RECEIVED: September 6, 2019

REVISED: November 29, 2019

ACCEPTED: January 1, 2020

PUBLISHED: January 15, 2020

Bosonic sectorized strings and the $(DF)^2$ theory

Thales Azevedo,^a Renann Lipinski Jusinkas^b and Matheus Lize^c

^a*Instituto de Física, Universidade Federal do Rio de Janeiro,
Av. Athos da Silveira Ramos 149, 21941-972, Rio de Janeiro, Brazil*

^b*Institute of Physics of the Czech Academy of Sciences & CEICO,
Na Slovance 2, 182 21, Prague, Czech Republic*

^c*Instituto de Física Teórica, Universidade Estadual Paulista,
Rua Dr. Bento Teobaldo Ferraz 271, 01140-070, São Paulo, Brazil*

E-mail: thales@if.ufrj.br, renannlj@fzu.cz, matheus.lize@unesp.br

ABSTRACT: In this work, we investigate the bosonic chiral string in the sectorized interpretation, computing its spectrum, kinetic action and 3-point amplitudes. As expected, the bosonic ambitwistor string is recovered in the tensionless limit.

We also consider an extension of the bosonic model with current algebras. In that case, we compute the effective action and show that it is essentially the same as the action of the mass-deformed $(DF)^2$ theory found by Johansson and Nohle. Aspects which might seem somewhat contrived in the original construction — such as the inclusion of a scalar transforming in some real representation of the gauge group — are shown to follow very naturally from the worldsheet formulation of the theory.

KEYWORDS: Bosonic Strings, BRST Quantization, String Field Theory, Scattering Amplitudes

ARXIV EPRINT: [1908.11371](https://arxiv.org/abs/1908.11371)

Contents

1	Introduction	1
2	The bosonic sectorized string	3
2.1	The Polyakov action in first-order form	3
2.2	The sectorized interpretation	4
2.3	Physical spectrum	5
2.4	Bosonic kinetic action and 3-point amplitudes	9
3	Extension of the sectorized model with current algebras	10
3.1	Physical spectrum	12
3.2	3-point amplitudes	13
3.3	Effective field theory: $(DF)^2 + \text{YM}$	15
3.3.1	Kinetic action	16
3.3.2	Cubic vertices and the effective action	17
3.4	Including the other gauge sector: $(DF)^2 + \text{YM} + \phi^3$	18
4	Conclusion	20
A	Current algebra CFT	21

---

1 Introduction

When Cachazo, He and Yuan (CHY) found their celebrated formulae for the tree-level scattering amplitudes of massless particles [1, 2], it seemed plausible that those expressions could be obtained from some worldsheet model. Indeed, it did not take long for Mason and Skinner to come up with such a model, dubbed ambitwistor strings [3], followed by a manifestly supersymmetric version using the pure spinor formalism [4]. The CHY formulae were later generalized to different theories [5, 6] and, again, different ambitwistor strings were proposed as their underlying worldsheet model [7].

By construction, ambitwistor strings are two-dimensional chiral theories which contain no dimensionful parameter. At first, they were thought to stem from an infinite tension limit of ordinary string theory, a belief motivated in part by the fact that the spectrum of the type II version of the model is identical to that of the corresponding supergravity. However, for the bosonic and heterotic versions it is clear that no such procedure is possible, since their spectra do not match their (super)gravity counterparts.

On the other hand, as observed in [8], ambitwistor strings are equivalent to the spinor moving frame formulation of the null superstring — therefore, tensionless. This idea was supported by Siegel in [9] and other similar results followed (e.g. [10]).

It was then noticed that the spectrum of tensionful chiral strings could contain a finite number of massive states [11], depending on the amount of spacetime supersymmetry. For the type II case, for instance, the physical spectrum is independent of the string tension. In this context, the so-called sectorized string model [12] plays an important role. It was introduced as an alternative to the above-mentioned pure spinor analogue of ambitwistor strings [4], motivated by some inconsistencies in its heterotic version and difficulties in coupling it to the $\mathcal{N} = 2$ supergravity background [13]. As such, it was supposed to be a theory for massless particles only. Nevertheless, it was later shown [14] that the heterotic sectorized model actually contains the $\mathcal{N} = 1$ supergravity states together with a single massive multiplet with the same quantum numbers as the first massive level of the (conventional) open superstring. This is possible thanks to a dimensionful parameter whose existence had been overlooked, since the chiral worldsheet action has no parameters. Moreover, when this parameter is taken to zero, corresponding to a tensionless limit, one recovers the heterotic ambitwistor string.

In this work, we analyze the bosonic incarnation of the sectorized model and show how the theory can be interpreted in terms of two sectors after a particular gauge-fixing is performed. As in the heterotic case, the two sectors emulate the left- and right-moving sectors of the usual string theory, but all worldsheet fields are holomorphic. Using methods similar to the ones used in [15] for the ambitwistor string — which in turn were based on [16] —, we compute its physical spectrum and the correspondent kinetic action. We also analyze the 3-point tree level functions. As expected, the bosonic ambitwistor string is recovered in the tensionless limit.

We then consider an extension of the bosonic model by including current algebras. As a main result of this work, the usual methods are shown to give rise to a worldsheet derivation of the so-called $(DF)^2 + \text{YM}$ theory found by Johansson and Nohle [17]. In particular, the scalar field transforming in some real representation of the gauge group, whose inclusion might seem somewhat contrived in the original construction, appears naturally in the sectorized-string formulation.

Theories whose Lagrangians include a $(DF)^2$ -type kinetic term were first introduced as a way of obtaining conformal (super)gravity amplitudes (R^2 gravity, in general) from color-kinematics duality [18], and were shown to admit CHY/ambitwistor representations in [19]. Like R^2 gravity, such theories contain “ghost” states which render them non-unitary. Moreover, in the particular model studied in this paper, a tachyon is also present. It is then natural to ask what physical interest the model might have.

The answer is that scattering amplitudes computed from the $(DF)^2$ theories have been recently found to play a crucial role in the double-copy construction of (ordinary) bosonic and heterotic string tree-level amplitudes [20] — see also [21, 22]. Indeed, just like open superstring amplitudes with external massless states can be expressed in a basis of integrals with coefficients which are nothing but super-Yang-Mills amplitudes [23], the corresponding bosonic open string amplitudes can also be expressed in the same basis, but with coefficients which come from $(DF)^2 + \text{YM}$. Note that the presence of the tachyon makes perfect sense in this context.

This paper is organized as follows. In section 2, we introduce the sectorized description of the bosonic chiral string, having the Polyakov action in first-order form as our starting point. We then investigate the physical spectrum of the model and analyze its tensionless limit. The kinetic part of its effective action and some results on the tree-level three-point amplitudes are also presented. In section 3, the bosonic model is extended with the inclusion of current algebras, and the effective field theory inferred from the three-point functions is shown to agree with the $(DF)^2 + \text{YM} + \phi^3$ theory of Johansson and Nohle. Finally, we present our conclusions and perspectives in section 4. The appendix includes further details on the CFT of current algebras that are relevant for this work.

2 The bosonic sectorized string

In this section we will rederive some known results for chiral bosonic strings using the sectorized description, including its physical spectrum and tensionless limit analysis.

2.1 The Polyakov action in first-order form

The Polyakov action is given by

$$S_P = \frac{\mathcal{T}}{2} \int d\tau d\sigma \sqrt{-g} \{g^{ij} \partial_i X^m \partial_j X_m\}, \quad (2.1)$$

where $\mathcal{T} > 0$ is the string tension, g_{ij} is the worldsheet metric (with inverse g^{ij}) and $g = \det(g_{ij})$, with i, j denoting the usual worldsheet coordinates τ and σ . Spacetime indices $m, n, \dots$ are raised and lowered with the (mostly plus) Minkowski metric η_{mn} .

In the first order formulation, one can define a classically equivalent action, given by

$$\begin{aligned} \tilde{S}_P = \int d\tau d\sigma \left\{ P_m \partial_\tau X^m - \frac{1}{4\mathcal{T}} e_+ (P_m + \mathcal{T} \partial_\sigma X_m) (P^m + \mathcal{T} \partial_\sigma X^m) \right. \\ \left. - \frac{1}{4\mathcal{T}} e_- (P_m - \mathcal{T} \partial_\sigma X_m) (P^m - \mathcal{T} \partial_\sigma X^m) \right\}, \end{aligned} \quad (2.2)$$

where $e_\pm$ denote the Weyl invariant Lagrange multipliers related to the worldsheet metric as

$$e_\pm \equiv \frac{1}{g^{\tau\tau} \sqrt{-g}} \mp \frac{g^{\tau\sigma}}{g^{\tau\tau}}. \quad (2.3)$$

Although not manifestly, the action $\tilde{S}_P$ is invariant under worldsheet reparametrizations, generated by

$$H_\pm \equiv (P_m \pm \mathcal{T} \partial_\sigma X_m) (P^m \pm \mathcal{T} \partial_\sigma X^m). \quad (2.4)$$

The corresponding gauge transformations are given by

$$\delta X^m = \frac{1}{2} c_+ (P^m + \mathcal{T} \partial_\sigma X^m) + \frac{1}{2} c_- (P^m - \mathcal{T} \partial_\sigma X^m), \quad (2.5a)$$

$$\delta P_m = \frac{\mathcal{T}}{2} \partial_\sigma [c_+ (P^m + \mathcal{T} \partial_\sigma X^m) - c_- (P^m - \mathcal{T} \partial_\sigma X^m)],$$

$$\delta e_+ = \partial_\tau c_+ + c_+ \partial_\sigma e_+ - e_+ \partial_\sigma c_+, \quad (2.5b)$$

$$\delta e_- = \partial_\tau c_- - c_- \partial_\sigma e_- + e_- \partial_\sigma c_-, \quad (2.5c)$$

where c_+ and c_- are local parameters.

2.2 The sectorized interpretation

The quantization of the action (2.2) is straightforward, and the usual conformal gauge is obtained when we choose $e_{\pm} = 1$. We want to discuss, instead, a particular case of the one-parameter (β) family of gauges introduced in [9], which can be cast as

$$e_+ = 1, \quad e_- = \frac{(1 - \beta)}{(1 + \beta)}. \quad (2.6)$$

For $\beta = 0$, the conformal gauge is recovered. We are interested in the singular gauge $\beta \rightarrow \infty$, leading to a chiral worldsheet action. In this limit, $e_{\pm} = \pm 1$. This singular gauge was proposed in the context of doubled-coordinate field theory in [24]. After a Wick rotation of the worldsheet coordinate τ , the gauge-fixed action can be written as

$$S = \frac{1}{2\pi} \int d^2z \{ P_m \bar{\partial} X^m + b_+ \bar{\partial} c_+ + b_- \bar{\partial} c_- \}, \quad (2.7)$$

where the gauge parameters $c_{\pm}$ have been promoted to anticommuting ghosts with corresponding antighosts $b_{\pm}$. All fields in S are holomorphic and the string tension $\mathcal{T}$ is now hidden.

