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Construct α' corrected or loop corrected solutions without curvature singularities

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ABSTRACT: For the bosonic gravi-dilaton system, we provide systematical approaches to construct non-perturbative string cosmological solutions without curvature singularities, which can match the perturbative solution to any order in α' expansion. When higher order α' corrections are calculated, they can be straightforwardly plugged in to generate compatible non-perturbative evolutions without curvature singularities. We also give a (phenomenological) map between α' corrected EOM and loop corrected EOM. This map enables us to easily generate a loop corrected solution from an α' corrected solution, and vice versa, therefore substantially enlarges the solution space.

KEYWORDS: Bosonic Strings, Nonperturbative Effects, Spacetime Singularities, String Duality

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1 Introduction

An important challenge for string theory is to show how the big-bang singularity could be resolved. In the Einstein gravity, the big-bang singularity is the initial singularity. Nevertheless, in the traditional (tree level) string cosmology, a “scale-factor” duality emerges [1–6]. This duality combined with time reversal yields a new phase: the pre-big-bang [7–10]. The big-bang singularity splits the pre-big-bang and post-big-bang into two disconnected regions. To be specific, we set the spacetime dimensionality to be $D = d + 1$ and work with bosonic string theory. The scale-factor duality turns out to be a special case of a more general symmetry, the $O(d, d)$ symmetry. This duality has no descendant in the Einstein gravity since the dilaton transform nontrivially.

Beyond the perturbative regime, the tree level¹ string effective action receives two kinds of corrections: the higher-derivative expansion, controlled by the squared string length α' , and the higher-genus expansion, controlled by the string coupling $g_s = e^{2\phi}$. Ignoring matter sources, the most general perturbative form of the string effective action has the following structure

$$\begin{aligned}
 I = \int d^{d+1}x \sqrt{-g} \Big\{ & e^{-2\phi} \left[\left(R + 4(\partial\phi)^2 - \frac{1}{12}\mathcal{H}^2 \right) + \frac{\alpha'}{4}(R_{\mu\nu\sigma\rho}R^{\mu\nu\sigma\rho} + \dots) + \mathcal{O}(\alpha'^2) \right] \\
 & + \left[(c_R^1 R + c_\phi^1 (\partial\phi)^2 + c_{\mathcal{H}}^1 \mathcal{H}^2) + \alpha' (c_{\alpha'R}^1 R_{\mu\nu\sigma\rho} R^{\mu\nu\sigma\rho} + \dots) + \mathcal{O}(\alpha'^2) \right] \\
 & + e^{2\phi} \left[(c_R^2 R + c_\phi^2 (\partial\phi)^2 + c_{\mathcal{H}}^2 \mathcal{H}^2) + \alpha' (c_{\alpha'R}^2 R_{\mu\nu\sigma\rho} R^{\mu\nu\sigma\rho} + \dots) + \mathcal{O}(\alpha'^2) \right] \\
 & + \dots \Big\}, \tag{1.1}
 \end{aligned}$$

where ϕ is the dilaton and $\mathcal{H}_{\mu\nu\rho} = 3\partial_{[\mu}b_{\nu\rho]}$ is the field strength of the antisymmetric Kalb-Ramond field $b_{\mu\nu}$. For simplicity, we set $b_{\mu\nu} = 0$ in this paper. All the coefficients $c_{[\dots]}^i$ are yet unknown. Each line contains a full expansion in α' . In terms of genus, the first

¹If not specified, “tree level” indicates the lowest order in both α' and loop.

line is the tree level terms with complete stringy contributions, the second line is the full one-loop contribution, and so on. Throughout this paper, we always working with FLRW background

$$ds^2 = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j, \quad (1.2)$$

with the Hubble parameter $H \equiv \dot{a}/a$. The traditional tree level cosmology does not take into account the α' and loop corrections, thus is valid only in the perturbative regime $g_s \rightarrow 0$ and $\alpha'H^2 \rightarrow 0$. As the universe approaches the big-bang region, there would be $g_s \rightarrow 1$, $\alpha'H^2 \rightarrow 1$ or both. It is then natural to anticipate the α' or loop corrections could regularize the big-bang singularity. Indeed, by implementing some non-local dilaton potentials which account for non-perturbative effects caused by the dilaton, the loop corrections could smooth out the singularity [7, 11–13].

However, there is not much progress on how to resolve the big-bang singularity with α' corrections. The main reason is that, the higher-derivative α' corrections usually would change the order of the differential equations in the equations of motion (EOM). At the tree level, the EOM are second order differential equations, at the first order in α' , the EOM become fourth order differential equations, and so on. In [14], by assuming the heterotic string admits non-singular constant curvature solutions in the Einstein frame, an $O(d, d)$ violating first order α' correction was chosen. In terms of the scale factor $a(t)$, the EOM, as expected, are fourth order differential equations. A carefully designed effective dilaton potential was further brought in to support a non-singular evolution from an early-time de-Sitter phase to a late-time Minkowski spacetime.

Notwithstanding little hope to conduct analysis on the higher order α' corrections, inspiringly, for the first order α' correction, by using some field redefinitions, it turns out that the fourth order derivatives can be eliminated [15]. Thus the EOM are still second order differential equations. This nice property enables the authors of ref. [16] to numerically verify that the perturbative string vacuum could connect with some “fixed-points”, at the cost of the scale-factor duality.

