General solutions in Chern-Simons gravity and $T\bar{T}$ -deformations

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ABSTRACT: We find the most general solution to Chern-Simons AdS_3 gravity in Fefferman-Graham gauge. The connections are equivalent to geometries that have a non-trivial curved boundary, characterized by a 2-dimensional vielbein and a spin connection. We define a variational principle for Dirichlet boundary conditions and find the boundary stress tensor in the Chern-Simons formalism. Using this variational principle as the departure point, we show how to treat other choices of boundary conditions in this formalism, such as, including the mixed boundary conditions corresponding to a $T\bar{T}$ -deformation.

KEYWORDS: AdS-CFT Correspondence, Chern-Simons Theories, Conformal Field Theory

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1 Introduction

Three-dimensional gravity is a topological theory with no local degrees of freedom. This feature makes it a useful framework to study aspects of quantum and classical gravity, avoiding some of the complications we encounter in higher dimensions. The topological character of AdS_3 gravity is made manifest by rewriting it as a Chern-Simons theory with gauge group $\text{SO}(2, 2) = \text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R})$ [1, 2]. The Chern-Simons formulation has many advantages: the Einstein field equations map to a flatness condition on the connection, diffeomorphisms may be easily interpreted as gauge transformations [2], BTZ black holes emerge naturally as topological defects around which the gauge fields have non-trivial holonomies. Moreover, the Chern-Simons formulation simplifies in a great manner the study of theories of higher spin gravity [3–7]. In this paper, we study 3d gravity in Chern-Simons formalism. This theory should not be confused with Einstein gravity plus a Chern-Simons term since both theories are sometimes referred to as Chern-Simons gravity.

Since there are no propagating degrees of freedom in the bulk, boundary conditions are especially relevant, because they encode the interesting physics. Boundary conditions have been extensively studied in the metric formulation [8–11]. The analysis is highly simplified by the fact that the most general asymptotically AdS_3 condition is known

$$ds^2 = \ell^2 \frac{dz^2}{z^2} + g_{ij}(x^k, z) dx^i dx^j, \quad g_{ij}(x^k, z) = \frac{g_{ij}^{(0)}(x^k)}{z^2} + g_{ij}^{(2)}(x^k) + z^2 g_{ij}^{(4)}(x^k), \quad (1.1)$$

where z is a radial coordinate, with boundary at $z \rightarrow 0$, and i, j, k are 2-dimensional boundary indexes. Equation (1.1) is known as the Fefferman-Graham expansion of the metric. The parameters $g_{ij}^{(2)}$, and $g_{ij}^{(4)}$ are related to $g_{ij}^{(0)}$ through the asymptotic equations of motion [12]. Different boundary conditions correspond to fixing different components of the asymptotic metric. For example, $g_{ij}^{(0)}$ can be fixed with Dirichlet boundary conditions, we could impose Neumann for $g_{ij}^{(2)}$, or consider a situation with mixed boundary conditions. These have a clear interpretation in the context of the $\text{AdS}_3/\text{CFT}_2$ correspondence: when $g_{ij}^{(0)}$ is fixed we can identify it with the CFT_2 metric [11], Neumann boundary conditions couple the boundary to 2d gravity, and mixed conditions are equivalent to double-trace deformations in CFT [13, 14]. However, a systematic analysis of the Fefferman-Graham expansion (1.1) in Chern-Simons formalism is missing in the literature. This could be very helpful since the gauge theory description of 3d gravity is generally simpler than the metric case. In this paper, we solve the Chern-Simons equations of motion in a systematic way for a set of boundary conditions that reproduce the Fefferman-Graham gauge (1.1). Generalized boundary conditions for Chern-Simons AdS_3 gravity were already discussed in [15, 16]. In section 4, we will comment on how our generalized connections relate to the results in [15, 16].

Double-trace deformations in CFT have recently obtained much attention in the context of holography. This is mainly due to the proposal [17] which relates $T\bar{T}$ -deformed CFT_2 to AdS_3 gravity with a finite bulk cutoff. This proposal was generalized to include matter in [18], higher dimensions [19–21], and entanglement entropy calculations [22–25]. However, the formulation of double-trace deformations in AdS_3 in Chern-Simons formalism has remained unstudied until now. We will use our new generalized connections to define a variational principle with Dirichlet boundary conditions as in the metric formalism. This allows us to interpret the boundary values of the gauge fields as sources and expectation values of the dual operators in the CFT. This identification will prove very useful because it constitutes a departure point for the analysis of double-trace deformations in Chern-Simons formalism. We focus on one particular type of deformation that will be dual to a $T\bar{T}$ -deformation in the boundary CFT. We would like to emphasize that our analysis is purely classical. The identification between the boundary terms of the Chern-Simons theory and the deformed CFT action is done using known results from the metric formalism [37]. The precise understanding of the subtleties that might arise at the quantum level is left for future work.

This article is organized as follows. We start in section 2 by reviewing the Chern-Simons formalism of 3d gravity. In section 3, we solve the $\text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R})$ Chern-Simons theory for a set of generalized boundary conditions, and construct a well-defined

variational principle. In section 4, we identify our connections with the most general solutions of asymptotically AdS_3 spaces in the Fefferman-Graham gauge and give a holographic interpretation to the boundary values of the connections. Finally, in section 5, we use the knowledge acquired in the previous sections to analyze double-trace deformations in the Chern-Simons formalism. We conclude in section 6, and in the appendices, we set our notation and review the vielbein formalism of general relativity, which is useful for our calculations.

