Five-dimensional Super-Yang-Mills and its Kaluza-Klein tower

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ABSTRACT: We compactify the abelian 6d (1,0) tensor multiplet on a circle bundle, thus reducing the theory down to 5d SYM while keeping all the KK modes. This abelian classical field theory, when interpreted suitably, has a nonlocal superconformal symmetry. Furthermore, a nonabelian generalization, where all the KK modes are kept, is possible for the nonlocal superconformal symmetry, whereas for the local superconformal symmetry we can only realize a subgroup.

KEYWORDS: Field Theories in Higher Dimensions, M-Theory, Supersymmetric Gauge Theory

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1 Introduction

One proposal for the 6d $(2,0)$ tensor multiplet compactified on a circle is that this is fully captured by the dimensionally reduced 5d maximally SYM [4, 5], where instanton particles are believed to play the role of all the KK particles. For abelian gauge group, a precise match between all instanton particles and all KK particles was found in [7, 12].

Another proposal has been to add a KK tower of fundamental field excitations to the 5d SYM. It has been shown that this can be done while preserving the 6d (1,0) Poincare supersymmetry in flat space with one compact circle direction [6, 11, 14, 15]. In this paper, we will generalize this construction and consider six-manifolds that are circle-bundles. On such manifolds, we will realize the 6d (1,0) superconformal symmetry $\mathcal{C}_{6d,(1,0)}$ on 5d SYM plus the KK tower. But not in a conventional way. This symmetry acts in a nonlocal way on the fields. Why we need a nonlocal variation is easy to understand. We assume the existence of a supersymmetry parameter that solves the 6d conformal Killing spinor equation

$$D_M \varepsilon = \Gamma_M \eta \quad (1.1)$$

Such a supersymmetry parameter depends in general on the location along the circle fiber. If we use this parameter to make a local variation of a zero mode field, that is, a field in the 5d SYM multiplet, then the variation of that field has to involve higher KK modes. If on the other hand we vary the 5d SYM field nonlocally, then we can do that without bringing in the KK modes into the variation. Thus with a nonlocal variation we can vary the zero mode fields in a closed way among themselves despite the supersymmetry parameter itself is not a zero mode. If we restrict ourselves to a local symmetry, then we can still realize a 5d restriction $\mathcal{C}_{5d,(1,0)}$ of $\mathcal{C}_{6d,(1,0)}$, where the supersymmetry parameter, in addition to satisfying (1.1), is constant along the circle fiber,

$$\partial_t \varepsilon(t) = 0 \quad (1.2)$$

Here we parametrize the position along the fiber by the time coordinate t . The time derivative coincides with the Lie derivative along the fiber since we use the standard circle bundle metric, and a reparametrization of t is not allowed because that will take us outside the standard circle bundle form for the metric. Alternatively, we may write the condition (1.2) in a coordinate independent way as

$$\mathcal{L}_V \varepsilon = 0$$

where V is the Killing vector field along the circle fiber. In [16] we showed that 5d SYM has the classical symmetry $\mathcal{C}_{5d,(1,0)}$.

2 The six-sphere

To show the existence of a solution to both (1.1) and (1.2) on a curved space, it seems that the round S^6 will be the easiest example to study. The metric on S^6 with radius r in polar coordinates is given by

$$ds_{S^6}^2 = r^2 (d\theta^2 + \sin^2 \theta ds_{S^5}^2)$$

where

$$ds_{S^5}^2 = (d\tau + \kappa_i dx^i)^2 + ds_{\mathbb{CP}^2}^2$$

is the metric on the equatorial S^5 , viewed as a circle-bundle over $\mathbb{CP}^2$ with fiber coordinate $\tau \sim \tau + 2\pi$ and graviphoton κ_i whose field strength $w_{ij} = \partial_i \kappa_j - \partial_j \kappa_i$ is proportional to the Kähler two-form on $\mathbb{CP}^2$. By making the coordinate transformation

$$R = 2r \tan \frac{\theta}{2}$$

we get

$$ds_{S^6}^2 = e^{2\sigma} (dR^2 + R^2 ds_{S^5}^2)$$

$$e^{\sigma(R)} = \frac{1}{1 + \frac{R^2}{4r^2}}$$

This shows that S^6 is conformally flat. We may write this metric in the standard circle-bundle form

$$ds_{S^6}^2 = e^{2\sigma(R)} R^2 (d\tau + \kappa_i dx^i)^2 + e^{2\sigma(R)} (dR^2 + R^2 ds_{\mathbb{CP}^2}^2) \quad (2.1)$$

We may also express this same metric as

$$ds_{S^6}^2 = e^{2\sigma} dx^M dx^M$$

$$e^\sigma = \frac{1}{1 + \frac{x^M x^M}{4r^2}}$$

where x^M are Euclidean coordinates on $(\mathbb{R}^6, \delta_{MN})$. In terms of these coordinates, the most general solution to (1.1) is given by [18]

$$\varepsilon = e^{\frac{\sigma}{2}} (\varepsilon_1 + x^M \Gamma_M \varepsilon_2)$$

where ε_1 and ε_2 are constant parameters, $\partial_M \varepsilon_{1,2} = 0$. The Killing spinor solution

$$\varepsilon_0 = e^{\frac{\sigma(R)}{2}} \varepsilon_1$$

survives the dimensional reduction along the Hopf fiber along the τ -direction since it does not depend on the coordinate τ . One may show that once a metric is in the standard circle-bundle form, the Lie derivative along the τ direction is simply given by $\mathcal{L}_{\frac{\partial}{\partial \tau}} \varepsilon = \partial_\tau \varepsilon$.

We note that σ depends on the coordinate $R = \sqrt{x^M x^M}$ in the metric (2.1), but it does not depend on τ . Thus we have now showed the existence of such a supersymmetry parameter on the curved space S^6 .

3 Superconformal algebra in curved space

Let us start with a realization of the $\mathcal{C}_{6d,(2,0)}$ symmetry. We assume an abelian gauge group. The supersymmetry parameter ε satisfies the conformal Killing spinor equation (1.1) and the 6d Weyl projection $\Gamma \varepsilon = -\varepsilon$. The (2,0) tensor multiplet consists of a Weyl fermion

of opposite chirality $\Gamma\Psi = \Psi$, five scalar fields ϕ^A and a two-form gauge field B_{MN} . The supersymmetry variations are

$$\begin{aligned}\delta\phi^A &= i\bar{\varepsilon}\tau^A\Psi \\ \delta B_{MN} &= i\bar{\varepsilon}\Gamma_{MN}\Psi \\ \delta\Psi &= \frac{1}{12}\Gamma^{MNP}\varepsilon H_{MNP} - \Gamma^M\tau^A\varepsilon\partial_M\phi^A - 4\tau^A\eta\phi^A\end{aligned}\quad (3.1)$$

The supersymmetric Lagrangian is given by

$$\mathcal{L} = \frac{1}{2\pi} \left(-\frac{1}{24}H_{MNP}^2 - \frac{1}{2}(D_M\phi^A)^2 - \frac{R}{10}(\phi^A)^2 + \frac{i}{2}\bar{\Psi}\Gamma^M D_M\Psi \right)$$

where R is the Ricci scalar of the six-manifold. More precisely, its supersymmetry variation is given by a total derivative

$$\delta\mathcal{L} = D_M \left(-\frac{i}{4\pi}\bar{\Psi} \left(\frac{1}{12}\Gamma^{RST}\Gamma^M\varepsilon H_{RST} - \Gamma^R\tau^A\Gamma^M\varepsilon D_R\phi^A - 4\tau^A\Gamma^M\eta\phi^A \right) \right)$$

If we assume the six-manifold is closed, then the action will be supersymmetric on any six-manifold where (1.1) has some solution. For instance on S^6 . However, the above Lagrangian is written in Lorentzian signature.

In a curved space, we shall distinguish between the Weyl weight $\mathcal{W}$ and the scaling dimension Δ . If the field has tensor indices in the spacetime directions, then these indices contribute to the scaling dimension but not to the Weyl weight. For instance, B_{MN} has Weyl weight $\mathcal{W} = 0$ and scaling dimension $\Delta = 2$. One way to see this is by looking at the action for a nonchiral tensor gauge field with field strength $H_{MNP} = (dB)_{MNP}$ in curved space,

$$-\frac{1}{12} \int d^6x \sqrt{-g} g^{MM'} g^{NN'} g^{PP'} H_{MNP} H_{M'N'P'}$$

This is invariant under the Weyl transformation

$$\begin{aligned}g'_{MN} &= e^{2\Omega} g_{MN} \\ B'_{MN} &= B_{MN}\end{aligned}$$

Since B_{MN} is not Weyl rescaled, we will say that it has Weyl weight zero, $\mathcal{W} = 0$. On the other hand, B_{MN} has scaling dimension $\Delta = 2$. The scaling dimension appears in the superconformal algebra in flat space. It does not appear in the superconformal algebra in curved space. What appears in the superconformal algebra in curved space, is the Weyl weight.

