supersymmetric field theories. Of special interest are SQM models with extended $\mathcal{N} = 4$ and $\mathcal{N} = 8$ worldline supersymmetries. Some of these $1D$ sigma models admit, e.g., hyper-Kähler manifolds as their bosonic target spaces and are capable to provide a good laboratory for analyzing various properties of the higher-dimensional theories associated with such kind of targets.

Until recently, the majority of SQM models (including those with $\mathcal{N} \geq 4$) were constructed under an assumption that the worldline (or super worldline in the case of superfield models) are “flat”, i.e. $1D$ supersymmetry is rigid. In the paper [8], using $\mathcal{N} = 4$, $1D$ harmonic superspace approach [9], we constructed a new type of $\mathcal{N} = 4$ supersymmetric mechanics involving $4n$ -dimensional Quaternion-Kähler (QK) $1D$ sigma models as the bosonic core. The basic distinguishing features of the new SQM models constructed are

local worldline $\mathcal{N} = 4$ supersymmetry and the presence of the appropriate $\mathcal{N} = 4, 1D$ supergravity multiplet for ensuring this local invariance.

When restricted to the $\mathbb{H}H^n$ or $\mathbb{H}P^n$ target manifolds, the bosonic sector of these models is identical to the dimensionally reduced homogeneous QK sigma models of refs. [10, 11], [12]. In ref. [8] the component action was also obtained for the fermionic sector, therefore the whole component action in an arbitrary gauge is available. With so much detailed information accessible it is worth trying some further insights into these models. These models have rather large gauge symmetries, so that the quantization in a covariant manner seems challenging. In this paper we will restrict ourselves to the study of the global symmetries which these models possess, restricting the gauge transformations to constant parameters.

We will start by recalling the pivotal points of the harmonic superfield formulation of general QK $\mathcal{N} = 4$ supersymmetric mechanics and presenting a short review of the QK component $\mathbb{H}H^n$ and $\mathbb{H}P^n$ models followed by a review of their transformation properties. Then we will concentrate on the global properties of these models, restricting the gauge transformations to constant parameters. After a somehow involved algebra we will show that the corresponding Noether charges are vanishing as a consequence of the equations of motion, and therefore give rise to the gauge constraints, in a similar fashion, e.g., to the appearance of Virasoro constraints in the string theory. The hamiltonian analysis of the bosonic sectors of these homogeneous models is performed as a prerequisite to their quantization.

2 QK supersymmetric mechanics in superspace

Here we recall the basic points of the construction in ref. [8].

2.1 $\mathcal{N} = 4, 1D$ harmonic superspace setup

One-dimensional $\mathcal{N} = 4$ supersymmetry admits a natural realization in $1D$ harmonic superspace [9]. In the analytic basis, this superspace is represented by the coordinate set

$$z := (t_A, \theta^+, \bar{\theta}^+, \theta^-, \bar{\theta}^-, w_i^\pm), \quad w_i^+ w_k^- - w_k^+ w_i^- = \varepsilon_{ik}, \quad (2.1)$$

where $w_i^\pm$ are harmonics parametrizing the automorphism $SU(2)$ group. The analytic harmonic superspace is a subset of (2.1)

$$\zeta := (t_A, \theta^+, \bar{\theta}^+, w_i^\pm). \quad (2.2)$$

In what follows, we will omit the subscript A of t_A . Both (2.1) and (2.2) are closed under the appropriate realization of $\mathcal{N} = 4$ supersymmetry.¹ These coordinate sets are also closed and real with respect to the generalized $\sim$ -conjugation,

$$\widetilde{t} = t, \quad \widetilde{\theta^\pm} = \bar{\theta}^\pm, \quad \widetilde{\bar{\theta}^\pm} = -\theta^\pm, \quad \widetilde{w_i^\pm} = w^{\pm i} = \varepsilon^{ik} w_k^\pm, \quad \widetilde{w^{\pm i}} = -w_i^\pm. \quad (2.3)$$

¹The $\mathcal{N} = 4$ transformations of the superspace coordinates, as well as the relations between the analytic and central bases of harmonic superspace can be found in [9] and in a more recent paper [13].

An important ingredient of the harmonic superspace formalism is the harmonic derivatives

$$D^{++} = \partial^{++} + 2i\theta^+\bar{\theta}^+\partial_t + \theta^+\partial_{\theta^-} + \bar{\theta}^+\partial_{\bar{\theta}^-}, \quad (2.4)$$

$$D^{--} = \partial^{--} + 2i\theta^-\bar{\theta}^-\partial_t + \theta^-\partial_{\theta^+} + \bar{\theta}^-\partial_{\bar{\theta}^+}, \quad (2.5)$$

$$[D^{++}, D^{--}] = D^0 = \partial^0 + \theta^+\partial_{\theta^+} + \bar{\theta}^+\partial_{\bar{\theta}^+} - \theta^-\partial_{\theta^-} - \bar{\theta}^-\partial_{\bar{\theta}^-}, \quad (2.6)$$

where

$$\partial^{\pm\pm} = w_i^{\pm} \frac{\partial}{\partial w_i^{\mp}}, \quad \partial^0 = w_i^+ \frac{\partial}{\partial w_i^+} - w_i^- \frac{\partial}{\partial w_i^-}. \quad (2.7)$$

The covariant derivative D^{++} is distinguished in that it preserves the analyticity: the result of its action on the analytic superfield $\Phi(\zeta)$, i.e. $D^{++}\Phi(\zeta)$, is again an analytic $1D$ superfield. The operator D^0 counts the external harmonic $U(1)$ charges of the harmonic $1D$ superfields. All superfields are assumed to possess a definite harmonic $U(1)$ charge and so are eigenfunctions of D^0 .

The QK supersymmetric mechanics is formulated in terms of two analytic superfields $q^{+a}(\zeta)$ and $2n$ analytic superfields $\hat{Q}^{+r}(\zeta)$, where $a = 1, 2$ and $r = 1, \dots, 2n$ are, respectively, the indices of the fundamental representations of some extra groups $Sp(1) \sim SU(2)$ and $Sp(n)$ commuting with supersymmetry. They are subjected to the tilde-reality conditions

$$\widetilde{q_a^+} = \varepsilon^{ab} q_b^+, \quad \widetilde{\hat{Q}_r^+} = \Omega^{rs} \hat{Q}_s^+, \quad (2.8)$$

where $\Omega^{rs}, \Omega^{rp}\Omega_{ps} = \delta_s^r$, are skew-symmetric constant $Sp(n)$ invariant “metrics”. The superfields q_a^+ and $\hat{Q}_s^+$ are $1D$ analogs of the compensating hypermultiplet and “matter” hypermultiplets [12, 14–16].

In the case of simplest $\mathbb{H}\mathbb{H}^n$ and $\mathbb{H}\mathbb{P}^n$ sigma models which will be the main subject of the present paper, these superfields are subject to the linear harmonic constraints

$$D^{++}q_a^+(\zeta) = 0, \quad D^{++}\hat{Q}_r^+(\zeta) = 0. \quad (2.9)$$

The general QK sigma model amounts to some nonlinear version of these constraints

$$D^{++}q^{+a} - \gamma \frac{1}{2} \frac{\partial}{\partial q_a^+} \left[\hat{\kappa}^2 (w^- \cdot q^+)^2 \mathcal{L}^{+4} \right] = 0, \quad \gamma = \pm 1, \quad (2.10)$$

$$D^{++}\hat{Q}^{+r} + \frac{1}{2} \frac{\partial}{\partial \hat{Q}_r^+} \left[\hat{\kappa}^2 (w^- \cdot q^+)^2 \mathcal{L}^{+4} \right] = 0, \quad (2.11)$$

$$\mathcal{L}^{+4} \equiv \mathcal{L}^{+4} \left(\frac{\hat{Q}^{+r}}{\hat{\kappa}(w^- \cdot q^+)}, \frac{q^{+a}}{(w^- \cdot q^+)}, w_i^- \right), \quad (w^- \cdot q^+) := w_a^- q^{+a}. \quad (2.12)$$

The object $\mathcal{L}^{+4}$ is just the QK potential introduced in [14] and derived in [16] within the pure geometric setting as the fundamental object of QK geometry; the parameter $\hat{\kappa}$ is the contraction parameter to the general HK case: when $\hat{\kappa}$ goes to zero, $\hat{\kappa}(w^- \cdot q^+) \rightarrow 1$, $\frac{q^{+a}}{(w^- \cdot q^+)} \rightarrow w^{+a}$ and, respectively, $\mathcal{L}^{+4} \rightarrow \mathcal{L}^{+4}(Q, w^+, w^-)$. In this limit (2.11) becomes the nonlinear constraint describing the most general HK $\mathcal{N} = 4, 1D$ sigma model [13]. The superfield q_a^+ fully decouples in this limit. It is important that the QK potential $\mathcal{L}^{+4}$

originally does not involve any explicit w^+ harmonics. The parameter $\gamma = \pm 1$ in (2.10) discriminates the cases with non-compact $\mathbb{H}\mathbb{H} = Sp(n, 1)/[Sp(1) \times Sp(n)]$ ($\gamma = 1$) and compact $\mathbb{H}\mathbb{P} = Sp(n+1)/[Sp(1) \times Sp(n)]$ ($\gamma = -1$) homogeneous projective manifolds as the “flattest” n -dimensional QK ones (see next subsection).

While solving the generic nonlinear constraints (2.10), (2.11) is a rather involved problem, it is much easier for their linear version (2.9). The explicit solution of (2.9) is as follows

$$q^{+a}(\zeta) = f^{ia}(t)w_i^+ + \theta^+\chi^a(t) - \bar{\theta}^+\bar{\chi}^a(t) - 2i\theta^+\bar{\theta}^+\dot{f}^{ia}(t)w_i^-, \quad (2.13)$$

$$\hat{Q}^{+r}(\zeta) = \hat{F}^{ir}(t)w_i^+ + \theta^+\chi^r(t) - \bar{\theta}^+\bar{\chi}^r(t) - 2i\theta^+\bar{\theta}^+\dot{\hat{F}}^{ir}(t)w_i^-. \quad (2.14)$$

The superfield reality conditions (2.8) imply the following reality properties for the component fields:

$$\widetilde{f_a^+} = f^{+a} \Leftrightarrow \overline{(f_{ia})} = f^{ia}, \quad \overline{(f^{ia})} = f_{ia}; \quad \overline{(\chi_a)} = \bar{\chi}^a, \quad \overline{(\chi^a)} = -\bar{\chi}_a \quad (2.15)$$

(and similar ones for $\hat{F}^{ir}, \chi^r$). It is assumed that the indices a and r are raised and lowered in the standard way by the skew-symmetric tensors $\varepsilon_{ab}, \varepsilon^{ab}$ and Ω_{rs}, Ω^{rs} . We observe that q^{+a} carries 4 real bosonic degrees of freedom and $\hat{Q}^{+r}$ - $4n$ such degrees, total of $4(n+1)$ bosonic degrees.