A few comments about the gauge fixing (2.6) are in order. For any finite β , a redefinition of the worldsheet coordinates can always bring the gauge fixed action to the conformal gauge. This is hardly surprising, since the physical model should be gauge independent. This was noted by Siegel in [9], but his construction of the chiral string involved another crucial ingredient related to a change in the boundary conditions of the action. At any rate, adopting the singular gauge ($\beta \rightarrow \infty$) is useful since then the delta functions realizing the scattering equations become explicit.

It was later noticed that the boundary condition leading to Siegel's new propagator for the target space coordinates could in fact be described by the usual string theory in the conformal gauge ($\beta = 0$), albeit with a different choice of vacuum [25]. In the ambitwistor context, this alternative vacuum was investigated in [10] (and further in [26]) and also arises naturally from the quantization of the action (2.7). As it turns out, this seems to be the only consistent vacuum in the singular gauge $\beta \rightarrow \infty$. It might look like a contradiction, but the key idea here is precisely that this is a *singular* gauge which effectively leads to a degenerate worldsheet metric. In other words, the action (2.7) is completely oblivious to the usual conformal gauge in string theory because this gauge choice is not invertible (hence, singular).

In spite of being chiral, the model can be interpreted in terms of two sectors, namely the “+” and the “−”, which partially emulate the left and right movers of the usual bosonic string. Each sector has its own *characteristic* energy-momentum tensor given by

$$T_+ = -\frac{1}{4\mathcal{T}} P_m^+ P_n^+ \eta^{mn} - 2b_+ \partial c_+ + c_+ \partial b_+, \quad (2.8a)$$

$$T_- = \frac{1}{4\mathcal{T}} P_m^- P_n^- \eta^{mn} - 2b_- \partial c_- + c_- \partial b_-, \quad (2.8b)$$

with

$$P_m^{\pm} \equiv P_m \pm \mathcal{T} \partial X_m. \quad (2.9)$$

The sectorization is manifest in the BRST charge Q :

$$Q = Q^+ + Q^-, \quad (2.10)$$

$$Q^\pm \equiv \oint \{c_\pm T_\pm - b_\pm c_\pm \partial c_\pm\}. \quad (2.11)$$

Nilpotency of Q requires the number of spacetime dimensions to be $d = 26$.

Note that the complete energy-momentum tensor is given by

$$\begin{aligned} T &= T_+ + T_- \\ &= -P_m \partial X^m - b \partial c - \partial(bc) - \tilde{b} \partial \tilde{c} - \partial(\tilde{b}\tilde{c}), \end{aligned} \quad (2.12)$$

and it is BRST exact, since $\{Q, (b_+ + b_-)\} = T$. In fact, if we define

$$\begin{aligned} c &\equiv \frac{1}{2}(c_+ + c_-), & \tilde{c} &\equiv \frac{1}{2\mathcal{T}}(c_- - c_+), \\ b &\equiv (b_+ + b_-), & \tilde{b} &\equiv \mathcal{T}(b_- - b_+), \end{aligned} \quad (2.13)$$

the action (2.7) becomes

$$S = \frac{1}{2\pi} \int d^2z \{P_m \bar{\partial} X^m + b \bar{\partial} c + \tilde{b} \bar{\partial} \tilde{c}\}, \quad (2.14)$$

while the BRST charge is rewritten as

$$Q = \oint \left\{ cT - bc\partial c + \frac{1}{2} \tilde{c} P^m P_m + \frac{\mathcal{T}^2}{2} \tilde{c} (\partial X^m \partial X_m - 2b\partial\tilde{c}) \right\}, \quad (2.15)$$

and the familiar Virasoro structure emerges. The tensionless limit of Q is now very clear: it is precisely the BRST operator introduced by Mason and Skinner for the bosonic ambitwistor string [3].

We will see, however, that the sectorized description is more advantageous in the cohomology analysis, for it leads to a natural splitting of the vertex operators in the different mass levels.

2.3 Physical spectrum

The BRST cohomology at ghost number zero is given by the identity operator. At ghost number one, the cohomology contains only the zero-momentum states mapped to the operators $c_+ P_m^+$ and $c_- P_m^-$.

Physical states will be defined as elements of the BRST cohomology with ghost number two and annihilated by the zero mode of b . The latter follows from the usual off-shell condition $(b_0 - \bar{b}_0) = 0$ on physical states, but adapted to the chiral model. The most general vertex operator with conformal weight zero satisfying these conditions can be written as

$$V = V_0 + V_+ + V_-, \quad (2.16)$$

where

$$V_0 = c_+ c_- P_m^+ P_n^- G^{mn} + \mathcal{T}(c_+ \partial^2 c_+ + c_- \partial^2 c_-) D + \mathcal{T}(c_+ \partial^2 c_+ - c_- \partial^2 c_-) E \\ + c_+ P_m^+ (\partial c_+ - \partial c_-) A_+^m + c_- P_m^- (\partial c_+ - \partial c_-) A_-^m, \quad (2.17)$$

$$V_+ = c_+ c_- P_m^+ P_n^+ H_+^{mn} + c_- P_m^+ (\partial c_+ - \partial c_-) B_+^m + c_+ c_- \partial P_m^+ C_+^m \\ + \mathcal{T} c_- \partial^2 c_+ F^+ + b_+ c_+ c_- (\partial c_+ - \partial c_-) G^+, \quad (2.18)$$

$$V_- = c_+ c_- P_m^- P_n^- H_-^{mn} + c_+ P_m^- (\partial c_+ - \partial c_-) B_-^m - c_+ c_- \partial P_m^- C_-^m \\ + \mathcal{T} c_+ \partial^2 c_- F^- + b_- c_+ c_- (\partial c_+ - \partial c_-) G^-. \quad (2.19)$$

Here, G^{mn} , $H_{\pm}^{mn}$, $A_{\pm}^m$, $B_{\pm}^m$, $C_{\pm}^m$, D , E , $F^{\pm}$ and $G^{\pm}$ are the X dependent fields. This splitting of the terms appearing in the vertex operator is motivated by their mass-level, as will become clear shortly.

In order to determine the physical degrees of freedom, we will analyze each of the vertices in (2.16) separately. For V_0 , the equations of motion imposed by BRST closedness are given by

$$A_+^m = \frac{1}{2} \partial_n G^{mn} - \frac{1}{2} \partial^m (D - E), \quad \square D = \partial_m (A_+^m + A_-^m), \\ A_-^m = \frac{1}{2} \partial_n G^{nm} - \frac{1}{2} \partial^m (D + E), \quad \square E = \partial_m (A_+^m - A_-^m), \quad (2.20) \\ \square G^{mn} = 2 \partial^m A_-^n + 2 \partial^n A_+^m.$$

These equations become more transparent if we rewrite them in terms of the fields

$$g^{mn} \equiv \frac{1}{2} (G^{mn} + G^{nm}), \quad (2.21a)$$

$$b^{mn} \equiv \frac{1}{2} (G^{mn} - G^{nm}), \quad (2.21b)$$

$$\phi \equiv \frac{\mathcal{T}}{2} G^{mn} \eta_{mn} - \mathcal{T} D, \quad (2.21c)$$

$$g^m \equiv A_+^m + A_-^m - \frac{1}{\mathcal{T}} \partial^m D, \quad (2.21d)$$

$$b^m \equiv A_+^m - A_-^m - \frac{1}{\mathcal{T}} \partial^m E, \quad (2.21e)$$

such that g^m and b^m have algebraic solutions, cf. (2.20),

$$g^m = \partial_n g^{mn} - \eta_{np} \partial^m g^{np} + \frac{2}{\mathcal{T}} \partial^m \phi, \quad (2.22a)$$

$$b^m = \partial_n b^{mn}, \quad (2.22b)$$

and

$$\square g^{mn} - \partial_p \partial^n g^{mp} - \partial_p \partial^m g^{np} + \eta_{pq} \partial^m \partial^n g^{pq} - \frac{2}{\mathcal{T}} \partial^m \partial^n \phi = 0, \quad (2.23a)$$

$$\square \phi = 0, \quad (2.23b)$$

$$\partial_p (\partial^p b^{mn} + \partial^m b^{np} + \partial^n b^{pm}) = 0. \quad (2.23c)$$

The gauge transformations, with parameters λ^m and ω^m , are simply

$$\delta\phi = 0, \quad \delta g^{mn} = \partial^{(m}\lambda^{n)}, \quad \delta b^{mn} = \partial^{[m}\omega^{n]}. \quad (2.24)$$

It is now easy to identify the field content of the massless sector described by the vertex V_0 : ϕ corresponds to the dilaton, b^{mn} is the Kalb-Ramond 2-form and g^{mn} is the graviton, satisfying the linearized equation of motion (2.23a).

For the vertices V_+ and V_- , the two sets of equations of motion are very similar to each other and can be displayed collectively as

$$\begin{aligned} B_{\pm}^m &= \partial_n H_{\pm}^{mn} - C_{\pm}^m - \frac{1}{2} \partial^m F^{\pm}, & \left(\frac{1}{4}\square \mp \mathcal{T}\right) C_{\pm}^m &= \mathcal{T} B_{\pm}^m + \frac{1}{2} \partial^m G^{\pm}, \\ G^{\pm} &= \frac{\mathcal{T}}{2} H_{\pm}^{mn} \eta_{mn} + \frac{1}{2} \partial_m C_{\pm}^m - \frac{3\mathcal{T}}{2} F^{\pm}, & \left(\frac{1}{4}\square \mp \mathcal{T}\right) F^{\pm} &= \frac{1}{2} \partial_m B_{\pm}^m \mp \frac{3}{2} G^{\pm}, \\ \left(\frac{1}{4}\square \mp \mathcal{T}\right) H_{\pm}^{mn} &= \frac{1}{4} \partial^m B_{\pm}^n + \frac{1}{4} \partial^n B_{\pm}^m \mp \frac{1}{4} \eta^{mn} G^{\pm}, \end{aligned} \quad (2.25)$$

Again, these equations become more transparent after the field redefinitions

$$\begin{aligned} h_{\pm}^{mn} &\equiv H_{\pm}^{mn} - \frac{1}{4\mathcal{T}} (\partial^n C_{\pm}^m + \partial^m C_{\pm}^n) \pm \frac{1}{20\mathcal{T}} (\partial^m \partial^n \pm \mathcal{T} \eta^{mn}) F^{\pm} \\ &\quad \mp \frac{1}{20\mathcal{T}} (\partial^m \partial^n \pm \mathcal{T} \eta^{mn}) H_{\pm}^{pq} \eta_{pq}, \end{aligned} \quad (2.26a)$$

$$f_{\pm} \equiv F^{\pm} - H_{\pm}^{mn} \eta_{mn}, \quad (2.26b)$$

$$c_{\pm}^m \equiv C_{\pm}^m \pm \frac{1}{10} \partial^m H_{\pm}^{np} \eta_{np} \mp \frac{1}{10} \partial^m F^{\pm}, \quad (2.26c)$$

which imply (using $d = 26$) that

$$\left(\frac{1}{4}\square \mp \mathcal{T}\right) h_{\pm}^{mn} = 0, \quad (2.27a)$$

$$\partial_n h_{\pm}^{mn} = 0, \quad (2.27b)$$

$$h_{\pm}^{mn} \eta_{mn} = 0, \quad (2.27c)$$

with gauge transformations

$$\delta h_{\pm}^{mn} = 0, \quad \delta f_{\pm} = \pm 5 \Sigma^{\pm}, \quad \delta c_{\pm}^m = \mathcal{T} \Pi_{\pm}^m. \quad (2.28)$$

The fields $f_{\pm}$ and $c_{\pm}^m$ are pure gauge, therefore $h_{\pm}^{mn}$ contain all the physical degrees of freedom, corresponding to spin 2 fields with $m^2 = \pm 4\mathcal{T}$.