In curved space, the closure relations are

$$\begin{aligned}\delta^2\phi^A &= -i\mathcal{L}_S\phi^A - 2i\mathcal{W}_\phi\bar{\varepsilon}\eta\phi^A - 4i\bar{\varepsilon}\tau^{AB}\eta\phi^B \\ \delta^2B_{MN} &= -i\mathcal{L}_SB_{MN} - 2i\mathcal{W}_B\bar{\varepsilon}\eta B_{MN} + 2\partial_M\Lambda_N \\ \delta^2\Psi &= -i\mathcal{L}_S\Psi - 2i\mathcal{W}_\psi\bar{\varepsilon}\eta\Psi - i\bar{\varepsilon}\tau^{AB}\eta\tau^{AB}\Psi \\ &\quad + \frac{3i}{8}S_N\Gamma^N\Gamma^M D_M\Psi \\ &\quad - \frac{i}{8}\bar{\varepsilon}\Gamma_N\tau^A\varepsilon\Gamma^N\tau^A\Gamma^M D_M\Psi\end{aligned}$$

where $S^M = \bar{\varepsilon}\Gamma^M\varepsilon$ is a conformal Killing vector, satisfying

$$D_MS_N + D_NS_M = \frac{1}{3}g_{MN}D^PS_P$$

Our spinor conventions are summarized in the appendix. The gauge parameters are

$$\Lambda_M = -iB_{MN}S^N + i\bar{\varepsilon}\Gamma_M\tau^A\varepsilon\phi^A$$

and the Weyl weights are

$$\begin{aligned}\mathcal{W}_\phi &= 2 \\ \mathcal{W}_B &= 0 \\ \mathcal{W}_\psi &= \frac{5}{2}\end{aligned}$$

From the above closure relations, we see that the superconformal algebra in curved space should contain the relation

$$\delta^2 = -i\mathcal{L}_S - 2i\mathcal{W}\bar{\varepsilon}\eta - 2\bar{\varepsilon}\tau^{AB}\eta S^{AB} \quad (3.2)$$

where the generator of SO(5) R-symmetry is represented as

$$\begin{aligned}(S^{AB})^{CD} &= 2i\delta^{AB,CD} \\ S^{AB} &= \frac{i}{2}\tau^{AB}\end{aligned}$$

We can recover from (3.2) the usual superconformal algebra relation on flat $\mathbb{R}^{5,1}$ where the most general solution to (1.1) is given by

$$\varepsilon = \varepsilon_0 + \Gamma_M\eta x^M$$

All the conformal transformations are encoded in the Lie derivative, and in flat space this Lie derivative becomes

$$S^M\mathcal{P}_M = \bar{\varepsilon}_0\Gamma^M\varepsilon_0\mathcal{P}_M + 2\bar{\varepsilon}_0\eta\mathcal{D} + 2\bar{\varepsilon}_0\Gamma^{MN}\eta\mathcal{L}_{MN} - \bar{\eta}\Gamma^M\eta\mathcal{K}_M$$

where

$$\begin{aligned}\mathcal{P}_M &= -i\partial_M \\ \mathcal{D} &= -ix^M\partial_M \\ \mathcal{L}_{MN} &= i(x_M\partial_N - x_N\partial_M) \\ \mathcal{K}_M &= -2x_Mx^N\partial_N + |x|^2\partial_M\end{aligned}$$

and (3.2) leads to

$$\delta^2 = \bar{\varepsilon}_0\Gamma^M\varepsilon_0\mathcal{P}_M + 2\bar{\varepsilon}_0\eta\mathcal{D} + 2\bar{\varepsilon}_0\Gamma^{MN}\eta\mathcal{L}_{MN} - \bar{\eta}\Gamma^M\eta\mathcal{K}_M + 2\bar{\varepsilon}_0\tau^{AB}\eta S^{AB}$$

which is part of the $(2, 0)$ superconformal algebra on flat space $\mathbb{R}^{1,5}$, where

$$\begin{aligned} P_M &= \mathcal{P}_M \\ D &= \mathcal{D} - i\Delta \\ L_{MN} &= \mathcal{L}_{MN} + S_{MN} \\ K_M &= \mathcal{K}_M - 2i\Delta x_M - S_{MN}x^N \end{aligned}$$

It is particularly interesting to note how the Lorentz generators L_{MN} appear on the right-hand side in this superconformal algebra. These Lorentz generators do not appear if we limit ourselves to the Poincare supercharges that gives us only the translational symmetries.

The advantage with working with the superconformal algebra on curved space is that we may keep a compact circle direction in the manifold and yet we may ask whether the theory has the superconformal symmetry. By leaving flat space, we may avoid the difficult question of how to take the decompactification limit of that circle and the question of Lorentz symmetry in that limit.

If we consider only the supersymmetry parameter $\varepsilon_0 = e^{\sigma/2}\varepsilon_1$ on S^6 and conformally map this to $\mathbb{R}^6$, then we get just the Poincare supersymmetries on $\mathbb{R}^6$ and this can only show us the translational symmetries. To see the Lorentz symmetries on $\mathbb{R}^6$ we need the full 6d superconformal algebra and for that we need to show the existence of all the higher KK towers of supercharges as well. On S^6 , those higher supercharges are corresponding to the other solution $\varepsilon = e^{\sigma/2}x^M\Gamma_M\varepsilon_2$.

4 Dictionary between 6d and 5d languages

We will assume the 6d metric is of the general circle-bundle form

$$ds^2 = -r^2(dt + \kappa_m dx^m)^2 + G_{mn}dx^m dx^n$$

We take time to be compact $t \sim t + 2\pi$ and we define

$$w_{mn} = \partial_m \kappa_n - \partial_n \kappa_m$$

To get a 5d formulation, we expand all the 6d fields in terms of modes

$$\Phi(t, x^m) = \Phi_0(x^m) + \sum_{n \in \mathbb{Z}} \Phi_n(x^m) e^{int}$$

To get to the 6d theory on the Euclidean space whose metric is

$$ds^2 = r^2(dt + \kappa_m dx^m)^2 + G_{mn}dx^m dx^n$$

what we need to do from the 5d perspective, is to replace the KK mode number n everywhere by in , but we will not do this here.

The 6d conformal Killing spinor equation expressed in terms of 5d quantities reads [8]

$$\begin{aligned} \tilde{D}_m \varepsilon &= \gamma_m \eta - \frac{r}{4} w_{mn} \gamma^n \varepsilon + \kappa_m \partial_0 \varepsilon \\ \eta &= \frac{1}{r} \partial_0 \varepsilon - \frac{r}{8} w_{mn} \gamma^{mn} \varepsilon + \frac{1}{2r} (\partial_m r) \gamma^m \varepsilon \end{aligned} \tag{4.1}$$

Here we put a tilde on 5d quantities, so for instance $\tilde{D}_m$ is the 5d spinor derivative that uses the 5d spin connection. The precise expressions for the derivatives acting on spinors are presented in the appendix.

As we want closure on a 6d Lie derivative but we use a 5d formulation, we first need to understand how to express the 6d Lie derivatives in terms of 5d quantities. First we relate the 6d quantity

$$S^M = \bar{\varepsilon} \Gamma^M \varepsilon$$

with the 5d quantities

$$\begin{aligned} s^m &= \bar{\varepsilon} \gamma^m \varepsilon \\ s &= -\bar{\varepsilon} \varepsilon \end{aligned}$$

By using corresponding relations between the 6d and 5d gamma matrices, we get the following relations

$$\begin{aligned} S^m &= s^m \\ S^0 &= -\frac{s}{r} - \kappa_m s^m \end{aligned}$$

and

$$\begin{aligned} S_m &= s_m + \kappa_m r s \\ S_0 &= r s \end{aligned}$$

Let us now consider the 6d Lie derivatives on a scalar, a two-form and a spinor,

$$\begin{aligned} \mathcal{L}_S \sigma &= S^M \partial_M \sigma \\ \mathcal{L}_S B_{MN} &= S^P \partial_P B_{MN} + (\partial_M S^P) B_{PN} + (\partial_M S^P) B_{MP} \\ \mathcal{L}_S \chi &= S^M D_M \chi + \frac{1}{4} (D_M S_N) \Gamma^{MN} \chi \end{aligned}$$

If we express these 6d Lie derivatives in terms of 5d quantities, they become

$$\begin{aligned} \mathcal{L}_S \sigma &= s^m (\partial_m \sigma - \kappa_m \partial_0 \sigma) - \frac{s}{r} \partial_0 \sigma \\ \mathcal{L}_S A_m &= s^p (\partial_p A_m - \kappa_p \partial_0 A_m) - \frac{s}{r} \partial_0 A_m + (\partial_m s^p) A_p \\ &\quad - \frac{1}{r} (\partial_0 s) A_m - \kappa_n (\partial_0 s^n) A_m - (\partial_0 s^n) B_{nm} \\ \mathcal{L}_S B_{mn} &= s^p (\partial_p B_{mn} - \kappa_p \partial_0 B_{mn}) + 2(\partial_{[m} s^p) B_{p|n]} - \frac{s}{r} \partial_0 B_{mn} \\ &\quad + 2\partial_m \left(\frac{s}{r} + \kappa_q s^q \right) A_n \\ \mathcal{L}_S \chi &= s^m (\tilde{D}_m \chi - \kappa_m \partial_0 \chi) - \frac{s}{r} \partial_0 \chi + \frac{1}{4} (\partial_m s_n + r s w_{mn}) \gamma^{mn} \chi \end{aligned}$$

Alternatively we can write

$$\begin{aligned}\mathcal{L}_S A_m &= s^p \partial_p A_m + (\partial_m s^p) A_p \\ &\quad - \partial_0 \left(s^p \kappa_p A_m + \frac{s}{r} A_m \right) - (\partial_0 s^n) B_{nm}\end{aligned}$$