One more relevant superfield (just specific for QK models) is the scalar real superfield $E(z)$ which collects the objects of $\mathcal{N} = 4, 1D$ “supergravity”. It lives on the whole harmonic superspace (2.1) and is subjected to the purely harmonic constraint

$$D^{++}E = 0, \quad \tilde{E} = E, \quad (2.16)$$

which means that E in the central basis does not depend on harmonics at all, and so is the standard harmonic-independent $\mathcal{N} = 4, 1D$ superfield. In the analytic basis, the component structure of E is as follows

$$\begin{aligned} E(z) = & h + \theta^+\theta^-\bar{M} - \bar{\theta}^+\bar{\theta}^-\bar{M} + \theta^+\bar{\theta}^-(\mu - i\dot{h}) + \bar{\theta}^+\theta^-(\mu + i\dot{h}) \\ & + 4i(\theta^+\bar{\theta}^+w_i^-w_k^- - \theta^+\bar{\theta}^-w_i^-w_k^+ - \theta^-\bar{\theta}^+w_i^-w_k^+ + \theta^-\bar{\theta}^-w_i^+w_k^+)L^{(ik)} \\ & + 4\theta^+\bar{\theta}^+\theta^-\bar{\theta}^-[D + 2\dot{L}^{(ik)}w_i^+w_k^-] \end{aligned} \quad (2.17)$$

$$\begin{aligned} & + (\theta^-w_i^+ - \theta^+w_i^-)\phi^i - (\bar{\theta}^-w_i^+ - \bar{\theta}^+w_i^-)\bar{\phi}^i + 4i\theta^-\bar{\theta}^-(\theta^+w_i^+\sigma^i - \bar{\theta}^+w_i^+\bar{\sigma}^i) \\ & + 2i\theta^+\bar{\theta}^+[\theta^-w_i^-(2\sigma^i - \dot{\phi}^i) - \bar{\theta}^-w_i^-(2\bar{\sigma}^i - \dot{\bar{\phi}}^i)]. \end{aligned} \quad (2.18)$$

It includes eight bosonic fields $h(t), M(t), \bar{M}(t), \mu(t), D(t), L^{(ik)}(t)$ and eight fermionic fields $\phi^i(t), \bar{\phi}^i(t), \sigma^i(t), \bar{\sigma}^i(t)$. The conjugation rules for all these fields read

$$\begin{aligned} \overline{(h)} &= h, & \overline{(M)} &= \bar{M}, & \overline{(\mu)} &= \mu, & \overline{(D)} &= D, & \overline{(L^{ik})} &= \varepsilon_{il}\varepsilon_{kj}L^{lj}, \\ \overline{(\phi_i)} &= \bar{\phi}^i, & \overline{(\sigma_i)} &= \bar{\sigma}^i. \end{aligned} \quad (2.19)$$

2.2 Superfield QK actions

The universal $1D$ superfield action describing $\mathcal{N} = 4$ supersymmetric QK mechanics looks very simple [8]

$$S_{QK} = \frac{1}{8} \int \mu_H \left[E \left(\gamma q^{+a} q_a^- - \hat{Q}^{+r} \hat{Q}_r^- \right) + \beta \sqrt{E} \right], \quad \beta := -\gamma \frac{2}{\hat{\kappa}^2}, \quad (2.20)$$

where $\mu_H = dt dwd^2\theta^+ d^2\theta^-$ is the measure of integration over the whole $\mathcal{N} = 4, 1D$ superspace and $q^{-a} := D^{--}q^{+a}$, $\hat{Q}^{-r} := D^{--}\hat{Q}^{+r}$. The specificity of the given model is concentrated in the harmonic constraints (2.10), (2.11). One or another choice of the QK potential $\mathcal{L}^{+4}$ generates one or another QK sigma model in the bosonic sector of (2.20).

An important object appearing in (2.20) is the harmonic-independent (in the central basis) supervielbein E incorporating fields of some non-minimal version of $\mathcal{N} = 4, 1D$ “supergravity”. Its presence secures invariance of (2.20) under the appropriate local extension of $\mathcal{N} = 4, 1D$ supersymmetry. The precise form of the superspace realization of this local supersymmetry is given in [8], the realization on the component fields will be presented in the next section. With the choice $\gamma = \pm 1$, the action (2.20) is invariant, respectively, under the extended global $Sp(n, 1)$ ($\gamma = 1$) or $Sp(n + 1)$ ($\gamma = -1$) groups realized as

$$\delta q^{+a} = -\gamma \Lambda^{ar} \hat{Q}_r^+, \quad \delta \hat{Q}^{+r} = \Lambda^{ar} q_a^+, \quad (2.21)$$

where Λ^{ar} are the coset $Sp(n, 1)/[Sp(1) \times Sp(n)]$ or $Sp(n + 1)/[Sp(1) \times Sp(n)]$ constant parameters. The $Sp(1)$ and $Sp(n)$ subgroups in both cases are realized as symplectic rotations with respect to the indices a and r , respectively. Though the action (2.20) is formally invariant under (2.21) at any choice of the QK potential $\mathcal{L}^{+4}$ in the harmonic constraints (2.10), (2.11), the constraints themselves are covariant only with $\mathcal{L}^{+4} = 0$, i.e. when they take the linear form (2.9). In this case for $\gamma = \pm 1$ the action (2.20) in the bosonic sector, after fixing some gauges with respect to local symmetries, describes $1D$ sigma models on the $\mathbb{H}^n$ and $\mathbb{H}P^n$ target spaces (see subsection 3.2). Thus the action (2.20) with the constraints (2.9) describes the “maximally symmetric” particular $\mathbb{H}^n$ or $\mathbb{H}P^n$ cases of $\mathcal{N} = 4$ supersymmetric QK mechanics,²

$$S_{HP} = \frac{1}{8} \int \mu_H \left[E \left(\gamma q^{+a} q_a^- - \hat{Q}^{+r} \hat{Q}_r^- \right) + \beta \sqrt{E} \right], \quad D^{++} q^{+a} = D^{++} \hat{Q}^{+k} = 0. \quad (2.22)$$

3 Component QK Lagrangians

In accord with the consideration in section 2, three classes of the superfields which appear in (2.22) have the following field contents:

- The vielbein E encompasses the multiplet $\mathbf{8} + \mathbf{8}$ of the “non-minimal” $\mathcal{N} = 4, 1D$ “supergravity”. Its bosonic sector consists of the dynamical field $h(t)$ (“graviton”) and the auxiliary fields $M(t), \bar{M}(t), \mu(t), D(t), L^{(ik)}(t)$. The fermionic sector involves the dynamical fermionic fields $\phi_i(t), \bar{\phi}_i(t)$, (“gravitino”) and the auxiliary fields $\sigma^i(t), \bar{\sigma}^i(t)$. The conjugation rules for the involved fields are given in (2.19).
- Two other superfields, q^{+a} and $\hat{Q}^{+r}$, describe $1D$ “matter” and both encompass the $\mathcal{N} = 4, 1D$ multiplets $(\mathbf{4}, \mathbf{4}, \mathbf{0})$. First of them contains the dynamical fields $f^{ia}, \chi^a, \bar{\chi}^a$. It is $1D$ analog of the “conformal compensator” q^+ superfield of the harmonic superspace formulation of $\mathcal{N} = 2, 4D$ supergravity [14, 15]. The set of

²QK mechanics corresponding to any other homogeneous QK manifolds are associated with some non-zero QK potentials in (2.10), (2.11). A wide list of such QK potentials was given in ref. [16].

($\mathbf{4}, \mathbf{4}, \mathbf{0}$) fields entering $\hat{Q}^{+r}$ contains the dynamical fields $\hat{F}^{ia}, \chi^r, \bar{\chi}^r$. The conjugation properties of the relevant fields are given in (2.15).

The precise way how all these fields enter the respective superfields was already presented in (2.13), (2.14).

We will be interested in the component Lagrangian $\mathbb{L}_{HP}$ which corresponds to the superfield one (2.22) and is obtained from the latter after integrating there over Grassmann and harmonic variables. It is a sum of the three Lagrangians:

- The gauge-covariantized kinetic terms of the bosonic compensator and matter fields:

$$\begin{aligned} \mathbb{L}_{HP}^b = & \frac{1}{2}h \left(\dot{\hat{F}}^{ir} \dot{\hat{F}}_{ir} - \gamma f^{ia} \dot{f}_{ia} \right) + L_{ik} \left(\hat{F}^{(ir} \dot{\hat{F}}_r^{k)} - \gamma f^{(ia} \dot{f}_a^{k)} \right) \\ & + \frac{1}{4}D \left(\gamma f^{ia} f_{ia} - \hat{F}^{ir} \hat{F}_{ir} + \frac{\beta}{\sqrt{h}} \right) \\ & + \frac{\beta}{4} \frac{1}{\sqrt{h}h} \left[L^{ik} L_{ik} - \frac{1}{8} (M\bar{M} + \mu^2 + \dot{h}^2) \right]. \end{aligned} \quad (3.1)$$

- The gauge covariantization of the kinetic terms of the fermionic compensator and matter fields:

$$\begin{aligned} \mathbb{L}_{HP}^{f(1)} = & \frac{i}{4}h \left[\gamma \left(\chi^a \dot{\bar{\chi}}_a - \dot{\chi}^a \bar{\chi}_a \right) - \chi^r \dot{\bar{\chi}}_r + \dot{\chi}^r \bar{\chi}_r \right] \\ & + \frac{i}{2}\phi_i \left(\gamma f^{ia} \bar{\chi}_a - \dot{\hat{F}}^{ir} \bar{\chi}_r \right) - \frac{i}{2}\bar{\phi}_i \left(\gamma f^{ia} \chi_a - \dot{\hat{F}}^{ir} \chi_r \right) \\ & + \frac{i}{2}\sigma_i \left(\gamma f^{ia} \bar{\chi}_a - \dot{\hat{F}}^{ir} \bar{\chi}_r \right) - \frac{i}{2}\bar{\sigma}_i \left(\gamma f^{ia} \chi_a - \dot{\hat{F}}^{ir} \chi_r \right) \\ & + \frac{M}{8} \left(\gamma \bar{\chi}^a \bar{\chi}_a - \bar{\chi}^r \bar{\chi}_r \right) - \frac{\bar{M}}{8} \left(\gamma \chi^a \chi_a - \chi^r \chi_r \right) \\ & + \frac{\mu}{4} \left(\gamma \bar{\chi}^a \chi_a - \bar{\chi}^r \chi_r \right). \end{aligned} \quad (3.2)$$

- The remaining Lagrangian that involves fermionic fields of the 1D “supergravity” multiplet and comes solely from the last term in (2.20):

$$\begin{aligned} \mathbb{L}_{HP}^{f(2)} = & \beta \frac{i}{32h^{3/2}} \left(\phi^i \dot{\bar{\phi}}_i - \bar{\phi}^i \dot{\phi}_i + 4\sigma^i \bar{\phi}_i - 4\bar{\sigma}^i \phi_i \right) \\ & + \beta \frac{3}{64h^{5/2}} \left(4iL^{ik} \phi_{(i} \bar{\phi}_{k)} + \frac{M}{2} \bar{\phi}^i \bar{\phi}_i - \frac{\bar{M}}{2} \phi^i \phi_i + \mu \phi^i \bar{\phi}_i \right) \\ & + \beta \frac{15}{64 \cdot 8} \frac{1}{h^{7/2}} (\phi^k \phi_k) (\bar{\phi}^i \bar{\phi}_i). \end{aligned} \quad (3.3)$$

The total off-shell Lagrangian is the sum of these three ones:

$$\mathbb{L}_{HP} = \mathbb{L}_{HP}^b + \mathbb{L}_{HP}^{f(1)} + \mathbb{L}_{HP}^{f(2)}. \quad (3.4)$$

3.1 Transformation properties of QK Lagrangian

The above Lagrangian is invariant under the local transformations with the parameters $b(t), \lambda^i(t), \tau_k^i(t)$ associated, respectively, with the time reparametrizations, local $\mathcal{N} = 4, 1D$ supersymmetry and local $SU(2)$ R -symmetry. Various sets of fields have the following transformation laws:

1. The fields $f^{ia}, \chi^a, \bar{\chi}^a$:

$$\delta_b f^{ia} = -2b \dot{f}^{ia} - \dot{b} f^{ia}, \quad \delta_b \chi^a = -2b \dot{\chi}^a - 2\dot{b} \chi^a, \quad \delta_b \bar{\chi}^a = -2b \dot{\bar{\chi}}^a - 2\dot{b} \bar{\chi}^a, \quad (3.5)$$

$$\delta_\lambda f^{ia} = -\lambda^i \chi^a + \bar{\lambda}^i \bar{\chi}^a, \quad \delta_\lambda \chi^a = 2i\partial_t(\bar{\lambda}^i f_i^a), \quad \delta_\lambda \bar{\chi}^a = 2i\partial_t(\lambda^i f_i^a), \quad (3.6)$$

$$\delta_\tau f^{ia} = \tau_k^i f^{ka}, \quad \delta_\tau \chi^a = \delta_\tau \bar{\chi}^a = 0. \quad (3.7)$$

2. The fields $\hat{F}^{ir}, \chi^r, \bar{\chi}^r$:

$$\delta_b \hat{F}^{ir} = -2b \dot{\hat{F}}^{ir} - \dot{b} \hat{F}^{ir}, \quad \delta_b \chi^r = -2b \dot{\chi}^r - 2\dot{b} \chi^r, \quad \delta_b \bar{\chi}^r = -2b \dot{\bar{\chi}}^r - 2\dot{b} \bar{\chi}^r, \quad (3.8)$$

$$\delta_\lambda \hat{F}^{ir} = -\lambda^i \chi^r + \bar{\lambda}^i \bar{\chi}^r, \quad \delta_\lambda \chi^r = 2i\partial_t(\bar{\lambda}^i \hat{F}_i^r), \quad \delta_\lambda \bar{\chi}^r = 2i\partial_t(\lambda^i \hat{F}_i^r), \quad (3.9)$$

$$\delta_\tau \hat{F}^{ir} = \tau_k^i \hat{F}^{kr}, \quad \delta_\tau \chi^r = \delta_\tau \bar{\chi}^r = 0. \quad (3.10)$$