Recently, the situation has changed by the remarkable work of Hohm and Zwiebach [17–19]. Early works in refs. [2–5] showed that for cosmological background, all orders in α' expansion possess an $O(d, d)$ symmetry. Moreover, to the first order in α' , the $O(d, d)$ matrix can maintain the standard form in term of α' corrected fields [20]. With a reasonable assumption that this property also holds for all orders in α' , Hohm and Zwiebach proved that for time dependent configurations, only first order time derivatives of the fields appear in the action. An immediate consequence of this striking simplification is that the EOM with complete α' corrections are still second order differential equations.

The Hohm-Zwiebach action paves the way to seriously address the non-perturbative features sourced by α' corrections. This remarkable result leads them to show that, in bosonic string theory, non-perturbative de-Sitter (dS) vacua are admitted by including complete α' corrections. In [21], the analogy in the Einstein frame is then discussed. In our recent work [22], we showed that similar stories occur for configurations depending on a single space coordinate, and non-perturbative Anti-de-Sitter (AdS) vacua are also allowed. Furthermore, we conjectured that the non-perturbative AdS and dS vacua might not be able to coexist in bosonic string theory.

The Hohm-Zwiebach action also sheds light on the resolution of the big-bang singularity. However, straightforward perturbative calculation does not work. As shown in ref. [19], the Hubble parameter and $O(d, d)$ invariant dilaton calculated order by order in α' are

$$\begin{aligned} H(t) &= \frac{1}{\sqrt{dt}} - \frac{5}{4} \frac{1}{d^{3/2}} \frac{\alpha'}{t^3} + h_2 \frac{\alpha'^2}{t^5} + h_3 \frac{\alpha'^3}{t^7} + \dots, \\ \Phi(t) &= -\frac{1}{2} \log(\gamma^2 t^2) - \frac{1}{2} \frac{t_0^2}{t^2} + \omega_2 \frac{t_0^4}{t^4} + \omega_3 \frac{t_0^6}{t^6} + \dots, \end{aligned} \quad (1.3)$$

where the coefficients h_i and ω_i are yet undetermined, γ is an integration constant and $t_0 \equiv \frac{\sqrt{\alpha'}}{\sqrt{2d}}$. The $O(d, d)$ invariant dilaton Φ is defined as $e^{-\Phi} = \sqrt{-g}e^{-2\phi}$. Obviously, higher order terms are more and more singular. After realizing the above solution is actually valid in the perturbative regime $t \rightarrow \infty$ ($\alpha' \rightarrow 0$), in [23], we constructed non-perturbative solutions which are non-singular (non-singular in this paper refers to the curvature, but not to the string coupling behaviour²) in the whole regime $t \in (-\infty, \infty)$ for nonvanishing α' . The term “non-perturbative” refers to that the domain of the solution covers the non-perturbative regime and all α' corrections are included. Those non-singular non-perturbative solutions are justified by matching the first two orders of the perturbative solution eq. (1.3) exactly and having the same expansion behaviors at higher orders, in the perturbative regime $t \rightarrow \infty$ ($\alpha' \rightarrow 0$).

Hitherto, in the perturbative α' expansion, orders higher than one are unknown. So the solutions constructed in [23] only need to match the first two orders. An inspiring question naturally arises: is there any guidance to construct non-singular non-perturbative solutions when orders higher than one are calculated in the future? The trial and error is a very inefficient method and becomes unpractical for high orders. One of the purposes of this paper is to do this job. We are going to provide two formulas to easily construct “more accurate” non-singular non-perturbative solutions, which can match the perturbative results to an arbitrary order. In contrast to the perturbative solution which is more and more singular at higher orders, every term in our solutions is non-singular. One more α' correction is provided, one more non-singular term is fixed. This process continues to any order.

Moreover, we find a very useful and suggestive (phenomenological) map between the EOM corrected by α' and the EOM corrected by loops. The effective dilaton potentials which represent loop corrections can be mapped to some functions of α' corrections. With this map, one can easily generate an α' corrected solution from a loop corrected solution, or vice versa. It turns out it is much easier to construct α' corrected solutions with our method than to find loop corrected solutions. Therefore, this map substantially enlarges the solution space of the traditional string cosmology, and one may analysis more scenarios. Furthermore, the new loop corrected solutions generated from the α' corrected solutions we constructed are more consistent and reasonable than those given in literature.

²We wish to address that the term “non-singular” in our previous work [23] and this paper means that the curvature and $O(d, d)$ dilaton Φ have no singularities. However, the string coupling which is controlled by the physical dilaton $g_s = e^{2\phi} = \sqrt{-g}e^{\Phi}$, blows up as $t \rightarrow \infty$ in our solutions. So, more precisely, what we provide are solutions without curvature singularities. We are indebted to the anonymous referee to help us clarify this confusion. In an upcoming paper, we will demonstrate that the string coupling can also be regularized by introducing a non-trivial Kalb-Ramond field $B(t)$ into the solutions [24].

The reminder of this paper is outlined as follows. In section 2, we show how to construct non-singular non-perturbative solutions to any order in α' expansion. In section 3, we present a (phenomenological) map between α' corrected EOM and loop corrected EOM. We also give some examples. Section 4 is the conclusion.