Let us also here note the following identities,

$$\begin{aligned}\partial_m s &= r w_{mn} s^n + \frac{s}{r} \partial_m r - \frac{1}{r} \partial_0 s_m + \kappa_m \partial_0 s \\ \partial_m (rs) + 4r \bar{\varepsilon} \gamma_m \eta &= \partial_0 s_m + r \kappa_m \partial_0 s \\ \partial_m s_n + 2 \bar{\varepsilon} \gamma_{mn} \eta &= \frac{rs}{2} w_{mn} + \kappa_m \partial_0 s_n\end{aligned}$$

The abelian 6d fermionic equation of motion

$$\Gamma^M D_M \chi = 0$$

becomes in the 5d language

$$\gamma^m (\tilde{D}_m \chi - \partial_0 \kappa_m \chi) - \frac{1}{r} \partial_0 \chi + \frac{1}{2r} (\partial_m r) \gamma^m \chi - \frac{r}{8} w_{mn} \gamma^{mn} \chi = 0$$

5 The (1, 0) supermultiplets

As was noted in [11], it is necessary to break supersymmetry by half in order to write down a non-abelian generalization. We impose the Weyl projection condition

$$\tau^5 \varepsilon = -\varepsilon$$

thus breaking $\text{SO}(5)$ R-symmetry down to $\text{SO}(4) = \text{SU}(2)_R \times \text{SU}(2)_F$, where the first factor $\text{SU}(2)_R$ is the resulting R symmetry and the second factor $\text{SU}(2)_F$ is a flavor symmetry. We then use the index notation $A = (i, 5)$ where $i = 1, 2, 3, 4$. This will reduce the amount of supersymmetry by half, to (1, 0), and it splits the (2, 0) tensor multiplet fermion into a (1, 0) tensor multiplet fermion χ and a hypermultiplet fermion ζ subject to

$$\begin{aligned}\tau^5 \chi &= -\chi \\ \tau^5 \zeta &= \zeta\end{aligned}$$

The supersymmetry variations for the abelian (1, 0) tensor multiplet in the 5d language were obtained in the appendix in [15]. They are given by

$$\begin{aligned}\delta \sigma &= -i \bar{\varepsilon} \chi \\ \delta A_m &= i r \bar{\varepsilon} \gamma_m \chi \\ \delta B_{mn} &= i \bar{\varepsilon} \gamma_{mn} \chi - 2 i r \kappa_{[m} \bar{\varepsilon} \gamma_{n]} \chi \\ \delta \chi &= \frac{1}{2r} \gamma^{mn} \varepsilon \mathcal{F}_{mn} + \gamma^m \varepsilon (\partial_m \sigma - i n \kappa_m \sigma) + \frac{i n}{r} \varepsilon \zeta + 4 \eta \sigma\end{aligned}$$

Here we A_m is the B_{mt} -component of B_{MN} . For the hypermultiplet one may likewise obtain

$$\begin{aligned}\delta\sigma^i &= i\bar{\varepsilon}\tau^i\zeta \\ \delta\zeta &= -\gamma^m\tau^i\varepsilon\left(\partial_m\sigma^i - in\kappa_m\sigma^i\right) - \frac{in}{r}\tau^i\varepsilon\sigma^i - 4\tau^i\eta\sigma^i\end{aligned}$$

All our fields carry a hidden KK mode index n in terms of which we represent ∂_0 as in where n is integer. For the abelian case we do not need a separate treatment of the zero modes $n = 0$ but the above variations apply to those zero modes as well. Now since these are nothing but a rewriting of the 6d supersymmetry variations, there must be the infinite tower of KK supercharges satisfying (4.1).

6 Abelian supersymmetry

As we will introduce a more general mode expansion below, let us now write the 6d supersymmetry variations for the tensor multiplet without any mode expansion where we use the time derivative ∂_0 instead of writing it in terms of the modes. Then we have

$$\begin{aligned}\delta\sigma &= -i\bar{\varepsilon}\chi \\ \delta A_m &= ir\bar{\varepsilon}\gamma_m\chi \\ \delta B_{mn} &= i\bar{\varepsilon}\gamma_{mn}\chi - 2ir\kappa_m\bar{\varepsilon}\gamma_n\chi \\ \delta\chi &= \frac{1}{2r}\gamma^{mn}\varepsilon\mathcal{F}_{mn} + \gamma^m\varepsilon(D_m\sigma - \kappa_m\partial_0\chi) + \frac{1}{r}\varepsilon\partial_0\sigma + 4\eta\sigma\end{aligned}$$

The closure relations are

$$\begin{aligned}\delta^2\sigma &= -i\mathcal{L}_S\sigma - 4i\bar{\varepsilon}\eta\sigma \\ \delta^2 A_m &= -i\mathcal{L}_S A_m + (\partial_m\Lambda_0 - \kappa_m\partial_0\Lambda_0) - \partial_0\tilde{\Lambda}_m \\ \delta^2 B_{mn} &= -i\mathcal{L}_S B_{mn} + 2\partial_m\left(\tilde{\Lambda}_n + \kappa_n\Lambda_0\right) \\ &\quad + is^r\mathcal{E}[B]_{rmn} \\ \delta^2\chi &= -i\mathcal{L}_S\chi - 5i\bar{\varepsilon}\eta\chi + \frac{ir}{8}s^{mn,ij}w_{mn}\tau^{ij}\chi \\ &\quad + \left(-\frac{3i}{8}s + \frac{3i}{8}s_p\gamma^p + \frac{i}{64}s_{pq}^{ij}\gamma^{pq}\tau^{ij}\right)\mathcal{E}[\chi]\end{aligned}$$

We thus have closure on-shell, that is when we put

$$\begin{aligned}\mathcal{E}[B]_{rmn} &= H_{rmn} + \frac{1}{2r}\mathcal{E}_{rmn}{}^{pq}\mathcal{F}_{pq} - 3\kappa_r\mathcal{F}_{mn} \\ \mathcal{E}[\chi] &= \gamma^m(\tilde{D}_m\chi - \kappa_m\partial_0\chi) - \frac{1}{r}\partial_0\chi + \frac{1}{2r}(\partial_mr)\gamma^m\chi - \frac{r}{8}w_{mn}\gamma^{mn}\chi\end{aligned}$$

to zero. We then have closure up to a gerbe gauge transformation, with the gauge parameters

$$\begin{aligned}\Lambda_0 &= i(s^m A_m - rs\sigma) \\ \tilde{\Lambda}_m &= \frac{i}{r}(sA_m - rs_m\sigma) + is^n(\kappa_n A_m - \kappa_m A_n) - iB_{mn}s^n\end{aligned}$$

The superconformal Lagrangian is given by [1, 8, 10, 11, 13]

$$\begin{aligned}
 \mathcal{L} = & \frac{1}{4r} \mathcal{F}^{mn} \mathcal{F}_{mn} + \frac{1}{8} \mathcal{E}^{mnpqr} \mathcal{F}_{mn} \partial_0^{-1} (\partial_r - \kappa_r \partial_0) \mathcal{F}_{pq} \\
 & + \frac{1}{2r^3} (\partial_0 \sigma)^2 - \frac{1}{2r} G^{mn} (\partial_m \sigma - \kappa_m \partial_0 \sigma) (\partial_n \sigma - \kappa_n \partial_0 \sigma) - \frac{K}{2r} \sigma^2 \\
 & + \frac{i}{2r^2} \bar{\chi} \partial_0 \chi - \frac{i}{2r} \bar{\chi} \gamma^m (\tilde{\mathcal{D}}_m \chi - \kappa_m \partial_0 \chi) + \frac{i}{16r} \bar{\chi} \gamma^{mn} \chi w_{mn}
 \end{aligned} \tag{6.1}$$

where $(\tilde{R})$ denotes the Ricci scalar on the base manifold)

$$K = \frac{\tilde{R}}{5} - \frac{r^2}{20} w_{mn}^2 - \frac{3}{5r} D_m^2 r$$

Here we introduce ∂_0^{-1} that we shall define as the inverse of ∂_0 for the KK modes. For the zero modes, what we really should do, is to separate this Lagrangian into a zero mode part (5d SYM) and a KK tower. Here we instead use a condensed notation where the fields include zero mode parts. But to make sense of our Lagrangian for the zero modes, we then need to define $\partial_0^{-1} = 0$ and $\partial_0^{-1} \partial_0 = 1$ when these operators act on a zero mode. This way we extract from the above Lagrangian the usual graviphoton term

$$-\frac{1}{8} \mathcal{E}^{mnpqr} f_{mn} f_{pq} \kappa_r$$

for the zero mode part.

If we vary A_m in this Lagrangian, then we get the Maxwell equation of motion, which is a consequence of the selfduality equation. If we vary B_{mn} for the KK modes then we actually get the selfduality equation of motion itself.

It is custom to rescale the matter fields in the Lagrangian when we reduce from 6d tensor multiplet fields to 5d SYM fields. Indeed, here we have also rescaled the fields to write down the Lagrangian. The fields that appear in the Lagrangian are related to the fields in our supersymmetry variations as $\sigma_{\text{Lagrangian}} = \sigma/(2\pi r)$, $\chi_{\text{Lagrangian}} = \chi/(2\pi r)$, while the gauge fields A_m and B_{mn} have not been rescaled. The unrescaled fields may also be thought of as 6d fields, although here we rescaled both zero mode fields as well as KK mode matter fields. Also we suppressed an overall factor of $1/(4\pi^2)$ that shall multiply the whole Lagrangian. This overall factor plays no role for classical supersymmetry, but is important in the quantum theory (for instance it appears in instanton factors).

