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Open topological recursion relations in genus 1 and integrable systems

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Abstract: The paper is devoted to the open topological recursion relations in genus 1, which are partial differential equations that conjecturally control open Gromov-Witten invariants in genus 1. We find an explicit formula for any solution analogous to the Dijkgraaf-Witten formula for a descendent Gromov-Witten potential in genus 1. We then prove that at the approximation up to genus 1 the exponent of an open descendent potential satisfies a system of explicitly constructed linear evolutionary PDEs with one spatial variable.

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1 Introduction

Total descendent potentials, also called *formal Gromov-Witten potentials*, are certain formal power series of the form

$$\mathcal{F}(t^*_*,\varepsilon)=\sum_{g\geq 0}\varepsilon^{2g}\mathcal{F}_g(t^*_*)\in\mathbb{C}[[t^*_*,\varepsilon]],$$

where $N\geq 1$, and t^α_a, $1\leq\alpha\leq N$, $a\geq 0$, and ε are formal variables, appearing in various curve counting theories in algebraic geometry including Gromov-Witten theory, Fan-Jarvis-Ruan-Witten theory, and the more recent theory of Gauged Linear Sigma Models. The number N is often called the *rank*. Typically, the coefficients of total descendent potentials are the integrals of certain cohomology classes over moduli spaces of closed Riemann surfaces with additional structures. The function $\mathcal{F}_g$ controls the integrals over the moduli spaces of Riemann surfaces of genus g. The simplest example of a total descendent potential is the Witten generating series $\mathcal{F}^W(t_0,t_1,t_2,\ldots,\varepsilon)$ of intersection numbers on the moduli space of stable Riemann surfaces of genus g with n marked points $\overline{\mathcal{M}}_{g,n}$. Note that here and below we omit the upper indices in the t-variables when the rank is 1.

There is a unified approach to total descendent potentials using the notion of a cohomological field theory (CohFT) [24] and the Givental group action [20–22]. Briefly speaking, the generating series of correlators of CohFTs form the space of *total ancestor potentials*, and then using the lower-triangular Givental group action one gets the whole space of total descendent potentials (see e.g. [28, section 2] and [19]).

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There is a remarkable and deep relation between total descendent potentials and the theory of nonlinear PDEs. One of its manifestations is the following system of PDEs for the descendent potential in genus 0 (see e.g. [28, section 2] and [19, Corollary 4.13]):

$$\frac{\partial \mathcal{F}_0}{\partial t^1_0} = \sum_{a\geq 0} t^\alpha_{a+1}\frac{\partial \mathcal{F}_0}{\partial t^\alpha_a} + \frac{1}{2}\eta_{\alpha\beta}t^\alpha_0 t^\beta_0, \tag{1.1}$$

$$\frac{\partial^3 \mathcal{F}_0}{\partial t^\alpha_{a+1}\partial t^\beta_b \partial t^\gamma_c} = \frac{\partial^2 \mathcal{F}_0}{\partial t^\alpha_a \partial t^\mu_0}\eta^{\mu\nu}\frac{\partial^3 \mathcal{F}_0}{\partial t^\nu_0 \partial t^\beta_b \partial t^\gamma_c}, \qquad 1\leq \alpha,\beta,\gamma\leq N, \qquad a,b,c\geq 0, \tag{1.2}$$

called the *string equation* and the *topological recursion relations in genus* 0, respectively. Here $(\eta_{\alpha\beta}) = \eta$ is an $N\times N$ symmetric nondegenerate matrix with complex coefficients, the constants $\eta^{\alpha\beta}$ are defined by $(\eta^{\alpha\beta}) := \eta^{-1}$, and we use the Einstein summation convention for repeated upper and lower Greek indices. Note that the system of equations (1.2) can be equivalently written as

$$d\left(\frac{\partial^2 \mathcal{F}_0}{\partial t^\alpha_{a+1}\partial t^\beta_b}\right) = \frac{\partial^2 \mathcal{F}_0}{\partial t^\alpha_a \partial t^\mu_0}\eta^{\mu\nu} d\left(\frac{\partial^2 \mathcal{F}_0}{\partial t^\nu_0 \partial t^\beta_b}\right), \quad 1\leq \alpha,\beta\leq N, \quad a,b\geq 0,$$

where $d(\cdot)$ denotes the full differential.

There are equations similar to (1.2) in genus 1 (see e.g. [18, equation (1.7)]):

$$\frac{\partial \mathcal{F}_1}{\partial t^\alpha_{a+1}} = \frac{\partial^2 \mathcal{F}_0}{\partial t^\alpha_a \partial t^\mu_0}\eta^{\mu\nu}\frac{\partial \mathcal{F}_1}{\partial t^\nu_0} + \frac{1}{24}\eta^{\mu\nu}\frac{\partial^3 \mathcal{F}_0}{\partial t^\mu_0 \partial t^\nu_0 \partial t^\alpha_a}, \quad 1\leq \alpha\leq N, \quad a\geq 0. \tag{1.3}$$

They are called the *topological recursion relations in genus* 1. These equations imply that

$$\mathcal{F}_1 = \frac{1}{24}\log\det(\eta^{-1}M) + G(v^1,\dots,v^N), \tag{1.4}$$

where the $N\times N$ matrix $M = (M_{\alpha\beta})$ is defined by $M_{\alpha\beta} := \frac{\partial^3 \mathcal{F}_0}{\partial t^1_0 \partial t^\alpha_0 \partial t^\beta_0}$, $v^\alpha := \eta^{\alpha\mu}\frac{\partial^2 \mathcal{F}_0}{\partial t^\mu_0 \partial t^1_0}$, and $G(t^1,\dots,t^N) := \mathcal{F}_1|_{t^*_{\geq 1}=0}$ [15] (see also [16, equation (1.16)]). Equations similar to (1.2) and (1.3) exist in all genera, but their complexity grow very rapidly with the genus (see e.g. [25] for some results in genus 2).

One can see that equations (1.1), (1.2), (1.3) are universal, meaning that they do not depend on a total descendent potential. On the other hand, there is a rich theory [17] of hierarchies of evolutionary PDEs with one spatial variable associated to total descendent potentials and containing the full information about these potentials. Conjecturally, for any total descendent potential $\mathcal{F}$ there exists a unique system of PDEs of the form

$$\frac{\partial w^\alpha}{\partial t^\beta_b} = P^\alpha_{\beta,b}, \quad 1\leq \alpha,\beta\leq N, \quad b\geq 0, \tag{1.5}$$

where $w^1,\dots,w^N \in \mathbb{C}[[t^*_*,\varepsilon]]$, $P^\alpha_{\beta,b}$ are differential polynomials in $w^1,\dots,w^N$, i.e., $P^\alpha_{\beta,b}$ are formal power series in ε with the coefficients that a polynomials in $w^\gamma_x, w^\gamma_{xx},\dots$ (we identify $x = t^1_0$) whose coefficients are formal power series in w^γ, such that a unique solution of the system (1.5) specified by the condition $w^\alpha|_{t^\beta_b=\delta^{\beta,1}\delta_{b,0}x} = \delta^{\alpha,1}x$ is given by $w^\alpha = \eta^{\alpha\mu}\frac{\partial^2\mathcal{F}}{\partial t^\mu_0\partial t^1_0}$.