2 Solutions without curvature singularities to an arbitrary order in α'

It is well known that, for FLRW background (1.2), the tree level string effective action can be put into an explicit $O(d, d)$ covariant form. This is also true at the first order in α' with appropriate field redefinitions [20]. Based on a reasonable assumption that, to all orders in α' , the standard $O(d, d)$ matrix can be maintained by field redefinitions, in [18, 19], a substantial simplification on the α' corrections is achieved:

$$I = \int d^D x \sqrt{-g} e^{-2\phi} \left(R + 4(\partial\phi)^2 + \frac{1}{4}\alpha' (R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} + \dots) + \alpha'^2(\dots) + \dots \right), \quad (2.1)$$

$$= \int dt e^{-\Phi} \left(-\dot{\Phi}^2 + \sum_{k=1}^{\infty} (\alpha')^{k-1} c_k \text{tr} \left(\dot{\mathcal{S}}^{2k} \right) \right), \quad (2.2)$$

where $\Phi(t) = 2\phi(t) - \log \sqrt{-g}$ is the $O(d, d)$ invariant dilaton. Kalb-Ramond field is set to be zero for simplicity. The first line is the classical action in a general background. The second line is the Hohm-Zwiebach action in FLRW metric. The $2d \times 2d$ standard $O(d, d)$ matrix $\mathcal{S}$ is defined as

$$\mathcal{S} = \begin{pmatrix} 0 & a^2(t) \\ a^{-2}(t) & 0 \end{pmatrix}. \quad (2.3)$$

Thus far, in the Hohm-Zwiebach action (2.2), only $c_1 = -\frac{1}{8}$ and $c_2 = \frac{1}{64}$ for the bosonic string theory ($c_2 = \frac{1}{128}$ for heterotic string and $c_2 = 0$ for type II strings) are calculated through the beta functions of the non-linear sigma model, and $c_{k \geq 3}$ are undetermined constants. The EOM (generalized Friedmann equations) of (2.2) are given by

$$\begin{aligned} \ddot{\Phi} + \frac{1}{2} H f(H) &= 0, \\ \dot{\Phi}^2 + g(H) &= 0, \\ \frac{d}{dt} (e^{-\Phi} f(H)) &= 0, \end{aligned} \quad (2.4)$$

with

$$\begin{aligned} H(t) &= \frac{\dot{a}(t)}{a(t)}, \\ f(H) &= d \sum_{k=1}^{\infty} (-\alpha')^{k-1} 2^{2(k+1)} k c_k H^{2k-1} = -2dH - 2d\alpha' H^3 + \mathcal{O}(\alpha'^2), \\ g(H) &= d \sum_{k=1}^{\infty} (-\alpha')^{k-1} 2^{2k+1} (2k-1) c_k H^{2k} = -dH^2 - \frac{3}{2} d\alpha' H^4 + \mathcal{O}(\alpha'^2), \end{aligned} \quad (2.5)$$

where $H(t)$ is the Hubble parameter. Note

$$g'(H) = Hf'(H), \quad \text{and} \quad g(H) = Hf(H) - \int_0^H f(x)dx, \quad (2.6)$$

where $f'(H) \equiv \frac{d}{dH}f(H)$. The Hohm-Zwiebach action can be recast as

$$I_{HZ} = \int dt e^{-\Phi} \left(-\dot{\Phi}^2 + g(H) - Hf(H) \right). \quad (2.7)$$

In the perturbative regime $|t| \rightarrow \infty$ ($\alpha' \rightarrow 0$), using (2.5), the EOM can be solved iteratively to arbitrary order in $\frac{\sqrt{\alpha'}}{t}$,

$$\begin{aligned} H(t) &= \frac{\sqrt{2}}{\sqrt{\alpha'}} \left[\frac{t_0}{t} - 160c_2 \frac{t_0^3}{t^3} + \frac{256(770c_2^2 + 19c_3)}{3} \frac{t_0^5}{t^5} \right. \\ &\quad \left. - \frac{2048(88232c_2^3 + 4644c_3c_2 + 41c_4)}{5} \frac{t_0^7}{t^7} + \mathcal{O}\left(\frac{t_0^9}{t^9}\right) \right], \quad t_0 \equiv \frac{\sqrt{\alpha'}}{\sqrt{2d}} \\ \Phi(t) &= -\frac{1}{2} \log \left(\beta^2 \frac{t^2}{t_0^2} \right) - 32c_2 \frac{t_0^2}{t^2} + \frac{256(44c_2^2 + c_3)}{3} \frac{t_0^4}{t^4} \\ &\quad - \frac{2048(6976c_2^3 + 352c_3c_2 + 3c_4)}{15} \frac{t_0^6}{t^6} + \mathcal{O}\left(\frac{t_0^8}{t^8}\right), \end{aligned} \quad (2.8)$$

and

$$\begin{aligned} f(H(t)) &= -2dH - 128c_2 d\alpha' H^3 + 768c_3 d\alpha'^2 H^5 - 4096c_4 d\alpha'^3 H^7 + \mathcal{O}(\alpha'^4 H^9), \\ &= \frac{\sqrt{d}}{t_0} \left[-\frac{2t_0}{t} + 64c_2 \frac{t_0^3}{t^3} - \frac{512(50c_2^2 + c_3)}{3} \frac{t_0^5}{t^5} \right. \\ &\quad \left. + \frac{4096(2632c_2^3 + 124c_3c_2 + c_4)}{5} \frac{t_0^7}{t^7} + \mathcal{O}\left(\frac{t_0^9}{t^9}\right) \right], \\ g(H(t)) &= -dH^2 - 96c_2 d\alpha' H^4 + 640c_3 d\alpha'^2 H^6 - 3584c_4 d\alpha'^3 H^8 + \mathcal{O}(\alpha'^4 H^{10}), \\ &= \frac{1}{t_0^2} \left[-\frac{t_0^2}{t^2} + 128c_2 \frac{t_0^4}{t^4} - \frac{2048(50c_2^2 + c_3)}{3} \frac{t_0^6}{t^6} \right. \\ &\quad \left. + \frac{8192(24448c_2^3 + 1136c_3c_2 + 9c_4)}{15} \frac{t_0^8}{t^8} + \mathcal{O}\left(\frac{t_0^{10}}{t^{10}}\right) \right], \end{aligned}$$