A term like $\bar{\psi} \gamma^m \psi \partial_m r = 0$ is identically zero. So such a term could never produce the second term $-\frac{1}{2r} (\partial_m r) \gamma^m \psi$ in the fermionic equation of motion. This shows that the prefactor $1/r$ in front of the Lagrangian is necessary for supersymmetry alone, once r has been made space-dependent. This in turn shows that the field rescaling of the matter fields is actually necessary, and not just for convenience. It is necessary to rescale the matter fields in order to write down a supersymmetric Lagrangian. However, we can do without such a rescaling of the fields as long as we are just closing the supersymmetry variations. We have chosen not to rescale our fields in the supersymmetry variations because it makes it slightly easier to check closure of the supersymmetry variations.

7 Mode expansion

We introduce a discrete basis of real-valued 2π periodic functions $\varphi^a(t)$ on the circle fiber, where $a \in \mathbb{Z}$. Since it is a basis, we can expand the derivative in the basis functions as

$$\partial_0 \varphi^a(t) = \varphi^b(t) T_b^a$$

for some matrix T_b^a . If we define the metric

$$h^{ab} = \int dt \varphi^a(t) \varphi^b(t)$$

then we find the constraints

$$T^{ab} + T^{ba} = 0$$

where

$$T^{ab} = T^a_c h^{cb}$$

We have the completeness relation

$$h_{ab} \varphi^a(t') \varphi^b(t) = \delta(t - t')$$

where h_{ab} denotes the inverse metric. Any function on the circle can be mode expanded as

$$f(t) = f_a \varphi^a(t)$$

where the coefficients are

$$f_a = \int dt f(t) \varphi_a(t)$$

If we have a function on the form

$$v(t) = v_{ab\dots} \varphi^a(t) \varphi^b(t) \dots$$

then we get

$$\partial_0 v(t) = T[v_{ab\dots}] \varphi^a(t) \varphi^b(t) \dots$$

where we define

$$T[v_{ab\dots}] = T_a^{a'} v_{a'b\dots} + T_b^{b'} v_{ab'b\dots} + \dots$$

In particular, we find that the metric is time translation invariant,

$$T[h_{ab}] = 0$$

We define

$$f_{abc\dots d} = \int dt \varphi_a(t) \varphi_b(t) \varphi_c(t) \dots \varphi_d(t)$$

8 A nonlocal time derivative operator

Let us consider a function $f(t)$ of time t . We may define the time derivative of this function as a limit

$$T[f(t)] = \lim_{\varepsilon \rightarrow 0} \frac{f(t + \varepsilon) - f(t)}{\varepsilon}$$

We will also use the conventional notation

$$T[f(t)] = \partial_t f(t)$$

The function as well as its time derivative are local, because they are defined locally at each point t . We may generalize the concept of a local function of time, to a nonlocal function of time $f(t', t)$ which depends on time through two time points t' and t that may or may not be equal. We would now like to define a time derivative of this nonlocal function. If we do not discriminate among t and t' , then the natural generalization from the local to the nonlocal time derivative, will be as the following limit

$$T[f(t', t)] = \lim_{\varepsilon \rightarrow 0} \frac{f(t + \varepsilon, t' + \varepsilon) - f(t, t')}{\varepsilon}$$

or in other words,

$$T[f(t', t)] = \partial_{t'} f(t', t) + \partial_t f(t', t)$$

We do not need to worry about reparametrization invariance of our definition of $T[f(t', t)]$, because for our circle bundle the time coordinate is fixed and can not be reparametrized without taking us outside the circle bundle form of the metric. For the special case that $f(t', t) = g(t')h(t)$ is a product of two local functions, we get

$$T[g(t')h(t)] = \partial_{t'} g(t')h(t) + g(t')\partial_t h(t)$$

and we see that the limit $t' \rightarrow t$ is smooth, and in the limit we recover the chain rule for the time derivative of a product of two functions,

$$T[g(t)h(t)] = \partial_t g(t)h(t) + g(t)\partial_t h(t)$$

Let us now consider an infinitesimal supersymmetry variation

$$\delta_{\varepsilon(t)} \Phi(t)$$

of some field $\Phi(t)$ depending locally on t . In the Lagrangian there may appear a time derivative of this field, and therefore we will need to understand what is a local variation of that time derivative as well. We define

$$\delta_{\varepsilon(t)} T[\Phi(t)] = T[\delta_{\varepsilon(t)} \Phi(t)]$$

To make this more concrete, let us assume that the variation depends linearly on $\varepsilon(t)$ as

$$\delta_{\varepsilon(t)} \Phi(t) = \varepsilon(t) \Psi(t)$$

We then have

$$\delta_{\varepsilon(t)}T[\Phi(t)] = T[\varepsilon(t)\Phi(t)] = \partial_t\varepsilon(t)\Phi(t) + \varepsilon(t)\partial_t\Phi(t) \quad (8.1)$$

In the last step we have used the chain rule for the time derivative.

We would now like to introduce a nonlocal supersymmetry variation

$$\delta_{\varepsilon(t')}\Phi(t) = \varepsilon(t')\Psi(t)$$

The Lagrangian is the same local Lagrangian, and it may contain the local time derivative $T[\Phi(t)] = \partial_t\Phi(t)$ of that field. Now we would like to ask ourselves how this will vary when we act by the nonlocal supersymmetry variation,

$$\delta_{\varepsilon(t')}T[\Phi(t)] = ?$$

If we define this as follows

$$\delta_{\varepsilon(t')}T[\Phi(t)] = \bar{\varepsilon}(t')\partial_t\Psi(t)$$

then we will run into the problem of having a discontinuity as t' approaches t . Namely, when $t' = t$, we have by the chain rule the result in (8.1), which means that our equations will be inconsistent, or we will need a different set of equations when $t' \neq t$ as compared to when $t' = t$. Therefore we will instead propose that we shall not view T that appears in the Lagrangian as the local time derivative ∂_t , but rather as a more general time derivative operator as we defined it above, and which reduces to the local time derivative when it acts on a local field. Thus our proposal is that for the nonlocal supersymmetry variation, the time derivative of a field will vary according to

$$\delta_{\varepsilon(t')}T[\Phi(t)] = T[\delta_{\varepsilon(t')}\Phi(t)] = \partial_{t'}\varepsilon(t')\Psi(t) + \bar{\varepsilon}(t')\partial_t\Psi(t)$$

This definition is now consistent in the sense that it applies to both the case that $t' \neq t$ as well as to the case when $t' = t$, where the usual chain rule of differentiation can be applied.

In terms of modes, this relation reads

$$\delta_a T[\Phi_b] = T[\delta_a \Phi_b] = T_a^c \delta_c \Phi_b + T_b^c \delta_a \Phi_c \quad (8.2)$$

where we define

$$\delta_a = \int dt' \varphi_a(t') \delta_{\varepsilon(t')}$$

9 Abelian nonlocal supersymmetry

We now expand the supersymmetry parameter as

$$\varepsilon(t) = \varepsilon_a \varphi^a(t)$$

The Killing spinor equation on the modes becomes

$$\begin{aligned}\tilde{D}_m \varepsilon_a &= \gamma_m \eta_a - \frac{r}{4} w_{mn} \gamma^n \varepsilon_a + \kappa_m T_a{}^b \varepsilon_b \\ \eta_a &= \frac{1}{r} T_a{}^b \varepsilon_b - \frac{r}{8} w_{mn} \gamma^{mn} \varepsilon_a + \frac{1}{2r} (\partial_m r) \gamma^m \varepsilon_a\end{aligned}$$

We derive the identities¹

$$\begin{aligned}\partial_m s_{ef} &= r w_{mn} s_{ef}^n + \frac{1}{r} (\partial_m r) s_{ef} \\ &\quad - \frac{1}{r} T[s_{m,ef}] + \kappa_m T[s_{ef}]\end{aligned}\tag{9.1}$$

$$\partial_m (r s_{ef}) + 4r \bar{\varepsilon}_e \gamma_m \eta_f = T[s_{m,ef}] + r \kappa_m T[s_{ef}]\tag{9.2}$$

$$\partial_{[m} s_{n],ef} + 2\bar{\varepsilon}_e \gamma_{mn} \eta_f = \frac{r}{2} s_{ef} w_{mn} + \kappa_{[m} T[s_{n],ef}]\tag{9.3}$$

Here we define

$$\begin{aligned}s_{ef} &= -\bar{\varepsilon}_{(e} \varepsilon_{f)} \\ s_{ef}^m &= \bar{\varepsilon}_{(e} \gamma^m \varepsilon_{f)}\end{aligned}$$

The nonlocal abelian supersymmetry variations read

$$\begin{aligned}\delta_a \sigma_b &= -i \bar{\varepsilon}_a \sigma_b \\ \delta_a A_{m,b} &= i r \bar{\varepsilon}_a \gamma_m \chi_b \\ \delta_a B_{mn,b} &= i \bar{\varepsilon}_a \gamma_{mn} \chi_b - 2i r \kappa_{[m} \bar{\varepsilon}_a \gamma_{n]} \chi_b \\ \delta_a \chi_b &= \frac{1}{2r} \gamma^{mn} \varepsilon_a \mathcal{F}_{mn,b} + \gamma^m \varepsilon_a (D_m \sigma_b - T_b{}^c \kappa_m \sigma_c) + \frac{1}{r} T_b{}^c \varepsilon_a \sigma_c + 4\eta_a \sigma_b\end{aligned}$$

The abelian action (6.1) is invariant under these nonlocal variations, and hence they are symmetry variations. But for this we must interpret the local time derivatives that appears in the action as time derivative operators with the property (8.2).