Tensionless limit. Evidently, in the tensionless limit all the physical states are massless. In fact, if we naively take the $\mathcal{T} \rightarrow 0$ limit of the vertex (2.16), it may seem that Mason and Skinner's results are recovered [3]. However, the analysis of such limit has to be done more carefully precisely because all the physical states become massless. In other words, the vertices (2.17), (2.18) and (2.19) should mix in the tensionless limit. Therefore, we should find a convenient combination of the fields G^{mn} , $H_{\pm}^{mn}$, $A_{\pm}^m$, $B_{\pm}^m$, $C_{\pm}^m$, D , E , $F^{\pm}$ and

$G^\pm$ in (2.16) such that the tensionless limit preserves the most general form of the vertex operator. The solution is

$$\begin{aligned}
 V = & c\tilde{c}P_mP_nG_{(1)}^{mn} + c\tilde{c}\partial X_m\partial X_nG_{(2)}^{mn} + c\tilde{c}P_m\partial X_nG_{(3)}^{mn} + c\tilde{c}P_m\partial X_nB^{mn} \\
 & + c\tilde{c}\partial^2X_mA_{(1)}^m + c\tilde{c}\partial P_mA_{(2)}^m + \partial\tilde{c}\tilde{c}P_mA_{(3)}^m + \partial\tilde{c}\tilde{c}\partial X_mA_{(4)}^m \\
 & + c\partial\tilde{c}P_mA_{(5)}^m + c\partial\tilde{c}\partial X_mA_{(6)}^m + bc\partial\tilde{c}\tilde{c}S_{(1)} + \partial^2ccS_{(2)} \\
 & + \partial^2\tilde{c}\tilde{c}S_{(3)} + \partial^2c\tilde{c}S_{(4)} + c\partial^2\tilde{c}S_{(5)} + \tilde{b}\tilde{c}c\partial\tilde{c}S_{(6)},
 \end{aligned} \tag{2.29}$$

with

$$\begin{aligned}
 G_{(1)}^{mn} &\equiv 2\mathcal{T}\left[\frac{1}{2}(G^{mn}+G^{mm})+H_+^{mn}+H_-^{mn}\right], & A_{(5)}^m &\equiv -2\mathcal{T}(A_+^m+B_+^m+A_-^m+B_-^m), \\
 G_{(2)}^{mn} &\equiv 2\mathcal{T}^3\left[H_+^{mn}+H_-^{mn}-\frac{1}{2}(G^{mn}+G^{mm})\right], & A_{(6)}^m &\equiv -2\mathcal{T}^2(A_+^m+B_+^m-A_-^m-B_-^m), \\
 G_{(3)}^{mn} &\equiv 4\mathcal{T}^2(H_+^{mn}-H_-^{mn}), & S_{(1)} &\equiv 2\mathcal{T}^2(G^++G^-), \\
 B^{mn} &\equiv -2\mathcal{T}^2(G^{mn}-G^{mm}), & S_{(2)} &\equiv -(2D+F^++F^-), \\
 A_{(1)}^m &\equiv 2\mathcal{T}^2(C_+^m+C_-^m), & S_{(3)} &\equiv -\mathcal{T}^2(2D-F^+-F^-), \\
 A_{(2)}^m &\equiv 2\mathcal{T}(C_+^m-C_-^m), & S_{(4)} &\equiv \mathcal{T}(2E-F^++F^-), \\
 A_{(3)}^m &\equiv -2\mathcal{T}^2(A_+^m-B_+^m-A_-^m+B_-^m), & S_{(5)} &\equiv -\mathcal{T}(2E+F^+-F^-), \\
 A_{(4)}^m &\equiv -2\mathcal{T}^3(A_+^m-B_+^m+A_-^m-B_-^m), & S_{(6)} &\equiv 2\mathcal{T}(G^--G^+).
 \end{aligned} \tag{2.30}$$

Here the notation for the fields was chosen so as to agree with the ambitwistor construction of [15], where it was demonstrated that the free field dynamics associated to the fields above involve higher derivative operators. This result follows naturally from our construction above. For example, we can show that the equation of motion for the fields $G_{(1)}^{mn}$, $G_{(2)}^{mn}$ and $G_{(3)}^{mn}$ can be obtained using (2.23a) and (2.27), and are given (in the gauge $c_\pm^m = f_\pm = 0$) by

$$G_{(2)}^{mn} = \frac{1}{4}\square G_{(3)}^{mn} - \mathcal{T}^2 G_{(1)}^{mn}, \tag{2.31a}$$

$$2G_{(3)}^{mn} = \square G_{(1)}^{mn} - \partial_p \partial^n G_{(1)}^{mp} - \partial_p \partial^m G_{(1)}^{np} + \eta_{pq} \partial^m \partial^n G_{(1)}^{pq} - \frac{2}{\mathcal{T}} \partial^m \partial^n \phi, \tag{2.31b}$$

$$(\square^2 - 16\mathcal{T}^2)G_{(3)}^{mn} = 0, \tag{2.31c}$$

with $G_{(3)}^{mn}\eta_{mn} = \partial_n G_{(3)}^{mn} = 0$.

Note that, by substituting (2.31b) into (2.31c), we get an equation involving $\square^3 G_{(1)}^{mn}$ which, in the tensionless limit, has the same form as the one found in [15]. Of course, this had to be the case since the vertex operator (2.29) preserves its form as $\mathcal{T} \rightarrow 0$, while the BRST operator reduces to the (bosonic) ambitwistor one, as is evident from (2.15). Indeed, all the other equations of motion can be reproduced in a similar way.

2.4 Bosonic kinetic action and 3-point amplitudes

As shown above, $G_{(2)}^{mn}$ and $G_{(3)}^{mn}$ can be seen as auxiliary fields¹ which effectively implement a higher derivative equation of motion for $G_{(1)}^{mn}$. This behavior can be better understood from another point of view, namely in terms of the effective action of the model and, in particular, its kinetic part.

Indeed, the kinetic terms associated to g^{mn} and $h_{\pm}^{mn}$ have opposite signs. Physically, this indicates an instability of the model (ghosts), in agreement with the results of [11]. Such ghosts can usually be described in terms of higher derivative theories and this is precisely what happens here.

Bosonic kinetic action. Inspired by Zwiebach’s closed string action [16], the kinetic action for ambitwistor strings was built in [15]. We will use the same prescription for the tensionful model and the kinetic action will be defined by

$$S = \frac{1}{2} \langle V | \partial c Q | V \rangle \quad (2.32)$$

where $|V\rangle$ is the state associated to the vertex operator (2.16), obtained from the identity state $|0\rangle$ through the state-operator map

$$|V\rangle = \lim_{z \rightarrow 0} V(z) |0\rangle, \quad (2.33)$$

and $\langle V|$ its BPZ conjugate. In order to simplify the calculations, we will fix the gauge $f_{\pm} = c_{\pm}^m = 0$ — cf. equations (2.26) and (2.28) — and use the auxiliary equations of motion in (2.21) to write the vertex operator (2.16) in terms of the fields g^{mn} , b^{mn} , ϕ and $h_{\pm}^{mn}$.

Now, using the usual ghost measure $\langle c_{\pm} \partial c_{\pm} \partial^2 c_{\pm} \rangle = 2$, it is straightforward to show that the free action can be cast as

$$S_{\text{bosonic}} = S_0 + S_+ + S_-, \quad (2.34)$$

where

$$\begin{aligned} S_0 = 2 \int d^{26}x \{ & g^{mn} \square g_{mn} + \partial_p g^{mp} \partial^q g_{mq} + 2(g + \phi) \partial_m \partial_n g^{mn} \\ & - (g + \phi) \square (g + \phi) + b^{mn} \square b_{mn} - b^{mn} \partial_m \partial^r b_{nr} \}, \end{aligned} \quad (2.35)$$

and

$$S_{\pm} = 4 \int d^{26}x \{ -h_{\pm}^{mn} (\square \mp 4\mathcal{T}) h_{\pm mn} + h_{\pm}^{mn} \partial_n \partial^r h_{\pm mr} - 2h_{\pm} \partial_m \partial_n h_{\pm}^{mn} + h_{\pm} (\square \mp 4\mathcal{T}) h_{\pm} \}, \quad (2.36)$$

with $g = g^{mn} \eta_{mn}$ and $h_{\pm} = h_{\pm}^{mn} \eta_{mn}$. As expected, the free field equations of motion derived from S_0 and $S_{\pm}$ precisely reproduce (2.23) and (2.27). The kinetic terms for g^{mn} and $h_{\pm}^{mn}$ have opposite signs, consistent with the ghost interpretation.

---

¹Here the word “auxiliary” should not be understood as “not propagating degrees of freedom,” but rather that the degrees of freedom represented by these fields can be incorporated in another one which satisfies a higher-derivative equation of motion — cf. equations (2.31) above.

3-point amplitudes. The 3-point tree level scattering amplitudes for the bosonic chiral string were obtained in [11]. However, it is instructive to redo this analysis here since our unintegrated vertex operators have a different structure and, in particular, do not give rise to a Koba-Nielsen factor. For higher point amplitudes, we would need integrated vertex operators but their definition is still unknown.

It will be convenient to gauge fix the vertex operators in (2.16) and work with momentum eigenstates, such that

$$V_0 = c_+ c_- P_m^+ P_n^- G^{mn} e^{ik \cdot X}, \quad V_\pm = c_+ c_- P_m^\pm P_n^\pm H_\pm^{mn} e^{ik \cdot X}, \quad (2.37)$$

where $G^{mn}, H_\pm^{mn}$ are now seen as polarization tensors satisfying $k_m G^{mn} = k_n G^{mn} = k_m H_\pm^{mn} = \eta_{mn} H_\pm^{mn} = 0$.

In order to compute the 3-point amplitudes, we have to evaluate its OPE reduction by contracting all $P_m^\pm$'s with one another and with the momentum exponentials $e^{ik \cdot X}$. We also need the ghost 3-point function, which has the usual form

$$\langle c_\pm(z) c_\pm(y) c_\pm(w) \rangle = (z - y)(y - w)(w - z). \quad (2.38)$$

By virtue of the sectorized description, it is easy to show that the amplitude factorizes into a product of two open string amplitudes (where $\mathcal{T} \mapsto -\mathcal{T}$ in the minus sector). With all this in mind, we can compute, for example, the 3-point amplitude involving only massless states. The result is

$$\langle V_0(z_1) V_0(z_2) V_0(z_3) \rangle = G_1^{mn} G_2^{pq} G_3^{rs} T_{mpr} \bar{T}_{nqs} \delta^{26} (k^1 + k^2 + k^3), \quad (2.39)$$

where

$$T_{mnp} \equiv k_m^2 k_n^3 k_p^1 + 2\mathcal{T} (k_m^2 \eta_{np} + k_n^3 \eta_{mp} + k_p^1 \eta_{mn}) \quad (2.40)$$

and $\bar{T}_{nqs}$ is equal to T_{nqs} with the sign of $\mathcal{T}$ flipped. The amplitude does not depend on the positions of the vertex operator insertions and is, therefore, $\text{SL}(2, \mathbb{C})$ invariant. This result is to some extent expected, since the vertex structure is completely analogous to the ordinary bosonic string and the Koba-Nielsen factors are just 1 for three massless vertices. However, the $\text{SL}(2, \mathbb{C})$ invariance can be shown for *any* 3-point tree level amplitude, even though the Koba-Nielsen factor is *always* 1 in the chiral model (there are no contractions between the momentum exponentials since the XX OPE is trivial). The amplitudes factorize in the plus and minus sectors, and there is a precise cancelation of the poles and zeros in $z_{ij} \equiv z_i - z_j$.