3. $\mathcal{N} = 4, 1D$ “supergravity” fields:

$$\begin{aligned} \delta_b h &= -2b \dot{h} + 4\dot{b} h, & \delta_b M &= -2b \dot{M} + 2\dot{b} M, & \delta_b \mu &= -2b \dot{\mu} + 2\dot{b} \mu, \\ \delta_b L^{(ik)} &= -2b \dot{L}^{(ik)} + 2\dot{b} L^{(ik)}, & \delta_b D &= -2b \dot{D} + 2\partial_t(\dot{b} h), \\ \delta_b \phi^i &= -2b \dot{\phi}^i + 3\dot{b} \phi^i, & \delta_b \sigma^i &= -2b \dot{\sigma}^i + \dot{b} \sigma^i + \dot{b} \phi^i, \end{aligned} \quad (3.11)$$

$$\begin{aligned} \delta_\lambda h &= \lambda^i \phi_i - \bar{\lambda}^i \bar{\phi}_i, & \delta_\lambda M &= 2i\dot{\lambda}^i \phi_i + i\bar{\lambda}^i(4\sigma_i - 2\dot{\phi}_i), \\ \delta_\lambda \bar{M} &= -2i\dot{\lambda}^i \bar{\phi}_i - i\lambda^i(4\bar{\sigma}_i - 2\dot{\bar{\phi}}_i), \\ \delta_\lambda \mu &= -i(\dot{\lambda}^i \phi_i + \dot{\bar{\lambda}}^i \bar{\phi}_i) - i[\lambda^i(2\sigma_i - \dot{\phi}_i) + \bar{\lambda}^i(2\bar{\sigma}_i - \dot{\bar{\phi}}_i)] \\ \delta_\lambda L^{(ik)} &= \bar{\lambda}^{(i} \dot{\sigma}^{k)} - \lambda^{(i} \dot{\sigma}^{k)} - [\dot{\bar{\lambda}}^{(i} \bar{\phi}^{k)} - \dot{\lambda}^{(i} \phi^{k)}], \\ \delta_\lambda D &= \lambda^i \dot{\sigma}_i - \bar{\lambda}^i \dot{\bar{\sigma}}_i - \dot{\lambda}^i \sigma_i + \dot{\bar{\lambda}}^i \bar{\sigma}_i + \partial_t(\dot{\lambda}^i \phi_i - \dot{\bar{\lambda}}^i \bar{\phi}_i), \\ \delta_\lambda \phi^i &= \lambda^i M + \bar{\lambda}^i(\mu + i\dot{h}) + 4i\bar{\lambda}^k L_k^i - 4i\dot{\bar{\lambda}}^i h, \\ \delta_\lambda \bar{\phi}^i &= \bar{\lambda}^i \bar{M} - \lambda^i(\mu - i\dot{h}) + 4i\lambda^k L_k^i - 4i\dot{\lambda}^i h, \\ \delta_\lambda \sigma^i &= \dot{\lambda}^i M + \dot{\bar{\lambda}}^i(\mu - i\dot{h}) + 2i\bar{\lambda}^k \dot{L}_k^i + i\bar{\lambda}^i D - 2i\dot{\bar{\lambda}}^i h, \\ \delta_\lambda \bar{\sigma}^i &= \dot{\bar{\lambda}}^i \bar{M} - \dot{\lambda}^i(\mu + i\dot{h}) + 2i\lambda^k \dot{L}_k^i + i\lambda^i D - 2i\dot{\lambda}^i h, \end{aligned} \quad (3.12)$$

$$\begin{aligned} \delta_\tau h &= \delta_\tau M = \delta_\tau \mu = 0, & \delta_\tau L^{(ik)} &= h\dot{\tau}^{(ik)} - 2\tau^{(im} L_m^{k)}, & \delta_\tau D &= -2\dot{\tau}^{(ik)} L_{(ik)}, \\ \delta_\tau \phi^i &= \tau_k^i \phi^k, & \delta_\tau \sigma^i &= \tau_k^i \sigma^k - \dot{\tau}^{(ik)} \phi_k. \end{aligned} \quad (3.13)$$

The standard rigid $\mathcal{N} = 4, 1D$ supersymmetry and R -symmetry $SU(2)$ transformations of the component fields are recovered upon choosing the constant parameters in (3.5)–(3.13), $\dot{\lambda}^i = \dot{b} = \dot{\tau}^{ik} = 0$. The $\dot{b} = 0$ transformations are just the constant time shifts.

The internal symmetry transformations (2.21) uniformly act in the evident way on the indices a and r of matter bosonic and fermionic fields, properly mixing f^{ia} with $\hat{F}^{ir}$ and χ^a with χ^r .

It is not so easy to check that the Lagrangian $\mathbb{L}_{HP}$ given by eq. (3.4) is indeed invariant, up to a total time derivative, under all these sets of the field transformations. Nevertheless, this can be done, in complete agreement with the corresponding invariances of the superfield action (2.20) proved in [8].

3.2 Passing to the explicit $\mathbb{H}^n$ or $\mathbb{H}P^n$ sigma model metrics

Though in what follows we will deal with the off-shell “ungauged” Lagrangians (3.1)–(3.3), it is instructive to see how the standard $\mathbb{H}^n$ or $\mathbb{H}P^n$ sigma model Lagrangians are recovered from (3.1).

First, we omit the auxiliary fields by their algebraic equations of motion, $M = \bar{M} = \mu = 0$ (up to some pure fermionic terms). After this we rescale the fields as

$$f^{ia} = h^{-1/4} \tilde{f}^{ia}, \quad \hat{F}^{ir} = h^{-1/4} \tilde{F}^{ir}. \quad (3.14)$$

Then, varying the auxiliary field D yields the constraint

$$\gamma \tilde{f}^2 - \tilde{F}^2 - \gamma \frac{2}{\hat{\kappa}^2} = 0. \quad (3.15)$$

The next step is to gauge away from $\tilde{f}^{ia}$ the symmetric part in the doublet indices using the local $SU(2)$ symmetry (3.7):

$$\tilde{f}^{ia}(t) \Rightarrow \sqrt{2} \varepsilon^{ai} \tilde{\omega}(t). \quad (3.16)$$

After substituting this into (3.15) we solve the latter for $\tilde{\omega}$ as

$$\tilde{\omega} = \frac{1}{\sqrt{2}|\hat{\kappa}|} \sqrt{1 + \gamma \frac{\hat{\kappa}^2}{2} \tilde{F}^2}. \quad (3.17)$$

Finally, we eliminate the auxiliary field L^{ik} by its algebraic equation of motion

$$L^{ik} = h \gamma \kappa^2 \tilde{F}^{(ir} \dot{\tilde{F}}_r^{k)}. \quad (3.18)$$

Plugging all these expressions into (3.1), we obtain

$$\mathbb{L}_{HP}^b = \frac{1}{2} h^{1/2} \left[\dot{\tilde{F}}^{ir} \dot{\tilde{F}}_{ir} + \gamma \hat{\kappa}^2 \dot{\tilde{F}}^{(ir} \tilde{F}_r^{k)} \dot{\tilde{F}}_{(i} \tilde{F}_{k)r} - \gamma \frac{\hat{\kappa}^2}{4} \frac{1}{1 + \gamma \frac{\hat{\kappa}^2}{2} \tilde{F}^2} (\tilde{F} \dot{\tilde{F}})(\tilde{F} \dot{\tilde{F}}) \right]. \quad (3.19)$$

Note that the term $\sim (\dot{h})^2$ drops out from the final Lagrangian. The Lagrangian takes a simpler form after redefining

$$\tilde{F}^{ir} = \frac{F^{ir}}{\sqrt{1 - \gamma \frac{\hat{\kappa}^2}{2} F^2}}, \quad (3.20)$$

$$\mathbb{L}_{HP}^b = \frac{1}{2} h^{1/2} \left[\frac{1}{1 - \gamma \frac{\hat{\kappa}^2}{2} F^2} (\dot{F} \dot{F}) + \gamma \frac{\hat{\kappa}^2}{(1 - \gamma \frac{\hat{\kappa}^2}{2} F^2)^2} (F^{ir} F_i^s) (\dot{F}_r^k \dot{F}_{ks}) \right], \quad (3.21)$$

whence one can read off the target metric as

$$\mathbb{L}_{HP}^b = \frac{1}{2} h^{1/2} g_{ir\,ks} \dot{F}^{ir} \dot{F}^{ks}, \quad (3.22)$$

$$g_{ir\,ks} = \varepsilon_{ik} \frac{1}{1 - \gamma \frac{\hat{\kappa}^2}{2} F^2} \left[\Omega_{rs} + \gamma \hat{\kappa}^2 \frac{1}{1 - \gamma \frac{\hat{\kappa}^2}{2} F^2} F_r^j F_{js} \right]. \quad (3.23)$$

The nonlinearly realized coset $Sp(1, n)/[Sp(1) \times Sp(n)]$ ($\gamma = 1$) and $Sp(1 + n)/[Sp(1) \times Sp(n)]$ ($\gamma = -1$) transformations leaving invariant (3.22) read

$$\delta F^{ir} = \Lambda^{ir} - \gamma \hat{\kappa}^2 \Lambda^{sk} F_k^r F_s^i. \quad (3.24)$$

Note that the Lagrangians (3.19) and (3.21) reveal an obvious analogy with the Lagrangians of the massless “relativistic particle” on the aforementioned QK spaces $\mathbb{H}H^n$ and $\mathbb{H}P^n$: in the gauge $h = 1$ they become the corresponding 1D sigma model Lagrangians, while varying with respect to the “einbein” h yields a sort of the particle $P^2 \simeq 0$ constraint on these n -dimensional manifolds (in the Hamiltonian approach, this is just vanishing of the relevant Hamiltonian, see section 5).

4 Equations of motion

It is straightforward to derive the classical equations of motion following from the action with the Lagrangian $\mathbb{L}_{HP}$, eq. (3.4). They are divided into sets of non-dynamical algebraic equations, as well as the dynamical equations, of the second order in ∂_t for bosonic fields and of the first order for fermionic fields.

1. Non-dynamical equations:

$$D : \quad \gamma f^2 - \hat{F}^2 + \frac{\beta}{h^{1/2}} = 0, \quad (4.1)$$

$$L^{ik} : \quad L^{ik} = -2 \frac{h^{3/2}}{\beta} \left[\hat{F}^{(ir} \dot{\hat{F}}_r^{k)} - \gamma f^{(ia} \dot{f}_a^{k)} \right] + \frac{3i}{8h} \bar{\phi}^{(i} \phi^{k)}, \quad (4.2)$$

$$\bar{M} : \quad M = \frac{4h^{3/2}}{\beta} \left(\chi^r \chi_r - \gamma \chi^a \chi_a \right) - \frac{3}{4h} \phi^i \phi_i,$$

$$M : \quad \bar{M} = \frac{4h^{3/2}}{\beta} \left(\gamma \bar{\chi}^a \bar{\chi}_a - \bar{\chi}^r \bar{\chi}_r \right) + \frac{3}{4h} \bar{\phi}^i \bar{\phi}_i,$$

$$\mu : \quad \mu = \frac{4h^{3/2}}{\beta} \left(\gamma \bar{\chi}^a \chi_a - \bar{\chi}^r \chi_r \right) + \frac{3}{4h} \phi^i \bar{\phi}_i, \quad (4.3)$$

$$\bar{\sigma}_i : \quad \phi^i = \frac{4h^{3/2}}{\beta} \left(\gamma f^{ia} \chi_a - \hat{F}^{ir} \chi_r \right), \quad \sigma_i : \quad \bar{\phi}^i = \frac{4h^{3/2}}{\beta} \left(\gamma f^{ia} \bar{\chi}_a - \hat{F}^{ir} \bar{\chi}_r \right). \quad (4.4)$$

2. Dynamical equations:

$$\begin{aligned} \hat{F}^{ir} : \quad & \partial_t (h \dot{\hat{F}}_{ir}) + \frac{1}{2} D \hat{F}_{ir} - 2 L_{ik} \dot{\hat{F}}_r^k - \dot{L}_{ik} \hat{F}_r^k - \frac{i}{2} (\phi_i \dot{\chi}_r - \bar{\phi}_i \dot{\bar{\chi}}_r) \\ & + \frac{i}{2} [(\sigma_i - \dot{\phi}_i) \bar{\chi}_r - (\bar{\sigma}_i - \dot{\bar{\phi}}_i) \chi_r] = 0 \end{aligned} \quad (4.5)$$