This system of PDEs (if it exists) is called the *Dubrovin-Zhang hierarchy* or the *hierarchy of topological type*. The conjecture is proved at the approximation up to ε^2 [16] and in the case when the Dubrovin-Frobenius manifold associated to the total descendent potential is semisimple [8, 9]. The Dubrovin-Zhang hierarchy corresponding to the Witten potential $\mathcal{F}^W$ is the Korteweg-de Vries (KdV) hierarchy

$$\begin{aligned}
\frac{\partial w}{\partial t_1} &= ww_x + \frac{\varepsilon^2}{12}w_{xxx},\\
\frac{\partial w}{\partial t_2} &= \frac{w^2 w_x}{2} + \varepsilon^2\left(\frac{ww_{xxx}}{12} + \frac{w_x w_{xx}}{6}\right) + \varepsilon^4\frac{w_{xxxxx}}{240},\\
&\vdots
\end{aligned}$$

This statement is equivalent to Witten's conjecture [31], proved by Kontsevich [23].

A more recent and less developed field of research is the study of the intersection theory on various moduli spaces of Riemann surfaces with boundary. Such a moduli space always comes with an associated moduli space of closed Riemann surfaces, and, thus, there is the corresponding total descendent potential $\mathcal{F}(t^*_*,\varepsilon) = \sum_{g\ge 0}\varepsilon^{2g}\mathcal{F}_g(t^*_*)$ of some rank N. There is a large class of examples [7, 12–14, 26, 29, 32] where the intersection numbers on the corresponding moduli space of Riemann surfaces with boundary of genus 0 are described by a formal power series $\mathcal{F}^o_0(t^*_*,s_*)\in\mathbb{C}[[t^*_*,s_*]]$ depending on an additional sequence of formal variable s_a, $a\ge 0$, and satisfying the relations

$$\frac{\partial\mathcal{F}^o_0}{\partial t^1_0} = \sum_{a\ge 0} t^\alpha_{a+1}\frac{\partial\mathcal{F}^o_0}{\partial t^\alpha_a} + \sum_{a\ge 0} s_{a+1}\frac{\partial\mathcal{F}^o_0}{\partial s_a} + s_0, \tag{1.6}$$

$$d\left(\frac{\partial\mathcal{F}^o_0}{\partial t^\alpha_{a+1}}\right) = \frac{\partial^2\mathcal{F}_0}{\partial t^\alpha_a\partial t^\mu_0}\eta^{\mu\nu}d\left(\frac{\partial\mathcal{F}^o_0}{\partial t^\nu_0}\right) + \frac{\partial\mathcal{F}^o_0}{\partial t^\alpha_a}d\left(\frac{\partial\mathcal{F}^o_0}{\partial s_0}\right),\qquad 1\le\alpha\le N,\qquad a\ge 0, \tag{1.7}$$

$$d\left(\frac{\partial\mathcal{F}^o_0}{\partial s_{a+1}}\right) = \frac{\partial\mathcal{F}^o_0}{\partial s_a}d\left(\frac{\partial\mathcal{F}^o_0}{\partial s_0}\right),\qquad a\ge 0. \tag{1.8}$$

Equation (1.6) is called the *open string equation*. Equations (1.7) and (1.8) are called the *open topological recursion relations in genus* 0. The function $\mathcal{F}^o_0$ is called the *open descendent potential in genus* 0.

Remark 1.1. The system of PDEs (1.6)–(1.8) implies that the function $\mathcal{F}^o_0|_{t^*_{\ge 1}=s_{\ge 1}=0}$ satisfies the open WDVV equations (see [6, section 4]), which actually appear in some of the papers mentioned above. However, in [3] the authors presented a construction of an open descendent potential starting from an arbitrary solution of the open WDVV equations.

Regarding higher genera, much less is known. However, conjecturally, the intersection theory on moduli spaces of Riemann surfaces with boundary of genus 1 is controlled by formal power series $\mathcal{F}^o_1(t^*_*,s_*)\in\mathbb{C}[[t^*_*,s_*]]$ satisfying the relations

$$\begin{aligned}
\frac{\partial\mathcal{F}^o_1}{\partial t^\alpha_{a+1}} &= \frac{\partial^2\mathcal{F}_0}{\partial t^\alpha_a\partial t^\mu_0}\eta^{\mu\nu}\frac{\partial\mathcal{F}^o_1}{\partial t^\nu_0} + \frac{\partial\mathcal{F}^o_0}{\partial t^\alpha_a}\frac{\partial\mathcal{F}^o_1}{\partial s_0} + \frac{1}{2}\frac{\partial^2\mathcal{F}^o_0}{\partial t^\alpha_a\partial s_0}, && 1\le\alpha\le N, \qquad a\ge 0,\\
\frac{\partial\mathcal{F}^o_1}{\partial s_{a+1}} &= \frac{\partial\mathcal{F}^o_0}{\partial s_a}\frac{\partial\mathcal{F}^o_1}{\partial s_0} + \frac{1}{2}\frac{\partial^2\mathcal{F}^o_0}{\partial s_a\partial s_0}, && a\ge 0,
\end{aligned}$$

called the *open topological recursion relations in genus* 1. In the case of the intersection theory on the moduli spaces of Riemann surfaces with boundary of genus g with k boundary marked points and l internal marked points $\overline{\mathcal{M}}_{g,k,l}$, these relations were conjectured by the authors of [26] and proved in [7, section 6.2.3] (a proof by other methods is obtained by J. P. Solomon and R. J. Tessler in a work in preparation). An evidence that the open topological recursion relations in genus 1 hold for the open r-spin theory is also given in [7, section 6.2.3].

An analog of the theory of Dubrovin-Zhang hierarchies for solutions of the system (1.6)–(1.8) was developed in [3]. Regarding higher genera, a very promising direction was opened by the series of papers [4, 5, 10, 26, 30] (see also [1]), where the authors studied the intersection numbers on the moduli spaces of Riemann surfaces with boundary of genus g with k boundary marked points and l internal marked points $\overline{\mathcal{M}}_{g,k,l}$. The main result of these works is the proof [10] of the Pandharipande-Solomon-Tessler conjecture [26] saying that the generating series

$$\mathcal{F}^{o,PST}(t_*, s_*, \varepsilon) = \sum_{g\geq 0} \varepsilon^g \mathcal{F}^{o,PST}_g(t_*, s_*) \in \mathbb{C}[[t_*, s_*, \varepsilon]]$$

of the intersection numbers satisfies the following system of PDEs:

$$\frac{\partial}{\partial t_p}\exp(\varepsilon^{-1}\mathcal{F}^{o,PST}) = \frac{\varepsilon^{-1}}{(2p+1)!!}\left(L^{p+\frac{1}{2}}\right)_+ \exp(\varepsilon^{-1}\mathcal{F}^{o,PST}), \qquad p \geq 0, \tag{1.9}$$

$$\frac{\partial}{\partial s_p}\exp(\varepsilon^{-1}\mathcal{F}^{o,PST}) = \frac{\varepsilon^{-1}}{2^{p+1}(p+1)!}L^{p+1} \exp(\varepsilon^{-1}\mathcal{F}^{o,PST}), \qquad p \geq 0, \tag{1.10}$$

where $L = (\varepsilon\partial_x)^2 + 2w$ is the Lax operator for the KdV hierarchy, and $w = \frac{\partial^2 \mathcal{F}^W}{\partial t_0^2}$.