3 Extension of the sectorized model with current algebras

In this section we will explore the extension of the bosonic sectorized model in a target space with dimension $d < 26$ and the introduction of current algebras, i.e. a gauge sector. To the action (2.7), we will add two extra pieces, S_C^+ and S_C^- , describing two current algebras. The new BRST charge preserves its form in (2.10) but now with

$$T_+ = -\frac{1}{4\mathcal{T}} P_m^+ P_n^+ \eta^{mn} - 2b_+ \partial c_+ + c_+ \partial b_+ + T_C^+, \quad (3.1)$$

$$T_- = \frac{1}{4\mathcal{T}} P_m^- P_n^- \eta^{mn} - 2b_- \partial c_- + c_- \partial b_- + T_C^-, \quad (3.2)$$

where $T_C^\pm$ denotes the energy-momentum tensor associated to different group manifolds with central charge

$$c^{(\pm)} = 26 - d. \quad (3.3)$$

For now we will focus on the “−” sector, which contains the tachyonic excitations. The inclusion of the “+” sector, which has an analogous structure, will be discussed in subsection 3.4.

Let us consider an affine Lie algebra associated to some group G , with structure constants f_{ab}^c ($a, b, \dots = 1$ to $\dim G$) and level k . The addition of S_C^- to the action allows us to define currents J_a which are primary conformal fields and satisfy the OPE

$$J_a(z) J_b(y) \sim \frac{k\delta_{ab}}{(z-y)^2} + if_{ab}^c \frac{J_c(y)}{(z-y)}. \quad (3.4)$$

Here the group generators have been orthonormalized such that the metric δ_{ab} corresponds to a Kronecker delta, and we will make no further distinction between upper and lower indices.

The energy-momentum tensor of the algebra can be obtained using the Sugawara construction and is given by

$$T_C^- \equiv \frac{1}{2(k+g)} (J_a, J_a), \quad (3.5)$$

where g is the dual Coxeter number, defined through

$$f_{acd}f_{bcd} = 2g\delta_{ab}. \quad (3.6)$$

We use the ordering prescription

$$(A, B)(y) \equiv \frac{1}{2\pi i} \oint \frac{dz}{(z-y)} A(z) B(y), \quad (3.7)$$

which can be understood as the product of two operators $A(z)$ and $B(y)$ in the limit $z \rightarrow y$, with singular terms removed.

It is then straightforward to compute the central charge of this model, which is given by

$$\begin{aligned} c^{(-)} &= \frac{k\Delta}{(k+g)}, \\ &\stackrel{!}{=} 26 - d, \end{aligned} \quad (3.8)$$

where

$$\Delta \equiv \delta^{ab}\delta_{ab} = \dim G. \quad (3.9)$$

The second equality in (3.8) comes from imposing the nilpotency of the BRST operator and constrains the group G and the level k of the current algebra. For example, for a target space with $d = 10$ one of the solutions is $G = \text{SO}(32)$ and $k = 1$, while for $d = 4$ we can have $G = \text{SU}(5)$ and $k = 55$, and so on. Further constraints on the group should arise from the analysis of anomalies but this will not be discussed in this work.

3.1 Physical spectrum

The BRST cohomology now includes additional states with corresponding vertex operators containing the currents J_a , expressed as

$$V_J = c_+c_-P_m^+J_aF_a^m + c_-(\partial c_+ - \partial c_-)J_aF^a + c_+c_-\partial J_aS_a \\ + c_+c_-P_m^-J_aG_a^m + c_+(\partial c_+ - \partial c_-)J_aG^a + c_+c_-J_\alpha\varphi_\alpha. \quad (3.10)$$

Here F_a^m , G_a^m , S^a , F_a , G_a and φ_α are target space fields. The index α belongs to a traceless-symmetric bi-adjoint representation of the group G (see appendix), with dimension

$$\Delta(\alpha) = \frac{\Delta(\Delta+1)}{2} - 1. \quad (3.11)$$

J_α is a primary conformal weight 2 operator defined as

$$J_\alpha \equiv (C^{-1})_{\alpha ab} J_{(ab)}, \quad (3.12)$$

where $J_{(ab)}$ is given by the traceless-symmetric ordered product of two currents, i.e.

$$J_{(ab)} \equiv \frac{1}{2}(J_a, J_b) + \frac{1}{2}(J_b, J_a) - \frac{2(k+g)}{\Delta}\delta_{ab}T_C^-, \quad (3.13)$$

and $(C^{-1})_{\alpha ab}$ are the inverse of the Clebsch-Gordan coefficients, $C_{\alpha ab}$. The properties of these coefficients will be discussed in the next subsection and in the appendix. Observe that we could have considered also the trace contribution in the vertex, e.g. $c_+c_-T_C^-\varphi$. However, the field φ couples only to the vertex V_- in subsection 2.3 and does not change the physical content of the model.

The BRST invariance of the vertex V_J implies the following equations of motion

$$(\square + 4\mathcal{T})\varphi_\alpha = 0, \quad (3.14a)$$

$$F_a = \frac{1}{2}\partial_m F_a^m, \quad (3.14b)$$

$$G_a = \frac{1}{2}\partial_m G_a^m - S_a, \quad (3.14c)$$

$$\partial_n(\partial^m F_a^m - \partial^n F_a^m) = 0, \quad (3.14d)$$

$$\partial_n(\partial^m G_a^m - \partial^n G_a^m) = 4\mathcal{T}G_a^m + 2\partial^m S_a, \quad (3.14e)$$

and the gauge transformations can be summarized as

$$\delta F_a^m = \partial^m \Lambda_a, \quad \delta G_a^m = \partial^m \Omega_a, \quad \delta S_a = -2\mathcal{T}\Omega_a. \quad (3.15)$$

Since S_a is pure gauge, the physical states described by the vertex (3.10) correspond to a massless vector F_a^m and two fields with negative mass-squared $m^2 = -4\mathcal{T}$ namely the scalar φ_α and the vector G_a^m .

In parallel to subsection 2.3, we can prepare the vertex V_J for the tensionless limit analysis. Considering the redefinitions of the worldsheet ghosts of (2.13), V_J can be rewritten as

$$\frac{1}{2\mathcal{T}}V_J = c\tilde{c}J_\alpha\varphi_\alpha + c\tilde{c}P_mJ_aA_a^m + c\tilde{c}\partial X_mJ_aB_a^m - c\partial\tilde{c}J_aA_a - \tilde{c}\partial\tilde{c}J_aB_a. \quad (3.16)$$

Here, the fields A_a , A_a^m , B_a and B_a^m are defined in terms of F_a^m , G_a^m , F_a and G_a as

$$\begin{aligned} A_a &\equiv F_a + G_a, & B_a &\equiv \mathcal{T}(F_a - G_a), \\ A_a^m &\equiv F_a^m + G_a^m, & B_a^m &\equiv \mathcal{T}(F_a^m - G_a^m), \end{aligned} \quad (3.17)$$

with gauge transformations $\delta A_a^m = \partial^m \Lambda_a$ and $\delta B_a^m = \mathcal{T} \partial^m \Lambda_a$.

Their equations of motion follow from (3.14) and are given by

$$\begin{aligned} A_a &= \frac{1}{2} \partial_m A_a^m, & \mathcal{T} A_a^m - \frac{1}{2} \partial_n F_a^{mn} &= B_a^m, \\ B_a &= \frac{1}{2} \partial_m B_a^m, & (\square + 4\mathcal{T}) \partial_n F_a^{mn} &= 0. \end{aligned} \quad (3.18)$$

Therefore, the physical spectrum can be described in terms of only two fields, φ_α and A_a^m . The vector B_a^m is auxiliary, helping to implement a quartic equation of motion for A_a^m , which carries the degrees of freedom of both the massless and the massive vector fields, F_a^m and G_a^m . Note, in particular, the tensionless limit renders a massless spectrum with equations of motion $\square \varphi_\alpha = \square^2 A_a^m = 0$. As in the bosonic model of section 2, this behavior can be easily observed when analyzing the effective field theory associated to the model, which will be done in subsection 3.3. The first step will be to determine the 3-point amplitudes using the vertex (3.10).

3.2 3-point amplitudes

In order to compute the 3-point amplitude

$$\mathcal{A}_3 \equiv \langle V_J(z) V_J(y) V_J(w) \rangle, \quad (3.19)$$

we need to provide further details on the current algebra CFT, in particular the OPE's involving the operator J_α defined in (3.12) and the properties of the Clebsch-Gordan coefficients $C_{\alpha ab}$.

The operator J_α satisfies the following OPE's:

$$T_C^-(z) J_\alpha(y) \sim \frac{2J_\alpha}{(z-y)^2} + \frac{\partial J_\alpha}{(z-y)}, \quad (3.20a)$$

$$J_a(z) J_\alpha(y) \sim C_{\alpha ab} \frac{J_b}{(z-y)^2} - (T_a)_{\alpha\beta} \frac{J_\beta}{(z-y)}, \quad (3.20b)$$

$$\begin{aligned} J_\alpha(z) J_\beta(y) &\sim \frac{k\delta_{\alpha\beta}}{(z-y)^4} - (T_a)_{\alpha\beta} \left\{ \frac{J_a}{(z-y)^3} + \frac{1}{2} \frac{\partial J_a}{(z-y)^2} + \frac{1}{6} \frac{\partial^2 J_a}{(z-y)} \right\} \\ &\quad + d_{\alpha\beta\gamma} \left\{ \frac{J_\gamma}{(z-y)^2} + \frac{1}{2} \frac{\partial J_\gamma}{(z-y)} \right\} \\ &\quad + d_{\alpha\beta abc} \frac{J_{(abc)}}{(z-y)} + d_{\alpha\beta[ab]} \frac{J_{[ab]}}{(z-y)} + e_{a\alpha\beta} \frac{(J_a, T_C^-)}{(z-y)}. \end{aligned} \quad (3.20c)$$

The first OPE states that J_α is a primary operator of conformal dimension 2. The second OPE is connected to the definition of the Clebsch-Gordan coefficients (quadratic pole) and

the group transformation of J_α (simple pole). $(T_a)_{\alpha\beta}$ denotes the group generators in the traceless bi-adjoint representation of the group G and satisfy

$$[T_a, T_b]_{\alpha\beta} = if_{abc} (T_c)_{\alpha\beta}, \quad (3.21a)$$

$$(T_a)_{\alpha\beta} \equiv 2if_{abc} C_{\alpha(ce)} (C^{-1})_{\beta(be)}, \quad (3.21b)$$

$$(T_a T_b)_{\alpha\alpha} = 2g(\Delta + 2)\delta_{ab}, \quad (3.21c)$$

$$(T_a T_a)_{\alpha\beta} = 4g\delta_{\alpha\beta} - 2f_{abc} f_{ade} C_{\alpha(ce)} (C^{-1})_{\beta(bd)}, \quad (3.21d)$$

The OPE (3.20c) can be used to define the 2-point and 3-point functions involving only J_α 's. Operators of conformal dimension 3 appear in the last line (with numerical coefficients $d_{\alpha\beta abc}$, $d_{\alpha\beta[ab]}$ and $e_{a\alpha\beta}$) but they do not contribute to $\mathcal{A}_3$. $J_{(abc)}$ is the totally symmetric traceless normal ordered product of J_a , J_b and J_c , and $J_{[ab]}$ is the antisymmetric product $(J_a, J_b) - (J_b, J_a)$.