where $\beta^2 = \gamma^2 t_0^2 = \gamma^2 \frac{\alpha'}{2d}$ is an integration constant, $t_0 \equiv \frac{\sqrt{\alpha'}}{\sqrt{2d}}$ and we used the universal $c_1 = -\frac{1}{8}$. Note for all solutions, their (scale-factor) dual solutions: $H(t) \rightarrow -H(t)$, $\Phi(t) \rightarrow \Phi(t)$, $f(t) \rightarrow -f(t)$ and $g(t) \rightarrow g(t)$ are always implied in this paper. This solution is obviously singular around the big-bang region $t = 0$. In a recent work [23] (where we set $\beta^2 = 4d$), we have constructed a pair of non-perturbative non-singular (scale-factor) dual solutions for the EOM (2.4), which exactly match the perturbative solution (2.8) in the

perturbative regime,

$$\begin{aligned}
H(t) &= -\frac{\sqrt{2}}{\sqrt{\alpha'}} \frac{(1-\tau^2)}{(1+\tau^2)^{3/2}}, & \tau \equiv \frac{t}{t_0} &= \frac{\sqrt{2d}}{\sqrt{\alpha'}} t, \\
\Phi(t) &= -\frac{1}{2} \log \beta^2 - \frac{1}{2} \log(1+\tau^2), \\
f(t) &= -\frac{2\sqrt{2}d}{\sqrt{\alpha'}} \frac{1}{\sqrt{1+\tau^2}} = -2dH - 2d\alpha' H^3 + \mathcal{O}(\alpha'^2), \\
g(t) &= -\frac{2d}{\alpha'} \frac{\tau^2}{(1+\tau^2)^2} = -dH^2 - \frac{3}{2}d\alpha' H^4 + \mathcal{O}(\alpha'^2).
\end{aligned} \tag{2.9}$$

After constructing the above solution which is consistent with the already known c_1 and c_2 , one may wonder when the coefficients $c_{k \geq 3}$ in eq. (2.5) of higher orders are available, what the compatible non-singular solutions would be? It would be very unpleasant if we have to do trial and error again and again. In particular, for orders very high, trial and error even becomes impossible. In the following, we are going to provide two methods to solve this problem.

Referring to the EOM (2.4), a very useful observation is that all other quantities could be determined by $\Phi(t)$:

$$\begin{aligned}
g(H(t)) &= -\dot{\Phi}^2, \\
f(H(t)) &= -\frac{2\sqrt{2}d}{\sqrt{\alpha'}} \beta e^{\Phi}, \\
H(t) &= \frac{\sqrt{\alpha'/2}}{\beta d} \frac{\ddot{\Phi}}{e^{\Phi}},
\end{aligned} \tag{2.10}$$

where the integration constant has been set to be consistent with the perturbative solution (2.8). Therefore, we only need to figure out a proper $\Phi(t)$ to make the solutions non-singular. As $t \rightarrow \infty$ ($\alpha' \rightarrow 0$), the ansatz $\Phi(t)$ must exactly match the perturbative solution (2.8), which ensures that $f(H)$ and $g(H)$ are identical to eq. (2.5). In addition, we should also check $H(t)$ is non-singular, since an inappropriate non-singular $\Phi(t)$ may lead to a singular $H(t)$. So, the core and most difficult part is to find the right ansatz for $\Phi(t)$. Fortunately, we already have a successful example eq. (2.9) to guide us to construct the following two solutions.

2.1 Solution A

The first solution is

$$\Phi(t) = \frac{1}{2} \log \left(\sum_{k=1}^{\infty} \frac{\lambda_k}{1+\tau^{2k}} \right), \quad \tau \equiv \frac{t}{t_0} = \frac{\sqrt{2d}}{\sqrt{\alpha'}} t. \tag{2.11}$$

The solution (2.9) is a special case with $\lambda_1 = 1/\beta^2$ and $\lambda_{k \geq 2} = 0$. From eqs. (2.10), we get

$$\begin{aligned}
 H(t) &= \frac{-\left(\sum_{k=1}^{\infty} \frac{2k\lambda_k\tau^{2k-1}}{(\tau^{2k}+1)^2}\right)^2 + \left(\sum_{k=1}^{\infty} \frac{\lambda_k}{\tau^{2k}+1}\right) \sum_{k=1}^{\infty} \left(\frac{8k^2\lambda_k\tau^{4k-2}}{(\tau^{2k}+1)^3} - \frac{2k(2k-1)\lambda_k\tau^{2k-2}}{(\tau^{2k}+1)^2}\right)}{\sqrt{2}\sqrt{\alpha'}\beta \left(\sum_{k=1}^{\infty} \frac{\lambda_k}{\tau^{2k}+1}\right)^{5/2}}, \\
 f(H(t)) &= -\frac{2\sqrt{2}\beta d}{\sqrt{\alpha'}} \sqrt{\sum_{k=1}^{\infty} \frac{\lambda_k}{\tau^{2k}+1}}, \\
 g(H(t)) &= -\frac{2d \left(\sum_{k=1}^{\infty} \frac{k\lambda_k\tau^{2k-1}}{(\tau^{2k}+1)^2}\right)^2}{\alpha' \left(\sum_{k=1}^{\infty} \frac{\lambda_k}{\tau^{2k}+1}\right)^2} \quad (2.12)
 \end{aligned}$$