The closure relations are

$$\begin{aligned}\frac{1}{2} \{\delta_f, \delta_e\} \sigma_b &= -i \mathcal{L}_{S,ef} \sigma_b - 4i \bar{\varepsilon}_e \eta_f \sigma_b \\ \frac{1}{2} \{\delta_f, \delta_e\} A_{m,b} &= -i \mathcal{L}_{S,ef} A_{m,b} + (\partial_m \Lambda_{0,efb} - \kappa_m T[\Lambda_{0,efb}]) - T[\tilde{\Lambda}_{m,efb}] \\ \frac{1}{2} \{\delta_f, \delta_e\} B_{mn,b} &= -i \mathcal{L}_{S,ef} B_{mn,b} + 2\partial_m (\tilde{\Lambda}_{n,efb} + \kappa_m \Lambda_{0,efb}) \\ &\quad + i s_{ef}^r \mathcal{E}[B_b]_{rmn} \\ \frac{1}{2} \{\delta_f, \delta_e\} \chi_b &= -i \mathcal{L}_{S,ef} \chi_b - 5i \bar{\varepsilon}_e \eta_f \chi_b + \frac{i r}{8} s_{ef}^{mn,ij} w_{mn} \tau^{ij} \chi_b \\ &\quad + \left(-\frac{3i}{8} s_{ef} + \frac{3i}{8} s_{p,ef} \gamma^p + \frac{i}{64} s_{pq,ef}^{ij} \gamma^{pq} \tau^{ij} \right) \mathcal{E}[\chi_b]\end{aligned}$$

¹Round brackets means symmetrization and square brackets means antisymmetrization, all with weight one.

where

$$\begin{aligned}\Lambda_{0,efb} &= i \left(s_{ef}^m A_{m,b} - r s_{ef} \sigma_b \right) \\ \tilde{\Lambda}_{m,efb} &= \frac{i}{r} (s_{ef} A_{m,b} - r s_{m,ef} \sigma_b) + i s_{ef}^n (\kappa_n A_{m,b} - \kappa_m A_{n,b}) - i B_{mn,b} s_{ef}^n\end{aligned}$$

Here $\mathcal{L}_{S,ef} = \mathcal{L}_{S_{ef}}$ denotes the 6d Lie derivative along the 6d vector field $S_{ef}^M = \bar{\varepsilon}_e \Gamma^M \varepsilon_f$. We can extract the 6d conformal Killing vector field $S^M(t) = \bar{\varepsilon}(t) \Gamma^M \varepsilon(t)$ by contracting with $\varphi^e(t) \varphi^f(t)$. Still the resulting conformal transformation, the Lie derivative acting on the field, will be nonlocal because the field has to be in general evaluated at a different time from the time t . Thus closure results in conformal transformations of the nonlocal type $\mathcal{L}_{S(t)} \Phi(t')$. This shall be contrasted with local variations, which result in closure relations on the form

$$\delta^2 \Phi_a = f_a^{efg} \mathcal{L}_{S,ef} \Phi_g$$

By contracting this relation with $\varphi^a(t)$ and noting the identity

$$\varphi^a(t) f_a^{efg} = \varphi^e(t) \varphi^f(t) \varphi^g(t)$$

we get

$$\delta^2 \Phi(t) = \mathcal{L}_{S(t)} \Phi(t)$$

where $S(t) = \bar{\varepsilon}(t) \Gamma^M \varepsilon(t)$. This is the usual local conformal transformation.

10 Nonabelian nonlocal supersymmetry

Having the set the ground, we are now ready to present an ansatz for the nonabelian generalization of the nonlocal superconformal variations. For the zero modes we take the nonlocal variations to be

$$\begin{aligned}\delta_a \phi &= -i \bar{\varepsilon}_a \chi \\ \delta_a a_m &= i r \bar{\varepsilon}_a \gamma_m \psi \\ \delta_a \psi &= \frac{1}{2r} \gamma^{mn} \varepsilon_a f_{mn} + \gamma^m \varepsilon_a D_m \phi + 4 \eta_a \phi\end{aligned}$$

As we advertised in the introduction, we see that with a nonlocal variation we are able to express these variations in a close form such that the variation only involves the zero mode fields. For a local variation we are instead forced to pick the zero mode ε_0 (for which $\partial_0 \varepsilon_0 = 0$) as the supersymmetry parameter and thereby we reduce the symmetry to $\mathcal{C}_{5d,(1,0)}$. It is only by allowing for the variation to be nonlocal that we can allow the supersymmetry parameter to carry a nonvanishing mode number ε_a whose time derivative $T_a{}^b \varepsilon_b$ is nonvanishing and then we can realize $\mathcal{C}_{6d,(1,0)}$.

For the KK modes, we make the ansatz

$$\begin{aligned}\delta_a \sigma_b &= -i\bar{\varepsilon}_a \chi_b \\ \delta_a A_{m,b} &= ir\bar{\varepsilon}_a \gamma_m \chi_b \\ \delta_a B_{mn,b} &= i\bar{\varepsilon}_a \gamma_{mn} \chi_b - 2ir\kappa_m \bar{\varepsilon}_a \gamma_n \chi_b + C_b^c ([\phi, \bar{\varepsilon}_a \gamma_{mn} \chi_c] - [\sigma_c, \bar{\varepsilon}_a \gamma_{mn} \psi]) + \tilde{C}_b^c [A_{n,c}, \bar{\varepsilon}_a \gamma_m \psi] \\ \delta_a \chi_b &= \frac{1}{2r} \gamma^{mn} \varepsilon_a \mathcal{F}_{mn,b} + \gamma^m \varepsilon_a (D_m \sigma_b - T_b^c \kappa_m \sigma_c) + \frac{1}{r} T_b^c \varepsilon_a \sigma_c + 4\eta_a \sigma_b - ir\varepsilon_a [\phi, \sigma_b]\end{aligned}$$

Let here illustrate how we determine the coefficients C_b^c and $\tilde{C}_b^c$. We then begin by considering

$$\delta_f \delta_e B_{mn,b} = -2ir\kappa_m \bar{\varepsilon}_e \gamma_n \delta_f \chi_b + C_b^c [\phi, \bar{\varepsilon}_e \gamma_{mn} \delta_f \chi_c] + \dots$$

where we insert the variation

$$\delta_f \chi_b = \gamma^m \varepsilon_f T_b^c \kappa_m \sigma_c - ir\varepsilon_f [\phi, \sigma_b] + \dots$$

We then get

$$\delta_f \delta_e B_{mn,b} = -2r^2 \kappa_m \bar{\varepsilon}_e \gamma_n \varepsilon_f [\phi, \sigma_b] - 2C_b^c T_c^d \kappa_m \bar{\varepsilon}_e \gamma_n \varepsilon_f [\phi, \sigma_d] + \dots$$

For these terms to cancel, we shall take

$$C_a^b = -r^2 (T^{-1})_a^b$$

Let us now look at a term

$$\delta_f \delta_e B_{mn,b} = -i\bar{\varepsilon}_e \gamma^r \varepsilon_f H_{rmn,b} + \dots$$

that arises when we use a selfduality equation of motion. Here we define

$$H_{rmn,b} = 3D_r B_{mn} - i(T^{-1})_b^c [A_{r,c}, f_{mn}] - 2i(T^{-1})_b^c [A_{m,c}, f_{nr}]$$

The last term gives rise to a term

$$\delta_f \delta_e B_{mn,b} = -2\bar{\varepsilon}_e \gamma^r \varepsilon_f (T^{-1})_b^c [A_{m,c}, f_{nr}] + \dots$$

that we need to cancel. We cancel it by varying the term in $\delta_e B_{mn,b}$ that is proportional to $\tilde{C}_b^c$, which will produce a term of the form

$$\delta_f \delta_e B_{mn,b} = -\frac{1}{r} \bar{\varepsilon}_e \gamma^r \varepsilon_f \tilde{C}_b^c [A_{m,c}, f_{nr}] + \dots$$