Remark 1.2. To be precise, we have presented a version of the Pandharipande-Solomon-Tessler conjecture, which is slightly different from the original one in two aspects. First of all, in [26] the authors considered a function $\widetilde{\mathcal{F}}^{o,PST}$ related to our function $\mathcal{F}^{o,PST}$ by $\widetilde{\mathcal{F}}^{o,PST} = \mathcal{F}^{o,PST}\big|_{s_{\geq 1}=0}$. The function $\mathcal{F}^{o,PST}$ can be reconstructed from the function $\widetilde{\mathcal{F}}^{o,PST}$ using the system of PDEs

$$\frac{\partial}{\partial s_p}\exp(\varepsilon^{-1}\mathcal{F}^{o,PST}) = \frac{\varepsilon^p}{(p+1)!}\frac{\partial^{p+1}}{\partial s_0^{p+1}}\exp(\varepsilon^{-1}\mathcal{F}^{o,PST}), \quad p \geq 1.$$

Second, the system of PDEs from [26, Conjecture 2] determining the function $\widetilde{\mathcal{F}}^{o,PST}$ does not have the form of a system of evolutionary PDEs with one spatial variable. The fact that the presented version of the Pandharipande-Solomon-Tessler conjecture is equivalent to the original one was observed in [5].

In this paper we study solutions of the open topological recursion relations in genus 1. First, we find an analog of formula (1.4). Then, using this formula, we construct a system of linear PDEs of the form similar to (1.9) and (1.10) such that the function $\exp(\mathcal{F}^o_0 + \varepsilon\mathcal{F}^o_1)$ satisfies it at the approximation up to ε. An expectation in higher genera and a relation with a Lax description of the Dubrovin-Zhang hierarchies are also discussed.

2 Closed and open descendent potentials in genus 0

In this section we recall the definitions of closed and open descendent potentials in genus 0 and the construction of associated to them systems of PDEs.

2.1 Differential polynomials

Consider formal variables v_i^α, $\alpha = 1, \dots, N$, $i = 0, 1, \dots$. Following [17] (see also [27]) we define the ring of *differential polynomials* $\mathcal{A}_{v^1,\dots,v^N}$ in the variables $v^1, \dots, v^N$ as the ring of polynomials in the variables v_i^α, $i > 0$, with coefficients in the ring of formal power series in the variables $v^\alpha := v_0^\alpha$:

$$\mathcal{A}_{v^1,\dots,v^N} := \mathbb{C}[[v^*]][v^*_{\geq 1}].$$

Remark 2.1. It is useful to think of the variables $v^\alpha = v_0^\alpha$ as the components $v^\alpha(x)$ of a formal loop $v\colon S^1 \to \mathbb{C}^N$ in the standard basis of $\mathbb{C}^N$. Then the variables $v_1^\alpha := v_x^\alpha, v_2^\alpha := v_{xx}^\alpha, \dots$ are the components of the iterated x-derivatives of the formal loop.

The *standard gradation* on $\mathcal{A}_{v^1,\dots,v^N}$, which we denote by deg, is introduced by $\deg v_i^\alpha := i$. The homogeneous component of $\mathcal{A}_{v^1,\dots,v^N}$ of standard degree d is denoted by $\mathcal{A}_{v^1,\dots,v^N;d}$. Introduce an operator $\partial_x \colon \mathcal{A}_{v^1,\dots,v^N} \to \mathcal{A}_{v^1,\dots,v^N}$ by

$$\partial_x := \sum_{i \geq 0} v_{i+1}^\alpha \frac{\partial}{\partial v_i^\alpha}.$$

It increases the standard degree by 1.

Consider the extension $\widehat{\mathcal{A}}_{v^1,\dots,v^N} := \mathcal{A}_{v^1,\dots,v^N}[[\varepsilon]]$ of the space $\mathcal{A}_{v^1,\dots,v^N}$ with a new variable ε of standard degree $\deg \varepsilon := -1$. Let $\widehat{\mathcal{A}}_{v^1,\dots,v^N;d}$ denote the subspace of degree d of $\widehat{\mathcal{A}}$. Abusing the terminology we still call elements of the space $\widehat{\mathcal{A}}_{v^1,\dots,v^N}$ *differential polynomials.*

2.2 Closed descendent potentials in genus 0

Let us fix $N \geq 1$, an $N \times N$ symmetric nondegenerate complex matrix $\eta = (\eta_{\alpha\beta})$, and an N-tuple of complex numbers $(A^1, \dots, A^N)$, not all equal to zero. We will use the notation

$$\frac{\partial}{\partial t_a^{1\!\!1}} := A^\alpha \frac{\partial}{\partial t_a^\alpha}, \quad a \geq 0.$$

Definition 2.2. A formal power series $\mathcal{F}_0 \in \mathbb{C}[[t_*^*]]$ is called a *descendent potential in genus* 0 if it satisfies the following system of PDEs:

$$\sum_{a \geq 0} t_{a+1}^\alpha \frac{\partial \mathcal{F}_0}{\partial t_a^\alpha} - \frac{\partial \mathcal{F}_0}{\partial t_0^{1\!\!1}} = -\frac{1}{2}\eta_{\alpha\beta} t_0^\alpha t_0^\beta, \tag{2.1}$$

$$\sum_{a \geq 0} t_a^\alpha \frac{\partial \mathcal{F}_0}{\partial t_a^\alpha} - \frac{\partial \mathcal{F}_0}{\partial t_1^{1\!\!1}} = 2\mathcal{F}_0, \tag{2.2}$$

$$\frac{\partial^3 \mathcal{F}_0}{\partial t_{a+1}^\alpha \partial t_b^\beta \partial t_c^\gamma} = \frac{\partial^2 \mathcal{F}_0}{\partial t_a^\alpha \partial t_0^\mu} \eta^{\mu\nu} \frac{\partial^3 \mathcal{F}_0}{\partial t_0^\nu \partial t_b^\beta \partial t_c^\gamma}, \qquad 1 \leq \alpha, \beta, \gamma \leq N, \qquad a, b, c \geq 0, \tag{2.3}$$

$$\frac{\partial^2 \mathcal{F}_0}{\partial t_{a+1}^\alpha \partial t_b^\beta} + \frac{\partial^2 \mathcal{F}_0}{\partial t_a^\alpha \partial t_{b+1}^\beta} = \frac{\partial^2 \mathcal{F}_0}{\partial t_a^\alpha \partial t_0^\mu} \eta^{\mu\nu} \frac{\partial^2 \mathcal{F}_0}{\partial t_0^\nu \partial t_b^\beta}, \qquad 1 \leq \alpha, \beta \leq N, \qquad a, b \geq 0.$$

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We will sometimes call a descendent potential in genus 0 a *closed* descendent potential in genus 0 in order to distinguish it from an open analog that we will discuss below.