One of the big advantages of the ansatz (2.11) is that as long as $\Phi(t)$ is non-singular, $H(t)$ is guaranteed to be non-singular. We therefore only need to care about the singularity of $\Phi(t)$. Another advantage of the ansatz (2.11) is that every individual term inside log is non-singular, in contrast to the perturbative solution where all terms are singular. Singularities appear if and only if

$$\sum_{k=1}^{\infty} \frac{\lambda_k}{1 + \tau^{2k}} = 0, \quad (2.13)$$

has real roots. In the perturbative regime $t \rightarrow \infty$ ($\alpha' \rightarrow 0$), the ansatz $\Phi(t)$ is expanded as,

$$\begin{aligned}
 \Phi(t/\sqrt{\alpha'} \rightarrow \infty) &= \frac{1}{2} \log \left(\frac{\lambda_1}{\tau^2} \right) + \frac{1}{2} \log \left(\sum_{k=1}^{\infty} \frac{1}{\tau^{2k-2}} \frac{\lambda_k/\lambda_1}{1 + 1/\tau^{2k}} \right) \\
 &= \frac{1}{2} \log \left(\frac{\lambda_1}{\tau^2} \right) + \frac{1}{2} \log \left(\frac{1}{1 + 1/\tau^2} + \sum_{k=2}^{\infty} \frac{1}{\tau^{2k-2}} \frac{\lambda_k/\lambda_1}{1 + 1/\tau^{2k}} \right) \\
 &= -\frac{1}{2} \log \left(\frac{\tau^2}{\lambda_1} \right) + \frac{\lambda_2 - \lambda_1}{2\lambda_1} \frac{1}{\tau^2} + \frac{\lambda_1^2 + 2(\lambda_2 + \lambda_3)\lambda_1 - \lambda_2^2}{4\lambda_1^2} \frac{1}{\tau^4} \\
 &\quad - \frac{\lambda_1^3 + 3(\lambda_2 - \lambda_3 - \lambda_4)\lambda_1^2 + 3\lambda_2(\lambda_2 + \lambda_3)\lambda_1 - \lambda_2^3}{6\lambda_1^3} \frac{1}{\tau^6} + \dots \quad (2.14)
 \end{aligned}$$

To match the perturbative solution (2.8), the coefficients λ_i are fixed:

$$\begin{aligned}
 \lambda_1 &= \frac{1}{\beta^2}, \quad \lambda_2 = 0, \quad \lambda_3 = \frac{4 + 512c_3}{3\beta^2}, \quad \lambda_4 = \frac{-4}{15\beta^2}(31 + 6272c_3 + 3072c_4), \\
 \lambda_5 &= \frac{8(1638400c_3^2 + 66688c_3 + 53248c_4 + 20480c_5 + 219)}{35\beta^2}, \\
 &\dots, \quad (2.15)
 \end{aligned}$$

where we used $c_2 = 1/64$. It is clear that λ_n is fixed by $c_{k \leq n}$. Guaranteed by the EOM, in terms of $H(t)$, $f(H)$ and $g(H)$ in (2.12) must be identical to eq. (2.5) after replacing λ_n by $c_{k \leq n}$. And it is easy to understand that matching $H(t)$ produces the same λ_i . The solution (2.11) is non-perturbative in the sense that it is defined in the whole regime

$t \in (-\infty, \infty)$ and α' does not need to approach zero. What we really show is that α' corrections do admit non-singular evolutions. Up to any order n , though $\lambda_{k \leq n}$ are fixed by the (in the future) known $c_{k \leq n}$, one always has freedom to choose $\lambda_{k > n}$ as any real value to violate the singular condition (2.13).

2.2 Solution B

Suppose the coefficients $c_{k \leq n}$ are known, another interesting ansatz is

$$\Phi(t) = -\frac{1}{2N} \log \left[\sum_{k=0}^N \rho_k \tau^{2k} \right], \quad \tau \equiv \frac{t}{t_0} = \frac{\sqrt{2d}}{\sqrt{\alpha'}} t, \quad (2.16)$$

where $N \geq n-1$ is some arbitrary integer and $\rho_0 > 0$, $\rho_N > 0$. Also from eqs. (2.10), we have

$$\begin{aligned} H(t) &= \frac{-\left(\sum_{k=0}^N \rho_k \tau^{2k}\right) \sum_{k=0}^N 2k(2k-1) \rho_k \tau^{2k-2} + \left(\sum_{k=0}^N 2k \rho_k \tau^{2k-1}\right)^2}{\sqrt{2} \sqrt{\alpha'} \beta N \left(\sum_{k=0}^N \rho_k \tau^{2k}\right)^{2-\frac{1}{2N}}}, \\ f(H(t)) &= -\frac{2\sqrt{2}\beta d}{\sqrt{\alpha'}} \left(\sum_{k=0}^N \rho_k \tau^{2k}\right)^{-\frac{1}{2N}}, \\ g(H(t)) &= -\frac{d \left(\sum_{k=0}^N 2k \rho_k \tau^{2k-1}\right)^2}{2\alpha' N^2 \left(\sum_{k=0}^N \rho_k \tau^{2k}\right)^2}. \end{aligned} \quad (2.17)$$