(equation for f^{ia} has the same form, with the evident substitution of indices $r \rightarrow a$).

$$h : \quad \partial_t^2(h^{-\frac{1}{2}}) = \frac{4}{\beta}(\dot{\hat{F}}^2 - \gamma f^2) + V, \quad (4.6)$$

$$\begin{aligned} V = & -\frac{D}{h^{3/2}} - \frac{3}{h^{5/2}} \left[L^{ik} L_{ik} - \frac{1}{8}(M\bar{M} + \mu^2 + \dot{h}^2) + \frac{i}{8}(\phi^i \dot{\bar{\phi}}_i - \bar{\phi}^i \dot{\phi}_i + 4\sigma^i \bar{\phi}_i - 4\bar{\sigma}^i \phi_i) \right] \\ & + \frac{2i}{\beta} \left[\gamma(\chi^a \dot{\bar{\chi}}_a - \dot{\chi}^a \bar{\chi}_a) - \chi^r \dot{\bar{\chi}}_r + \dot{\chi}^r \bar{\chi}_r \right] - \frac{15}{16h^{7/2}} \left(4iL^{ik} \phi_i \bar{\phi}_k + \frac{M}{2} \bar{\phi}^i \bar{\phi}_i - \frac{\bar{M}}{2} \phi^i \phi_i \right. \\ & \left. + \mu \phi^i \bar{\phi}_i \right) - \frac{7 \times 15}{128h^{9/2}} \phi^k \phi_k \bar{\phi}^i \bar{\phi}_i. \end{aligned} \quad (4.7)$$

Then follow the equations for matter fermions:

$$\begin{aligned} \bar{\chi}^r : \quad & \dot{\chi}^r + \frac{1}{h} \left[-\phi_i \dot{\hat{F}}_r^i - \sigma_i \hat{F}_r^i + \frac{i}{2} M \bar{\chi}_r + \frac{i}{2} \mu \chi_r + \frac{1}{2} \dot{h} \chi_r \right] = 0, \\ \chi^r : \quad & \dot{\bar{\chi}}^r + \frac{1}{h} \left[-\bar{\phi}_i \dot{\hat{F}}_r^i - \bar{\sigma}_i \hat{F}_r^i + \frac{i}{2} \bar{M} \chi_r - \frac{i}{2} \mu \bar{\chi}_r + \frac{1}{2} \dot{h} \bar{\chi}_r \right] = 0 \end{aligned} \quad (4.8)$$

(and similar equations for χ_a and $\bar{\chi}_a$).

Finally, we obtain the equations for the “gravitino” ϕ^i and $\bar{\phi}^i$:

$$\begin{aligned} \bar{\phi}^i : \quad & \dot{\phi}_i - 2\sigma_i + \frac{3i}{4} h^{-1} [4iL_i^k \phi_k + M \bar{\phi}_i + (\mu + i\dot{h})\phi_i] - \frac{8}{\beta} h^{3/2} (\gamma f_i^a \chi_a - \dot{\hat{F}}_i^r \chi_r) \\ & + \frac{15i}{16} h^{-2} \bar{\phi}_i (\phi^k \phi_k) = 0, \\ \phi^i : \quad & \dot{\bar{\phi}}_i - 2\bar{\sigma}_i - \frac{3i}{4} h^{-1} [-4iL_i^k \bar{\phi}_k - \bar{M} \phi_i + (\mu - i\dot{h})\bar{\phi}_i] - \frac{8}{\beta} h^{3/2} (\gamma f_i^a \bar{\chi}_a - \dot{\hat{F}}_i^r \bar{\chi}_r) \\ & - \frac{15i}{16} h^{-2} \phi_i (\bar{\phi}^k \bar{\phi}_k) = 0. \end{aligned} \quad (4.9)$$

5 Noether charges

Noether charges are calculated in the standard way for the transformations with constant parameters.

1. R-symmetry. We start with the $SU(2)$ current associated with the parameters τ_{ik} . We define

$$\delta_\tau \Phi_A \frac{\partial \mathbb{L}}{\partial \dot{\Phi}_A} = \tau^{ik} J_{ik}$$

whence

$$\begin{aligned} J_{ik} = & h [\hat{F}_{(ir} \dot{\hat{F}}_{k)}^r - \gamma f_{(ia} f_{k)}^a] + \frac{i}{2} [\phi_{(i} (\hat{F}_{k)}^r \bar{\chi}_r - \gamma f_{k)}^a \bar{\chi}_a) - \bar{\phi}_{(i} (\hat{F}_{k)}^r \chi_r - \gamma f_{k)}^a \chi_a)] \\ & - \frac{1}{2} (\hat{F}^2 - \gamma f^2) L_{ik} + i\beta \frac{1}{16h^{3/2}} \phi_{(i} \bar{\phi}_{k)}. \end{aligned} \quad (5.1)$$

Using the bosonic constraint (4.1) and the on-shell expression for L_{ik} and $\phi_i, \bar{\phi}_i$ (from the non-dynamical equations (4.2) and (4.4)), it is easy to show that on shell

$$J_{ik} = 0. \quad (5.2)$$

2. Supersymmetry. Next we construct supercharges associated with the constant parameters λ^i . We define

$$\delta\lambda\mathbb{L}_{HP} = \partial_t U, \quad \delta\lambda\Phi_A \frac{\partial\mathbb{L}_{HP}}{\partial\dot{\Phi}_A} - U = \lambda^k Q_k + \bar{\lambda}^k \bar{Q}_k, \quad (5.3)$$

with

$$\begin{aligned} U = & \lambda^k \left\{ \frac{h}{2} (\dot{\hat{F}}_k^r \chi_r - \gamma \dot{f}_k^a \chi_a) + L_k^i (\hat{F}_i^r \chi_r - \gamma f_i^a \chi_a) - \frac{1}{4} \sigma_k (\hat{F}^2 - \gamma f^2) \right. \\ & - \frac{i}{4} \phi_k (\chi^r \bar{\chi}_r - \gamma \chi^a \bar{\chi}_a) + \frac{i}{4} \bar{\phi}_k (\chi^r \chi_r - \gamma \chi^a \chi_a) \\ & + \frac{\beta}{4h^{1/2}} \left[\sigma_k - \frac{i}{8h} (M \bar{\phi}_k + \mu \phi_k - i \dot{h} \phi_k + 4i L_k^i \phi_i) - \frac{3i}{32h^2} \bar{\phi}_k \phi^i \phi_i \right] \Big\} \\ & + \bar{\lambda}^k \left\{ -\frac{h}{2} (\dot{\hat{F}}_k^r \bar{\chi}_r - \gamma \dot{f}_k^a \bar{\chi}_a) - L_k^i (\hat{F}_i^r \bar{\chi}_r - \gamma f_i^a \bar{\chi}_a) + \frac{1}{4} \bar{\sigma}_k (\hat{F}^2 - \gamma f^2) \right. \\ & - \frac{i}{4} \bar{\phi}_k (\chi^r \bar{\chi}_r - \gamma \chi^a \bar{\chi}_a) + \frac{i}{4} \phi_k (\bar{\chi}^r \bar{\chi}_r - \gamma \bar{\chi}^a \bar{\chi}_a) \\ & + \frac{\beta}{4h^{1/2}} \left[-\bar{\sigma}_k + \frac{i}{8h} (\bar{M} \phi_k - \mu \bar{\phi}_k - i \dot{h} \bar{\phi}_k + 4i L_k^i \bar{\phi}_i) - \frac{3i}{32h^2} \phi_k \bar{\phi}^i \bar{\phi}_i \right] \Big\}, \quad (5.4) \end{aligned}$$

whence

$$\begin{aligned} Q_k = & h (\dot{\hat{F}}_k^r \chi_r - \gamma \dot{f}_k^a \chi_a) + \frac{1}{4} \sigma_k (\hat{F}^{jr} \hat{F}_{jr} - \gamma f^{ak} f_{ak}) \\ & - \frac{i}{4} \phi_k (\chi^r \bar{\chi}_r - \gamma \chi^a \bar{\chi}_a) + \frac{i}{4} \bar{\phi}_k (\chi^r \chi_r - \gamma \chi^a \chi_a) \\ & + \frac{\beta}{16h^{3/2}} \left[-4h\sigma_k - \dot{h}\phi_k - 4L_k^l \phi_l + \frac{3i}{8h} \bar{\phi}_k (\phi^l \phi_l) \right], \quad (5.5) \end{aligned}$$

$$\begin{aligned} \bar{Q}_k = & h (\dot{\hat{F}}_k^r \bar{\chi}_r - \gamma \dot{f}_k^a \bar{\chi}_a) + \frac{1}{4} \bar{\sigma}_k (\hat{F}^{jr} \hat{F}_{jr} - \gamma f^{ak} f_{ak}) \\ & + \frac{i}{4} \bar{\phi}_k (\chi^r \bar{\chi}_r - \gamma \chi^a \bar{\chi}_a) - \frac{i}{4} \phi_k (\bar{\chi}^r \bar{\chi}_r - \gamma \bar{\chi}^a \bar{\chi}_a) \\ & - \frac{\beta}{16h^{3/2}} \left[4h\bar{\sigma}_k + \dot{h}\bar{\phi}_k + 4L_k^l \bar{\phi}_l + \frac{3i}{8h} \phi_k (\bar{\phi}^l \bar{\phi}_l) \right]. \quad (5.6) \end{aligned}$$

Let us show that $Q_i = 0$ on shell. We identically rewrite the expression in the first term in (5.5) as

$$\begin{aligned} \dot{\hat{F}}_k^r \chi_r - \gamma \dot{f}_k^a \chi_a = & \partial_t (\hat{F}_k^r \chi_r - \gamma f_k^a \chi_a) - (\hat{F}_k^r \dot{\chi}_r - \gamma f_k^a \dot{\chi}_a) \\ = & -\beta \left(\frac{1}{4h^{3/2}} \dot{\phi}_k - \frac{3}{8} \frac{1}{h^{5/2}} \dot{h} \phi_i \right) - \frac{1}{h} \phi_j (\hat{F}_k^r \dot{\hat{F}}_r^j - \gamma f_k^a \dot{f}_a^j) \\ & + \frac{\beta}{2h^{3/2}} \sigma_k - i \frac{\beta}{8h^{5/2}} (M \bar{\phi}_k + \mu \phi_k + i \dot{h} \phi_k), \quad (5.7) \end{aligned}$$

where we made use of the bosonic constraint (4.1), eq. (4.8), together with the analogous one for χ_a and (a few times) of the fermionic constraint (4.4). As the next steps we represent

$$-\frac{1}{h} \phi_j (\hat{F}_k^r \dot{\hat{F}}_r^j - \gamma f_k^a \dot{f}_a^j) = -\frac{\beta}{8h^{5/2}} \dot{h} \phi_k + \frac{1}{h} \phi^j [\hat{F}_{(k}^r \dot{\hat{F}}_{j)r} - \gamma f_{(k}^a \dot{f}_{j)a}],$$

substitute it in (5.7) and then use the equation of motion for ϕ_i , eq. (4.9). After some algebra, we obtain

$$\begin{aligned} \dot{\hat{F}}_k^r \chi_r - \gamma \dot{f}_k^a \chi_a &= \frac{\beta}{16h^{5/2}} \dot{h} \phi_k - \frac{1}{h} \phi^j \left[\hat{F}_{(k}^r \dot{\hat{F}}_{j)r} - \gamma f_{(k}^a \dot{f}_{j)a} \right] \\ &\quad - i \frac{\beta}{16h^{5/2}} \left[M \bar{\phi}_k + \mu \phi_k + 12i L_k^l \phi_l + \frac{15}{4h} \bar{\phi}_k (\phi^l \phi_l) \right]. \end{aligned} \quad (5.8)$$

Finally, using (4.2), we obtain

$$\begin{aligned} \dot{\hat{F}}_k^r \chi_r - \gamma \dot{f}_k^a \chi_a &= \frac{\beta}{16h^{5/2}} \dot{h} \phi_k \\ &\quad - i \frac{\beta}{16h^{5/2}} \left[M \bar{\phi}_k + \mu \phi_k + 4i L_k^l \phi_l + \frac{3}{2h} \bar{\phi}_k (\phi^l \phi_l) \right], \end{aligned} \quad (5.9)$$

and this expression exactly cancels the remaining terms in (5.5), taking into account that the total coefficient of σ_i in (5.5) is vanishing as a consequence of the constraint (4.1).