Remark 2.3. Doing a linear change of variables, one can make $\frac{\partial}{\partial t^{1\!\!1}_a} = \frac{\partial}{\partial t^1_a} \Leftrightarrow A^\alpha = \delta^{\alpha,1}$ in equations (2.1) and (2.2). That is why authors often assume that $A^\alpha = \delta^{\alpha,1}$.

Remark 2.4. For any total descendent potential $\mathcal{F} = \sum_{g\ge 0} \varepsilon^{2g}\mathcal{F}_g$ the function $\mathcal{F}_0$ is a descendent potential in genus 0. However, describing precisely which descendent potentials in genus 0 can be extended to total descendent potentials is an interesting open problem.

Define differential polynomials $\Omega^{[0]}_{\alpha,a;\beta,b} \in \mathcal{A}_{v^1,\dots,v^N;0}$, $1 \le \alpha,\beta \le N$, $a,b \ge 0$, by

$$\Omega^{[0]}_{\alpha,a;\beta,b} := \left.\frac{\partial^2 \mathcal{F}_0}{\partial t^\alpha_a \partial t^\beta_b}\right|_{t^\gamma_c = \delta_{c,0} v^\gamma},$$

and let

$$(v^{\mathrm{top}})^\alpha := \eta^{\alpha\mu} \frac{\partial^2 \mathcal{F}_0}{\partial t^\mu_0 \partial t^{1\!\!1}_0} \in \mathbb{C}[[t^*_*]], \quad 1 \le \alpha \le N.$$

Then we have (see e.g. [9, Proposition 3])

$$\frac{\partial^2 \mathcal{F}_0}{\partial t^\alpha_a \partial t^\beta_b} = \left.\Omega^{[0]}_{\alpha,a;\beta,b}\right|_{v^\gamma = (v^{\mathrm{top}})^\gamma}. \tag{2.4}$$

This implies that the N-tuple of functions $(v^{\mathrm{top}})^\alpha|_{t^\gamma_0 \mapsto t^\gamma_0 + A^\gamma x}$ is a solution of the following system of PDEs:

$$\frac{\partial v^\alpha}{\partial t^\beta_b} = \eta^{\alpha\mu} \partial_x \Omega^{[0]}_{\mu,0;\beta,b}, \quad 1 \le \alpha,\beta \le N, \quad b \ge 0,$$

which is called the *principal hierarchy* associated to the potential $\mathcal{F}_0$.

2.3 Open descendent potentials in genus 0

Let us fix a closed descendent potential in genus 0 $\mathcal{F}_0$.

Definition 2.5. An *open descendent potential in genus* 0 $\mathcal{F}^o_0 \in \mathbb{C}[[t^*_*, s_*]]$ is a solution of the following system of PDEs:

$$\sum_{b\ge 0} t^\beta_{b+1} \frac{\partial \mathcal{F}^o_0}{\partial t^\beta_b} + \sum_{a \ge 0} s_{a+1} \frac{\partial \mathcal{F}^o_0}{\partial s_a} - \frac{\partial \mathcal{F}^o_0}{\partial t^{1\!\!1}_0} = -s_0, \tag{2.5}$$

$$\sum_{b\ge 0} t^\beta_b \frac{\partial \mathcal{F}^o_0}{\partial t^\beta_b} + \sum_{a \ge 0} s_a \frac{\partial \mathcal{F}^o_0}{\partial s_a} - \frac{\partial \mathcal{F}^o_0}{\partial t^{1\!\!1}_1} = \mathcal{F}^o_0,$$

$$d\left(\frac{\partial \mathcal{F}^o_0}{\partial t^\alpha_{p+1}}\right) = \frac{\partial^2 \mathcal{F}_0}{\partial t^\alpha_p \partial t^\mu_0} \eta^{\mu\nu} d\left(\frac{\partial \mathcal{F}^o_0}{\partial t^\nu_0}\right) + \frac{\partial \mathcal{F}^o_0}{\partial t^\alpha_p} d\left(\frac{\partial \mathcal{F}^o_0}{\partial s_0}\right), \tag{2.6}$$

$$d\left(\frac{\partial \mathcal{F}^o_0}{\partial s_{p+1}}\right) = \frac{\partial \mathcal{F}^o_0}{\partial s_p} d\left(\frac{\partial \mathcal{F}^o_0}{\partial s_0}\right). \tag{2.7}$$

Consider a new formal variable ϕ. Similarly to the differential polynomials $\Omega^{[0]}_{\alpha,a;\beta,b}$, let us introduce differential polynomials $\Gamma^{[0]}_{\alpha,a}, \Delta^{[0]}_a \in \mathcal{A}_{v^1,\dots,v^N,\phi;0}$, $1 \le \alpha \le N$, $a \ge 0$, by

$$\Gamma^{[0]}_{\alpha,a} := \left.\frac{\partial \mathcal{F}^o_0}{\partial t^\alpha_a}\right|_{\substack{t^\gamma_c=\delta_{c,0}v^\gamma \\ s_c=\delta_{c,0}\phi}}, \qquad \Delta^{[0]}_a := \left.\frac{\partial \mathcal{F}^o_0}{\partial s_a}\right|_{\substack{t^\gamma_c=\delta_{c,0}v^\gamma \\ s_c=\delta_{c,0}\phi}},$$

and let

$$\phi^{\mathrm{top}} := \frac{\partial \mathcal{F}^o_0}{\partial t^1_0} \in \mathbb{C}[[t^*_*, s_*]].$$

We have the following properties, analogous to the property (2.4) ([3, section 4.4], [2, Proposition 2.2]):

$$\frac{\partial \mathcal{F}^o_0}{\partial t^\alpha_a} = \Gamma^{[0]}_{\alpha,a}\Big|_{\substack{v^\gamma=(v^{\mathrm{top}})^\gamma \\ \phi=\phi^{\mathrm{top}}}}, \qquad \frac{\partial \mathcal{F}^o_0}{\partial s_a} = \Delta^{[0]}_a\Big|_{\substack{v^\gamma=(v^{\mathrm{top}})^\gamma \\ \phi=\phi^{\mathrm{top}}}}.$$

This implies that the $(N+1)$-tuple of functions $\left((v^{\mathrm{top}})^1,\dots,(v^{\mathrm{top}})^N,\phi^{\mathrm{top}}\right)\Big|_{t^\gamma_0\mapsto t^\gamma_0+A^\gamma x}$ satisfies the following system of PDEs:

$$\begin{aligned} \frac{\partial v^\alpha}{\partial t^\beta_b} &= \partial_x \eta^{\alpha\mu}\Omega^{[0]}_{\mu,0;\beta,b}, & \frac{\partial v^\alpha}{\partial s_b} &= 0,\\ \frac{\partial \phi}{\partial t^\beta_b} &= \partial_x \Gamma^{[0]}_{\beta,b}, & \frac{\partial \phi}{\partial s_b} &= \partial_x \Delta^{[0]}_b, \end{aligned}$$

which we call the *extended principal hierarchy* associated to the pair of potentials $(\mathcal{F}_0, \mathcal{F}^o_0)$.