The solution (2.9) is a special case with $N=1$ and $\rho_0 = \rho_1 = \beta^2$. This solution has the same advantages as solution A: every single term inside log is non-singular; $\Phi(t)$ and $H(t)$ share the same singularity if and only if

$$\sum_{k=0}^N \rho_k \tau^{2k} = 0, \quad (2.18)$$

has real roots. In the perturbative regime $t \rightarrow \infty$ ($\alpha' \rightarrow 0$), the ansatz $\Phi(t)$ in (2.16) is expanded as,

$$\begin{aligned} \Phi(t) &= -\frac{1}{2} \log(\tau^2 \rho_N^{1/N}) - \frac{1}{2N} \\ &\times \left\{ \frac{\rho_{N-1}}{\rho_N} \frac{1}{\tau^2} + \frac{2\rho_N \rho_{N-2} - \rho_{N-1}^2}{2\rho_N^2} \frac{1}{\tau^4} + \frac{3\rho_N^2 \rho_{N-3} - 3\rho_N \rho_{N-1} \rho_{N-2} + \rho_{N-1}^3}{3\rho_N^3} \frac{1}{\tau^6} \right. \\ &\left. + \frac{4\rho_N^3 \rho_{N-4} - 2\rho_N^2 \rho_{N-2}^2 - 4\rho_N^2 \rho_{N-1} \rho_{N-3} + 4\rho_N \rho_{N-1}^2 \rho_{N-2} - \rho_{N-1}^4}{4\rho_N^4} \frac{1}{\tau^8} + \mathcal{O}\left(\frac{1}{\tau^{10}}\right) \right\}. \end{aligned} \quad (2.19)$$

To match the perturbative solution (2.8), the coefficients ρ_i are fixed:

$$\rho_N = \beta^{2N}, \quad \rho_{N-1} = N\beta^{2N}, \quad \rho_{N-2} = \frac{N\beta^{2N}}{6} (3N - 1024c_3 - 11), \quad \dots, \quad (2.20)$$

where we used $c_2 = 1/64$. It should be noted that only $\rho_N, \rho_{N-1} \cdots \rho_{N-n+1}$ are fixed by the known coefficients $c_1, c_2 \cdots c_n$. Other parameters $\rho_0, \rho_1 \cdots \rho_{N-n}$ can take any real numbers to violate the singular condition (2.18). In particular, we should set $\rho_0 > 0$ to avoid $t = 0$ becoming a singularity. Again, guaranteed by the EOM, in terms of $H(t)$, $f(H)$ and $g(H)$ in (2.17) must be identical to eq. (2.5) after replacing $\rho_{k \geq N-n+1}$ by $c_{k \leq n}$.

We close this section by rewriting the EOM (2.4) in another form, which will be used to derive a (phenomenological) map between α' corrected EOM and loop corrected EOM in next section. Note from eq. (2.6), we have

$$\dot{g}(H) = g'(H)\dot{H}(t) = Hf'(H)\dot{H}(t) = H\dot{f}(H), \quad (2.21)$$

Then one can easily verify that the EOM (2.4) can be recast as

$$\begin{aligned} 2\ddot{\Phi} - 2df(H)^2 + \frac{d}{dt} \left[g(H) + df(H)^2 \right] \frac{f(H)}{\dot{f}(H)} &= 0, \\ \dot{\Phi}^2 - df(H)^2 + \left[g(H) + df(H)^2 \right] &= 0, \\ \dot{f}(H) - f(H)\dot{\Phi} &= 0. \end{aligned} \quad (2.22)$$

3 A map between α' corrected EOM and loop corrected EOM

It was discovered long time ago that the big-bang singularity could be regularized by loop corrections. Referring to the complete string effective action (1.1), setting $\alpha' = 0$, we are left with a purely loop corrected theory. All higher genus corrections have the same structure as the tree level, but with unknown coefficients and different couplings. In the context of discussing singularity resolution, it is sufficient to implement some effective dilaton potentials to stand for loop corrections. However, since the physical dilaton ϕ is not an $O(d, d)$ scalar, a generalized non-local dilaton is introduced to keep the $O(d, d)$ symmetry [7, 11],

$$e^{-\Phi(x)} = \int d^{d+1}x' \sqrt{-g(x')} e^{-2\phi(x')} \sqrt{4|g^{\mu\nu} \partial_\mu \phi(x') \partial_\nu \phi(x')|} \delta(2\phi(x') - 2\phi(x)), \quad (3.1)$$

which reduces to the $O(d, d)$ dilaton in the FLRW background (1.2),

$$e^{-\Phi(t)} = V_d \int dt' \left| \frac{d(2\phi)}{dt'} \right| \sqrt{-g(t')} e^{-2\phi(t')} \delta(2\phi(t) - 2\phi(t')) = V_d \sqrt{-g(t)} e^{-2\phi(t)}. \quad (3.2)$$

A phenomenological loop corrected effective theory then is

$$\begin{aligned} I_{\text{Loop}} &= \int d^{d+1}x \sqrt{-g} e^{-2\phi} \left[R + 4(\partial_\mu \phi)^2 - V(e^{-\Phi(x)}) \right], \\ &= \int dt e^{-\Phi} \left[-\dot{\Phi} + dH^2 - V(e^{-\Phi}) \right], \end{aligned} \quad (3.3)$$

where in the second line, we applied the FLRW background. The EOM is [7, 11],

$$\begin{aligned} 2\ddot{\Phi} - 2dH^2 - \frac{\partial V}{\partial \Phi} &= 0, \\ \dot{\Phi}^2 - dH^2 - V &= 0, \\ \dot{H} - H\dot{\Phi} &= 0. \end{aligned} \quad (3.4)$$