2 Review: AdS_3 as a Chern-Simons theory

There exists an alternative description of pure 3d gravity in terms of Chern-Simons (CS) gauge connections. In this section, we briefly review this formalism, and for further details, we refer the reader to the original articles [1, 2] and more recently to, e.g., [26, 27]. Let us start by considering the Chern-Simons action:

$$S_{\text{CS}}[A] = \frac{k}{4\pi} \int_{\mathcal{M}} \text{Tr} \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right), \quad (2.1)$$

where A is a gauge field that lives in a manifold $\mathcal{M}$. The trace $\text{Tr}(\dots)$ is a shortcut notation for the contraction using the Killing forms of the algebra. The gauge group is chosen to match the symmetry group of AdS_3 , which in Lorentzian signature is $\text{SO}(2, 2) \cong \text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R})$. As a result, we consider the gauge connections $A, \bar{A}$ valued in two independent copies of $\mathfrak{sl}(2, \mathbb{R})$, which are defined as

$$A = \left(\Omega^m + \frac{1}{\ell} E^m \right) L_m, \quad \bar{A} = \left(\Omega^m - \frac{1}{\ell} E^m \right) \bar{L}_m, \quad (2.2)$$

where E^m , and Ω^m , and L_m are respectively the 3-dimensional vielbein, spin connection, and $\mathfrak{sl}(2, \mathbb{R})$ generators defined in section B.2. With the definition in (2.2), the following actions are equivalent:

$$S_{\text{EH}}[E, \Omega] = S_{\text{CS}}[A] - S_{\text{CS}}[\bar{A}], \quad (2.3)$$

where S_{EH} is the Einstein-Hilbert action with negative cosmological constant, written in vielbein formalism. The relation (2.3) is only true if the gravitational constant is related to the Chern-Simons level via

$$k = \frac{\ell}{4G_3}. \quad (2.4)$$

Moreover, one can show that the Einstein field equations are equivalent the Chern-Simons equations of motion

$$dA + A \wedge A = 0, \quad d\bar{A} + \bar{A} \wedge \bar{A} = 0. \quad (2.5)$$

The space-time metric can be recovered directly from the connections through

$$g_{\mu\nu} = 2\text{Tr}(E_\mu E_\nu) = \frac{1}{2} \text{Tr} \left((A - \bar{A})_\mu (A - \bar{A})_\nu \right), \quad (2.6)$$

where we are taking the trace in the fundamental representation of $\mathfrak{sl}(2, \mathbb{R})$, with the conventions in (B.13). This provides a map between the solutions of the Chern-Simons theory with $\text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R})$ and geometries in AdS_3 .

The construction of locally AdS_3 solutions in the CS formalism is discussed in [26], whose results we summarize here. We start by gauge fixing the radial component of the connection to

$$A_\rho = L_0, \quad \bar{A}_\rho = -L_0. \quad (2.7)$$

With this gauge choice, the connections can be parametrized without losing any generality [7] by

$$A = b(\rho)^{-1} \left(a(x^+, x^-) + d \right) b(\rho), \quad \bar{A} = b(\rho) \left(\bar{a}(x^+, x^-) + d \right) b(\rho)^{-1}, \quad b(\rho) = e^{-\rho L_0}. \quad (2.8)$$

Here ρ is the holographic radial direction, and x^+, x^- are boundary coordinates. The connections (2.8) are a solution of the Chern-Simons equations of motion (2.5) as long as

$$da + a \wedge a = 0, \quad d\bar{a} + \bar{a} \wedge \bar{a} = 0. \quad (2.9)$$

The advantage of the parametrization in (2.8) is that we can isolate $(a, \bar{a})$, which act as the boundary values of the connections, and interpret the radial direction as emergent from a gauge transformation. The set of boundary conditions proposed in [26] leads to

$$\begin{aligned} a_{x^+} &= \left(L_+ - \frac{2\pi \mathcal{L}(x^+)}{k} L_- \right), & a_{x^-} &= 0, \\ \bar{a}_{x^+} &= 0, & \bar{a}_{x^-} &= - \left(L_- - \frac{2\pi \bar{\mathcal{L}}(x^-)}{k} L_+ \right), \end{aligned} \quad (2.10)$$

where $\mathcal{L}(x^+)$, and $\bar{\mathcal{L}}(x^-)$ are generic functions will always fulfil the equations of motion (2.9). The connections described by (2.8) with (2.10) are known as *Bañados solutions*. These connections parametrize the space of all solutions that are asymptotically AdS_3 with a trivial flat boundary.

3 Generalized solutions in Chern-Simons theory

The Bañados solutions in (2.7)–(2.10) analysed in the previous section have boundary conditions $A_{x^-} = \bar{A}_{x^+} = 0$. In this section, we depart from this choice and find solutions to $\text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R})$ Chern-Simons gauge theories with generic boundary conditions. We also find a variational principle with Dirichlet boundary conditions for the gauge field. Here, all the analysis is purely done in the Chern-Simons formalism of 3d gravity. However, in section 4, we will show that our solutions exactly reproduce the Fefferman-Graham expansion in metric formalism (4.1).

3.1 The connections

Here we assume the same gauge choice of the radial direction as in (2.7), but we will go away from the solution (2.10) with trivial boundary. We consider from now on $\ell = 1$. We propose the following general boundary conditions for a and $\bar{a}$ is:

$$a_i = 2e_i^+ L_+ - f_i^- L_- + \omega_i L_0, \quad \bar{a}_i = f_i^+ L_+ - 2e_i^- L_- + \omega_i L_0, \quad (3.1)$$

where the index $i = x^+, x^-$. The only choice we made is to have equal L_0 components for both a and $\bar{a}$. The equations of motion (2.5), for each component in the Lie algebra indices, impose the following constraints for the parameters in (3.1)

$$d\omega - 4e^+ \wedge f^- = 0, \quad d\omega - 4f^+ \wedge e^- = 0 \quad (3.2)$$

$$de^+ - \omega \wedge e^+ = 0, \quad df^+ - \omega \wedge f^+ = 0, \quad (3.3)$$

$$dt^- + \omega \wedge f^- = 0, \quad de^- + \omega \wedge e^- = 0. \quad (3.4)$$

The left column corresponds to the constraints deduced from a , and the left column from $\bar{a}$. We can write the previous equations more compactly, adding and subtracting different combinations of them

$$d\omega - 2\varepsilon_{ab}e^a \wedge f^b = 0, \quad e^a \wedge f_a = 0 \quad (3.5)$$

$$de^a - \varepsilon_b^a e^b \wedge \omega = 0, \quad df^a - \varepsilon_b^a f^b \wedge \omega = 0 \quad (3.6)$$

Here, we have defined a new 2-dimensional flat index that has values $a, b = \{+, -\}$. The symbol ε_{ab} is the Levi-Civita tensor defined in (B.1).¹ The appearance of this 2-dimensional indexing is a consequence of the gauge choice (2.7), where the radial dependence is taken care of L_0 . Its origin will be clear in section 4.2. It is interesting to use (2.8), together with (B.14), to analyse the full solution:

$$A_i = \frac{2e_i^+}{z}L_+ - z f_i^- L_- + \omega_i L_0, \quad \bar{A}_i = z f_i^+ L_+ - \frac{2e_i^-}{z}L_- + \omega_i L_0, \quad (3.7)$$

where we have used the change $z = e^{-\rho}$ in the radial coordinate, with boundary located at $z \rightarrow 0$. We would like to remark that the solution (3.7) is independent of the parametrization (2.8). Alternatively, one could solve the equations of motion (2.5) for A and $\bar{A}$, with the gauge choice (2.7), and the most generic solution would also lead to (3.7), and truncate at some powers of z . The connections (3.7) have generalized boundary conditions respect to the Bañados solutions (2.10). In section 4, we will show that they reproduce the Fefferman-Graham expansion (4.1) in Chern-Simons formalism.