3 Open descendent potentials in genus 1

Here we introduce the notion of an open descendent potential in genus 1 and prove two main results of our paper: theorems 3.2 and 3.5.

3.1 Open descendent potentials in genus 1

Let us fix a pair $(\mathcal{F}_0, \mathcal{F}^o_0)$ of closed and open potentials in genus 0.

Definition 3.1. An *open descendent potential in genus* 1 $\mathcal{F}^o_1 \in \mathbb{C}[[t^*_*, s_*]]$ is a solution of the following system of PDEs:

$$\frac{\partial \mathcal{F}^o_1}{\partial t^\alpha_{a+1}} = \frac{\partial^2 \mathcal{F}_0}{\partial t^\alpha_a \partial t^\mu_0}\eta^{\mu\nu}\frac{\partial \mathcal{F}^o_1}{\partial t^\nu_0} + \frac{\partial \mathcal{F}^o_0}{\partial t^\alpha_a}\frac{\partial \mathcal{F}^o_1}{\partial s_0} + \frac{1}{2}\frac{\partial^2 \mathcal{F}^o_0}{\partial t^\alpha_a \partial s_0}, \qquad 1 \le \alpha \le N, \qquad a \ge 0, \tag{3.1}$$

$$\frac{\partial \mathcal{F}^o_1}{\partial s_{a+1}} = \frac{\partial \mathcal{F}^o_0}{\partial s_a}\frac{\partial \mathcal{F}^o_1}{\partial s_0} + \frac{1}{2}\frac{\partial^2 \mathcal{F}^o_0}{\partial s_a \partial s_0}, \qquad a \ge 0. \tag{3.2}$$

Consider an open descendent potential in genus 1 $\mathcal{F}^o_1$. Define a formal power series $G^o \in \mathbb{C}[[v^*, \phi]]$ by

$$G^o := \mathcal{F}^o_1\big|_{\substack{t^\alpha_a=\delta_{a,0}v^\alpha \\ s_a=\delta_{a,0}\phi}}.$$

Theorem 3.2. *We have*

$$\mathcal{F}_1^o = \frac{1}{2}\log\frac{\partial^2\mathcal{F}_0^o}{\partial t_0^{1\!\!1}\partial s_0} + G^o|_{\substack{v^\gamma=(v^{\text{top}})^\gamma\\ \phi=\phi^{\text{top}}}}\,. \tag{3.3}$$

Proof. Note that equation (2.5) implies that

$$\left.\frac{\partial^2\mathcal{F}_0^o}{\partial t_0^{1\!\!1}\partial s_0}\right|_{t^*_{\ge 1}=s_{\ge 1}=0} = 1.$$

Therefore, the logarithm $\log\frac{\partial^2\mathcal{F}_0^o}{\partial t_0^{1\!\!1}\partial s_0}$ is a well-defined formal power series in the variables t^*_* and s_*. Also, equations (2.1) and (2.5) imply that

$$\left.(v^{\text{top}})^\alpha\right|_{t^*_{\ge 1}=0} = t_0^\alpha, \qquad \left.\phi^{\text{top}}\right|_{t^*_{\ge 1}=s_{\ge 1}=0} = s_0.$$

Therefore, equation (3.3) is true when $t^*_{\ge 1} = s_{\ge 1} = 0$.

Using the linear differential operators

$$\begin{aligned}
P^1_{\alpha,a} &:= \frac{\partial}{\partial t^\alpha_{a+1}} - \frac{\partial^2\mathcal{F}_0}{\partial t^\alpha_a\partial t^\mu_0}\eta^{\mu\nu}\frac{\partial}{\partial t^\nu_0} - \frac{\partial\mathcal{F}_0^o}{\partial t^\alpha_a}\frac{\partial}{\partial s_0}, && 1\le\alpha\le N, && a\ge 0,\\
P^2_a &:= \frac{\partial}{\partial s_{a+1}} - \frac{\partial\mathcal{F}_0^o}{\partial s_a}\frac{\partial}{\partial s_0}, && && a\ge 0,
\end{aligned}$$

equations (3.1) and (3.2) can be equivalently written as

$$\begin{aligned}
P^1_{\alpha,a}\mathcal{F}_1^o &= \frac{1}{2}\frac{\partial^2\mathcal{F}_0^o}{\partial t^\alpha_a\partial s_0}, && 1\le\alpha\le N, && a\ge 0,\\
P^2_a\mathcal{F}_1^o &= \frac{1}{2}\frac{\partial^2\mathcal{F}_0^o}{\partial s_a\partial s}, && && a\ge 0.
\end{aligned}$$

This system of PDEs uniquely determines the function $\mathcal{F}_1^o$ starting from the initial condition $\mathcal{F}_1^o|_{t^*_{\ge 1}=s_{\ge 1}=0} = G^o(t_0^1,\ldots,t_0^N,s_0)$. By equations (2.3), (2.6), and (2.7), we have

$$P^1_{\alpha,a}(v^{\text{top}})^\beta = P^1_{\alpha,a}\phi^{\text{top}} = P^2_a(v^{\text{top}})^\beta = P^2_a\phi^{\text{top}} = 0.$$

Therefore, it remains to check that

$$P^1_{\alpha,a}\log\frac{\partial^2\mathcal{F}_0^o}{\partial t_0^{1\!\!1}\partial s_0} = \frac{\partial^2\mathcal{F}_0^o}{\partial t^\alpha_a\partial s_0}, \tag{3.4}$$

$$P^2_a\log\frac{\partial^2\mathcal{F}_0^o}{\partial t_0^{1\!\!1}\partial s_0} = \frac{\partial^2\mathcal{F}_0^o}{\partial s_a\partial s_0}. \tag{3.5}$$

To prove equation (3.4), we compute

$$
\begin{aligned}
P^1_{\alpha,a}\log\frac{\partial^2\mathcal{F}^o_0}{\partial t^{\mathbb{1}}_0\partial s_0} &= \frac{1}{\frac{\partial^2\mathcal{F}^o_0}{\partial t^{\mathbb{1}}_0\partial s_0}}\left(\frac{\partial^3\mathcal{F}^o_0}{\partial t^\alpha_{a+1}\partial t^{\mathbb{1}}_0\partial s_0}-\frac{\partial^2\mathcal{F}_0}{\partial t^\alpha_a\partial t^\mu_0}\eta^{\mu\nu}\frac{\partial^3\mathcal{F}^o_0}{\partial t^\nu_0\partial t^{\mathbb{1}}_0\partial s_0}-\frac{\partial\mathcal{F}^o_0}{\partial t^\alpha_a}\frac{\partial^3\mathcal{F}^o_0}{\partial s_0\partial t^{\mathbb{1}}_0\partial s_0}\right)\\
&= \frac{1}{\frac{\partial^2\mathcal{F}^o_0}{\partial t^{\mathbb{1}}_0\partial s_0}}\left[\frac{\partial}{\partial s_0}\underline{\left(\frac{\partial^2\mathcal{F}^o_0}{\partial t^\alpha_{a+1}\partial t^{\mathbb{1}}_0}-\frac{\partial^2\mathcal{F}_0}{\partial t^\alpha_a\partial t^\mu_0}\eta^{\mu\nu}\frac{\partial^2\mathcal{F}^o_0}{\partial t^\nu_0\partial t^{\mathbb{1}}_0}-\frac{\partial\mathcal{F}^o_0}{\partial t^\alpha_a}\frac{\partial^2\mathcal{F}^o_0}{\partial s_0\partial t^{\mathbb{1}}_0}\right)}\right.\\
&\qquad\qquad\left.+\frac{\partial^2\mathcal{F}^o_0}{\partial t^\alpha_a\partial s_0}\frac{\partial^2\mathcal{F}^o_0}{\partial t^{\mathbb{1}}_0\partial s_0}\right]\\
&= \frac{\partial^2\mathcal{F}^o_0}{\partial t^\alpha_a\partial s_0},
\end{aligned}
$$

where the vanishing of the underlined expression follows from equation (2.6). The proof of equation (3.5) is analogous. The theorem is proved. □