In a similar fashion or just by conjugation we obtain:

$$\begin{aligned} \dot{\hat{F}}_k^r \bar{\chi}_r - \gamma \dot{f}_k^a \bar{\chi}_a &= \frac{\beta}{16h^{5/2}} \dot{h} \bar{\phi}_k \\ &\quad - i \frac{\beta}{16h^{5/2}} \left[\bar{M} \phi_k - \mu \bar{\phi}_k + 4i L_k^l \bar{\phi}_l - \frac{3}{2h} \phi_k (\bar{\phi}^l \bar{\phi}_l) \right], \end{aligned} \quad (5.10)$$

Thus on shell

$$Q_k = \bar{Q}_k = 0. \quad (5.11)$$

It is worth mentioning that, using the identities (5.9), (5.10) deduced above, the equations for the auxiliary fields $\sigma_i, \bar{\sigma}_i$ (4.9) can be simplified:

$$\begin{aligned} \dot{\phi}_i - 2\sigma_i + \frac{i}{4h} [4i L_i^k \phi_k + M \bar{\phi}_i + (\mu + i\dot{h}) \phi_i] + \frac{3i}{16} h^{-2} \bar{\phi}_i (\phi^k \phi_k) &= 0, \\ \dot{\bar{\phi}}_i - 2\bar{\sigma}_i - \frac{i}{4h} [-4i L_i^k \bar{\phi}_k - \bar{M} \phi_i + (\mu - i\dot{h}) \bar{\phi}_i] - \frac{3i}{16} h^{-2} \phi_i (\bar{\phi}^k \bar{\phi}_k) &= 0. \end{aligned} \quad (5.12)$$

3. Time translations. We define the conserved charge associated with b transformations (Hamiltonian) as

$$\delta_b \mathbb{L}_{HP} = -2b \mathbb{L}_{HP}, \quad \dot{\Phi}_A \frac{\partial \mathbb{L}_{HP}}{\partial \dot{\Phi}_A} - \mathbb{L}_{HP} = \mathbb{H}, \quad (5.13)$$

whence

$$\mathbb{H} = \mathbb{H}^b + \mathbb{H}^{f(1)} + \mathbb{H}^{f(2)},$$

where

$$\begin{aligned} \mathbb{H}^b &= \frac{1}{2} h \left(\dot{\hat{F}}^{ir} \dot{\hat{F}}_{ir} - \gamma \dot{f}^{ia} \dot{f}_{ia} \right) - \frac{1}{4} D \left(\gamma f^{ia} f_{ia} - \hat{F}^{ir} \hat{F}_{ir} + \frac{\beta}{\sqrt{h}} \right) \\ &\quad - \frac{\beta}{4} \frac{1}{h^{3/2}} \left[L^{ik} L_{ik} - \frac{1}{8} (M \bar{M} + \mu^2 - \dot{h}^2) \right], \end{aligned} \quad (5.14)$$

$$\begin{aligned}
 \mathbb{H}^{f(1)} = & -\frac{i}{2}\sigma_i \left(\gamma f^{ia} \bar{\chi}_a - \hat{F}^{ir} \bar{\chi}_r \right) + \frac{i}{2}\bar{\sigma}_i \left(\gamma f^{ia} \chi_a - \hat{F}^{ir} \chi_r \right) \\
 & - \frac{M}{8} \left(\gamma \bar{\chi}^a \bar{\chi}_a - \bar{\chi}^r \bar{\chi}_r \right) + \frac{\bar{M}}{8} \left(\gamma \chi^a \chi_a - \chi^r \chi_r \right) \\
 & - \frac{\mu}{4} \left(\gamma \bar{\chi}^a \chi_a - \bar{\chi}^r \chi_r \right),
 \end{aligned} \tag{5.15}$$

$$\begin{aligned}
 \mathbb{H}^{f(2)} = & -\beta \frac{3}{64h^{5/2}} \left(4iL^{ik} \phi_{(i} \bar{\phi}_{k)} + \frac{M}{2} \bar{\phi}^i \bar{\phi}_i - \frac{\bar{M}}{2} \phi^i \phi_i + \mu \phi^i \bar{\phi}_i \right) \\
 & -\beta \frac{i}{8h^{3/2}} \left(\sigma^i \bar{\phi}_i - \bar{\sigma}^i \phi_i \right) - \beta \frac{15}{64 \cdot 8} \frac{1}{h^{7/2}} (\phi^k \phi_k) (\bar{\phi}^i \bar{\phi}_i).
 \end{aligned} \tag{5.16}$$

Putting these formulas together and using some equations for the auxiliary fields we get:

$$\begin{aligned}
 \mathbb{H} = & \frac{1}{2}h \left(\dot{\hat{F}}^{ir} \dot{\hat{F}}_{ir} - \gamma f^{ia} \dot{f}_{ia} \right) - \frac{\beta}{4h^{3/2}} \left[L^{ik} L_{ik} - \frac{1}{8} (M\bar{M} + \mu^2 - \dot{h}^2) \right] \\
 & - \frac{3i\beta}{16h^{5/2}} L^{ik} \phi_i \bar{\phi}_k - \frac{15\beta}{64 \cdot 8h^{7/2}} (\phi^k \phi_k) (\bar{\phi}^i \bar{\phi}_i).
 \end{aligned} \tag{5.17}$$

It still remains to show that

$$\mathbb{H} = 0 \tag{5.18}$$

on shell.

The proof of (5.18) is more involved compared to (5.11). The basic step is to represent the first term in the first line of (5.14) as

$$\frac{h}{2} \left(\dot{\hat{F}}^{ir} \dot{\hat{F}}_{ir} - \gamma f^{ia} \dot{f}_{ia} \right) = \frac{1}{4} \partial_t [h \partial_t (\hat{F}^2 - \gamma f^2)] - \frac{1}{2} [\hat{F}^{ir} \partial_t (h \dot{\hat{F}}_{ir}) - \gamma f^{ia} \partial_t (h \dot{f}_{ia})]. \tag{5.19}$$

Then, making use of the equations of motion, we obtain:

$$\frac{1}{2}h \left(\dot{\hat{F}}^{ir} \dot{\hat{F}}_{ir} - \gamma f^{ia} \dot{f}_{ia} \right) = -\frac{\beta}{4} \partial_t^2 (h^{1/2}) + W, \tag{5.20}$$

where

$$\begin{aligned}
 W = & \frac{1}{2} \hat{F}^{ir} \left\{ \frac{1}{2} D \hat{F}_{ir} - 2L_{ik} \dot{\hat{F}}_r^k - \dot{L}_{ik} \hat{F}_r^k - \frac{i}{2} (\phi_i \dot{\chi}_r - \bar{\phi} \dot{\chi}_r) + \frac{i}{2} [(\sigma_i - \dot{\phi}_i) \bar{\chi}_r - (\bar{\sigma}_i - \dot{\bar{\phi}}_i) \chi_r] \right\} \\
 & - \gamma \frac{1}{2} f^{ia} \left\{ \frac{1}{2} D f_{ia} - 2L_{ik} \dot{f}_a^k - \dot{L}_{ik} f_a^k - \frac{i}{2} (\phi_i \dot{\chi}_a - \bar{\phi} \dot{\chi}_a) + \frac{i}{2} [(\sigma_i - \dot{\phi}_i) \bar{\chi}_a - (\bar{\sigma}_i - \dot{\bar{\phi}}_i) \chi_a] \right\}.
 \end{aligned} \tag{5.21}$$

As the next step, we cast (4.6) in the form:

$$-\frac{1}{h} \partial_t^2 (h^{1/2}) + \frac{\dot{h}^2}{2h^{5/2}} = \frac{4}{\beta} (\dot{\hat{F}}^2 - \gamma \dot{f}^2) + V. \tag{5.22}$$

Using this equation, we rewrite (5.20) as the following identity:

$$\frac{h}{2} \left(\dot{\hat{F}}^{ir} \dot{\hat{F}}_{ir} - \gamma f^{ia} \dot{f}_{ia} \right) = -\frac{\beta h}{4} V + \frac{\beta}{8h^{3/2}} \dot{h}^2 - W. \tag{5.23}$$

Now, the strategy will be following: the terms which contain $\dot{h}$ are going to eventually cancel, the other terms with time derivatives, or matter fermions, can be replaced by using the equations of motion. In this way we end up with an expression for $\mathbb{H}$ which contains only terms related to the supergravity multiplet and its auxiliary fields ordered by odd integer powers of $h^{-\frac{1}{2}}$, and the coefficients of each individual power of this sort must vanish. This can be rather easily checked for the terms with $h^{-\frac{1}{2}}$ and $h^{-\frac{3}{2}}$. More involved calculation shows that all terms with the higher inverse degrees of h also vanish.

Note that the vanishing of the (super)charges associated with the rigid world-line symmetries is in accordance with the 1D version of the second Noether theorem (see [17, 18] for a modern discussion of this theorem and its implications). Moreover, since $\mathcal{N} = 4$ supersymmetry combine these (super)charges into a supermultiplet, it would be enough to show the on-shell vanishing only for one of them, say for $Q_i, \bar{Q}^i$. Nevertheless, the above explicit checks provide a good verification of the self-consistency of our approach. To be firmly confident of the correctness of the expressions for Noether (super)charges is important for the construction of the Hamiltonian formalism (actually, its purely bosonic truncation) in the next sections.

4. $Sp(n, 1)$ and $Sp(n + 1)$ symmetries. Finally, we will discuss the rigid isometries. The general definition of the relevant conserved current is

$$\delta_\Lambda \Phi_A \frac{\partial \mathbb{L}}{\partial \dot{\Phi}_A} = \Lambda^{ar} J_{ar}.$$

Then

$$\begin{aligned} J_{ar} = & f_a^i \left[h \dot{\hat{F}}_{ir} - \frac{i}{2} (\phi_i \bar{\chi}_r - \bar{\phi}_i \chi_r) \right] + \hat{F}_r^i \left[h \dot{f}_{ia} - \frac{i}{2} (\phi_i \bar{\chi}_a - \bar{\phi}_i \chi_a) \right] \\ & - 2L_{ik} f_a^i \hat{F}_r^k + \frac{i}{2} h (\chi_r \bar{\chi}_a - \bar{\chi}_r \chi_a). \end{aligned} \quad (5.24)$$

The $Sp(1)$ and $Sp(n)$ currents are calculated analogously. All these currents are conserved, but non-vanishing on shell, because they correspond to the internal global symmetries as opposed to the previously presented currents associated with the worldline symmetries.

6 Canonical momenta

The canonical momenta are calculated straightforwardly:

$$\begin{aligned} \mathcal{P}_{ir}^{(F)} &= \frac{\partial \mathbb{L}}{\partial \dot{\hat{F}}_{ir}} = h \dot{\hat{F}}_{ir} - L_{ik} \hat{F}_r^k - \frac{i}{2} (\phi_i \bar{\chi}_r - \bar{\phi}_i \chi_r), \\ \mathcal{P}_{ia}^{(f)} &= \frac{\partial \mathbb{L}}{\partial \dot{f}_{ia}} = -\gamma h \dot{f}_{ia} + \gamma L_{ik} f_a^k + \frac{i}{2} \gamma (\phi_i \bar{\chi}_a - \bar{\phi}_i \chi_a), \\ \mathcal{P}^{(h)} &= \frac{\partial \mathbb{L}}{\partial \dot{h}} = -\frac{\beta}{16h^{3/2}} \dot{h}, \end{aligned} \quad (6.1)$$

$$\begin{aligned}
 \mathcal{P}_r^{(\chi)} &= \frac{\partial \mathbb{L}}{\partial \dot{\chi}^r} = -\frac{i}{4} h \bar{\chi}_r, & \mathcal{P}_r^{(\bar{\chi})} &= \frac{\partial \mathbb{L}}{\partial \dot{\bar{\chi}}^r} = \frac{i}{4} h \chi_r, \\
 \mathcal{P}_a^{(\chi)} &= \frac{\partial \mathbb{L}}{\partial \dot{\chi}^a} = \frac{i}{4} \gamma h \bar{\chi}_a, & \mathcal{P}_a^{(\bar{\chi})} &= \frac{\partial \mathbb{L}}{\partial \dot{\bar{\chi}}^a} = -\frac{i}{4} \gamma h \chi_a, \\
 \mathcal{P}_i^{(\phi)} &= \frac{\partial \mathbb{L}}{\partial \dot{\phi}^i} = i\beta \frac{1}{32h^{3/2}} \bar{\phi}_i, & \mathcal{P}_i^{(\bar{\phi})} &= \frac{\partial \mathbb{L}}{\partial \dot{\bar{\phi}}^i} = -i\beta \frac{1}{32h^{3/2}} \phi_i,
 \end{aligned} \tag{6.2}$$

where the derivatives with respect to $\dot{\chi}^r, \dot{\bar{\chi}}^r, \dot{\chi}^a, \dot{\bar{\chi}}^a, \dot{\phi}^i$ and $\dot{\bar{\phi}}^i$ are understood as the right ones.