We see that for the two terms to cancel, we shall take

$$\tilde{C}_a^b = -2r(T^{-1})_a^b$$

We have now found that we should have

$$\begin{aligned}\delta_a B_{mn,b} &= i\bar{\varepsilon}_a \gamma_{mn} \chi_b - 2ir\kappa_m \bar{\varepsilon}_a \gamma_n \chi_b \\ &\quad + r^2 (T^{-1})_b^c ([\sigma_c, \bar{\varepsilon}_a \gamma_{mn} \psi] - [\phi, \bar{\varepsilon}_a \gamma_{mn} \chi_c]) - 2r(T^{-1})_b^c [A_{n,c}, \bar{\varepsilon}_a \gamma_m \psi]\end{aligned}$$

and then the closure relations become

$$\begin{aligned}
 \frac{1}{2}\{\delta_f, \delta_e\}\sigma_b &= -i\mathcal{L}_{S,ef}\sigma_b - 4i\tilde{\varepsilon}_e\eta_f\sigma_b - i[\sigma_b, \lambda_{ef}] \\
 \frac{1}{2}\{\delta_f, \delta_e\}A_{m,b} &= -i\mathcal{L}_{S,ef}A_{m,b} + (D_m\Lambda_{0,efb} - \kappa_m T[\Lambda_{0,efb}]) - T[\tilde{\Lambda}_{m,efb}] - i[A_{m,b}, \lambda_{ef}] \\
 \frac{1}{2}\{\delta_f, \delta_e\}B_{mn,b} &= -i\mathcal{L}_{S,ef}B_{mn,b} \\
 &\quad + 2D_{[m}(\tilde{\Lambda}_{n],efb} + \kappa_n\Lambda_{0,efb}) - i[B_{mn,b}, \lambda_{ef}] + i[f_{mn}, T^{-1}[\Lambda_{0,efb}]] \\
 &\quad + i s^r \mathcal{E}[B]_{rmn,b} \\
 \frac{1}{2}\{\delta_f, \delta_e\}\chi_b &= -i\mathcal{L}_{S,ef}\chi_b - 5i\tilde{\varepsilon}_e\eta_f\chi_b + \frac{ir}{8}s_{ef}^{mn,ij}w_{mn}\tau^{ij}\chi_b \\
 &\quad - i[\chi_b, \lambda_{ef}] \\
 &\quad + \left(-\frac{3i}{8}s_{ef} + \frac{3i}{8}s_{p,ef}\gamma^p + \frac{i}{64}s_{pq,ef}^{ij}\gamma^{pq}\tau^{ij}\right)\mathcal{E}[\chi_b]
 \end{aligned}$$

where

$$\begin{aligned}
 \mathcal{E}[B]_{rmn,b} &= H_{rmn,b} \\
 &\quad + \frac{1}{2r}\mathcal{E}_{rmn}{}^{pq}\left(\mathcal{F}_{pq} - ir^2(T^{-1})_b{}^c([\sigma_c, f_{pq}] - [\phi, \mathcal{F}_{pq,c}]) + \frac{r^3}{2}(T^{-1})_b{}^c\{\bar{\psi}\gamma_{pq}\chi_c\}\right) \\
 &\quad - 3\kappa_r\mathcal{F}_{mn,b} \\
 \mathcal{E}[\chi_b] &= \gamma^m(\tilde{D}_m\chi_b - \kappa_m T_b{}^c\chi_c) - \frac{1}{r}T_b{}^c\chi_c + \frac{1}{2r}(\partial_m r)\gamma^m\chi_b - \frac{r}{8}w_{mn}\gamma^{mn}\chi_b \\
 &\quad - ir[\chi_b, \phi] + 2ir[\psi, \sigma_b]
 \end{aligned}$$

and

$$\begin{aligned}
 \lambda_{ef} &= i\left(s_{ef}^q a_q - r s_{ef}\phi\right) \\
 \Lambda_{0,efb} &= i\left(s_{ef}^q A_{q,b} - r s_{ef}\sigma_b\right) \\
 \tilde{\Lambda}_{m,efb} &= -iB_{mn,b}s_{ef}^n - i\left(s_{m,ef}\sigma_b - \frac{s_{ef}}{r}A_{m,b}\right) \\
 &\quad + i s_{ef}^n(\kappa_n A_{m,b} - \kappa_m A_{n,b}) - rT^{-1}[\phi, r s_{m,ef}\sigma_b - s_{ef}A_{m,b}]
 \end{aligned}$$

11 Discussion

One application that our result might have is to resolve the conflict between the two proposals [4, 5] and [6, 11, 15]. The conflict may get resolved by utilizing the nonlocal superconformal transformation. We have seen that it seems impossible to realize the $\mathcal{C}_{6d,(1,0)}$ symmetry locally in a classical field theory. There may be other ways this symmetry can get realized. One possibility is that $\mathcal{C}_{6d,(1,0)}$ emerges at quantum level by means of instanton particles and enhance the classical symmetry $\mathcal{C}_{5d,(1,0)}$ that was shown to be present at the classical level in 5d SYM in [16]. This would be similar to how $\mathcal{N} = 8$ emerges at quantum level by means of monopole operators from $\mathcal{N} = 6$ ABJM theory. What plays the role

of ABJM for M2 would then be 5d SYM for M5. Another possibility, which need not be in conflict with this instanton-particle proposal, is if there is a classical field theory in which $\mathcal{C}_{6d,(1,0)}$ is realized in a nonlocal way. If we demand an ordinary local field theory description of the 6d (1,0) tensor multiplet, then the only possibility seems to be a 5d SYM, or something similar (for example lightcone reduction of 6d theory), that has a subgroup of $\mathcal{C}_{6d,(1,0)}$ realized at a classical level. This does not rule out the possibility that there may exist ways to realize $\mathcal{C}_{6d,(1,0)}$ classically, only that we may not be able to realize it as a local symmetry in a local field theory.

Regarding the proposal in [6, 11, 15], we would now like to argue that this proposal corresponds to a nonlocal field theory. This can be seen from three places. Gauge symmetry is nonlocal, the selfduality equation of motion is nonlocal. But even if that is not convincing enough (one may perhaps object by saying that neither H_{rmn} nor the gauge symmetry are really needed in the description of the theory?), then we have now provided a third indication that the theory is nonlocal. Namely even if the theory is an ordinary local field theory on S^6 , it would necessarily become nonlocal if we apply a nonlocally realized conformal transformation to map this theory to $\mathbb{R}^6$. Then we will get a nonlocal theory on $\mathbb{R}^6$ because the transformation is nonlocal, and in particular that means that 6d Lorentz symmetry will be realized in a nonlocal way on $\mathbb{R}^6$.

Let us discuss the question of how to obtain (2, 0) supersymmetry. We have failed with a manifestly $SO(5)$ covariant ansatz for the supersymmetry with a KK tower (unless of course the gauge group is abelian). We then need a unit vector v^A that selects a direction on S^4 and breaks (2, 0) supersymmetry down to (1, 0). We may hopefully be able to extract a nonabelian action $S[v^A, \phi^A]$ from our results (this would be rather straightforward) from which we may get an $SO(5)$ invariant action by integrating over $v^A \in S^4$,

$$\int_{S^4} dv e^{-S[v, \phi]} = e^{-S_{\text{eff}}[\phi]}$$

Then the question is if S_{eff} will also be (2, 0) supersymmetric. This seems to be a difficult question.

Another interesting application of the circle-bundle formulation of the 6d tensor multiplet is to singular fibrations [3, 9, 17]. The circle fiber may shrink to zero size at a submanifold of dimensions 0, 2 and 4 respectively [3]. We can see those dimensions appear from the following sequence of conformally flat six-manifolds, S^6 , $H^2 \times S^4$ and $H^4 \times S^2$. Odd dimensional spheres have the Hopf fibration and are regular fibrations. But even dimensional spheres necessarily have singular fibrations. It will be interesting to derive the corresponding theories that we may need to supplement to 5d SYM and that live at those singular points or submanifolds. Perhaps one may use results from [2] that provides a machinery to derive supersymmetric boundary theories. Here we have singular loci, and they are not quite boundaries though.

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A Closure on σ_b

We get

$$\begin{aligned} \frac{1}{2}\{\delta_f, \delta_e\}\sigma_b &= -i\bar{\varepsilon}_e\gamma^m\varepsilon_f(D_m\sigma_b - T_b^c\kappa_m\sigma_c) - \frac{i}{r}\bar{\varepsilon}_e\varepsilon_f T_b^c\sigma_c \\ &\quad - 4i\bar{\varepsilon}_e\eta_f\sigma_b \\ &\quad - r\bar{\varepsilon}_e\varepsilon_f[\phi, \sigma_b] \end{aligned}$$

We will write this as

$$\frac{1}{2}\{\delta_f, \delta_e\}\sigma_b = -i\mathcal{L}_{S,ef}\sigma_b - 4i\bar{\varepsilon}_e\eta_f\sigma_b - i[\sigma_b, \lambda_{ef}]$$

where

$$\lambda_{ef} = i(s_{ef}^m a_m - r s_{ef}\phi)$$

B Closure on $A_{m,b}$

We divide the computation into three pieces,

$$\delta_f\delta_e A_{m,b} = ir\bar{\varepsilon}_e\gamma_m(\delta_f\chi_b + \delta'_f\chi_b + \delta''_f\chi_b)$$

where

$$\begin{aligned} \delta_a\chi_b &= \frac{1}{2r}\gamma^{mn}\varepsilon_a\mathcal{F}_{mn,b} \\ \delta'_a\chi_b &= \gamma^m\varepsilon_a(D_m\sigma_b - T_b^c\kappa_m\sigma_c) + \frac{1}{r}T_b^c\varepsilon_a\sigma_c + 4\eta_a\sigma_b \\ \delta''_a\chi_b &= -ir\varepsilon_a[\phi, \sigma_b] \end{aligned}$$

We get

$$\delta_f\delta_e A_{m,b} = i s_{ef}^q \mathcal{F}_{mq,b}$$

where we define

$$\mathcal{F}_{mn,a} = 2D_m A_{n,a} + T_a^b B_{mn,b}$$

Then

$$\begin{aligned} \delta_f\delta_e A_{m,b} &= -i\mathcal{L}_{S,ef}A_{m,b} + D_m\left(is_{ef}^q A_{q,b}\right) \\ &\quad - T\left[\frac{i}{r}s_{ef}A_{m,b} + is_{ef}^q\kappa_q A_{m,b} + is_{ef}^q B_{qm}\right] \\ &\quad + [A_{m,b}, s_{ef}^q] \end{aligned}$$