The Clebsch-Gordan coefficients $C_{\alpha ab}$ are defined in such a way that

$$C_{\alpha ab} (C^{-1})_{\beta ab} = \delta_{\alpha\beta}, \quad (3.22a)$$

$$C_{\alpha ab} (C^{-1})_{\alpha cd} = \delta_{(ab)(cd)}, \quad (3.22b)$$

$$C_{\alpha ab} C_{\alpha cd} = \Delta_{(ab)(cd)} + 2k\delta_{(ab)(cd)}, \quad (3.22c)$$

$$C_{\alpha ab} C_{\beta ab} = f_{ade} f_{bce} C_{\beta ab} (C^{-1})_{\alpha cd} + 2k\delta_{\alpha\beta}, \quad (3.22d)$$

with

$$\delta_{(ab)(cd)} \equiv \frac{1}{2}\delta_{ac}\delta_{bd} + \frac{1}{2}\delta_{ad}\delta_{bc} - \frac{1}{\Delta}\delta_{ab}\delta_{cd}, \quad (3.23a)$$

$$\Delta_{(ab)(cd)} \equiv \frac{1}{2}f_{ade}f_{bce} + \frac{1}{2}f_{ace}f_{bde} - \frac{2g}{\Delta}\delta_{ab}\delta_{cd}. \quad (3.23b)$$

Finally, the coefficient $d_{\alpha\beta\gamma}$ is defined as

$$d_{\alpha\beta\gamma} \equiv (C^{-1})_{\beta ab} \left[(T_a T_b)_{\alpha\gamma} + 2C_{\alpha ac} C_{\gamma bc} \right], \quad (3.24)$$

or

$$C_{\beta ab} d_{\alpha\beta\gamma} = \frac{1}{2} (T_a T_b)_{\alpha\gamma} + C_{\alpha ae} C_{\gamma be} + (a \leftrightarrow b) - \text{trace}. \quad (3.25)$$

Although not manifestly, $d_{\alpha\beta\gamma}$ is traceless, i.e. $d_{\alpha\alpha\gamma} = 0$, and completely symmetric in the exchange of any pair of indices.

The 2-point amplitudes involving the gauge currents can be easily determined through the OPE's (3.4), (3.20b) and (3.20c), and are given by

$$\langle J_a(z) J_b(y) \rangle = \frac{k\delta_{ab}}{(z-y)^2}, \quad (3.26a)$$

$$\langle J_a(z) J_\alpha(y) \rangle = 0, \quad (3.26b)$$

$$\langle J_\alpha(z) J_\beta(y) \rangle = \frac{k\delta_{\alpha\beta}}{(z-y)^4}. \quad (3.26c)$$

The 3-point amplitudes are now straightforward to compute. They can be summarized as

$$\langle J_a(z)J_b(y)J_c(w) \rangle = -ikf_{abc}(z-y)^{-1}(y-w)^{-1}(w-z)^{-1}, \quad (3.27a)$$

$$\langle J_\alpha(z)J_a(y)J_b(w) \rangle = kC_{\alpha ab}(z-y)^{-2}(w-z)^{-2}, \quad (3.27b)$$

$$\langle J_\alpha(z)J_\beta(y)J_a(w) \rangle = k(T_a)_{\alpha\beta}(z-y)^{-3}(y-w)^{-1}(w-z)^{-1}, \quad (3.27c)$$

$$\langle J_\alpha(z)J_\beta(y)J_\gamma(w) \rangle = kd_{\alpha\beta\gamma}(z-y)^{-2}(y-w)^{-2}(w-z)^{-2}. \quad (3.27d)$$

As one last step before evaluating (3.19), it will be convenient to fix the gauge degrees of freedom of V_J . Using the gauge transformations (3.15), we will choose $S_a = 0$. In this gauge, $\partial_m G_a^m = 0$ as a consequence of the equations of motion. We can use the remaining parameter to fix the transversal gauge for the massless vector, such that the vertex is simplified to

$$V_J = c_+c_-P_m^+J_aF_a^m + c_+c_-P_m^-J_aG_a^m + c_+c_-J_\alpha\varphi_\alpha. \quad (3.28)$$

Using the tree level measure for the ghosts (2.38), the 3-point amplitude (3.19) can be computed to be

$$\begin{aligned} \mathcal{A}_3 = & kd_{\alpha\beta\gamma} \langle \varphi_\alpha \varphi_\beta \varphi_\gamma \rangle - 3k(T_a)_{\alpha\beta} \langle \varphi_\alpha \partial_m \varphi_\beta (F_a^m + G_a^m) \rangle \\ & - 3kC_{\alpha ab} \langle \partial_m \partial_n \varphi_\alpha (F_a^m + G_a^m) (F_b^n + G_b^n) \rangle \\ & - ikf_{abc} \langle \partial_p (F_a^m + G_a^m) \partial_m (F_b^n + G_b^n) \partial_n (F_c^p + G_c^p) \rangle \\ & + 6k\mathcal{T}C_{\alpha ab}\eta_{mn} \langle \varphi_\alpha (F_a^m - G_a^m) (F_b^n + G_b^n) \rangle \\ & - 6ik\mathcal{T}f_{abc}\eta_{mn} \langle (F_a^m - G_a^m) \partial_p (F_b^n + G_b^n) (F_c^p + G_c^p) \rangle. \end{aligned} \quad (3.29)$$

Observe that $\mathcal{A}_3$ is at most linear in $\mathcal{T}(F_a^m - G_a^m)$. If we look at the vertex (3.16), this is easy to understand because the 3-point amplitudes with two or three B_a^m 's vanish trivially.

In principle, 4-point amplitudes can be computed using the results of Siegel in [9]. Currently, however, there is no clear definition of the integrated vertex operators and higher point amplitudes cannot be *directly* computed from the chiral model. This problem will be dealt with in a separate paper by one of the authors.

In the next subsection we will propose an effective field theory action for the field content of the previous subsection.

3.3 Effective field theory: $(DF)^2 + \text{YM}$

As the main result of this paper, we would like to argue that the effective field theory action corresponding to this extension of the bosonic sectorized model is precisely the action of the $(DF)^2 + \text{YM}$ theory constructed in [17]. Indeed, we have already shown the spectrum to be the same. The action can be decomposed as

$$S_{\text{eff}} = S_J^0 + S_J^{\text{int}}, \quad (3.30)$$

where S_J^0 is the kinetic part of the action and S_J^{int} corresponds to the interactions.

For the kinetic part, we will proceed like in subsection 2.4. For the interaction part, we will analyze the possible vertices that give rise to the 3-point amplitudes displayed in (3.29). Next, we will require the non-linear gauge invariance of the resulting model in order to finally propose its effective action.

3.3.1 Kinetic action

As stated above, we will define the kinetic action as

$$S_J^0 \equiv \langle V_J | \partial c Q | V_J \rangle, \quad (3.31)$$

up to normalization.

In order to further simplify the computation, we will consider the algebraic solutions (3.14b) and (3.14c), such that

$$\begin{aligned} \partial c[Q, V_J] = & \frac{1}{4\mathcal{T}} c_+ c_- \partial c_+ \partial c_- J_a P_m^+ [\partial_n (\partial^n F_a^m - \partial^m F_a^n)] \\ & + \frac{1}{4\mathcal{T}} c_+ c_- \partial c_+ \partial c_- J_a P_m^- [\partial_n (\partial^n G_a^m - \partial^m G_a^n) + 4\mathcal{T} G_a^m] \\ & + \frac{1}{4\mathcal{T}} c_+ c_- \partial c_+ \partial c_- \partial J_a [2\mathcal{T} \partial_m G_a^m] \\ & + \frac{1}{4\mathcal{T}} c_+ c_- \partial c_+ \partial c_- J_\alpha [\square \varphi_\alpha + 4\mathcal{T} \varphi_\alpha]. \end{aligned} \quad (3.32)$$

It is then straightforward to show that

$$\begin{aligned} S_J^0 = & \int d^d x \{ \varphi_\alpha (\square \varphi_\alpha + 4\mathcal{T} \varphi_\alpha) - 2\mathcal{T} F_{ma} (\square F_a^m - \partial^m \partial_n F_a^n) \\ & + 2\mathcal{T} G_{ma} (\square G_a^m + 4\mathcal{T} G_a^m - \partial^m \partial_n G_a^n) \}. \end{aligned} \quad (3.33)$$

Note that the kinetic terms of the fields F_a^m and G_a^m have opposite sign in S_J^0 . Technically, the sign difference can be traced back to the OPE's of P_m^+ and P_m^- with themselves. As discussed previously, this indicates an instability of the model and we can again reinterpret it in terms of a higher derivative theory. In fact, as we will now show, this behavior is more transparent if we rewrite the action in terms of the vectors A_a^m and B_a^m defined in (3.17). The kinetic action can then be cast as

$$S_J^0 = \int d^d X \{ \varphi_\alpha (\square \varphi_\alpha + 4\mathcal{T} \varphi_\alpha) + 2B_{ma} \partial_n F_a^{mn} + 2(B_a^m - \mathcal{T} A_a^m) (B_{ma} - \mathcal{T} A_{ma}) \}, \quad (3.34)$$

with

$$F_a^{mn} \equiv \partial^m A_a^n - \partial^n A_a^m. \quad (3.35)$$

Ignoring for now the interaction terms, observe that the equation of motion for B_a^m is algebraic, given by

$$B_a^m = \mathcal{T} A_a^m + \frac{1}{2} \partial_n F_a^{nm}. \quad (3.36)$$

If we replace this solution back in the action, we obtain

$$S_J^0|_B = \int d^d X \left\{ \varphi_\alpha (\square \varphi_\alpha + 4\mathcal{T} \varphi_\alpha) + \mathcal{T} F_a^{mn} F_{mna} - \frac{1}{2} \partial_n F_a^{mn} \partial^p F_{mpa} \right\}. \quad (3.37)$$

This action can be identified with the kinetic part of the $(DF)^2 + \text{YM}$ theory constructed in [17]. Note that the propagator of A_a^m is given in momentum space by

$$G_{ab}^{mn}(p) = \frac{i\eta^{mn}\delta_{ab}}{p^2(p^2 - 4\mathcal{T})}. \quad (3.38)$$

The pole structure of this propagator agrees with the interpretation given after equation (3.18) that A_a^m effectively describes the massless and the massive vector fields, F_a^m and G_a^m .