3.2 Boundary terms

In pure 3-dimensional gravity, the boundary conditions are especially important because we do not have bulk degrees of freedom and, therefore, all the interesting information is located at the boundary. In section 3.1, we have fixed a radial gauge for our solution, and now we need to make sure we can tie it to a well defined variational principle at the boundary. This is crucial in the context of the AdS/CFT correspondence, since AdS action on-shell is related to the CFT partition function. We start by varying the CS action in (2.3), which on-shell is

$$\delta S_{\text{CS}}[A] - \delta S_{\text{CS}}[\bar{A}] = -\frac{k}{4\pi} \int_{\partial\mathcal{M}} \text{Tr} \left(A \wedge \delta A - \bar{A} \wedge \delta \bar{A} \right), \quad (3.8)$$

¹In this text, we use labels $m, n, l \dots$ for the $sl(2, \mathbb{R})$ indices that run over $\{0, +, -\}$. The letters $a, b, c \dots$ refer to the two-dimensional flat index with possible values $\{+, -\}$.

where $\partial\mathcal{M}$ is the boundary of the manifold $\mathcal{M}$. We consider light-cone coordinates (x^+, x^-) for the boundary as in the previous sections, and we can expand the wedge products as:

$$\delta S_{\text{CS}}[A] - \delta S_{\text{CS}}[\bar{A}] = -\frac{k}{4\pi} \int dx^+ dx^- \text{Tr} \left(A_{x^+} \delta A_{x^-} - A_{x^-} \delta A_{x^+} - \bar{A}_{x^+} \delta \bar{A}_{x^-} + \bar{A}_{x^-} \delta \bar{A}_{x^+} \right). \quad (3.9)$$

For the Bañados connections in (2.10), we have $A_{x^+} = 0$, and $\bar{A}_{x^-} = 0$. Therefore, the variation of the action (3.9) is zero if we also consider δA_{x^+} and $\delta \bar{A}_{x^-}$ fixed at the boundary, and hence we would have a well-defined variational principle. However, this argument does not hold for (3.7), because we have all non-zero components of the connection at the boundary. We need to define a new variational principle for these background solutions. We begin by expressing the variation of the action (3.8) in terms of the parameters in (3.7)

$$\delta S_{\text{CS}}[A] - \delta S_{\text{CS}}[\bar{A}] = -\frac{k}{2\pi} \int_{\partial\mathcal{M}} \varepsilon_{ab} \left(e^a \wedge \delta f^b - f^a \wedge \delta e^b \right), \quad (3.10)$$

where the parameter ω_i cancels out due to the negative sign between the barred and non-barred sector of the actions. It is also remarkable that there is no explicit dependence on the radial coordinate z , unlike in the metric formalism, where the action needs to be renormalized, due to divergences at the boundary (see section 4.1). This is due to the gauge choice (2.7), and the subsequent gauge reparametrization (2.8). Note, however, that the connections (3.7) still diverge at the boundary with

$$A_i = \frac{2e_i^+}{z} L_+ + \dots, \quad \bar{A}_i = -\frac{2e_i^-}{z} L_- + \dots, \quad (3.11)$$

where the dots stand for less dominant terms as $z \rightarrow 0$. We define a variational principle with Dirichlet boundary conditions for the connections, which will remain fixed at the boundary. This means that the variation of parameter e^a is null, and f^a is free to vary. With these considerations, the variation (3.10) is not zero, but we have the freedom to add a new boundary term to the action that accomplish it. We propose

$$S = S_{\text{CS}}[A] - S_{\text{CS}}[\bar{A}] + S_{\text{bd}}, \quad (3.12)$$

where S_{bd} is defined as:

$$S_{\text{bd}} = -\frac{k}{2\pi} \int_{\partial\mathcal{M}} \epsilon_{ab} \left(A^a \wedge A^b + \bar{A}^a \wedge \bar{A}^b \right). \quad (3.13)$$

The term S_{bd} is designed such that the total action has zero variation when e^a is held fixed, and, in fact, we see that

$$\delta S = \frac{k}{\pi} \int_{\partial\mathcal{M}} \varepsilon_{ab} f^a \wedge \delta e^b. \quad (3.14)$$

We have found then a variational principle for the connections (3.7) with Dirichlet boundary conditions. This is necessary to interpret the connections holographically. The AdS/CFT dictionary tells us that we can identify (3.14) with the variation of the CFT the action, which schematically is $\int d^2x O(x) \delta J(x)$ [28]. Therefore, we see that the vielbein e^a acts as

a source, whose variation vanishes at the boundary, and f^a as the expectation value the dual operators.

We would like to remark that our variational principle is different from the one used for the Bañados solutions (2.10). As explained in (3.9), this is due to the form of the connections (3.7) which have $A_{x-}, \bar{A}_{x+} \neq 0$ at the boundary. In [29], they also analyse the variation of the Chern-Simons action (3.8) for connections with all components not null at the boundary. However, their definition of variational principal is different from ours, and they do not need to add any counterterm such our (3.13). In higher spin gravity, it is as well usual to consider connections with no null boundary conditions for $a_{x-}, \bar{a}_{x+}$. Our identification of sources and vevs also differs from the one that is normally used in the higher spin literature. See, for example, [30].

4 Solutions with non-trivial curved boundary

In this section, we show that the Chern-Simons connections (3.7) represent geometries in the Fefferman-Graham expansion, which are the most generic solutions with boundaries that are asymptotically AdS_3 . We first review these solutions in metric formalism in section 4.1. In section 4.2, we identify the Chern-Simons parameters in (3.7) as CFT quantities: e_i , ω_i , and f_i are related to a 2-dimensional vielbein, spin connection, and stress tensor, respectively.