The supercharges can be expressed in terms of the bosonic canonical momenta as

$$\begin{aligned}
 Q_k &= \mathcal{P}_k^{(F)r} \chi_r + \mathcal{P}_k^{(f)a} \chi_a + \mathcal{P}^{(h)} \phi_k + \frac{3i\beta}{16 \cdot 8h^{5/2}} \bar{\phi}_k (\phi^l \phi_l) \\
 &\quad + \frac{i}{4} [\phi_k (\bar{\chi}^r \chi_r - \gamma \bar{\chi}^a \chi_a) - \bar{\phi}_k (\chi^r \chi_r - \gamma \chi^a \chi_a)],
 \end{aligned} \tag{6.3}$$

$$\begin{aligned}
 \bar{Q}_k &= \mathcal{P}_k^{(F)r} \bar{\chi}_r + \mathcal{P}_k^{(f)a} \bar{\chi}_a + \mathcal{P}^{(h)} \bar{\phi}_k - \frac{3i\beta}{16 \cdot 8h^{5/2}} \phi_k (\bar{\phi}^l \bar{\phi}_l) \\
 &\quad + \frac{i}{4} [\phi_k (\bar{\chi}^r \bar{\chi}_r - \gamma \bar{\chi}^a \bar{\chi}_a) - \bar{\phi}_k (\bar{\chi}^r \chi_r - \gamma \bar{\chi}^a \chi_a)].
 \end{aligned} \tag{6.4}$$

We used here the algebraic equations (4.1) and (4.4). Analogously, one can reexpress the $SU(2)$ current and the Hamiltonian

$$J_{kl} = \mathcal{P}_{(k}^{(F)r} \hat{F}_{l)r} + \mathcal{P}_{(k}^{(f)a} f_{l)a} + \frac{i\beta}{16h^{3/2}} \phi_k \bar{\phi}_l, \tag{6.5}$$

$$\begin{aligned}
 \mathbb{H} &= \frac{1}{2h} \left[(\mathcal{P}^{(F)})^2 - \gamma (\mathcal{P}^{(f)})^2 - \frac{16h^{5/2}}{\beta} (\mathcal{P}^{(h)})^2 \right] \\
 &\quad - \frac{i\beta}{4h^{5/2}} L^{(ik)} \phi_i \bar{\phi}_k - \frac{\beta}{32h^{3/2}} (M\bar{M} + \mu^2) - \frac{3\beta}{8 \cdot 64h^{7/2}} (\phi^i \phi_i) (\bar{\phi}^i \bar{\phi}_i) \\
 &\quad + \frac{1}{h} L^{ik} J_{ik} + \frac{i}{2h} (\phi_i \bar{Q}^i - \bar{\phi}_i Q^i).
 \end{aligned} \tag{6.6}$$

Since $J_{ik} = Q^i = \bar{Q}^i = 0$ on shell, the last line in the expression (6.6) can be suppressed. The auxiliary fields $M, \bar{M}$ and μ can be replaced by their on-shell expressions.

Finally, the internal symmetry current J_{ar} , (5.24), is expressed as

$$J_{ar} = f_a^i \mathcal{P}_{ir}^{(F)} - \gamma \hat{F}_r^i \mathcal{P}_{ia}^{(f)} + \frac{i}{2} h (\chi_r \bar{\chi}_a - \bar{\chi}_r \chi_a). \tag{6.7}$$

Now one can define the Poisson brackets, quantize them and find quantum expressions for the (super)charges. The wave function $|\Phi\rangle$ should satisfy the conditions

$$J_{ik} |\Phi\rangle = Q^i |\Phi\rangle = \bar{Q}^i |\Phi\rangle = \mathbb{H} |\Phi\rangle = 0 \tag{6.8}$$

and, after solving these equations, be expressed in terms of irreps of $Sp(n+1)$ (for $\gamma = -1$) or $Sp(1, n)$ (for $\gamma = 1$).

Due to the first-class constraints $J_{kl} \simeq 0$, $Q^i \simeq 0$, $\bar{Q}_i \simeq 0$, $\mathbb{H} \simeq 0$ the considered system, prior to quantization, should be exposed to an accurate Hamiltonian analysis, which for the bosonic sector will be performed in the next section. We will finish this section by presenting the set of Poisson brackets for the dynamical variables.

Bosonic brackets.

$$\{\hat{F}^{ir}, \mathcal{P}_{ks}^{(F)}\} = \delta_k^i \delta_s^r, \quad \{f^{ia}, \mathcal{P}_{kb}^{(f)}\} = \delta_k^i \delta_b^a, \quad \{h, \mathcal{P}^{(h)}\} = 1. \quad (6.9)$$

Fermionic brackets. To unambiguously define the brackets involving fermionic fields we need to apply to the Dirac method. As is seen from the expressions for the fermionic momenta (6.2), there is a set of second-class constraints

$$\begin{aligned} \varphi_r^{(\chi)} &:= \mathcal{P}_r^{(\chi)} + \frac{i}{4} h \bar{\chi}_r \simeq 0, & \bar{\varphi}_r^{(\chi)} &:= \mathcal{P}_r^{(\bar{\chi})} - \frac{i}{4} h \chi_r \simeq 0, \\ \varphi_a^{(\chi)} &:= \mathcal{P}_a^{(\chi)} - \frac{i}{4} \gamma h \bar{\chi}_a \simeq 0, & \bar{\varphi}_a^{(\chi)} &:= \mathcal{P}_a^{(\bar{\chi})} + \frac{i}{4} \gamma h \chi_a \simeq 0, \\ \varphi_i^{(\phi)} &:= \mathcal{P}_i^{(\phi)} - i\beta \frac{1}{32h^{3/2}} \bar{\phi}_i \simeq 0, & \bar{\varphi}_i^{(\phi)} &:= \mathcal{P}_i^{(\bar{\phi})} + i\beta \frac{1}{32h^{3/2}} \phi_i \simeq 0 \end{aligned} \quad (6.10)$$

with the following non-zero canonical brackets:

$$\{\varphi_r^{(\chi)}, \bar{\varphi}_s^{(\chi)}\} = \frac{i}{2} h \Omega_{rs}, \quad \{\varphi_a^{(\chi)}, \bar{\varphi}_b^{(\chi)}\} = -\frac{i}{2} \gamma h \varepsilon_{ab}, \quad \{\varphi_i^{(\phi)}, \bar{\varphi}_k^{(\phi)}\} = -i\beta \frac{1}{16h^{3/2}} \varepsilon_{ik}. \quad (6.11)$$

Then the standard Dirac procedure yields the following non-zero brackets involving the fermionic variables:

$$\begin{aligned} \{\chi_r, \bar{\chi}_s\}_D &= 2i \frac{1}{h} \Omega_{rs}, & \{\chi_a, \bar{\chi}_b\}_D &= -2i\gamma \frac{1}{h} \varepsilon_{ab}, & \{\phi_i, \bar{\phi}_k\}_D &= -16i \frac{h^{3/2}}{\beta} \varepsilon_{ik}, \\ \{\mathcal{P}^{(h)}, \chi_r\}_D &= \frac{1}{2h} \chi_r, & \{\mathcal{P}^{(h)}, \bar{\chi}_r\}_D &= \frac{1}{2h} \bar{\chi}_r, \\ \{\mathcal{P}^{(h)}, \chi_a\}_D &= \frac{1}{2h} \chi_a, & \{\mathcal{P}^{(h)}, \bar{\chi}_a\}_D &= \frac{1}{2h} \bar{\chi}_a, \\ \{\mathcal{P}^{(h)}, \phi_i\}_D &= -\frac{3}{4h} \phi_i, & \{\mathcal{P}^{(h)}, \bar{\phi}_i\}_D &= -\frac{3}{4h} \bar{\phi}_i. \end{aligned} \quad (6.13)$$

In what follows, we will omit the index “ D ” on these brackets. Using them, one can calculate the brackets between J_{kl} , Q^i , $\bar{Q}_i$, and $\mathbb{H}$ and convince oneself that they form a closed superalgebra. Here we present the brackets between the $Sp(n,1)(Sp(n+1))$ currents J_{ar} :

$$\{J_{ar}, J_{bs}\} = -2\gamma \left[\Omega_{rs} J_{(ab)} + \varepsilon_{ab} J_{(rs)} \right], \quad (6.14)$$

where

$$J_{(ab)} = f_{(a}^i \mathcal{P}_{ib)}^{(f)} - \frac{i}{2} \gamma h \chi_{(a} \bar{\chi}_{b)}, \quad J_{(rs)} = \hat{F}_{(r}^i \mathcal{P}_{is)}^{(F)} + \frac{i}{2} h \chi_{(r} \bar{\chi}_{s)} \quad (6.15)$$

are $Sp(1)$ and $Sp(n)$ currents. Note that the brackets of $\mathcal{P}^{(h)}$ with the internal symmetry currents J_{ar} , $J_{(ab)}$ and $J_{(rs)}$ are vanishing, as it should be. The same concerns the brackets with the Hamiltonian $\mathbb{H}$. It is rather easy to check that J_{ar} , $J_{(ab)}$ and $J_{(rs)}$ have zero bracket with the quadratic combination of the currents

$$C^{(2)} = \gamma J^{bs} J_{bs} - 2 \left[J^{(ab)} J_{(ab)} + J^{(rs)} J_{(rs)} \right], \quad (6.16)$$

which can thus be identified with the second-order Casimir of $Sp(n,1)$ (for $\gamma = 1$) or $Sp(n+1)$ (for $\gamma = -1$).

Actually, besides the dynamical variables, the full ungauged (super)charges involve the auxiliary fields, bosonic and fermionic, which have no kinetic terms in the Lagrangian and so have vanishing conjugate momenta. This produces new Hamiltonian constraints, which, for the time being, are difficult to analyze in a full generality.³ For this reason, in this paper we limit our consideration to the bosonic sector of the whole system.

Our goal is to weigh the way covariant quantization works. We did this by determining the global charges corresponding to the relevant symmetries, which vanish as a consequence of the equations of motion, as a short way to generate some of the constraints which will appear in the full constrained Hamiltonian approach. Subsequently we perform the full analysis of the bosonic sector which shows that our approach is correct.

7 Dirac analysis of the bosonic model

In what follows we will restrict our attention to the bosonic part (3.1):

$$\begin{aligned} \mathbb{L}_{HP}^b = & \frac{1}{2}h\left(\dot{\hat{F}}^{ir}\dot{\hat{F}}_{ir} - \gamma\dot{f}^{ia}\dot{f}_{ia}\right) + L_{ik}\left(\dot{\hat{F}}^{(ir}\dot{\hat{F}}_r^{k)} - \gamma\dot{f}^{(ia}\dot{f}_a^{k)}\right) \\ & + \frac{1}{4}D\left(\gamma\dot{f}^{ia}\dot{f}_{ia} - \dot{\hat{F}}^{ir}\dot{\hat{F}}_{ir} + \frac{\beta}{\sqrt{h}}\right) + \frac{\beta}{4}\frac{1}{\sqrt{h}h}\left(L^{ik}L_{ik} - \frac{1}{8}\dot{h}^2\right), \end{aligned} \quad (7.1)$$

where we ignored the fields $M, \bar{M}$ and μ which fully decouple in the absence of fermionic variables. We will perform the Dirac analysis of the above Lagrangian.

We prefer to deal with the “ungauged” Lagrangian (7.1) instead of the nonlinear sigma model Lagrangian (3.22), (3.23), because the former involves all the original fields (including the auxiliary ones) and the corresponding unfixed gauge invariances suggested by the original harmonic superspace formulation. We hope that this approach, being less nonlinear, will translate into a more acceptable quantum version, with less ordering problems. Moreover, it is rather non-trivial task to solve the constraints introduced by the field $D(t)$ for the more involved cases with non-vanishing QK potential, $\mathcal{L}^{+4} \neq 0$, including all inhomogeneous QK cases [12]. We hope that our “ungauged” approach could help to avoid (or at least relax) this problem as well. Anyway, to the best of our knowledge, nobody considered the Hamiltonian approach and the relevant quantization either for (7.1) or for (3.22), (3.23) before.

Digression: relativistic particle.