3.3.2 Cubic vertices and the effective action

As it turns out, the procedure of integrating B_a^m out can be partially extended to interactions. We say “partially” because in this paper we consider only unintegrated vertex operators, therefore only 3-point tree level amplitudes. We expect this integration to hold for higher point vertices as well.

By looking at $\mathcal{A}_3$ in (3.29), it is easy to show that the 3-point vertices in terms of the vectors A_a^m and B_a^m can be schematically expressed as

$$\begin{aligned}\varphi^3 &\sim d_{\alpha\beta\gamma}\varphi_\alpha\varphi_\beta\varphi_\gamma, & \varphi A^2 &\sim C_{\alpha ab}\varphi_\alpha\partial_n A_a^m\partial_m A_b^n, \\ \varphi^2 A &\sim (T_a)_{\alpha\beta}\varphi_\alpha\partial_m\varphi_\beta A_a^m, & A^3 &\sim if_{abc}\partial_p A_a^m\partial_m A_b^n\partial_n A_c^p, \\ \varphi AB &\sim C_{\alpha ab}\eta_{mn}\varphi_\alpha B_a^m A_b^n, & A^2 B &\sim if_{abc}\eta_{mn}B_a^m\partial_p A_b^n A_c^p.\end{aligned}\quad (3.39)$$

The idea now is to analyze the possible gauge invariant interactions that can generate these vertices after integrating out B_a^m , which is at most linear in the expressions above. The equation of motion for B_a^m in (3.36) gets modified to

$$B_a^m = \mathcal{T}A_a^m + \frac{1}{2}\partial_n F_a^{nm} + c_\# C_{\alpha ab}\varphi_\alpha A_{mb} + id_\# f_{abc}\eta_{mn}\partial_p A_b^n A_c^p + \dots, \quad (3.40)$$

where $c_\#$ and $d_\#$ are numerical constants and the dots contain other terms necessary to generate the correct gauge transformation for B_a^m (remember that the onshell 3-point amplitude $\mathcal{A}_3$ was computed using gauge-fixed vertex operators). Taking this into consideration and replacing B_a^m in the action, we can show that *all* 3-point vertices come from the operators

$$C_{\alpha ab}\varphi_\alpha F_a^{mn}F_{mnb}, \quad (D\varphi)^2, \quad (DF)^2, \quad F^3, \quad F^2, \quad d_{\alpha\beta\gamma}\varphi_\alpha\varphi_\beta\varphi_\gamma,$$

where F_a^{mn} was redefined to be the non-Abelian field strength

$$F_a^{mn} \equiv (\partial^m A_a^n - \partial^n A_a^m) + igf_{abc}A_b^m A_c^n, \quad (3.41)$$

with coupling constant g , and D^m denotes the covariant derivative with respect to the vector A_a^m . The form of the higher point vertices (4, 5 and 6) is severely restricted by the non-linear gauge invariance of the effective action. Some contributions naturally appear after integrating out B_a^m and we expect them to combine with the input coming from higher-point amplitudes, which involve integrated vertex operators.

Finally, we propose the effective field theory action of the model to be

$$\begin{aligned}S_{\text{eff}} = \int d^d x \Big\{ & \frac{1}{2}(D_n F_a^{mn})^2 - \mathcal{T}F_a^{mn}F_{mna} + \frac{1}{2}D_m\varphi_\alpha D^m\varphi_\alpha - 2\mathcal{T}(\varphi^\alpha)^2 \\ & + \frac{g}{3}f_{abc}F_{na}^m F_{pb}^n F_{mc}^p + \frac{g}{2}C_{\alpha ab}\varphi_\alpha F_a^{mn}F_{mnb} + \frac{g}{3!}d_{\alpha\beta\gamma}\varphi_\alpha\varphi_\beta\varphi_\gamma \Big\},\end{aligned}\quad (3.42)$$

where g is the coupling constant. This action describes the $(DF)^2 + \text{YM}$ theory of [17].

Moreover, if we include the “+” sector mentioned in the beginning of this section, the effective field theory action describes a more general model with a mirrored set of fields. In particular, if we restrict the gauge symmetry of the “+” sector to be instead a global symmetry, the effective action describes the $(DF)^2 + \text{YM} + \phi^3$ theory. This will be shown next.

3.4 Including the other gauge sector: $(DF)^2 + \text{YM} + \phi^3$

We will consider for the “+” sector an affine Lie algebra associated to a group $\hat{G}$ (with structure constants $\hat{f}_{AB}{}^C$) and level $\hat{k}$. Apart from the central charge constraint (3.3), $\{\hat{G}, \hat{k}\}$ are independent of $\{G, k\}$, from the “−” sector. The new currents, $\hat{J}_A$, are completely analogous to the ones discussed there, e.g. they satisfy the OPE

$$\hat{J}_A(z) \hat{J}_B(y) \sim \frac{\hat{k} \delta_{AB}}{(z-y)^2} + i \hat{f}_{AB}{}^C \frac{\hat{J}_C(y)}{(z-y)}, \quad (3.43)$$

when conveniently normalized. Here, δ_{AB} is a Kronecker delta.

In order to analyze the physical spectrum, we can start with the hatted version of (3.10), defined by

$$\begin{aligned} V_{\hat{J}} = & c_+ c_- P_m^+ \hat{J}_A \hat{G}_A^m + c_- (\partial c_+ - \partial c_-) \hat{J}_A \hat{G}^A + c_+ c_- \partial \hat{J}_A \hat{S}_A \\ & + c_+ c_- P_m^- \hat{J}_A \hat{F}_A^m + c_+ (\partial c_+ - \partial c_-) \hat{J}_A \hat{F}^A + c_+ c_- \hat{J}_{\hat{\alpha}} \hat{\varphi}^{\hat{\alpha}}. \end{aligned} \quad (3.44)$$

It is easy to see that the fields appearing in this vertex operator will satisfy essentially the same equations of motion and gauge transformations as their counterparts in the “−” sector, albeit with one important difference: the replacement $\mathcal{T} \rightarrow -\mathcal{T}$. By going through the same steps as in subsection 3.1, we find that the physical spectrum in this sector contains a “mirror image” of the physical spectrum in the “−” sector, but with opposite mass-squared.

In addition, we can build a new type of vertex operator involving currents from both sectors. It has the form

$$V_{\phi} = c_+ c_- J_a \hat{J}_A \phi^{aA}, \quad (3.45)$$

where ϕ^{aA} is a bi-adjoint scalar transforming in the adjoint representation of both gauge groups. BRST closedness implies the equation of motion

$$\square \phi^{aA} = 0, \quad (3.46)$$

whence ϕ^{aA} is a massless field.

Following the same method used in subsection 3.3, the kinetic part of the effective action involving the group indices can be cast as

$$S^0 = S_J^0 + S_{\hat{J}}^0 + S_{\phi}^0, \quad (3.47)$$

where S_J^0 was given in (3.37) and $S_{\hat{J}}^0$ is its hatted analogue, and

$$S_{\phi}^0 = k \hat{k} \int d^d X \{ \phi_{aA} \square \phi^{aA} \}. \quad (3.48)$$

As for the interacting part, it clearly contains the corresponding part in (3.42) and its hatted version. Moreover, note that cubic vertices mixing the fields in V_J with those in

$V_{\hat{J}}$ can only appear through $\langle V_\phi V_J V_{\hat{J}} \rangle$, since the three-point functions involving $\langle J \hat{J} \hat{J} \rangle$ or $\langle J J \hat{J} \rangle$ vanish. The non-vanishing three-point functions with insertions of V_ϕ are given by:

$$\langle V_\phi(z) V_\phi(y) V_\phi(w) \rangle = k\hat{k} f^{abc} \hat{f}^{ABC} \left\langle \phi^{aA} \phi^{bB} \phi^{cC} \right\rangle, \quad (3.49a)$$

$$\begin{aligned} \langle V_\phi(z) V_\phi(y) V_J(w) \rangle &= -ik\hat{k} f_{abc} \left\langle \phi^{aA} \partial_m \phi^{bA} (F_c^m + G_c^m) \right\rangle \\ &\quad - k\hat{k} C_{\alpha ab} \left\langle \phi^{aA} \phi^{bA} \varphi^\alpha \right\rangle, \end{aligned} \quad (3.49b)$$

$$\begin{aligned} \langle V_\phi(z) V_\phi(y) V_{\hat{J}}(w) \rangle &= -ik\hat{k} f_{ABC} \left\langle \phi^{aA} \partial_m \phi^{aB} (\hat{F}_C^m + \hat{G}_C^m) \right\rangle \\ &\quad - k\hat{k} \hat{C}_{\hat{\alpha} AB} \left\langle \phi^{aA} \phi^{aB} \hat{\varphi}^{\hat{\alpha}} \right\rangle, \end{aligned} \quad (3.49c)$$

$$\begin{aligned} \langle V_\phi(z) V_J(y) V_{\hat{J}}(w) \rangle &= \frac{1}{2} k\hat{k} \eta_{mn} \left\langle (F_a^m + G_a^m) (\hat{F}_A^n + \hat{G}_A^n) \square \phi^{aA} \right\rangle \\ &\quad - k\hat{k} \eta_{mn} \left\langle \phi^{aA} \partial_p (F_a^m + G_a^m) \partial^p (\hat{F}_A^n + \hat{G}_A^n) \right\rangle \\ &\quad + k\hat{k} \left\langle \phi^{aA} \partial_n (F_a^m + G_a^m) \partial^m (\hat{F}_A^n + \hat{G}_A^n) \right\rangle. \end{aligned} \quad (3.49d)$$

Thus, defining

$$\hat{A}_A^m \equiv \hat{F}_A^m + \hat{G}_A^m, \quad \hat{F}_A^{mn} \equiv \partial^m \hat{A}_A^n - \partial^n \hat{A}_A^m + ig \hat{f}_{ABC} \hat{A}_B^m \hat{A}_C^n, \quad (3.50)$$

and following arguments similar to the ones given in the previous subsection, we can write the effective action as

$$S_{\text{eff}} = S[A, \varphi] + S[\hat{A}, \hat{\varphi}] + S[A, \hat{A}, \phi], \quad (3.51)$$

where $S[A, \varphi]$ is the right-hand side of (3.42), $S[\hat{A}, \hat{\varphi}]$ is its hatted version and

$$\begin{aligned} S[A, \hat{A}, \phi] \equiv \int d^d x \left\{ \frac{\hat{k}}{2} (D_m \phi^{aA})^2 + \frac{g\hat{k}}{3!} f_{abc} \hat{f}_{ABC} \phi^{aA} \phi^{bB} \phi^{cC} + \frac{g}{2} C_{\alpha ab} \varphi^\alpha \phi^{aA} \phi^{bA} \right. \\ \left. + \frac{g}{2} \hat{C}_{\hat{\alpha} AB} \hat{\varphi}^{\hat{\alpha}} \phi^{aA} \phi^{aB} + g \phi^{aA} F_a^{mn} \hat{F}_{mnA} \right\}, \end{aligned} \quad (3.52)$$

where the covariant derivative of ϕ^{aA} with respect to both gauge fields is given by

$$D^m \phi^{aA} = \partial^m \phi^{aA} - ig f_{abc} A_b^m \phi^{cA} - ig \hat{f}_{ABC} \hat{A}_B^m \phi^{aC}. \quad (3.53)$$