Using the third equation, we have

$$\frac{\partial V}{\partial \Phi} = \frac{dV(\Phi)}{dt} \frac{1}{\dot{\Phi}} = \frac{dV}{dt} \frac{H(t)}{\dot{H}(t)}. \quad (3.5)$$

Therefore, the EOM (3.4) can be rewritten as

$$\begin{aligned} 2\ddot{\Phi} - 2dH^2 - \frac{dV}{dt} \frac{H(t)}{\dot{H}(t)} &= 0, \\ \dot{\Phi}^2 - dH^2 - V &= 0, \\ \dot{H} - H\dot{\Phi} &= 0. \end{aligned} \quad (3.6)$$

Comparing with the α' corrected EOM (2.22), we immediately identify a map between the loop corrected EOM and the α' corrected EOM,

$$\begin{aligned} \alpha' \text{ EOM (2.22)} : \quad & \text{Loop EOM (3.6)} : \\ g(H_{\alpha'}) + df(H_{\alpha'})^2 &\longleftrightarrow -V_L, \\ f(H_{\alpha'}) &\longleftrightarrow H_L, \\ \Phi_{\alpha'} &\longleftrightarrow \Phi_L + \Phi_0, \end{aligned} \quad (3.7)$$

where Φ_0 is a constant and the subscripts L and α' indicate to what corrections the quantities belong. It should be noted that in order to match the perturbative solution, we need to rescale $f(H_{\alpha'}) = -2dH_{\alpha'} + \dots$ by dividing $-2d$ after the mapping. This effectively can be accomplished by the constant Φ_0 . We want to stress that this does not mean there must exist such a map between the *true* complete loop corrections and complete α' corrections, since they might not share the same solution $\Phi(t)$ and the action (3.3) is a greatly simplified model. However, this *phenomenological* but instructive map is still very useful to mutually generate new solutions for either of them.

Generate α' corrected solutions from loop corrected solutions. In [11, 12], a class of phenomenological loop corrected solutions was constructed,

$$\begin{aligned} \Phi_L^{(n)}(t) &= \frac{1}{2n} \log \left(\frac{\sigma_n^{2n}}{1 + (m_n t)^{2n}} \right), \\ H_L^{(n)}(t) &= \frac{1}{\sqrt{d}} \frac{m_n}{\sigma_n} e^{\Phi_L^{(n)}(t)} = \frac{m_n}{\sqrt{d}} \left[\frac{1}{1 + (m_n t)^{2n}} \right]^{1/2n}. \end{aligned} \quad (3.8)$$

with a potential

$$V_L^{(n)} = \left(\frac{m_n}{\sigma_n} \right)^2 e^{2\Phi_L^{(n)}(t)} \left[\left(1 - \sigma_n^{-2n} e^{2n\Phi_L^{(n)}(t)} \right)^{\frac{2n-1}{n}} - 1 \right]. \quad (3.9)$$

where n is any positive integer and σ_n is a dimensionless coefficient. Since e^Φ roughly plays the role of a “dimensionally reduced” coupling constant, the parameter n is effectively a “loop counting” parameter and the potential (3.9) could be interpreted as the non-perturbative contributions from n th loop.

Using the identification (3.7), it is straightforward to generate a class of α' corrected solutions. By matching the perturbative α' corrected solution (2.8), the parameters n , σ_n and m_n are fixed

$$n = 1, \quad \sigma_1 = \frac{1}{\beta}, \quad m_1 = \sqrt{\frac{2d}{\alpha'}}. \quad (3.10)$$

Happily, the generated α' corrected solution is nothing but the solution (2.9), which was constructed in [23].

Generate loop corrected solutions from α' corrected solutions A. In section 2, we constructed a general class of α' corrected solutions (2.11) and (2.12), with the parameters λ_i fixed by matching the perturbative solution as in eq. (2.15). Applying the identification (3.7), we obtain a general class of loop corrected solutions

$$\begin{aligned} \Phi_L(t) &= \frac{1}{2} \log \left(\sum_{n=1}^{\infty} \frac{\sigma_n^{2n}}{1 + (m_n t)^{2n}} \right) = \frac{1}{2} \log \left(\sum_{n=1}^{\infty} e^{2n\Phi_L^{(n)}} \right), \\ H_L(t) &= \frac{1}{\sqrt{d}} \frac{m_1}{\sigma_1} \sqrt{\sum_{n=1}^{\infty} \frac{\sigma_n^{2n}}{1 + (m_n t)^{2n}}} = \frac{1}{\sqrt{d}} \frac{m_1}{\sigma_1} \sqrt{\sum_{n=1}^{\infty} e^{2n\Phi_L^{(n)}}}, \\ V_L(\Phi_L^{(n)}) &= \left(\frac{\sum_{n=1}^{\infty} n \dot{\Phi}_L^{(n)} e^{2n\Phi_L^{(n)}}}{\sum_{n=1}^{\infty} e^{2n\Phi_L^{(n)}}} \right)^2 - \left(\frac{m_1}{\sigma_1} \right)^2 \sum_{n=1}^{\infty} e^{2n\Phi_L^{(n)}}, \end{aligned} \quad (3.11)$$