3.2 Differential operators and PDEs

Consider a differential operator L of the form

$$L=\sum_{i\ge 0}L_i(v^*_*,\varepsilon)(\varepsilon\partial_x)^i,\qquad L_i\in\widehat{\mathcal{A}}_{v^1,\dots,v^N;0}.$$

Let f be a formal variable and consider the PDE

$$\frac{\partial}{\partial t}\exp(\varepsilon^{-1}f)=\varepsilon^{-1}L\exp(\varepsilon^{-1}f). \tag{3.6}$$

Note that

$$\frac{(\varepsilon\partial_x)^i\exp(\varepsilon^{-1}f)}{\exp(\varepsilon^{-1}f)}=Q_i(f_*,\varepsilon),\quad i\ge 0,$$

where $Q_i\in\widehat{\mathcal{A}}_f$ can be recursively computed by the relation

$$Q_i=\begin{cases}1, & \text{if } i=0,\\ f_xQ_{i-1}+\varepsilon\partial_xQ_{i-1}, & \text{if } i\ge 1.\end{cases}$$

Remark 3.3. Note that Q_i does not depend on f and is a polynomial in the derivatives $f_x, f_{xx},\dots$ and ε. Moreover, if we introduce a new formal variable ψ and substitute $f_{i+1}=\psi_i$, $i\ge 0$, then Q_i becomes a differential polynomial of degree 0.

We see that PDE (3.6) is equivalent to the following PDE:

$$\frac{\partial f}{\partial t}=\sum_{i\ge 0}L_i(v^*_*,\varepsilon)Q_i(f_*,\varepsilon). \tag{3.7}$$

Let us look at this PDE in more details at the approximation up to ε.

Lemma 3.4. *We have* $Q_i=f_x^i+\varepsilon\frac{i(i-1)}{2}f_x^{i-2}f_{xx}+O(\varepsilon^2)$.

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Proof. The formula is clearly true for $i=0$. We proceed by induction:

$$\begin{aligned}Q_{i+1}&=f_xQ_i+\varepsilon\partial_xQ_i=f_x^{i+1}+\varepsilon\left(f_x\frac{i(i-1)}{2}f_x^{i-2}f_{xx}+\partial_x\left(f_x^i\right)\right)+O(\varepsilon^2)=\\&=f_x^{i+1}+\varepsilon\frac{(i+1)i}{2}f_x^{i-1}f_{xx}+O(\varepsilon^2).\end{aligned}$$

□

Consider the expansion

$$L_i(v_*^*,\varepsilon)=\sum_{j\geq 0}L_i^{[j]}(v_*^*)\varepsilon^j,\quad L_i^{[j]}\in\mathcal{A}_{v^1,\dots,v^N;j}.$$

We see that equation (3.7) has the form

$$\frac{\partial f}{\partial t}=\sum_{i\geq 0}L_i^{[0]}f_x^i+\varepsilon\sum_{i\geq 0}\left(L_i^{[1]}f_x^i+L_i^{[0]}\frac{i(i-1)}{2}f_x^{i-2}f_{xx}\right)+O(\varepsilon^2). \tag{3.8}$$

3.3 A linear PDE for an open descendent potential up to genus 1

Define differential operators $L^{\mathrm{int}}_{\alpha,a}$, $1\leq\alpha\leq N$, $a\geq 0$, and L^{boun}_a, $a\geq 0$, by

$$\begin{aligned}L^{\mathrm{int}}_{\alpha,a}&:=\sum_{i\geq 0}\left(L^{\mathrm{int};[0]}_{\alpha,a,i}+\varepsilon L^{\mathrm{int};[1]}_{\alpha,a,i}\right)(\varepsilon\partial_x)^i,\\L^{\mathrm{boun}}_{a}&:=\sum_{i\geq 0}\left(L^{\mathrm{boun};[0]}_{a,i}+\varepsilon L^{\mathrm{boun};[1]}_{a,i}\right)(\varepsilon\partial_x)^i,\end{aligned}$$

where

$$\begin{aligned}L^{\mathrm{int};[0]}_{\alpha,a,i}&:=\mathrm{Coef}_{\phi^i}\Gamma^{[0]}_{\alpha,a}\in\mathcal{A}_{v^1,\dots,v^N;0},\\L^{\mathrm{int};[1]}_{\alpha,a,i}&:=\mathrm{Coef}_{\phi^i}\left[\left(\frac{\partial G^o}{\partial\phi}\frac{\partial\Gamma^{[0]}_{\alpha,a}}{\partial v^\beta}-\frac{\partial G^o}{\partial v^\beta}\frac{\partial\Gamma^{[0]}_{\alpha,a}}{\partial\phi}+\frac{1}{2}\frac{\partial^2\Gamma^{[0]}_{\alpha,a}}{\partial v^\beta\partial\phi}\right)v^\beta_x\right.\\&\qquad\left.+\frac{\partial G^o}{\partial v^\beta}\eta^{\beta\gamma}\partial_x\Omega^{[0]}_{\gamma,0;\alpha,a}\right]\in\mathcal{A}_{v^1,\dots,v^N;1},\\L^{\mathrm{boun};[0]}_{a,i}&:=\mathrm{Coef}_{\phi^i}\Delta^{[0]}_a\in\mathcal{A}_{v^1,\dots,v^N;0},\\L^{\mathrm{boun};[1]}_{a,i}&:=\mathrm{Coef}_{\phi^i}\left[\left(\frac{\partial G^o}{\partial\phi}\frac{\partial\Delta^{[0]}_a}{\partial v^\beta}-\frac{\partial G^o}{\partial v^\beta}\frac{\partial\Delta^{[0]}_a}{\partial\phi}+\frac{1}{2}\frac{\partial^2\Delta^{[0]}_a}{\partial v^\beta\partial\phi}\right)v^\beta_x\right]\in\mathcal{A}_{v^1,\dots,v^N;1}.\end{aligned}$$

Theorem 3.5. *The formal power series* $v^\beta=(v^{\mathrm{top}})^\beta\big|_{t^\gamma_0\mapsto t^\gamma_0+A^\gamma x}$ *and* $f=(\mathcal{F}^o_0+\varepsilon\mathcal{F}^o_1)|_{t^\gamma_0\mapsto t^\gamma_0+A^\gamma x}$ *satisfy the system of PDEs*

$$\frac{\partial}{\partial t^\alpha_a}\exp(\varepsilon^{-1}f)=\varepsilon^{-1}L^{\mathrm{int}}_{\alpha,a}\exp(\varepsilon^{-1}f),\qquad 1\leq\alpha\leq N,\qquad a\geq 0, \tag{3.9}$$

$$\frac{\partial}{\partial s_a}\exp(\varepsilon^{-1}f)=\varepsilon^{-1}L^{\mathrm{boun}}_{a}\exp(\varepsilon^{-1}f),\qquad a\geq 0. \tag{3.10}$$

at the approximation up to ε.