We would like to comment that general boundary conditions for AdS_3 Chern-Simons theories have been also studied in [15]. However, their solutions do not have a regular Fefferman-Graham expansion (1.1) when translated to the metric formalism, and the metric contains off-diagonal terms such as $g_{x\pm z}$. They find that the asymptotic symmetry algebra associated to their gauge condition consists of two copies of affine $\mathfrak{sl}(2)_k$ -algebras. Moreover, similar boundary conditions to (3.1) have been discussed in [16]. They also relate the dominant term in their connections to a 2-dimensional vielbein and find a well-defined variational principle at the boundary. However, our identification of the parameters in the connections (3.7) as CFT quantities is more systematic, which will be useful for future applications. For example, the relation we find between f_i and the boundary stress tensor will be crucial in section 5 for the analysis of double-trace deformations.

4.1 Review: metric formulation

Let us start by considering a set of coordinates where the metric is in *Fefferman-Graham gauge*:

$$ds^2 = \ell^2 \frac{dz^2}{z^2} + g_{ij}(x^k, z) dx^i dx^j. \quad (4.1)$$

Solutions to Einstein's field equations with negative cosmological constant truncate in three dimensions [12], which gives

$$g_{ij}(x^k, z) = \frac{g_{ij}^{(0)}(x^k)}{z^2} + g_{ij}^{(2)}(x^k) + z^2 g_{ij}^{(4)}(x^k), \quad (4.2)$$

where $g_{ij}^{(0)}$ is interpreted as the metric of the boundary theory. The elements of $g_{ij}^{(4)}$ are fully determined in terms of $g_{ij}^{(2)}$, and $g_{ij}^{(0)}$ by:

$$g_{ij}^{(4)} = \frac{1}{4} g_{ik}^{(2)} g^{(0)kl} g_{lj}^{(2)}. \quad (4.3)$$

Moreover, $g_{ij}^{(2)}$ fulfils the following conditions:

$$g^{(2)i}{}_i = \text{tr} \left((g^{(0)})^{-1} g^{(2)} \right) = \frac{\ell^2}{2} R^{(0)}, \quad \nabla^{(0)i} g_{ij}^{(2)} = \nabla_j^{(0)} g^{(2)i}{}_i. \quad (4.4)$$

The indices have been raised and lowered with the metric $g_{ij}^{(0)}$, and $R^{(0)}$ and $\nabla_j^{(0)}$, are the Ricci scalar, and the covariant derivative in the metric $g_{ij}^{(0)}$. In [11], they interpret these geometries holographically. For that, it is necessary to define a variational principle with Dirichlet boundary condition for the metric: when $g_{ij}^{(0)}$ is held fixed, and the sub-leading terms can vary at the boundary. Initially, the metric (4.2) is not an extremum of the Einstein-Hilbert action, because its variation contains linear terms in $\delta g_{ij}^{(2)}$, and divergences at the boundary. To circumvent this problem, it is necessary to add an appropriate counterterm:

$$S_{\text{ct}} = -\frac{1}{8\pi G_3} \int_{\partial\mathcal{M}} d^2x \sqrt{-g}. \quad (4.5)$$

For a review on this analysis, see e.g. [31]. After adding (4.5) to the Einstein-Hilbert action, with a Gibbons-Hawking term, the on-shell variation of the total action is:

$$\delta S = \frac{1}{2} \int_{\partial\mathcal{M}} d^2x \sqrt{-g^{(0)}} T^{ij} \delta g_{ij}^{(0)}, \quad (4.6)$$

where the boundary stress tensor T_{ij} is defined as:

$$T_{ij} = \frac{1}{8\pi G_3 \ell} \left(g_{ij}^{(2)} - g^{(2)k}{}_k g_{ij}^{(0)} \right). \quad (4.7)$$

It is interesting to see that the counterterm (4.5) is divergent at the boundary when we consider the expansion (4.2), as opposed to what happened in the boundary terms in the Chern-Simons formalism studied in section 3.2. The quantity T_{ij} is a symmetric tensor that obeys the properties of stress-energy tensor on the CFT [11]. For example, using (4.4) we find

$$\nabla^i T_{ij} = 0, \quad T_i^i = -\frac{\ell}{16\pi G_3} R^{(0)}. \quad (4.8)$$

The first equality shows that T_{ij} is conserved, and the second recovers the CFT Weyl anomaly, $T_i^i = -\frac{c}{24\pi} R$, from which we can deduce the central charge $c = \frac{3\ell}{2G_3}$. This illustrates that $g_{ij}^{(0)}$ is the metric of the CFT₂ living on the boundary of the AdS₃.

4.2 Connections with asymptotically curved boundary

In this section, we show that the asymptotically AdS₃ metric solutions presented in section 4.1 are equivalent to the Chern-Simons connections studied in section 3.1. The first

indication is that we have chosen the same gauge conditions: the choice (2.7) is equivalent to the Fefferman-Graham gauge in (4.1). (ω). One can easily see this using (2.6) to relate the metric and the connections. Moreover, we will now show that the parameters e_i and ω_i in (3.7) represent the vielbein and the spin connection of the theory living on the curved boundary. We start seeing this, identifying $g^{(0)}$ in (2.6) from the expansion (4.2):

$$g_{ij}^{(0)} = -2(e_i^+ e_j^- + e_i^- e_j^+) = 2 e_i \cdot e_j, \quad (4.9)$$

where in the second equality we have defined the scalar product as an abbreviation for the contraction with the metric η_{ab} , defined in (B.1). Equation (4.9) resembles the definition of a vielbein (A.1), up to a normalizations of the overall flat metric. Since $g_{ij}^{(0)}$ is the boundary metric, we identify e_i^a as a *boundary vielbein*. The boundary vielbein e_i^a should not be confused with the 3-dimensional bulk vielbein E_μ^m used in section 2, see appendix B. We can see that e_i controls the dominant term of E_i , as defined from (2.2):

$$E_i = \frac{A_i - \bar{A}_i}{2} = \left(e_i^+ L_+ + e_i^- L_- \right) \frac{1}{z} + \dots \quad (4.10)$$

where the dots mean we ignored lower orders in z . Moreover, the right constraint in (3.6) is equivalent to the torsion equation (A.4) in 2-dimensions, where the spin connection (B.2) has only one independent component. Therefore, if e_i is the boundary vielbein, the equation in the right of (3.6) defines the parameter ω_i as the *boundary spin connection*. The Ricci scalar in 2 dimensions is (B.9), which shows that the solution (3.7) describes theories with non-trivial curved boundaries, in contrast with Bañados connections (2.10).