As a warmup, we replay the Dirac formalism in the application to the well known case of the massive relativistic particle:

$$L = -m\left(-(\dot{\mathbf{X}})^2\right)^{1/2}, \quad (\dot{\mathbf{X}})^2 = -(\dot{X}^0)^2 + (\vec{\dot{X}})^2. \quad (7.2)$$

We have:

$$\mathbb{P} = \frac{\partial L}{\partial \dot{\mathbf{X}}} = \frac{m\dot{\mathbf{X}}}{\sqrt{-\dot{\mathbf{X}}^2}}, \Rightarrow \mathbb{P}^2 = -(P_0)^2 + \vec{P}^2 = \frac{m^2(\dot{\mathbf{X}})^2}{-\dot{\mathbf{X}}^2} = -m^2, \quad (7.3)$$

³The Hamiltonian analysis of the simplest $\mathcal{N} = 1, 1D$ “supergravity” system, with the worldline multiplet consisting of a real “graviton” and one real “gravitino”, was accomplished in [19].

that is, we arrive at the primary constraint:

$$\Phi(t) = \mathbb{P}^2 + m^2 \simeq 0. \quad (7.4)$$

The corresponding canonical Hamiltonian vanishes and so (7.4) is the only first-class constraint. The Hamiltonian becomes:

$$\mathbb{H} = \frac{1}{2}\ell \left(\mathbb{P}^2 + m^2 \right). \quad (7.5)$$

We consider now the Hamiltonian form of the Lagrangian for the relativistic massive particle:

$$L = \dot{\mathbb{X}}\mathbb{P} - \frac{\ell}{2}(\mathbb{P}^2 + m^2). \quad (7.6)$$

This Lagrangian has a local symmetry:

$$\delta\mathbb{X} = \alpha(t)\mathbb{P}, \quad \delta\mathbb{P} = 0, \quad \delta\ell = \dot{\alpha}(t), \quad (7.7)$$

for which reason we naturally expect first-class constraints. Indeed, $\mathbb{X}$ and $\mathbb{P}$ are conjugate variables, while the momentum corresponding to the variable ℓ vanishes:

$$\Pi_\ell = \frac{\partial L}{\partial \dot{\ell}} = 0 \Rightarrow \Phi_\ell = \Pi_\ell \simeq 0, \quad (7.8)$$

so that $\Phi_\ell \simeq 0$ is a primary constraint. Introducing the Hamiltonian:

$$\mathbb{H} = \dot{\mathbb{X}}\mathbb{P} - L = \frac{\ell}{2}(\mathbb{P}^2 + m^2), \quad (7.9)$$

together with the relevant non-vanishing equal-time Poisson brackets,

$$\{\mathbb{X}, \mathbb{P}\} = 1, \quad \{\ell, \Pi_\ell\} = 1, \quad (7.10)$$

and then imposing the condition of conservation of the constraint Φ_ℓ ,

$$\dot{\Phi}_\ell = \{\Phi_\ell, \mathbb{H}\} = -\frac{1}{2}(\mathbb{P}^2 + m^2) \simeq 0, \quad (7.11)$$

we arrive at the mass-shell condition $(\mathbb{P}^2 + m^2) \simeq 0$ as a secondary first-class constraint. This is the complete set of constraints, and the quantum theory is obtained by replacing the non-vanishing Poisson brackets by the commutators:

$$[\mathbb{X}, \mathbb{P}] = i, \quad [\ell, \Pi_\ell] = i. \quad (7.12)$$

Thus $\mathbb{X}, \ell$ can be viewed as standard multiplication operators, while the corresponding momenta are $\mathbb{P} = \partial/(i\partial\mathbb{X})$, $\Pi_\ell = \partial/(i\partial\ell)$. These operators act on the space of wave functions $\Psi(\mathbb{X}, \ell)$, and the corresponding first-class constraints tell us that the wave function does not depend on ℓ and obeys the Klein Gordon equation:

$$(\mathbb{P}^2 + m^2)\Psi(\mathbb{X}) = 0. \quad (7.13)$$

Getting back to (7.6), we can eliminate $\mathbb{P}$:

$$\dot{\mathbb{X}} = \ell \mathbb{P}, \quad \Rightarrow \quad L = \frac{\dot{\mathbb{X}}^2}{2\ell} - \frac{\ell m^2}{2} \quad (7.14)$$

and, furthermore, eliminate ℓ :

$$-\frac{\dot{\mathbb{X}}^2}{2\ell^2} - \frac{m^2}{2} = 0, \quad \Rightarrow \quad L = -m \left(-(\dot{\mathbb{X}})^2 \right)^{1/2}. \quad (7.15)$$

Then it becomes clear that the Lagrangian in the Hamiltonian form (7.6) is more general than (7.2), as it also permits the case $m = 0$. For the massless particle the configuration space action involves an additional field ℓ . The auxiliary field ℓ is necessary to guarantee the gauge invariance:

$$\delta \mathbb{X} = \frac{\alpha(t)}{\ell} \dot{\mathbb{X}}, \quad \delta \ell = \dot{\alpha}(t). \quad (7.16)$$

This auxiliary field cannot be eliminated as this would lead to a vanishing action. Its role is to impose the gauge invariant mass shell constraint (this is nicely explained in [20]). One can then fix the gauge to obtain a simpler action. In what follows we will proceed for (7.1) in a similar fashion.

Back to (7.1).

A first glance at (7.1) tells us that it is not in the Hamiltonian form analogous to (7.6). Exploiting the Dirac method, we will arrive at the corresponding Hamiltonian form.

We start by defining the canonical momenta:

$$\begin{aligned} \mathcal{P}_{ir}^{(F)} &= \frac{\partial \mathbb{L}_{HP}}{\partial \dot{\hat{F}}_{ir}} = h \dot{\hat{F}}_{ir} - L_{ik} \hat{F}_r^k, & \mathcal{P}_{ia}^{(f)} &= \frac{\partial \mathbb{L}_{HP}}{\partial \dot{f}_{ia}} = -\gamma h \dot{f}_{ia} + \gamma L_{ik} f_a^k, \\ \mathcal{P}^{(h)} &= \frac{\partial \mathbb{L}_{HP}}{\partial \dot{h}} = -\frac{\beta}{16h^{3/2}} \dot{h}, & \mathcal{P}_{ik}^{(L)} &= \frac{\partial \mathbb{L}_{HP}}{\partial \dot{L}_{ik}} = 0, & \mathcal{P}^{(D)} &= \frac{\partial \mathbb{L}}{\partial \dot{D}} = 0. \end{aligned} \quad (7.17)$$

An inspection of the above formulas tells us that we have now two primary sets of commuting constraints:

$$\Phi_{ik}^{(L)} := \mathcal{P}_{ik}^{(L)} \simeq 0, \quad \Phi^{(D)} := \mathcal{P}^{(D)} \simeq 0. \quad (7.18)$$

Indeed, using the non-vanishing Poisson brackets:

$$\begin{aligned} \left\{ \hat{F}^{ir}, \mathcal{P}_{js}^{(F)} \right\} &= \delta_j^i \delta_s^r, & \left\{ f^{ia}, \mathcal{P}_{jb}^{(f)} \right\} &= \delta_j^i \delta_b^a, & \left\{ D, \mathcal{P}^{(D)} \right\} &= 1, \\ \left\{ L^{ik}, \mathcal{P}_{jl}^{(L)} \right\} &= \frac{1}{2} \left(\delta_j^i \delta_l^k + \delta_l^i \delta_j^k \right), & \text{and} & & \left\{ h, \mathcal{P}^{(h)} \right\} &= 1, \end{aligned} \quad (7.19)$$

we can calculate:

$$\left\{ \Phi_{ik}^{(L)}, \Phi^{(D)} \right\} = 0. \quad (7.20)$$

Introducing the Hamiltonian of our system as

$$\mathbb{H}' = \dot{\hat{F}}^{ir} \mathcal{P}_{ir}^{(F)} + \dot{f}^{ia} \mathcal{P}_{ia}^{(f)} + \dot{h} \mathcal{P}^{(h)} + [(\dot{L}^{ik} \mathcal{P}_{ik}^{(L)} + \dot{D} \mathcal{P}^{(D)}) = 0] - \mathbb{L}_{HP}^b,$$

we obtain:

$$\begin{aligned} \mathbb{H}' = & \frac{1}{2h} \left(\mathcal{P}_{ir}^{(F)} + L_{ik} \hat{F}_r^k \right) \left(\mathcal{P}^{(F)ir} - L^{ik'} \hat{F}_{k'}^r \right) - \frac{\gamma}{2h} \left(\mathcal{P}_{ia}^{(f)} - \gamma L_{ik} f_a^k \right) \left(\mathcal{P}^{(f)ia} + \gamma L^{ik'} f_{k'}^a \right) \\ & - \frac{8h^{3/2}}{\beta} (\mathcal{P}^{(h)})^2 - \frac{D}{4} \left(\gamma f^2 - \hat{F}^2 + \frac{\beta}{h^{3/2}} \right) - \frac{\beta}{4h^{3/2}} L^{ij} L_{ij}. \end{aligned} \quad (7.21)$$

Imposing the requirement of weak vanishing of the Poisson bracket of the primary constraints (7.18) with the Hamiltonian (7.21), we obtain two new sets of first-class constraints:

$$\Phi^{F-f} := \gamma f^2 - \hat{F}^2 + \frac{\beta}{h^{1/2}} \simeq 0, \quad \Phi_{ll'}^{F\mathcal{P}^{(F)}} := \hat{F}_{(l}^r \mathcal{P}_{l')r}^{(F)} + f_{(l}^a \mathcal{P}_{l')a}^{(f)} \simeq 0. \quad (7.22)$$

Using the definitions (7.17) in (7.21) and taking into account the constraints themselves, one gets:

$$\mathbb{H}' \simeq \mathbb{H} \Big|_{(M, \bar{M}, \mu, \phi_i, \bar{\phi}_k)=0}, \quad (7.23)$$

where $\mathbb{H}$ is given by (5.17). Using eq. (7.17) in $\Phi_{ll'}^{F\mathcal{P}^{(F)}}$ we get:

$$\Phi_{ik}^{F\mathcal{P}^{(F)}} = -J_{ik} \Big|_{(\phi_i, \bar{\phi}_j)=0} = h \hat{F}_{(i}^r \dot{\hat{F}}_{k)r} + \frac{1}{2} \hat{F}^2 L_{ik},$$

where J_{ik} is given by eq. (5.1). This means that the constraint $\Phi_{ll'}^{F\mathcal{P}^{(F)}}$ is just the $SU(2)$ - R charge discussed earlier.

Further, one calculates:

$$\begin{aligned} \left\{ \Phi^{F-f}, \Phi_{ll'}^{F\mathcal{P}^{(F)}} \right\} &= 0, \\ \left\{ \Phi_{ll'}^{F\mathcal{P}^{(F)}}, \Phi_{kk'}^{F\mathcal{P}^{(F)}} \right\} &= \frac{1}{2} \left(\epsilon_{lk} \Phi_{l'k'}^{F\mathcal{P}^{(F)}} + \epsilon_{l'k} \Phi_{lk'}^{F\mathcal{P}^{(F)}} + \epsilon_{lk'} \Phi_{l'k}^{F\mathcal{P}^{(F)}} + \epsilon_{l'k'} \Phi_{lk}^{F\mathcal{P}^{(F)}} \right). \end{aligned}$$

At this stage, we can use these constraints to simplify the Hamiltonian (7.21):

$$\begin{aligned} \mathbb{H}' = & \frac{1}{2h} \left[(\mathcal{P}^{(F)})^2 - \frac{1}{\gamma} (\mathcal{P}^{(f)})^2 - \frac{16h^{5/2}}{\beta} (\mathcal{P}^{(h)})^2 \right] - \frac{1}{4} \left(\frac{L^{ij} L_{ij}}{h} + D \right) \Phi^{F-f} \\ & - 2L^{ij} \Phi_{ij}^{F\mathcal{P}^{(F)}} \simeq H, \end{aligned} \quad (7.24)$$

with

$$H = \frac{1}{2h} \left[(\mathcal{P}^{(F)})^2 - \frac{1}{\gamma} (\mathcal{P}^{(f)})^2 - \frac{16h^{5/2}}{\beta} (\mathcal{P}^{(h)})^2 \right]. \quad (7.25)$$

Then, $\Phi_{ll'}^{F\mathcal{P}^{(F)}}$ commutes with H , while commuting Φ^{F-f} with H gives rise to the new constraint:

$$\Phi^{\mathcal{P}^{(f)}f} := \frac{1}{h} \left(\mathcal{P}^{(f)ia} f_{ia} + \mathcal{P}^{(F)ir} \hat{F}_{ir} \right) - 4\mathcal{P}^h \simeq 0, \quad (7.26)$$

Poisson bracket of which with H defined in (7.25) is equal to:

$$\left\{ \Phi^{\mathcal{P}^{(f)}f}, H \right\} \simeq -\frac{2}{h} H.$$

We also have:

$$\left\{ \Phi^{\mathcal{P}^{(f)}f}, \Phi_{ll'}^{F\mathcal{P}^{(f)}} \right\} = 0, \quad \left\{ \Phi^{\mathcal{P}^{(f)}f}, \Phi^{F-f} \right\} = -\frac{2}{h} \Phi^{F-f}.$$

Taking into account that the new constraint H is weakly equivalent to $\mathbb{H}'$, which in its turn is weakly equivalent to $\mathbb{H}|_{M, \bar{M}, \mu, \phi_i, \bar{\phi}_k=0}$, the properly restricted (5.17) is weakly equivalent to the new constraint H . It is thereby established that, in the present approach, the generators of global symmetries of our model correspond to some first-class constraints.