Proof. Abusing notations let us denote the formal powers series $\mathcal{F}_0^o|_{t_0^\gamma\mapsto t_0^\gamma+A^\gamma x}$, $\mathcal{F}_1^o|_{t_0^\gamma\mapsto t_0^\gamma+A^\gamma x}$, $(v^{\mathrm{top}})^\alpha|_{t_0^\gamma\mapsto t_0^\gamma+A^\gamma x}$, and $\phi^{\mathrm{top}}|_{t_0^\gamma\mapsto t_0^\gamma+A^\gamma x}$ by $\mathcal{F}_0^o$, $\mathcal{F}_1^o$, v^α, and ϕ, respectively. We can then write the statement of Theorem 3.2 as

$$\mathcal{F}_1^o=\frac{1}{2}\log\phi_s+G^o.$$

Let us prove equation (3.9) at the approximation up to ε. We have

$$\partial_x\left(\mathcal{F}_0^o+\varepsilon\mathcal{F}_1^o\right)=\phi+\varepsilon\left(\frac{1}{2}\frac{\phi_{xs}}{\phi_s}+\partial_x G^o\right).$$

Therefore, by equation (3.8), we have to check that

$$\begin{aligned}
&\frac{\partial}{\partial t_a^\alpha}(\mathcal{F}_0^o+\varepsilon\mathcal{F}_1^o)=\sum_{i\geq0}L^{\mathrm{int};[0]}_{\alpha,a,i}\phi^i+\varepsilon\sum_{i\geq0}L^{\mathrm{int};[1]}_{\alpha,a,i}\phi^i+\\
&\qquad\qquad+\varepsilon\sum_{i\geq0}L^{\mathrm{int};[0]}_{\alpha,a,i}\left(\frac{i(i-1)}{2}\phi^{i-2}\phi_x+i\phi^{i-1}\left(\frac{1}{2}\frac{\phi_{xs}}{\phi_s}+\partial_x G^o\right)\right)\Leftrightarrow\\
&\Leftrightarrow\frac{\partial}{\partial t_a^\alpha}(\mathcal{F}_0^o+\varepsilon\mathcal{F}_1^o)=\Gamma^{[0]}_{\alpha,a}+\varepsilon\left(\sum_{i\geq0}L^{\mathrm{int};[1]}_{\alpha,a,i}\phi^i+\frac{1}{2}\frac{\partial^2\Gamma^{[0]}_{\alpha,a}}{\partial\phi^2}\phi_x+\frac{1}{2}\frac{\partial\Gamma^{[0]}_{\alpha,a}}{\partial\phi}\frac{\phi_{xs}}{\phi_s}+\frac{\partial\Gamma^{[0]}_{\alpha,a}}{\partial\phi}\partial_x G^o\right)\Leftrightarrow\\
&\Leftrightarrow\frac{\partial\mathcal{F}_1^o}{\partial t_a^\alpha}=\frac{1}{2}\frac{\partial\Gamma^{[0]}_{\alpha,a}}{\partial\phi}\frac{\phi_{xs}}{\phi_s}+\left(\frac{1}{2}\frac{\partial^2\Gamma^{[0]}_{\alpha,a}}{\partial\phi^2}+\frac{\partial\Gamma^{[0]}_{\alpha,a}}{\partial\phi}\frac{\partial G^o}{\partial\phi}\right)\phi_x+\frac{\partial\Gamma^{[0]}_{\alpha,a}}{\partial\phi}\frac{\partial G^o}{\partial v^\beta}v^\beta_x+\sum_{i\geq0}L^{\mathrm{int};[1]}_{\alpha,a,i}\phi^i.
\end{aligned}$$

Using the definition of $L^{\mathrm{int};[1]}_{\alpha,a,i}$, we see that the last equation is equivalent to

$$\begin{aligned}
\frac{\partial\mathcal{F}_1^o}{\partial t_a^\alpha}&=\frac{1}{2}\frac{\partial\Gamma^{[0]}_{\alpha,a}}{\partial\phi}\frac{\phi_{xs}}{\phi_s}+\left(\frac{1}{2}\frac{\partial^2\Gamma^{[0]}_{\alpha,a}}{\partial\phi^2}+\frac{\partial\Gamma^{[0]}_{\alpha,a}}{\partial\phi}\frac{\partial G^o}{\partial\phi}\right)\phi_x+\\
&\quad+\left(\frac{\partial G^o}{\partial\phi}\frac{\partial\Gamma^{[0]}_{\alpha,a}}{\partial v^\beta}+\frac{1}{2}\frac{\partial^2\Gamma^{[0]}_{\alpha,a}}{\partial v^\beta\partial\phi}\right)v^\beta_x+\frac{\partial G^o}{\partial v^\beta}\eta^{\beta\gamma}\partial_x\Omega^{[0]}_{\gamma,0;\alpha,a}\Leftrightarrow\\
\Leftrightarrow\frac{\partial\mathcal{F}_1^o}{\partial t_a^\alpha}&=\frac{1}{2}\frac{\partial\Gamma^{[0]}_{\alpha,a}}{\partial\phi}\frac{\phi_{xs}}{\phi_s}+\frac{1}{2}\partial_x\frac{\partial\Gamma^{[0]}_{\alpha,a}}{\partial\phi}+\frac{\partial G^o}{\partial\phi}\partial_x\Gamma^{[0]}_{\alpha,a}+\frac{\partial G^o}{\partial v^\beta}\eta^{\beta\gamma}\partial_x\Omega^{[0]}_{\gamma,0;\alpha,a}.
\end{aligned}\tag{3.11}$$