We have seen that the connections (3.7) and the metric (4.2) are both in Fefferman-Graham gauge, and allow for boundary theories with non-trivial metric. However, to ultimately check that the two solutions are the same but in different formalisms, we now prove that the equations of motion (3.5)–(3.6) are equivalent to the constraints (4.3)–(4.4). Using (2.6), we can find $g^{(2)}$, and $g^{(4)}$ in terms of the parameters of the connections:

$$g_{ij}^{(2)} = -2e_i \cdot f_j = -2f_i \cdot e_j, \quad g_{ij}^{(4)} = \frac{1}{2} f_i \cdot f_j, \quad (4.11)$$

where we have used the right equation in (3.5) to write the expression for $g_{ij}^{(2)}$. We start reproducing the constraint (4.3) with

$$\frac{1}{4} g_{ik}^{(2)} g^{(0)kl} g_{lj}^{(2)} = 2 \varepsilon^{kk'} \varepsilon^{ll'} (f_i \cdot e_k) (e_{k'} \cdot e_{l'}) (e_l \cdot f_j) = \frac{1}{2} f_i \cdot f_j = g_{ij}^{(4)}, \quad (4.12)$$

where ε^{ij} is the levi-civita tensor with curved indices defined in (B.4). In the first equality, we used the expression in (B.6) for the inverse of $g_{ij}^{(0)}$. For the second equality, we used the relation (B.7). The left constraint in (4.4) is recovered by

$$\text{tr} \left((g^{(0)})^{-1} g^{(2)} \right) = -4 \varepsilon^{ii'} \varepsilon^{jj'} (e_{i'} \cdot e_{j'}) (e_i \cdot f_j) = -\varepsilon^{ij} \partial_i \omega_j = \frac{R^{(0)}}{2}. \quad (4.13)$$

For the second equality we used (B.7), and the equation in the left of (3.5), and for the last equality, the definition of the Ricci scalar in (B.9) in two dimensions. We recover the

first constraint in (4.4) with $\ell = 1$. The right constraint in (3.5) can be written in terms of the covariant derivative (A.2) as:

$$\nabla_{[i} f_{j]}^a = 0. \quad (4.14)$$

The previous equation, together with (A.3), are enough to reproduce the differential condition over $g^{(2)}$ in (4.4).

In section 3.2, we found a variational principle for the solutions (3.7) that keeps the boundary value of the CS connections fixed. We will now show that this variational principle is equivalent to the one in the metric formulation reviewed in section 4.1. To start the comparison, we translate the term (4.6) to the vielbein formalism using (4.9):

$$\delta S = 4 \int_{\partial \mathcal{M}} d^2 x e T_a^i \delta e_i^a, \quad (4.15)$$

where we used the relation between determinants below (B.4). Comparing (3.14) with (4.15), we can identify the components of the stress-energy tensor. Expanding the wedge product, its compact expression turns out to be

$$T_a^i = \frac{k}{\pi} \varepsilon_{ab} \varepsilon^{ij} f_j^b. \quad (4.16)$$

It is interesting to compute the trace of the stress tensor:

$$T_i^i = T_a^i e_i^a = \frac{k}{2\pi} \varepsilon^{ij} \partial_i \omega_j = -\frac{1}{16\pi G_3} R^{(0)} \quad (4.17)$$

where we have used the left constraint in (3.6), and identification of Chern-Simons level (2.4), and with $\ell = 1$. This is the CFT Weyl anomaly we have recovered in the metric formulation in (4.8). Moreover, we can show that (4.7) is equal to (4.16) by changing to curved indices appropriately contracting with the vielbein. Therefore, the definitions of the variational principle coincide in the metric and Chern-Simons solutions. To finish, we would like to comment that, as expected, the Bañados connections (2.10) represent solutions with flat metric at the boundary, and have null Weyl anomaly.

5 $T\bar{T}$ -deformation on the boundary of Chern-Simons AdS_3 gravity

In this section, we use the general AdS_3 solutions (3.7) in Fefferman-Graham gauge, and the variational principle defined in (3.14) to study double-trace operators in the dual theories of $\text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R})$ Chern-Simons gauge theory. We will analyze a specific deformation in the boundary of the Chern-Simons theory that will turn out to be $T\bar{T}$ -deformations in dual CFT.

5.1 Mixed boundary conditions in Chern-Simons formalism

In section 3.2, we found a variational principle with Dirichlet boundary conditions for the CS connections (3.7). This allows us to interpret the Chern-Simons boundary action (3.14) in the CFT, with the quantities e^a and f^a acting as sources and expectation values of the

dual operator, respectively. In this section, we will use this knowledge to study the addition of double-trace combinations of f^a to the boundary action. We focus on double-trace deformations of the type:

$$S_{f+f^-} = \frac{1}{2} \left(\frac{k}{\pi} \right)^2 \int_{\partial\mathcal{M}} f^- \wedge f^+. \quad (5.1)$$

So far, we have not motivated the choice of this double-trace operator, but it will become clear in section 5.2. It is well known [13, 14] that performing double-trace deformations in a CFT with large N is equivalent to a change of the boundary conditions of the dual bulk quantities. From the variational principle, one can find modified relations between the asymptotic value of the bulk fields, and the sources and the expectation values [32, 33]. For a more recent review of the procedure, see [34]. We start by writing the variation of the action in the deformed theory as the variation of a new on-shell action, where the sources and operators depend on the deformation. In our case, this reads

$$\delta S(0) + \lambda \delta S_{f+f^-}(0) = \delta S(\lambda) = \frac{k}{\pi} \int_{\partial\mathcal{M}} \varepsilon_{ab} f^a(\lambda) \wedge \delta e^b(\lambda), \quad (5.2)$$

where δS is the action of the original theory (3.14), and λ is the deformation parameter. The label inside the parenthesis indicates if the quantities belong to the original or the deformed theory. We need to solve (5.2) for $f^a(\lambda)$ and $e^b(\lambda)$ to find how the vevs and sources change in term of the deformation parameter. This is equivalent to solving the following equation:

$$\frac{\partial}{\partial \lambda} \delta S(\lambda) = -\delta S_{f+f^-}(\lambda), \quad (5.3)$$