where we used eq. (3.8) to express quantities in term of the n th loop contributions. Although it is straightforward to verify that $\dot{\Phi}_L^{(n)}$ can be expressed in term of $e^{\Phi_L^{(n)}}$ from eq. (3.8), we keep $\dot{\Phi}_L^{(n)}$ to leave the freedom of the constants in $\Phi_L^{(n)}$. Thus the potential $V_L(\Phi_L^{(n)})$ is a function of all n th loop contributions. In practice, since σ_n 's are free constants, all m_n can be set to be the same as m_1 without losing generality. m_1/σ_1 is going to be fixed by the tree level solution up to an integration constant. σ_2 is going to be fixed by the one loop correction, σ_3 is going to be fixed by the two loop correction, and so on. It is difficult to find this solution directly from the loop corrected EOM (3.4). We thus generate infinitely many new solutions for loop corrections.

Some remarks are in order. Let us first expand $\Phi_L^{(n)}$ in eq. (3.8) as $|t| \rightarrow \infty$

$$\Phi_L^{(n)} = -\frac{1}{2} \log \frac{t^2}{t_0^2} + \mathcal{O} \left(\frac{t_0^{2n}}{t^{2n}} \right), \quad |t| \rightarrow \infty. \quad (3.12)$$

For loop corrections, we only know the tree level results, and coefficients of higher loops are still out of reach. Therefore, when constructing loop corrected solutions, one only needs to match the tree level perturbative solution $\Phi_L = -\frac{1}{2} \log(t^2/t_0^2) + \dots$. This is why in eq. (3.8) n can be any positive integer. But this is not consistent, since the loop corrections should be introduced order by order as one loop, two loop, and so on. The solution (3.11) we constructed is much more reasonable, since all loops are included and when higher loop corrections are given, more σ_i 's are fixed.

On the other hand, we now know much more information about the α' corrections. Not only the coefficient of the first order in α' , the behaviors of the higher orders are also

determined by the Hohm-Zwiebach action. These information selects (3.10), particularly $n = 1$, out of other numbers.

Generate loop corrected solutions from α' corrected solutions B. We can also generate loop corrected solutions from solution B (2.16) and (2.17) with the identification (3.7),

$$\begin{aligned}\Phi_L(t) &= -\frac{1}{2N} \log \left[\sum_{n=0}^N (m_n t)^{2n} \right] = -\frac{1}{2N} \log \left[\sum_{n=0}^N e^{-2n\Phi_L^{(n)}} \right], \\ H_L(t) &= \frac{m_N}{\sqrt{d}} \left[\sum_{n=0}^N (m_n t)^{2n} \right]^{-1/2N} = \frac{m_N}{\sqrt{d}} \left[\sum_{n=0}^N e^{-2n\Phi_L^{(n)}} \right]^{-1/2N}, \\ V_L(\Phi^{(n)}) &= \left(\frac{\sum_{n=0}^N n \dot{\Phi}_L^{(n)} e^{-2n\Phi_L^{(n)}}}{N \sum_{n=0}^N e^{-2n\Phi_L^{(n)}}} \right)^2 - m_N^2 \left[\sum_{n=0}^N e^{-2n\Phi_L^{(n)}} \right]^{-1/N},\end{aligned}\quad (3.13)$$

where we absorbed various constants into $\Phi_L^{(n)}$. Again, we used (3.8) to express quantities in term of the n th loop contributions. m_N is an integration constant. m_{N-1} is going to be fixed by the one loop correction, m_{N-2} is going to be fixed by the two loop correction, and so on. It is also not easy to find this solution directly from the loop corrected EOM (3.4).

4 Conclusion

In this paper, we provided two formulas to construct α' corrected cosmological solutions without curvature singularities, for bosonic gravi-dilaton system. Once the coefficient $c_{n>2}$ of the n th order in α' expansion is provided, more accurate solutions can be constructed straightforwardly. We can always make the solution non-singular by adjusting $\lambda_{k>n}$ ($\rho_{k<N-n}$) freely. We also gave a phenomenological map between the α' corrected EOM and loop corrected EOM. Although this map is based on considerably simplified loop corrections, one can use it to generate new solutions. Especially the loop corrected solutions generated from the α' corrected solutions are more reasonable than those in literature.

We addressed vacuum scenario in this work and set $b_{\mu\nu} = 0$. Since the theory is supposed to be $O(d, d)$ invariant, one might rotate time dependent $b_{\mu\nu}(t)$ into the evolution to get some new features. For example, nontrivial $b_{\mu\nu}(t)$ could stabilize the string coupling as $t \rightarrow \infty$, as showed in [25] for loop corrected solutions. Also matter sources in an $O(d, d)$ fashion are expected to lead to more realistic configurations.

Though looks quite phenomenological, in some sense, the map between the α' corrected EOM and loop corrected EOM we found is actually suggestive. In a previous work [26], we conjectured a possible correspondence between genus expansion and α' expansion by noting that, in terms of Riemann normal coordinate, the α' expansion of a string propagating in AdS matches exactly the genus expansion in the Gopakumar-Vafa formula, order by order. So there might exist some deep connection between α' expansion and loop expansion. To gain more insight, we need more information about the loop expansion.

In this paper, we used the map to construct new solutions. It is reasonable to expect there are more applications, at least phenomenologically. In coming works, we will address some inspiring applications.

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