Next

$$\delta'_f\delta_e A_{m,b} = -ir s_{ef}(D_m\sigma_b - T_b^c\kappa_m\sigma_c) + i s_{m,ef}T_b^c\sigma_c + 4ir\bar{\varepsilon}_e\gamma_m\eta_f\sigma_b$$

We extract a total derivative,

$$\begin{aligned}\delta'_f \delta_e A_{m,b} &= D_m (-rs_{ef}\sigma_b) + irs_{ef}T_b^c \sigma_c \kappa_m + is_{m,ef}T_b^c \sigma_c \\ &\quad + i\partial_m(rs_{ef}) + 4ir\bar{\varepsilon}_e \gamma_m \eta_f \sigma_b\end{aligned}$$

Now we apply (9.2) on the second line and get

$$\delta'_f \delta_e A_{m,b} = D_m (-rs_{ef}\sigma_b) - \kappa_m (-irT[s_{ef}\sigma_b]) - T[-is_{m,ef}\sigma_b]$$

Let us finally extract all commutators. One is from $\delta_f \chi_b$ and another is from $\delta''_f \chi_b$,

$$\begin{aligned}(\delta_f \delta_e A_{m,b})_{comm} &= [A_{m,b}, s_{ef}^n a_n] + r^2 s_{m,ef} [\phi, \sigma_b] \\ &= [A_{m,b}, s_{ef}^n a_n - rs_{ef}\phi] - r[\phi, s_{ef}A_{m,b} - rs_{m,ef}\sigma_b] \\ &= -i[A_{m,b}, \lambda_{ef}] - T[\tilde{\Lambda}_{m,efb}]\end{aligned}$$

where

$$\tilde{\Lambda}_{m,efb} = rT^{-1}[\phi, s_{ef}A_{m,b} - rs_{m,ef}\sigma_b]$$

C Closure on A_m , local computation

We here present part of the closure computation for local $\mathcal{C}_{5d,(1,0)}$ superconformal symmetry variations where $\partial_0 \varepsilon = 0$ and the variations are

$$\begin{aligned}\delta\sigma &= -i\bar{\varepsilon}\chi \\ \delta A_m &= ir\bar{\varepsilon}\gamma_m\chi \\ \delta B_{mn} &= i\bar{\varepsilon}\gamma_{mn}\chi - 2ir\kappa_{[m}\bar{\varepsilon}\gamma_{n]}\chi + \frac{ir^2}{n}([\sigma, \bar{\varepsilon}\gamma_{mn}\psi] - [\phi, \bar{\varepsilon}\gamma_{mn}\chi]) - \frac{2ir}{n}[A_n, \bar{\varepsilon}\gamma_m\psi] \\ \delta\chi &= \frac{1}{2r}\gamma^{mn}\varepsilon\mathcal{F}_{mn} + \gamma^m\varepsilon(D_m\sigma - in\kappa_m\sigma) + \frac{in}{r}\varepsilon\sigma + 4\eta\sigma - ir\varepsilon[\phi, \sigma]\end{aligned}$$

This way we can see that the corresponding nonlocal computation is almost completely analogous, and that in gets replaced by T . We get

$$\delta^2 A_m = is^n \mathcal{F}_{mn} - irs(D_m\sigma - in\kappa_m\sigma) - ns_m\sigma + 4ir\bar{\varepsilon}\gamma_m\eta\sigma + r^2\bar{\varepsilon}\gamma_m\varepsilon s_m[\phi, \sigma]$$

We rewrite this as

$$\begin{aligned}\delta^2 A_m &= -i\mathcal{L}_S A_m + D_m(i(s^q A_q - rs\sigma)) \\ &\quad + ns^n \kappa_n A_m - n\kappa_m rs\sigma \\ &\quad - n\left(B_{mn}s^n + s_m\sigma - \frac{s}{r}A_m - \frac{1}{n}r^2 s_m[\phi, \sigma]\right) + [A_m, s^q a_q] \\ &\quad + i\partial_m(rs)\sigma + 4ir\bar{\varepsilon}\gamma_m\eta\sigma\end{aligned}$$

where we have extracted the 6d Lie derivative. We note that the last line is zero as a consequence of the 5d Killing spinor equation. We add and subtract $[A_m, rs\phi]$ and get

$$\begin{aligned}\delta^2 A_m &= -i\mathcal{L}_S A_m + D_m (i(s^q A_q - rs\sigma)) \\ &\quad + ns^n \kappa_n A_m - nr s \kappa_m \sigma \\ &\quad - n \left(B_{mn} s^n + s_m \sigma - \frac{s}{r} A_m + \frac{1}{n} r [\phi, s A_m - r s_m \sigma] \right) + [A_m, s^q a_q - rs\phi]\end{aligned}$$

We can write this in the form

$$\delta^2 A_m = -i\mathcal{L}_S A_m + D_m \Lambda_0 - in \kappa_m \Lambda_0 - in \tilde{\Lambda}_m - i[A_m, \lambda]$$

where

$$\begin{aligned}\lambda &= i(s^q a_q - rs\phi) \\ \Lambda_0 &= i(s^q A_q - rs\sigma) \\ \tilde{\Lambda}_m &= -is^n (\kappa_m A_n - \kappa_n A_m) - iB_{mn} s^n + \frac{i}{r} (s A_m - r s_m \sigma) \\ &\quad - \frac{ir}{n} [\phi, s A_m - r s_m \sigma]\end{aligned}\tag{C.1}$$

D Closure on B_{mn} , local computation

We get

$$\begin{aligned}\delta^2 B_{mn} &= \frac{is}{r} \mathcal{F}_{mn} + \frac{i}{2r} s^r \mathcal{E}_{mnr}{}^{pq} \mathcal{F}_{pq} - 2i\kappa_m s^q \mathcal{F}_{nq} \\ &\quad + 2D_m (-i(s_n + r s \kappa_n) \sigma) \\ &\quad + \underbrace{-2r^2 \kappa_m s_n \varepsilon[\phi, \sigma]}_W + \underbrace{\frac{ir^2}{n} [\sigma, \bar{\varepsilon} \gamma_{mn} \delta\psi]}_Y - \underbrace{\frac{ir^2}{n} [\phi, \bar{\varepsilon} \gamma_{mn} \delta\chi]}_Z - \underbrace{\frac{2ir}{n} [A_n, \bar{\varepsilon} \gamma_m \delta\psi]}_X\end{aligned}$$

The contributions that appear in the first and the third lines are obvious, but to understand the contribution on the second line require some computation. We get this result from

$$(\delta^2 B_{mn})_{2\text{nd}} = i\bar{\varepsilon} \gamma_{mn} \delta\chi - 2ir\kappa_m \bar{\varepsilon} \gamma_n \delta\chi$$

where we plug in $\delta\chi = \gamma^p \varepsilon (D_p \sigma - \kappa_p \partial_0 \sigma) + 4\eta\sigma + \dots$ and omit the dots. Then we get

$$\begin{aligned}(\delta^2 B_{mn})_{2\text{nd}} &= 2D_m (-is\sigma) + 2i(\partial_m s_n) \sigma + 4i\bar{\varepsilon} \gamma_{mn} \eta \sigma \\ &\quad + 2ir\kappa_m (s D_n \sigma - 4\bar{\varepsilon} \gamma_n \eta \sigma)\end{aligned}$$

Now we use (9.2) and (9.3) to get after some computation the result

$$(\delta^2 B_{mn})_{2\text{nd}} = 2D_m (-is_n \sigma - ir\kappa_n s \sigma)$$

which is the second line.

Let us now consider the first line,

$$\delta^2 B_{mn} = \frac{i}{r} s \mathcal{F}_{mn} + \frac{i}{2r} s^r \mathcal{E}_{rmn}{}^{pq} \mathcal{F}_{pq} - 2i\kappa_m s^r \mathcal{F}_{nr}$$

We assume some selfduality relation

$$\frac{1}{2r} \mathcal{E}_{rmn}{}^{pq} \mathcal{F}_{pq} = -H_{rmn} + 3\kappa_r \mathcal{F}_{mn} + X_{rmn}$$

where X_{rmn} will be determined from requiring on-shell closure. We then get

$$\delta^2 B_{mn} = \frac{is}{r} \mathcal{F}_{mn} - is^r H_{rmn} + is^r \kappa_r \mathcal{F}_{mn} + is^r X_{rmn}$$

Then we use the definition

$$H_{rmn} = 3D_r B_{mn} - \frac{3}{n} [f_{mn}, A_r]$$

and we get

$$\delta^2 B_{mn} = -i\mathcal{L}_S B_{mn} + 2D_m \Lambda_n + [s^r a_r, B_{mn}] + \frac{is^r}{n} [f_{mn}, A_r] + \frac{2is^r}{n} [f_{rm}, A_n] + is^r X_{rmn}$$

We can write this in the form

$$\begin{aligned} \delta^2 B_{mn} = & -i\mathcal{L}_S B_{mn} + 2D_m \Lambda_n - i[B_{mn}, \lambda] + \frac{1}{n} [f_{mn}, \Lambda_0] \\ & + \frac{2is^r}{n} [f_{rm}, A_n] + is^r X_{rmn} \end{aligned}$$

where

$$\begin{aligned} \Lambda'_m &= \frac{i}{r} (sA_m - r s_m \sigma) - is^r B_{mr} + is^r \kappa_r A_m - ir \kappa_m s \sigma \\ \Lambda'_0 &= is^r A_r \\ \lambda' &= is^r a_r \end{aligned}$$

We now notice that

$$\Lambda'_m = \tilde{\Lambda}_m + \kappa_m \Lambda_0$$

where $\tilde{\Lambda}_m$ and Λ_0 are given in (C.1).