which gives

$$\begin{aligned} \varepsilon_{ab} \frac{\partial e^a}{\partial \lambda} \wedge \delta f^b + \varepsilon_{ab} e^a \wedge \delta \left(\frac{\partial f^b}{\partial \lambda} \right) &= -\frac{k}{2\pi} \left(\delta f^- \wedge f^+ + f^- \wedge \delta f^+ \right) \\ &= \frac{k}{2\pi} \varepsilon_{ab} f^a \wedge \delta f^b, \end{aligned} \quad (5.4)$$

where e^a , and f^a depend on the deformation parameter λ , but we leave the dependence implicit in this equation for simplicity in the notation. The second line in (5.4) allows us to see that (5.3) is solved by:

$$\frac{\partial e_i^a(\lambda)}{\partial \lambda} = \frac{k}{2\pi} f_i^a(\lambda), \quad \frac{\partial f_i^a(\lambda)}{\partial \lambda} = 0. \quad (5.5)$$

These are the equations for the flow of the CS parameters under a $T\bar{T}$ -deformation. The flow of the spin-connection ω can be found from the constraints in (3.6) using the flow of the other CS parameters. As we found, the left equation in (3.6) is the torsionless condition (A.4) in 2 dimensions in vielbein formalism, whose solution we know for the spin connection [35]:

$$\omega_i(\lambda) = e_{ia}(\lambda) (*de^a(\lambda)), \quad (5.6)$$

where the symbol $*de^a$ represents the Hodge dual of the 2-form de^a .

Now, we consider the following initial conditions at the beginning of the flow:

$$e_i^a(0) = e_i^a, \quad f_i^a(0) = f_i^a, \quad \omega_i(0) = 0, \quad (5.7)$$

which means we have choose a flat theory at the boundary, because the Ricci scalar (4.17) is zero. It is easy to solve the equations (5.5), and with this boundary conditions:

$$e_i^a(\lambda) = e_i^a + \lambda \frac{k}{2\pi} f_i^a, \quad f_i^a(\lambda) = f_i^a. \quad (5.8)$$

We see that the parameter $f^a(\lambda)$ is constant under the flow, and $e^a(\lambda)$ has a linear behaviour in λ . As expected, after the addition of the deformation, the boundary conditions mix the sources and expectations of the expectation values. We can deduce the flow of spin-connection using (5.6) with (5.8):

$$\omega_i(\lambda) = \left(2e_{ai} + \lambda \frac{k}{2\pi} f_{ai} \right) \left(*de^a + \lambda \frac{k}{2\pi} (*df^a) \right) = 0, \quad (5.9)$$

where in the last equality we have used that $de^a = df^a = 0$, since the initial parameters e^a and f^a follow the constraints (3.6) for $\omega_i = 0$. We see that if we start with a theory with a flat boundary theory, it will remain flat after evolution under a $f^+ f^-$ -deformation.

5.2 $f^+ f^-$ as a $T\bar{T}$ -deformation

In section 5.1, we analysed the flow of under the double trace deformation (5.1). Now, we will see that $f^+ f^-$ is equivalent to the usual $T\bar{T}$ operator in CFT. Remember the definition [36]:

$$T\bar{T} \equiv T^{ij}T_{ij} - T^2 = -2\varepsilon^{ab}\varepsilon_{ij}T_a^i T_b^j, \quad (5.10)$$

where in the second equality we have conveniently written it in terms of the flat indices using the vielbein. We consider the action of this double-trace operator

$$S_{T\bar{T}} = - \int_{\partial\mathcal{M}} d^2x e \varepsilon^{ab}\varepsilon_{ij}T_a^i T_b^j, \quad (5.11)$$

and using the holographic expression for the stress tensor in (4.16) to write the this action in terms of the Chern-Simons parameters we obtain

$$S_{T\bar{T}} = - \left(\frac{k}{\pi} \right)^2 \int_{\partial\mathcal{M}} d^2x e \varepsilon_{ab}\varepsilon^{ij} f_i^a f_j^b = \frac{1}{2} \left(\frac{k}{\pi} \right)^2 \int_{\partial\mathcal{M}} f^- \wedge f^+. \quad (5.12)$$

Therefore the $T\bar{T}$ -deformation is equivalent to the is equivalent to the $f^+ f^-$ -deformations (5.1).

It is instructive to compare with the metric formalism. In [37], they use the variational principle method to analyse the flow of operators and sources after a $T\bar{T}$ -deformation in a CFT. They then interpret this results holographically, and find that the metric field has mixed boundary conditions after the deformation. Here, we will see that the mixed boundary conditions found in [37], are equivalent to (5.8). For that we use (4.9), which we write as

$$g_{ij}^{(0)}(\lambda) = 2e_i(\lambda) \cdot e_j(\lambda) = 2e_i \cdot e_j + 2\frac{k\lambda}{\pi} f_i \cdot e_j + \frac{1}{2} \left(\frac{k\lambda}{\pi} \right)^2 f_i \cdot f_j. \quad (5.13)$$

Using the expressions (4.9) and (4.11), we can rewrite in terms of metric quantities in the original theory:

$$g_{ij}^{(0)}(\lambda) = g_{ij}^{(0)} - \frac{k\lambda}{\pi} g_{ij}^{(2)} + \left(\frac{k\lambda}{\pi}\right)^2 g_{ij}^{(4)}. \quad (5.14)$$

Comparing with (4.2), we see that (5.13) can be interpreted as an induced metric at a surface of constant $z = z_c$, where $z = z_c$ is defined as

$$z_c^2 = -\frac{k\lambda}{\pi} = -\frac{\lambda}{4\pi G_3} \quad (5.15)$$

where we have used (2.4) with $\ell = 1$. The interpretation as induced metric at $z = z_c$ is only valid for when $\mu < 0$. With this, we recovered exactly the result found in [37]. Before moving on, we observe that the boundary values of the connections (3.7) flow as:

$$A|_{z \rightarrow 0} = \left(e^+ + \lambda \frac{k}{2\pi} f^+\right) \frac{2L_+}{z}, \quad \bar{A}|_{z \rightarrow 0} = -\left(e^- + \lambda \frac{k}{2\pi} f^-\right) \frac{2L_-}{z}. \quad (5.16)$$

We cannot find a way to naturally rewrite (5.16) as an induced connections at a certain cutoff. It is surprising there is not a similar argument to (5.14) to deduce the radial cutoff from the Chern-Simons boundary connections.