The algebra of the constraints $(\Phi^{F-f}, \Phi_{ll'}^{F\mathcal{P}^{(f)}}, \Phi^{\mathcal{P}^{(f)}f}, H)$, closes in the weak sense and the Hamiltonian finally becomes:

$$\mathcal{H} = H_0(t)H - A^{ll'}(t)\Phi_{ll'}^{F\mathcal{P}^{(f)}} + B(t)\Phi^{\mathcal{P}^{(f)}f} - D(t)\Phi^{F-f}.$$

Now we are able to write the action in the Hamiltonian form:

$$S = \int_0^{t_1} dt \left[\dot{\hat{F}}^{ir}(t) \mathcal{P}_{ir}^{(F)}(t) + \dot{f}^{ia}(t) \mathcal{P}_{ia}^{(f)}(t) + \dot{h}(t) \mathcal{P}^{(h)}(t) - \mathcal{H}(t) \right]. \quad (7.27)$$

Using the definitions (7.17) in (7.26), one can show that

$$\Phi^{\mathcal{P}^{(f)}f} = -\frac{1}{2} \frac{d}{dt} \Phi^{F-f}. \quad (7.28)$$

Therefore, integrating by parts in the action, one can absorb the constraint $\Phi^{\mathcal{P}^{(f)}f}$ into a redefinition of the Lagrange multiplier D . We will not further pursue this approach, because, when we deduce the corresponding gauge transformations, some of them may turn out singular on the surface of constraints. The action (7.27) should have the corresponding gauge invariances specified by the transformations which are generated by the first-class constraints incorporated in this action. In what follows we will spell the local transformations which should not be singular on the surface of constraints.

A. Transformations corresponding to the constraint H (7.25):

$$\begin{aligned} \delta \hat{F}^{ir} &= \frac{b(t)}{h} \mathcal{P}^{(F)ir}, & \delta f^{ia} &= -\frac{b(t)}{\gamma h} \mathcal{P}^{(f)ia}, & \delta h &= -b \frac{16h^{3/2}}{\beta} \mathcal{P}^{(h)}, \\ \delta \mathcal{P}^{(h)} &= \frac{b(t)}{h} H + \frac{20b(t)h^{1/2}}{\beta} (\mathcal{P}^{(h)})^2, & \delta H_0 &= \dot{b} + \frac{2b}{h} B, \\ \delta B &= -2bD - \frac{16bh^{1/2}}{\beta} \mathcal{P}^{(h)} B. \end{aligned} \quad (7.29)$$

B. Transformations corresponding to the constraint $\Phi_{ll'}^{F\mathcal{P}^{(f)}}$ in (7.22):

$$\begin{aligned} \delta \hat{F}^{ir} &= \tau^{il} \hat{F}_l^r, & \delta \mathcal{P}_{ir}^F &= \tau_i^l \mathcal{P}_{rl}^{(F)}, & \delta f^{ia} &= \tau^{il} f_l^a, & \delta \mathcal{P}^{(f)ia} &= \tau_i^l \mathcal{P}^{(f)}_{al}, \\ \delta A^{ll'} &= -\dot{\tau}^{ll'} - \tau_k^l A^{kl'} - \tau_k^{l'} A^{kl}. \end{aligned} \quad (7.30)$$

C. Next, transformations corresponding to the constraint Φ^{F-f} in (7.22):

$$\delta \mathcal{P}_{ir}^F = 2\alpha \hat{F}_{ir}, \quad \delta \mathcal{P}_{ia}^f = -2\gamma \alpha f_{ia}, \quad \delta \mathcal{P}^{(h)} = \frac{\beta \alpha}{2h^{3/2}}, \quad (7.31)$$

$$\delta D = -\dot{\alpha} - \frac{2\alpha}{h} B, \quad \delta B = 2\alpha H_0. \quad (7.32)$$

D. Finally, transformations corresponding to the constraint $\Phi^{\mathcal{P}^{(f)}f}$ in (7.26):

$$\begin{aligned}\delta \hat{F}^{ir} &= \frac{c}{h} \hat{F}^{ir}, & \delta \mathcal{P}_{ir}^{(F)} &= -\frac{c}{h} \mathcal{P}_{ir}^{(F)}, & \delta f^{ia} &= \frac{c}{h} f^{ia}, \\ \delta \mathcal{P}_{ia}^{(f)} &= -\frac{c}{h} \mathcal{P}_{ia}^{(f)}, & \delta h &= -4c, \\ \delta \mathcal{P}^{(h)} &= \frac{c}{h^2} (f^{ia} \mathcal{P}^{(f)} + \hat{F}^{ir} \mathcal{P}_{ir}^{(F)}), & \delta H_0 &= -\frac{2c}{h} H_0, \\ \delta B &= \dot{c} + \frac{16h^{1/2}c}{\beta} \mathcal{P}^{(h)} H_0, & \delta D &= -\frac{2c}{h} D.\end{aligned}$$

We thus have established that the action (7.27) is invariant under the local transformations listed above. We have shown that this action has six gauge invariances, while for the original action (7.1) we exhibited only four explicit gauge invariances, *viz.*, the local $SU(2)$, and time reparametrizations. An interesting question is as to whether the action (7.1) also exhibits the remaining two gauge invariances. However, it may happen that such additional gauge invariances are specific just to the Hamiltonian form of the action. As in the case of the Hamiltonian form of the action for the massive particle, we expect the presence of the relevant first-class constraints. Indeed, $\mathcal{P}_{ir}^{(F)}(t)$, $\mathcal{P}_{ia}^{(f)}(t)$, $\mathcal{P}^{(h)}(t)$ are variables canonically conjugated to $\hat{F}^{ir}(t)$, $f^{ia}(t)$, $h(t)$, while the momenta corresponding to H_0 , $A^{ll'}$, B , D , vanish, so that the standard Poisson brackets restore the previous four constraints as secondary first-class ones. The quantization of the above system is problematic as at the moment we are not aware of the general solution to the first-class constraints on the space of wave functions $\Psi(\hat{F}^{ir}, f^{ia}, h)$. Gauge fixing might simplify the above constraints, but we are interested to glimpse the covariant quantization.

It remains to prove the equivalence with the original action. To this end, like for the massive particle in (7.14), we substitute the corresponding momenta in (7.27) by their explicit expressions:

$$\begin{aligned}\mathcal{P}^{(F)ir} &= \frac{h}{H_0} \left(\dot{\hat{F}}^{ir} + A^{il} \hat{F}_l^r - \frac{B}{h} \hat{F}^{ri} \right), & \mathcal{P}^{(f)ia} &= -\frac{\gamma h}{H_0} \left(\dot{f}^{ia} + A^{il} f_l^a - \frac{B}{h} f^{ia} \right), \\ \mathcal{P}^{(h)} &= -\frac{\beta}{16H_0 h^{3/2}} \left(\dot{h} + 4B \right).\end{aligned}$$

After the redefinition $L^{ij} = hA^{ij}$, some rearrangements and integrations by parts we represent the Lagrangian in (7.27) as:

$$L = \frac{1}{H_0} \left[\mathbb{L}_{HP}^b - \frac{D}{4} \Phi^{F-f} \right] + \left[\frac{4H_0 + 1}{4H_0} D - \frac{1}{2} \frac{d}{dt} \left(\frac{B}{H_0} \right) - \frac{B^2}{2hH_0} - \frac{L^{ij} L_{ij}}{4hH_0} \right] \Phi^{F-f}.$$

If we eliminate H_0 from the above expression we obtain that the Lagrangian is weakly vanishing. This can be easily confirmed using the available constraints. Like in the case of the massless particle, it is a step which can be done but it does not appear to be useful. To obtain our original action it is in fact enough to fix the gauge with respect to the transformations (7.29) by the condition $H_0 = 1$. In order to be convinced that this gauge choice is permissible, we assume that $H_0 = 1 + \epsilon$ and, using (7.29), determine the

infinitesimal transformation parameter to be:

$$b = \left(\int^t dt'' \epsilon(t'') \exp \left\{ 2 \int^{t''} du \frac{B}{h}(u) \right\} \right) \exp \left\{ -2 \int^t dt' \frac{B}{h}(t') \right\}.$$

Finally, in the gauge $H_0 = 1$ we redefine the auxiliary field D as

$$\frac{D'}{4} = \frac{5D}{4} - \frac{1}{2} \frac{dB}{dt} - \frac{B^2}{2h} - \frac{L^{ij} L_{ij}}{4h}$$

and come back to the initial Lagrangian (7.1). We therefore conclude that the action (7.27) and the action corresponding to (7.1) are equivalent.

8 Concluding remarks

In this paper we continued the study of the new class of $\mathcal{N} = 4$ supersymmetric mechanics models introduced in [8], the Quaternion-Kähler (QK) ones. We limited our attention to their simplest representatives, with the $1D$ sigma models on the homogeneous projective manifolds $\mathbb{H}H^n$ or $\mathbb{H}P^n$ as the bosonic core. We started from the total off-shell component actions of these supersymmetric models, wrote down the local gauge transformations leaving these actions invariant, and explicitly presented the corresponding global invariances, together with the Noether currents associated with the latter. The full set of the equations of motion for different fields, involving both dynamical and algebraic equations, was accurately written down. The currents corresponding to global symmetries the gauging of which yields the total local symmetries of the action, were found to be vanishing on the shell of the equations of motion, while those related to the global isometries $Sp(n+1)$ (or $Sp(n,1)$) do not vanish. The vanishing of the first type of currents is quite analogous of the on-shell vanishing of Virasoro currents in bosonic string theory and/or the vanishing of $\mathcal{N} = 4$ supercurrents in the spinning particle coupled to a non-propagating $\mathcal{N} = 4, 1D$ supergravity (see, e.g., [21]).⁴ The vanishing of these currents is associated with the local worldline $\mathcal{N} = 4, 1D$ supersymmetry of the models considered and so this property should be equally valid for the most general $\mathcal{N} = 4$ QK mechanics model which also respects this local supersymmetry and the superfield action of which (as well as the bosonic component action) were given in [8]. For the same reason, the hamiltonian analysis of section 8 should also be directly applicable to the general case. We hope to address, from this point of view, examples of more general QK mechanics models (with the reduced isometry groups and $\mathcal{L}^{+4} \neq 0$) elsewhere. An important property of $\mathcal{N} = 4$ QK mechanics models is the possibility to add, to the sigma-model type action, the locally $\mathcal{N} = 4$ supersymmetric Wess-Zumino term. It would be interesting to see how the inclusion of such terms (even in the simplest $\mathbb{H}H^n$ or $\mathbb{H}P^n$ cases) will affect the analysis carried out in the present

⁴An important difference from the spinning particle is that the models of $\mathcal{N} = 4$ QK mechanics are coupled to an extended (“non-minimal”) worldline $\mathcal{N} = 4, 1D$ supergravity multiplet involving, besides the gauge fields for local $1D$ reparametrizations, local $\mathcal{N} = 4$ supersymmetry and local $SU(2)$ R-symmetry, also some auxiliary fields. One of them (the field D) is a Lagrange multiplier for the important bosonic constraint (see (3.1)) ensuring the correct number of physical bosonic fields in the theory.

paper. In [8] we conjectured the simplest harmonic superspace actions describing QKT (“Quaternion-Kähler with Torsion”) [22] $\mathcal{N} = 4$ mechanics models. It would be desirable to extend our analysis to this non-trivial case too.

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