On the other hand, we compute

$$\begin{aligned}
\frac{\partial\mathcal{F}_1^o}{\partial t_a^\alpha}&=\left(\frac{1}{2}\log\phi_s+G^o\right)_{t_a^\alpha}=\frac{1}{2}\frac{\left(\phi_{t_a^\alpha}\right)_s}{\phi_s}+\frac{\partial G^o}{\partial v^\beta}\eta^{\beta\gamma}\partial_x\Omega^{[0]}_{\gamma,0;\alpha,a}+\frac{\partial G^o}{\partial\phi}\partial_x\Gamma^{[0]}_{\alpha,a}=\\
&=\frac{1}{2}\frac{\partial_x\left(\Gamma^{[0]}_{\alpha,a}\right)_s}{\phi_s}+\frac{\partial G^o}{\partial v^\beta}\eta^{\beta\gamma}\partial_x\Omega^{[0]}_{\gamma,0;\alpha,a}+\frac{\partial G^o}{\partial\phi}\partial_x\Gamma^{[0]}_{\alpha,a}=\\
&=\frac{1}{2}\frac{\partial_x\left(\frac{\partial\Gamma^{[0]}_{\alpha,a}}{\partial\phi}\phi_s\right)}{\phi_s}+\frac{\partial G^o}{\partial v^\beta}\eta^{\beta\gamma}\partial_x\Omega^{[0]}_{\gamma,0;\alpha,a}+\frac{\partial G^o}{\partial\phi}\partial_x\Gamma^{[0]}_{\alpha,a}=\\
&=\frac{1}{2}\partial_x\frac{\partial\Gamma^{[0]}_{\alpha,a}}{\partial\phi}+\frac{1}{2}\frac{\partial\Gamma^{[0]}_{\alpha,a}}{\partial\phi}\frac{\phi_{xs}}{\phi_s}+\frac{\partial G^o}{\partial v^\beta}\eta^{\beta\gamma}\partial_x\Omega^{[0]}_{\gamma,0;\alpha,a}+\frac{\partial G^o}{\partial\phi}\partial_x\Gamma^{[0]}_{\alpha,a},
\end{aligned}$$

which proves equation (3.11) and, hence, equation (3.9) at the approximation up to ε.

The proof of equation (3.10) is analogous. □

4 Expectation in higher genera

Consider a total descendent potential $\mathcal{F}(t^*_*,\varepsilon)=\sum_{g\geq 0}\varepsilon^{2g}\mathcal{F}_g(t^*_*)$ of some rank N.

Expectation 4.1. *Under possibly some additional assumptions, there exists a reasonable geometric construction of an open descendent potential in all genera* $\mathcal{F}^o(t^*_*,s^*,\varepsilon)=\sum_{g\geq 0}\varepsilon^g\mathcal{F}^o_g(t^*_*,\varepsilon)$ *satisfying the following properties:*

- *The functions* $\mathcal{F}^o_0$ *and* $\mathcal{F}^o_1$ *are open descendents potentials in genus* 0 *and* 1*, respectively (according to Definitions 2.5 and 3.1).*
- *The function* $\mathcal{F}^o$ *satisfies the open string equation in all genera*
$$\sum_{b\geq 0}t^\beta_{b+1}\frac{\partial\mathcal{F}^o}{\partial t^\beta_b}+\sum_{a\geq 0}s_{a+1}\frac{\partial\mathcal{F}^o}{\partial s_a}-\frac{\partial\mathcal{F}^o}{\partial t^{1\!\!1}_0}=-s_0+C\varepsilon,$$
where C *is some constant.*
- *Consider formal variables* $w^1,\ldots,w^N$*. Then there exist differential operators* $L^{\mathrm{full,int}}_{\alpha,a}$*,* $1\leq\alpha\leq N$*,* $a\geq 0$*, and* $L^{\mathrm{full,boun}}_a$*,* $a\geq 0$*, of the form*
$$\begin{aligned}
L^{\mathrm{full,int}}_{\alpha,a}&=\sum_{i\geq 0}L^{\mathrm{full,int}}_{\alpha,a,i}(w^*_*,\varepsilon)(\varepsilon\partial_x)^i, && L^{\mathrm{full,int}}_{\alpha,a,i}\in\widehat{\mathcal{A}}_{w^1,\ldots,w^n;0},\\
L^{\mathrm{full,boun}}_{a}&=\sum_{i\geq 0}L^{\mathrm{full,boun}}_{a,i}(w^*_*,\varepsilon)(\varepsilon\partial_x)^i, && L^{\mathrm{full,boun}}_{a,i}\in\widehat{\mathcal{A}}_{w^1,\ldots,w^n;0},\\
L^{\mathrm{full,int}}_{\alpha,a,i}&=L^{\mathrm{int}}_{\alpha,a,i}\Big|_{v^\beta_b=w^\beta_b}+O(\varepsilon^2),\\
L^{\mathrm{full,boun}}_{a,i}&=L^{\mathrm{boun}}_{a,i}\Big|_{v^\beta_b=w^\beta_b}+O(\varepsilon^2),
\end{aligned}$$
such that the formal power series $w^\beta=\eta^{\beta\mu}\frac{\partial^2\mathcal{F}}{\partial t^\mu_0\partial t^{1\!\!1}_0}\Big|_{t^\gamma_0\mapsto t^\gamma_0+A^\gamma x}$ *and* $f=\mathcal{F}^o|_{t^\gamma_0\mapsto t^\gamma_0+A^\gamma x}$ *satisfy the system of PDEs*
$$\frac{\partial}{\partial t^\alpha_a}\exp(\varepsilon^{-1}f)=\varepsilon^{-1}L^{\mathrm{full,int}}_{\alpha,a}\exp(\varepsilon^{-1}f),\qquad 1\leq\alpha\leq N,\qquad a\geq 0,\tag{4.1}$$
$$\frac{\partial}{\partial s_a}\exp(\varepsilon^{-1}f)=\varepsilon^{-1}L^{\mathrm{full,boun}}_{a}\exp(\varepsilon^{-1}f),\qquad a\geq 0.\tag{4.2}$$

Suppose that there exists a Dubrovin-Zhang hierarchy corresponding to our total descendent potential $\mathcal{F}$ (this is true when, for example, the associated Dubrovin-Frobenius manifold is semisimple). It is easy to show that if Expectation 4.1 is true, then the flows $\frac{\partial}{\partial t^\alpha_a}$ and $\frac{\partial}{\partial s_b}$ pairwise commute, which means that

$$\begin{aligned}
\frac{\partial L^{\mathrm{full,int}}_{\alpha,a}}{\partial t^\beta_b}-\frac{\partial L^{\mathrm{full,int}}_{\beta,b}}{\partial t^\alpha_a}+\varepsilon^{-1}\left[L^{\mathrm{full,int}}_{\alpha,a},L^{\mathrm{full,int}}_{\beta,b}\right]&=0, && 1\leq\alpha,\beta\leq N, && a,b\geq 0,\\
\frac{\partial L^{\mathrm{full,boun}}_{a}}{\partial t^\beta_b}+\varepsilon^{-1}\left[L^{\mathrm{full,boun}}_{a},L^{\mathrm{full,int}}_{\beta,b}\right]&=0, && 1\leq\beta\leq N, && a,b\geq 0,\\
\left[L^{\mathrm{full,boun}}_{a},L^{\mathrm{full,boun}}_{b}\right]&=0, && && a,b\geq 0,
\end{aligned}\tag{4.3}$$

where the derivatives $\frac{\partial L^{\text{full,int}}_{\alpha,a}}{\partial t^\beta_b}$ and $\frac{\partial L^{\text{full,boun}}_a}{\partial t^\beta_b}$ are computed using the flows of the Dubrovin-Zhang hierarchy. Note that equation (4.3) potentially gives a Lax description of the Dubrovin-Zhang hierarchy (see an alternative approach in [11]).

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