5.3 $T\bar{T}$ -deformation in terms of A , and $\bar{A}$

We would like to write (5.1) in terms of the full connections (3.7). In this way, we can analyse if the $T\bar{T}$ -deformation has a natural interpretation in the CS formulation of 3d gravity. To recover (5.1), we need combination of f^+ , and f^- , and we should consider a term that combines A , and $\bar{A}$. Even though this is a non-standard choice in the Chern-Simons formulation of 3d gravity, there is nothing that holds us from doing so.² Let me start by considering the simplest combination of A , and $\bar{A}$:

$$\frac{k}{4\pi} \int_{\partial\mathcal{M}} \text{Tr} \left(A \wedge \bar{A} \right) = -\frac{k}{\pi} \frac{1}{z^2} \int_{\partial\mathcal{M}} e^+ \wedge e^- - \frac{k}{4\pi} z^2 \int_{\partial\mathcal{M}} f^- \wedge f^+. \quad (5.17)$$

The first term in (5.17) is divergent at the boundary. It is interesting to note that the integrand in the first term is the determinant of the vielbein, i.e. $e^+ \wedge e^- = 2e \, d^2x$. Therefore, it is equivalent to the divergent part of the counterterm (4.5) added in the metric formalism, upon the expansion (4.2). Analogously to the metric formalism, we get cancel the divergence by adding an appropriate counterterm:

$$\frac{k}{4\pi} \int_{\partial\mathcal{M}} \text{Tr} \left(A \wedge \bar{A} \right) - \frac{k}{4\pi} \int_{\partial\mathcal{M}} \varepsilon_{ab} A^a \wedge \bar{A}^b = -\frac{k}{2\pi} z^2 \int_{\partial\mathcal{M}} f^- \wedge f^+. \quad (5.18)$$

This term is similar to (5.1), but it depends in the radial coordinate z . To have a proper interpretation of (5.18) as a $T\bar{T}$ -deformation in the boundary, this should not be the case. We could use the identification $z = z_c$ with (5.15) to avoid this problem. Again, the symbol λ in z_c , would act as deformation. However, as explained in (5.16), the identification of this boundary cutoff is not natural from the point of view of the CS connections.

²See [38], for an example of boundary condition that mixes both sectors.

Another candidate to reproduce the $T\bar{T}$ deformation in terms of the connections A , and $\bar{A}$:

$$\begin{aligned} S_{T\bar{T}} &= \int_{\partial\mathcal{M}} e d^2x \epsilon^{ik} \epsilon^{jl} \text{tr}(A_i A_j) \text{tr}(\bar{A}_k \bar{A}_l) \\ &= \int_{\partial\mathcal{M}} d^2x \frac{1}{e} \eta_{ab} \eta_{cd} (A^a \wedge \bar{A}^c) (A^b \wedge \bar{A}^d). \end{aligned} \quad (5.19)$$

This term is designed to exactly reproduce (5.1), up to an overall normalization constant, for (3.7) with $\omega_i = 0$. The advantage of this term is that we do not have explicit z -dependence, so it is natural to consider it as a pure boundary deformation. However, this advantage brings a more complicated combination of A , and $\bar{A}$, as a downside. In particular, we need an explicit dependence in the determinant of the vielbein. This is similar to the what happens the Chern-Simons $U(1)$ gauge field in [39], where we need to couple to the boundary metric to find a well variation principle. However, it is strange from the point of view of Chern-Simons as a gravitational theory, since all the information about the metric should be encoded in the connections.

We have proposed two terms that could perform a $T\bar{T}$ -deformation in terms of the connections A , and $\bar{A}$. However, it is not clear how natural they are from the point of view of the Chern-Simons variables.

6 Conclusions

In this article, we studied $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$ Chern-Simons gauge theories in AdS_3 , and we presented two main results. The first one is the solution of the Chern-Simons equations of motion with generalized boundary conditions (3.7). We have shown that these connections are equivalent to the Fefferman-Graham expansion (4.2) in metric formalism. Moreover, we proved that we can interpret the different components of the gauge field as CFT quantities: e_i , ω_i , and f_i are related to a 2-dimensional vielbein, spin connection, and stress tensor, respectively. We have done this by proposing a variational principle with Dirichlet boundary conditions for the gauge connections. This allowed us to study how the boundary of the Chern-Simons theory behaves under a specific type double-trace deformation introduced in (5.1). Using the variational principle method, we have found the mixed boundary after the deformation (5.8) in the Chern-Simons formalism. This is our second result. We showed that this deformation corresponds to a $T\bar{T}$ -deformation from the point of view of the CFT.

Higher spin theories of gravity are very straight-forwardly generalized from the description of AdS_3 gravity as Chern-Simons gauge theory. The generalization amounts to extend the gauge group from $SL(2, \mathbb{R})$ to $SL(N, \mathbb{R})$. In [7], they find solutions to these theories that are considered asymptotically AdS_3 with flat trivial boundary. Assuming (2.8) these are

$$a_{x^+} = L_1 + \sum_{s=2}^N J_{(s)}(x^+) W_{-s+1}^{(s)}, \quad \bar{a}_{x^-} = -L_{-1} + \sum_{s=2}^N \bar{J}_{(s)}(x^-) W_{s-1}^{(s)}, \quad (6.1)$$

where $J_{(s)}(x^+)$, and $\bar{J}_{(s)}(x^-)$ are any arbitrary function, and $a_{x-} = \bar{a}_{x+} = 0$. Here $\{L_0, L_{\pm 1}\}$ are the generators of the $sl(2, \mathbb{R})$ subalgebra in $sl(N, \mathbb{R})$, and $W_j^{(s)}$ are the spin- s generators with $j = -(s-1), \dots, (s-1)$. It would be interesting to generalise these solutions to have more generic boundary conditions of the type (3.7), as we proposed in this paper for the $sl(2, \mathbb{R})$ case. This, together with a well-defined variational principle, could allow us to formulate double-trace deformations involving spinning fields in Chern-Simons theory. We leave this idea as material for future work.

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A Vielbein formalism

This appendix is a very short review, and collection of useful identities, of the vielbein formulation of general relativity. For some more context, see, for example [40].