Thus we have found the complete effective action in the gauge sector of the model. Now we would like to make contact with the scalar extension of the $(DF)^2 + \text{YM}$ theory which was introduced by Johansson and Nohle [17]. There, the group $\hat{G}$ (with indices $A, B, \dots$) is viewed instead as a global symmetry group.² In the present chiral string formulation, we can turn off the gauge field $\hat{A}_A^m$ and the scalar $\hat{\varphi}^{\hat{\alpha}}$, effectively taking $S[\hat{A}, \hat{\varphi}] \rightarrow 0$ and turning the group $\hat{G}$ into a global symmetry at tree level. Moreover, we are free to rescale the field ϕ in order to eliminate $\hat{k}$ from its kinetic term. However, a factor of $\lambda \equiv \sqrt{\hat{k}}$ would

²In the context of the double-copy construction found in [20], this would be the heterotic string group.

still be present in the cubic term (with $\lambda > 0$). After performing these modifications, we can finally write the effective Lagrangian in the same form as in [17]:

$$\begin{aligned} \mathcal{L}_{(DF)^2+YM+\phi^3} = & \frac{1}{2}(D_n F_a^{mn})^2 + \frac{1}{2}(D_m \varphi^\alpha)^2 + \frac{1}{2}(D_m \phi_{aA})^2 + \frac{1}{2}m^2(\varphi^\alpha)^2 + \frac{1}{4}m^2(F_a^{mn})^2 \\ & + \frac{g}{3}F^3 + \frac{g}{2}C_{\alpha ab}\varphi^\alpha F^{mna}F_{mn}^b + \frac{g}{3!}d_{\alpha\beta\gamma}\varphi^\alpha\varphi^\beta\varphi^\gamma + \frac{g}{2}C_{\alpha ab}\varphi^\alpha\phi^{aA}\phi^{bA} \\ & + \frac{g\lambda}{3!}f_{abc}\hat{f}_{ABC}\phi^{aA}\phi^{bB}\phi^{cC}, \end{aligned} \quad (3.54)$$

where $m^2 = -4\mathcal{T}$.

4 Conclusion

In the first part of this work, we reexamined the bosonic chiral string, now in the sectorized interpretation, deriving a few novel results. By considering the action of the BRST operator on the most general vertex operator, we confirmed the physical spectrum found in [11], namely a massless level identical to that of the ordinary bosonic string and two traceless-symmetric fields $h_\pm^{mn}$ with mass-squared $m^2 = \pm 4\mathcal{T}$. Moreover, we showed that the extra (massive) states can be seen as auxiliary fields (cf. footnote 1) leading to a higher derivative gravity theory, which in the tensionless limit ($\mathcal{T} \rightarrow 0$) reduces to the recent results of [15] for the bosonic ambitwistor string. In [27] the massive spin-2 states were determined to be ghosts via a 4-point amplitude analysis based on a “twisted” Kawai-Lewellen-Tye formula. This fact is manifest in the quadratic action we construct from the vertex operator.

In the second part, we showed that the current algebra extension of the bosonic model effectively leads to the $(DF)^2 + YM + \phi^3$ Lagrangian of [17], with all its fields and couplings coming naturally from standard string (field) theory techniques. The emergence of the higher derivative term $(DF)^2$ from two vector fields of the physical spectrum is particularly interesting. In addition, we would like to point out that the group constants $C_{\alpha ab}$ and $d_{\alpha\beta\gamma}$, their relations and properties emerge naturally in our model and are valid for a generic level k of the algebra. In [17], on the other hand, such relations are obtained by demanding that the gluon amplitudes satisfy the Bern-Carrasco-Johansson relations [18] and our results agree when we take $k \rightarrow 0$. This limit corresponds to a projection to the single-trace amplitude sector, which is where we expect our results to match. The multitrace sector of the worldsheet model is “contaminated” by the gravity theory described in section 2, much like the Berkovits-Witten twistor string necessarily includes conformal gravity [28, 29].

It is amusing to notice that the amplitudes of $(DF)^2$ theories, which enter as double-copy constituents of (ordinary) bosonic and heterotic string amplitudes [20], can themselves be described in terms of a worldsheet model. Since the chiral string we studied is a closed string, it is plausible that its amplitudes can be written in double-copy form. The latter would probably involve the Z-theory investigated in [30–32], which is related to the basis of integrals mentioned in the introduction. We plan to address this question in future work.

Acknowledgments

We would like to thank Nathan Berkovits and Fei Teng for useful discussions. TA would like to thank Marco Chiodaroli, Henrik Johansson and Oliver Schlotterer for collaboration on related topics. RLJ would like to thank the Czech Science Foundation — GAČR for financial support under the grant 19-06342Y. ML would like to thank FAPESP grant 2016/16824-0 for financial support.

A Current algebra CFT

In this appendix we will discuss some general properties of the CFT of gauge sector of section 3.

As mentioned in the text, we are using the ordering prescription (3.7), which can be understood as the product of two operators $A(z)$ and $B(y)$ in the limit $z \rightarrow y$ with the removal of singular terms. Note that this prescription is neither commutative nor associative:

$$(A, B) \neq (B, A), \quad (\text{A.1})$$

$$((A, B), C) \neq (A, (B, C)). \quad (\text{A.2})$$

The energy-momentum tensor of the algebra can be obtained using the Sugawara construction and it is defined by

$$T \equiv A(J_a, J_a), \quad (\text{A.3})$$

where A is a numerical constant to be determined by imposing the OPE

$$J_a(z) T(y) \sim \frac{J_a}{(z-y)^2}. \quad (\text{A.4})$$

In order to do that, we can compute first

$$\begin{aligned} J_a(z) (J_b, J_c)(y) &\sim ik \frac{f_{abd} \delta_{dc}}{(z-y)^3} - f_{abd} f_{dce} \frac{J_e}{(w-y)^2} \\ &+ \frac{k \delta_{ac} J_b}{(z-y)^2} + \frac{i f_{acd}}{(z-y)} (J_b, J_d) \\ &+ \frac{k \delta_{ab} J_c}{(z-y)^2} + \frac{i f_{abd}}{(z-y)} (J_d, J_c). \end{aligned} \quad (\text{A.5})$$

It implies that

$$J_a(z) T(y) \sim 2Ak \frac{J_a}{(z-y)^2} + A f_{acd} f_{bcd} \frac{J_b}{(w-y)^2}. \quad (\text{A.6})$$

Now we introduce the dual Coxeter number, g , defined through

$$f_{acd} f_{bcd} = 2g \delta_{ab}. \quad (\text{A.7})$$

Therefore we can fix A to

$$A = \frac{1}{2(k+g)}. \quad (\text{A.8})$$

Now we can compute the central charge of the model through the OPE

$$T(z)T(y) \sim \frac{c/2}{(z-y)^4} + \frac{2T}{(z-y)^2} + \frac{\partial T}{(z-y)}. \quad (\text{A.9})$$

The result is

$$c = \frac{k\Delta}{(k+g)}. \quad (\text{A.10})$$

This is the central charge of the gauge sector.

Building additional primary operators. One of the operators we need for the computation of 3-point amplitudes is related to the ordered product of two currents, (J_a, J_b) . Observe, however, that this product is not symmetric. In fact, we can show that

$$(J_a, J_b) - (J_b, J_a) = if_{abc}\partial J_c. \quad (\text{A.11})$$

Therefore, we can define the operator $J_{ab} = J_{ba}$ as

$$J_{ab} \equiv \frac{1}{2}(J_a, J_b) + \frac{1}{2}(J_b, J_a), \quad (\text{A.12})$$

$$= (J_a, J_b) - \frac{i}{2}f_{ab}{}^c\partial J_c, \quad (\text{A.13})$$

which can be further decomposed in two irreducible pieces: its trace, proportional to T , and a traceless part.

Observe that any rank two tensor T_{ab} can automatically generate a symmetric traceless tensor $T_{(ab)}$ via a multiplication by the projector

$$\delta_{(ab)(cd)} \equiv \frac{1}{2}\delta_{ac}\delta_{bd} + \frac{1}{2}\delta_{ad}\delta_{bc} - \frac{1}{\Delta}\delta_{ab}\delta_{cd}.$$

It acts as an identity operator for the indices (ab) , as

$$\delta_{(ab)(ef)}\delta_{(ef)(cd)} = \delta_{(ab)(cd)}. \quad (\text{A.14})$$

The pair (ab) is an explicit realization of the index α introduced in section 3, labeling the field φ_α of the vertex operator (3.10).

As it turns out, the symmetric traceless projection picks only the primary part of the operator (J_a, J_b) :

$$T(z)\delta_{(ab)(cd)}(J_c, J_d)(y) \sim \delta_{(ab)(cd)}\frac{2(J_c, J_d)}{(z-y)^2} + \delta_{(ab)(cd)}\frac{\partial(J_c, J_d)}{(z-y)}. \quad (\text{A.15})$$

This is the only dimension 2 primary operator that can be build out of the currents J_a . In addition, we will define the operator

$$\Delta_{(ab)(cd)} \equiv \frac{1}{2}f_{ade}f_{bce} + \frac{1}{2}f_{ace}f_{bde} - \frac{2g}{\Delta}\delta_{ab}\delta_{cd}, \quad (\text{A.16})$$

which is also symmetric and traceless in the index pairs (ab) and (cd) , and the power series

$$C_{(ab)(cd)} = \delta_{(ab)(cd)} - 2\sum_{n=1}^{\infty}(-1)^n\frac{(2n-2)!}{(8k)^nn!}(\Delta^n)_{(ab)(cd)}, \quad (\text{A.17})$$

satisfying

$$C_{(ab)(ef)}C_{(ef)(cd)} = \delta_{(ab)(cd)} + \frac{1}{2k}\Delta_{(ab)(cd)}. \quad (\text{A.18})$$

This is a realization of the Clebsch-Gordan coefficients, $C_{\alpha ab}$, introduced earlier. By construction,

$$(C^{-1})_{(ab)(cd)} \equiv \delta_{(ab)(cd)} + \sum_{n=1}^{\infty} (-1)^n \frac{(2n)!}{(8k)^n n!} (\Delta^n)_{(ab)(cd)}. \quad (\text{A.19})$$

Let us now define the dimension 2 primary operator

$$J_{(ab)} \equiv (C^{-1})_{(ab)(cd)} (J_c, J_d), \quad (\text{A.20})$$

which satisfies the OPE

$$J_a(z) J_{(bc)}(y) \sim 2k C_{(ad)(bc)} \frac{J_d}{(z-y)^2} - (T_a)_{(bc)(de)} \frac{J_{(de)}}{(z-y)},$$

where

$$(T_a)_{(bc)(de)} \equiv -2i (C^{-1})_{(bc)(fg)} f_{afh} C_{(gh)(de)} \quad (\text{A.21})$$