In the vielbein formalism, the choice of coordinates x^i on a manifold is replaced by the election of a local basis. This basis is formed by the d -dimensional vectors, $e^a = e_i^a dx^i$, known as vielbeins. We can relate the metric of a curved manifold g_{ij} to a flat (non-coordinate) metric η_{ab} using the vielbein:

$$g_{ij} = \eta_{ab} e_i^a e_j^b. \quad (\text{A.1})$$

The vielbein is used to change the indices of the tensors from curved to flat, or viceversa. For covariant derivatives of tensors with flat indices, the role of the Christoffel symbol is replaced by the spin connection, characterized by $\omega_i^a{}_b$. For example, the necessary contractions in the case of a tensor with mixed indices X_j^a are:

$$\nabla_i X_j^a = \partial_i X_j^a + \omega_i^a{}_b X_j^b - \Gamma_{ij}^\lambda X_\lambda^a. \quad (\text{A.2})$$

The following identity is required by construction of the vielbein formalism:

$$\nabla_i e_j^a = 0, \quad (\text{A.3})$$

which is known as the tetrad postulate. Notice that (A.3) defines the spin connection in terms of the Christoffel symbols, and the vielbein. Moreover, the zero torsion condition, i.e., $\Gamma_{ij}^\lambda = \Gamma_{ji}^\lambda$ can be written in vielbein formalism as:

$$de^a + \omega^a{}_b \wedge e^b = 0. \quad (\text{A.4})$$

The condition over the flat metric $\nabla_i \eta_{ab} = 0$ imposes the so-called metricity condition:

$$\omega_{iab} = -\omega_{iba}, \quad (\text{A.5})$$

where $\omega_{iab} = \eta_{ac} \omega_i^c{}_b$. It is also useful to write the curvature tensor with flat indexes as:

$$R^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b, \quad (\text{A.6})$$

where we defined $R^a{}_b = R^a{}_{bij} dx^i \wedge dx^j$.

B Conventions and useful identities

The review of vielbein formalism made in appendix A is generic to any dimension. However, in this paper we have considered two different specific types of vielbein theories: a 3-dimensional one representing the bulk, and 2-dimensional one in the boundary. This appendix summarizes our conventions for both cases, and collects identities that are relevant for our manipulations.

B.1 Vielbein formalism in 2 dimensions

In the main text we described a 2-dimensional theory in vielbein formalism, whose curved indices are represented by $\{i, j\} = x^+, x^-$, and flat indices $\{a, b\} = +, -$. Our conventions for the flat metric η_{ab} are

$$\eta_{+-} = \eta_{-+} = -1, \quad (\text{B.1})$$

and zero otherwise. In two dimensions, the metricity condition (A.5) imposes that

$$\omega_i^{ab} = \varepsilon^{ab} \omega_i, \quad (\text{B.2})$$

where ω_i is the only independent component of the 2-dimensional spin connection. The symbol ε^{ab} is the Levi-Civita defined as

$$\varepsilon_{+-} = -\varepsilon_{-+} = 1. \quad (\text{B.3})$$

One can raise and lower indices of the Levi-Civita tensor with the flat metric, as, for example: $\varepsilon_b^a = \eta^{ab'} \varepsilon_{b'b}$. It is useful also to define ε^{ij} as Levi-Civita tensor with curved indices:

$$\varepsilon^{x^+ x^-} = -\varepsilon^{x^- x^+} = \frac{1}{\sqrt{-g}} = \frac{1}{2e}, \quad (\text{B.4})$$

where $g = \det(g)$ and $e = \det(e)$ are the determinants of metric and the vielbein, which are related via $\det(g) = -4 \det(e)^2$. The definition of determinant we used, for a generic tensor X_i^a :

$$\det(X) = e \varepsilon_{ab} \varepsilon^{ij} X_i^a X_j^b = X_{x^+}^+ X_{x^-}^- - X_{x^-}^+ X_{x^+}^-. \quad (\text{B.5})$$

We use the Levi-Civita symbols to write the inverse of the metric as

$$g^{ij} = \varepsilon^{ii'} \varepsilon^{jj'} g_{i'j'}. \quad (\text{B.6})$$

Other useful relations using the Levi-Civita symbols are the following contractions of vielbeins:

$$\varepsilon^{ij} e_i^a e_j^b = -\frac{\varepsilon^{ab}}{2}, \quad e_a^i e^j_b \varepsilon^a_b = -2\varepsilon^{ij}. \quad (\text{B.7})$$

We can use them, for example, to write find a compact expression for the Ricci scalar from the Riemann curvature tensor (A.6). In 2d, using (B.2), we have

$$R_b^a = \epsilon_b^a d\omega \quad \Leftrightarrow \quad R_{bij}^a = \epsilon_b^a \partial_{[i} \omega_{j]}. \quad (\text{B.8})$$

We find the trace of R_{bij}^a by contracting its indices with the vielbein:

$$R = e_a^i e^{jb} R_{bij}^a = -2\varepsilon^{ij} \partial_i \omega_j \quad (\text{B.9})$$

where we have used the contraction of vielbein in the right of (B.7).

B.2 Vielbein formalism in 3 dimensions

In 3-dimensions, we used the following definitions for the vielbein and spin connection:

$$E^m = E_\mu^m dx^\mu, \quad \Omega^m = \frac{1}{2} \varepsilon^{mnl} \Omega_{\mu nl} dx^\mu, \quad (\text{B.10})$$

where the capital the letters E and, Ω to make a distinction with the 2-dimensional case described in section B.1. To avoid confusion, we used also different nomenclature for the 3-dimensional coordinate indices $\{\mu, \nu\}$, and the flat indices $\{m, n, l\} = 0, \pm 1$, which in this case are directly related to the $sl(2, \mathbb{R})$ algebra:

$$[L_0, L_\pm] = \mp L_\pm, \quad [L_1, L_{-1}] = 2L_0. \quad (\text{B.11})$$

Our conventions for the fundamental representation of $sl(2, \mathbb{R})$ are

$$L_0 = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}, \quad L_1 = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \quad L_{-1} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \quad (\text{B.12})$$

The Lie algebra metric reads

$$\text{Tr}(L_0 L_0) = \frac{1}{2}, \quad \text{Tr}(L_+ L_-) = \text{Tr}(L_- L_+) = -1. \quad (\text{B.13})$$

A useful identity between the generators of the algebra is:

$$e^{-L_0 \alpha} L_m e^{L_0 \alpha} = L_m e^{m \alpha}. \quad (\text{B.14})$$

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