Observe that $(T_a)_{(bc)(de)}$ constitutes a representation of the group generator, as

$$\begin{aligned} [T_a, T_b]_{(de)(fg)} &\equiv (T_a)_{(de)(hi)} (T_b)_{(hi)(fg)} - (T_b)_{(cd)(gh)} (T_a)_{(gh)(ef)} \cdot \\ &= i f_{abc} (T_c)_{(de)(fg)}. \end{aligned} \quad (\text{A.22})$$

In addition, it satisfies

$$(T_a T_a)_{(bc)(de)} = 2g \delta_{bd} \delta_{ce} + 2g \delta_{be} \delta_{cd} - f_{bdf} f_{cef} - f_{bef} f_{cdf}, \quad (\text{A.23a})$$

$$(T_a T_b)_{(cd)(cd)} = 2g(\Delta + 2) \delta_{ab}. \quad (\text{A.23b})$$

At the next conformal level, there are only two primary operators that can be build out of J_a , defined as

$$J_{[ab]} \equiv \frac{1}{2} (J_a, \partial J_b) - \frac{1}{2} (J_b, \partial J_a) - \frac{i}{3} f_{abc} \partial^2 J_c + i C f_{abc} (J_c, T), \quad (\text{A.24a})$$

$$J_{(abc)} = J_{abc} - C [\delta_{bc} (J_a, T) + \delta_{ac} (J_b, T) + \delta_{ab} (J_c, T)], \quad (\text{A.24b})$$

where

$$J_{abc} \equiv \frac{1}{3} (J_a, J_{bc}) + \frac{1}{3} (J_b, J_{ac}) + \frac{1}{3} (J_c, J_{ab}), \quad (\text{A.25a})$$

$$C = \frac{2(k+g)}{k\Delta + 2(k+g)}. \quad (\text{A.25b})$$

They are naturally generated in the OPE algebra. For example,

$$\begin{aligned}
J_{(bc)}(z) J_{(ad)}(y) \sim & 2k^2 \frac{\delta_{(ad)(bc)}}{(z-y)^4} + \frac{4k(k+g)}{\Delta} \delta_{(ad)(bc)} \left\{ \frac{2T}{(z-y)^2} + \frac{\partial T}{(z-y)} \right\} \\
& - k (T_e)_{(bc)(ad)} \left\{ \frac{2J_e}{(z-y)^3} + \frac{\partial J_e}{(z-y)^2} + \frac{1}{3} \frac{\partial^2 J_e}{(z-y)} \right\} \\
& + \frac{1}{2} (C^{-1})_{(ad)(gh)} (T_g T_h)_{(bc)(ef)} \left\{ \frac{2J_{(ef)}}{(z-y)^2} + \frac{J_{(ef)}}{(z-y)} \right\} \\
& + D_{d(bc)(ef)} \frac{J_{(aef)}}{(z-y)} + D_{a(bc)(ef)} \frac{J_{(def)}}{(z-y)} \\
& + D_{(bc)(ad)[ef]} \frac{J_{[ef]}}{(z-y)} + E_{e(bc)(ad)} \frac{(J_e, T)}{(z-y)}, \tag{A.26}
\end{aligned}$$

where $D_{a(bc)(de)}$, $D_{(ab)(cd)[ef]}$ and $E_{a(bc)(de)}$ are given in terms of the structure constants of the group, but their precise expression will not be needed here. The OPE above was presented in the main text with the indices α, β in equation (3.20c).

Open Access. This article is distributed under the terms of the Creative Commons Attribution License ([CC-BY 4.0](https://creativecommons.org/licenses/by/4.0/)), which permits any use, distribution and reproduction in any medium, provided the original author(s) and source are credited.

References

- [1] F. Cachazo, S. He and E.Y. Yuan, *Scattering of Massless Particles in Arbitrary Dimensions*, *Phys. Rev. Lett.* **113** (2014) 171601 [[arXiv:1307.2199](https://arxiv.org/abs/1307.2199)] [[INSPIRE](#)].
- [2] F. Cachazo, S. He and E.Y. Yuan, *Scattering of Massless Particles: Scalars, Gluons and Gravitons*, *JHEP* **07** (2014) 033 [[arXiv:1309.0885](https://arxiv.org/abs/1309.0885)] [[INSPIRE](#)].
- [3] L. Mason and D. Skinner, *Ambitwistor strings and the scattering equations*, *JHEP* **07** (2014) 048 [[arXiv:1311.2564](https://arxiv.org/abs/1311.2564)] [[INSPIRE](#)].
- [4] N. Berkovits, *Infinite Tension Limit of the Pure Spinor Superstring*, *JHEP* **03** (2014) 017 [[arXiv:1311.4156](https://arxiv.org/abs/1311.4156)] [[INSPIRE](#)].
- [5] F. Cachazo, S. He and E.Y. Yuan, *Einstein-Yang-Mills Scattering Amplitudes From Scattering Equations*, *JHEP* **01** (2015) 121 [[arXiv:1409.8256](https://arxiv.org/abs/1409.8256)] [[INSPIRE](#)].
- [6] F. Cachazo, S. He and E.Y. Yuan, *Scattering Equations and Matrices: From Einstein To Yang-Mills, DBI and NLSM*, *JHEP* **07** (2015) 149 [[arXiv:1412.3479](https://arxiv.org/abs/1412.3479)] [[INSPIRE](#)].
- [7] E. Casali, Y. Geyer, L. Mason, R. Monteiro and K.A. Roehrig, *New Ambitwistor String Theories*, *JHEP* **11** (2015) 038 [[arXiv:1506.08771](https://arxiv.org/abs/1506.08771)] [[INSPIRE](#)].
- [8] I. Bandos, *Twistor/ambitwistor strings and null-superstrings in spacetime of $D = 4, 10$ and 11 dimensions*, *JHEP* **09** (2014) 086 [[arXiv:1404.1299](https://arxiv.org/abs/1404.1299)] [[INSPIRE](#)].
- [9] W. Siegel, *Amplitudes for left-handed strings*, [arXiv:1512.02569](https://arxiv.org/abs/1512.02569) [[INSPIRE](#)].
- [10] E. Casali and P. Tourkine, *On the null origin of the ambitwistor string*, *JHEP* **11** (2016) 036 [[arXiv:1606.05636](https://arxiv.org/abs/1606.05636)] [[INSPIRE](#)].
- [11] Y.-t. Huang, W. Siegel and E.Y. Yuan, *Factorization of Chiral String Amplitudes*, *JHEP* **09** (2016) 101 [[arXiv:1603.02588](https://arxiv.org/abs/1603.02588)] [[INSPIRE](#)].

- [12] R.L. Jusinkas, *Notes on the ambitwistor pure spinor string*, *JHEP* **05** (2016) 116 [[arXiv:1604.02915](#)] [[INSPIRE](#)].
- [13] O. Chandía and B.C. Vallilo, *Ambitwistor pure spinor string in a type-II supergravity background*, *JHEP* **06** (2015) 206 [[arXiv:1505.05122](#)] [[INSPIRE](#)].
- [14] T. Azevedo and R.L. Jusinkas, *Connecting the ambitwistor and the sectorized heterotic strings*, *JHEP* **10** (2017) 216 [[arXiv:1707.08840](#)] [[INSPIRE](#)].
- [15] N. Berkovits and M. Lize, *Field theory actions for ambitwistor string and superstring*, *JHEP* **09** (2018) 097 [[arXiv:1807.07661](#)] [[INSPIRE](#)].
- [16] B. Zwiebach, *Closed string field theory: Quantum action and the B-V master equation*, *Nucl. Phys. B* **390** (1993) 33 [[hep-th/9206084](#)] [[INSPIRE](#)].
- [17] H. Johansson and J. Nohle, *Conformal Gravity from Gauge Theory*, [arXiv:1707.02965](#) [[INSPIRE](#)].
- [18] Z. Bern, J.J.M. Carrasco and H. Johansson, *New Relations for Gauge-Theory Amplitudes*, *Phys. Rev. D* **78** (2008) 085011 [[arXiv:0805.3993](#)] [[INSPIRE](#)].
- [19] T. Azevedo and O.T. Engelund, *Ambitwistor formulations of R^2 gravity and $(DF)^2$ gauge theories*, *JHEP* **11** (2017) 052 [[arXiv:1707.02192](#)] [[INSPIRE](#)].
- [20] T. Azevedo, M. Chiodaroli, H. Johansson and O. Schlotterer, *Heterotic and bosonic string amplitudes via field theory*, *JHEP* **10** (2018) 012 [[arXiv:1803.05452](#)] [[INSPIRE](#)].
- [21] S. He, F. Teng and Y. Zhang, *String amplitudes from field-theory amplitudes and vice versa*, *Phys. Rev. Lett.* **122** (2019) 211603 [[arXiv:1812.03369](#)] [[INSPIRE](#)].
- [22] S. He, F. Teng and Y. Zhang, *String Correlators: Recursive Expansion, Integration-by-Parts and Scattering Equations*, *JHEP* **09** (2019) 085 [[arXiv:1907.06041](#)] [[INSPIRE](#)].
- [23] C.R. Mafra, O. Schlotterer and S. Stieberger, *Complete N-Point Superstring Disk Amplitude I. Pure Spinor Computation*, *Nucl. Phys. B* **873** (2013) 419 [[arXiv:1106.2645](#)] [[INSPIRE](#)].
- [24] O. Hohm, W. Siegel and B. Zwiebach, *Doubled α' -geometry*, *JHEP* **02** (2014) 065 [[arXiv:1306.2970](#)] [[INSPIRE](#)].
- [25] K. Lee, S.-J. Rey and J.A. Rosabal, *A string theory which isn't about strings*, *JHEP* **11** (2017) 172 [[arXiv:1708.05707](#)] [[INSPIRE](#)].
- [26] E. Casali, Y. Herfray and P. Tourkine, *The complex null string, Galilean conformal algebra and scattering equations*, *JHEP* **10** (2017) 164 [[arXiv:1707.09900](#)] [[INSPIRE](#)].
- [27] M.M. Leite and W. Siegel, *Chiral Closed strings: Four massless states scattering amplitude*, *JHEP* **01** (2017) 057 [[arXiv:1610.02052](#)] [[INSPIRE](#)].
- [28] E. Witten, *Perturbative gauge theory as a string theory in twistor space*, *Commun. Math. Phys.* **252** (2004) 189 [[hep-th/0312171](#)] [[INSPIRE](#)].
- [29] N. Berkovits and E. Witten, *Conformal supergravity in twistor-string theory*, *JHEP* **08** (2004) 009 [[hep-th/0406051](#)] [[INSPIRE](#)].
- [30] J.J.M. Carrasco, C.R. Mafra and O. Schlotterer, *Abelian Z-theory: NLSM amplitudes and α' -corrections from the open string*, *JHEP* **06** (2017) 093 [[arXiv:1608.02569](#)] [[INSPIRE](#)].
- [31] C.R. Mafra and O. Schlotterer, *Non-abelian Z-theory: Berends-Giele recursion for the α' -expansion of disk integrals*, *JHEP* **01** (2017) 031 [[arXiv:1609.07078](#)] [[INSPIRE](#)].
- [32] J.J.M. Carrasco, C.R. Mafra and O. Schlotterer, *Semi-abelian Z-theory: NLSM + ϕ^3 from the open string*, *JHEP* **08** (2017) 135 [[arXiv:1612.06446](#)] [[INSPIRE](#)].