Let us now study the term

$$X := -\frac{2ir}{n} [A_n, \bar{\varepsilon} \gamma_m \delta \psi]$$

where

$$\delta \psi = \frac{1}{2r} \gamma^{pq} \varepsilon f_{pq} + \gamma^p \varepsilon D_p \phi + 4\eta \phi$$

We get

$$X = -\frac{2i}{n} [A_n, s^q f_{mq}] - \frac{2i}{n} [A_n, r \bar{\varepsilon} \varepsilon D_m \phi] - \frac{8ir}{n} [A_n, \bar{\varepsilon} \gamma_m \eta \phi]$$

By using the 5d Killing spinor equation this can be recast in the form

$$X = -\frac{2i}{n}[A_n, s^q f_{mq}] - \frac{2i}{n}[A_n, D_m(r\bar{\varepsilon}\varepsilon\phi)]$$

We have two more terms

$$\begin{aligned} Y = \frac{ir^2}{n}[\sigma, \bar{\varepsilon}\gamma_{mn}\delta\psi] &= \frac{irs}{n}[\sigma, f_{mn}] + \frac{irs^r}{2n}\mathcal{E}_{rmn}{}^{pq}[\sigma, f_{pq}] \\ &+ \frac{2ir^2}{n}s_m[\sigma, D_n\phi] + \frac{4ir^2}{n}\bar{\varepsilon}\gamma_{mn}\eta[\sigma, \phi] \end{aligned}$$

and

$$\begin{aligned} Z = -\frac{ir^2}{n}[\phi, \bar{\varepsilon}\gamma_{mn}\delta\chi] &= -\frac{irs}{n}[\phi, \mathcal{F}_{mn}] - \frac{irs^r}{2n}\mathcal{E}_{rmn}{}^{pq}[\phi, \mathcal{F}_{pq}] \\ &- \frac{2ir^2}{n}s_m[\phi, D_n\sigma] - 2r^2s_m\kappa_n[\phi, \sigma] - \frac{4ir^2}{n}\bar{\varepsilon}\gamma_{mn}\eta[\phi, \sigma] \end{aligned}$$

Let us first notice that the fourth term in Z beautifully cancels against the term we called W above. Next we collect the two terms

$$\frac{2ir^2}{n}s_m([\sigma, D_n\phi] - [\phi, D_n\sigma]) = \frac{2ir^2}{n}s_m D_n([\sigma, \phi])$$

We also note that two terms add up to give

$$\frac{8ir^2}{n}\bar{\varepsilon}\gamma_{mn}\eta[\sigma, \phi]$$

and finally by expanding out the first term in Z and adding to it the last term in X , we get

$$D_m\left(-\frac{2irs}{n}[\phi, A_n]\right) + [rs\phi, B_{mn}]$$

We can now summarize what we have got so far as

$$\begin{aligned} X + Y + Z + W &= \frac{irs}{n}[\sigma, f_{mn}] + \frac{irs^r}{2n}\mathcal{E}_{rmn}{}^{pq}[\sigma, f_{pq}] \\ &+ D_m\left(-\frac{2irs}{n}[\phi, A_n]\right) + [rs\phi, B_{mn}] - \frac{irs^r}{2n}\mathcal{E}_{rmn}{}^{pq}[\phi, \mathcal{F}_{pq}] \\ &+ \frac{2ir^2}{n}s_m D_n([\sigma, \phi]) + \frac{8ir^2}{n}\bar{\varepsilon}\gamma_{mn}\eta[\sigma, \phi] \\ &- \frac{2i}{n}[A_n, s^q f_{mq}] \end{aligned}$$

By using the 5d Killing spinor equation, we can write the third line above as

$$\frac{2ir^2}{n}s_m D_n([\sigma, \phi]) + \frac{8ir^2}{n}\bar{\varepsilon}\gamma_{mn}\eta[\sigma, \phi] = D_n\left(\frac{2ir^2}{n}s_m[\sigma, \phi]\right)$$

so that we get

$$\begin{aligned}
X + Y + X + W &= \frac{irs}{n}[\sigma, f_{mn}] + [rs\phi, B_{mn}] \\
&+ \frac{irs^r}{2n}\mathcal{E}_{rmn}{}^{pq}([\sigma, f_{pq}] - [\phi, \mathcal{F}_{pq}]) \\
&+ D_m \left(-\frac{2irs}{n}[\phi, A_n] - \frac{2ir^2}{n}s_n[\sigma, \phi] \right) \\
&- \frac{2i}{n}[A_n, s^q f_{mq}]
\end{aligned}$$

We shall choose

$$X_{rmn} = \frac{r}{2n}\mathcal{E}_{rmn}{}^{pq}([\sigma, f_{pq}] - [\phi, \mathcal{F}_{pq}])$$

and then the selfduality constraint reads

$$H_{rmn} = -\frac{1}{2r}\mathcal{E}_{rmn}{}^{pq} \left(\mathcal{F}_{pq} - \frac{r^2}{n}([\sigma, f_{pq}] - [\phi, \mathcal{F}_{pq}]) \right) + 3\kappa_r \mathcal{F}_{mn}$$

We now should also obtain the fermionic contribution to this selfdual equation. But that will be the same as in flat space since the variation of those fields is the same as for flat space case. That is so for δA_m whose supersymmetry variation does not contain any curvature correction proportional to say κ_m . We can therefore borrow the result for the fermionic contribution directly from [15] that was obtained in flat space.

By adding the various contributions, we obtain the result presented in the main text.

E Relations between 6d and 5d quantities

We have the following 6d gamma matrices as expressed in terms of 5d gamma matrices (decorated with tilde)

$$\begin{aligned}
\Gamma_m &= \tilde{\Gamma}_m - r\kappa_m \Gamma^{\hat{0}} \\
\Gamma_0 &= -r\Gamma^{\hat{0}} \\
\Gamma^m &= \tilde{\Gamma}^m \\
\Gamma^0 &= \frac{1}{r}\Gamma^{\hat{0}} - \kappa_m \tilde{\Gamma}^m \\
\Gamma^{m0} &= \frac{1}{r}\tilde{\Gamma}^m \Gamma^{\hat{0}} - \kappa_n \tilde{\Gamma}^{mn} \\
\Gamma_{m0} &= -r\tilde{\Gamma}_m \Gamma^{\hat{0}}
\end{aligned}$$

We represent the 11d gammas as

$$\begin{aligned}
\Gamma^{\hat{0}} &= i\sigma^2 \otimes 1 \otimes 1 \\
\Gamma^{\hat{m}} &= \sigma^1 \otimes \gamma^m \otimes 1 \\
\Gamma^A &= \sigma^3 \otimes 1 \otimes \tau^A
\end{aligned}$$

and the 6d chirality matrix as

$$\Gamma = \sigma^3 \otimes 1 \otimes 1$$

The 6d covariant derivatives are related to 5d covariant derivatives as follows,

$$\begin{aligned} D_0 \chi &= in\chi - \frac{r^2}{8} w_{mn} \gamma^{mn} \chi - \frac{1}{2} (\partial_m r) \gamma^m \chi \\ D_m \chi &= \tilde{D}_m \chi - in\kappa_m \chi + \kappa_m D_0 \chi - \frac{r}{4} w_{mn} \gamma^n \chi \end{aligned}$$

For the supercharges, of opposite 6d chirality, there are some sign changes,

$$\begin{aligned} D_0 \varepsilon &= in\varepsilon - \frac{r^2}{8} w_{mn} \gamma^{mn} \varepsilon + \frac{1}{2} (\partial_m r) \gamma^m \varepsilon \\ D_m \varepsilon &= \tilde{D}_m \varepsilon - in\kappa_m \varepsilon + \kappa_m D_0 \varepsilon + \frac{r}{4} w_{mn} \gamma^n \varepsilon \end{aligned}$$

In 5d we shall have

$$(\gamma^m)^T = s C \gamma^m C^{-1}$$

with $s = +1$. This can be understood as follows. For either sign $s = \pm 1$, we get

$$\begin{aligned} (\gamma^5)^T &= (\gamma^{1234})^T \\ &= C \gamma^{1234} C^{-1} \\ &= C \gamma^5 C^{-1} \end{aligned}$$

and so by $\text{SO}(5)$ covariance, this should be true for all γ^m ($m = 1, 2, 3, 4, 5$).

For 11d gamma matrices we have on the other hand

$$(\Gamma^M)^T = -C_{11d} \Gamma^M C_{11d}^{-1}$$

and this is not in conflict with the fact that

$$\Gamma^{10} = \Gamma^{0123456789}$$

since here we find 5 exchanges and $(-1)^5 = -1$, so any sign s is fine by that argument. An explicit realization is

$$C_{11d} = \epsilon \otimes C \times C'$$

where we break $\text{SO}(1, 10) \rightarrow \text{SO}(1, 5) \times \text{SO}(5)$.

An explicit realization of the $\text{SO}(5)_R$ gammas is

$$\begin{aligned} \tau^I &= \sigma^1 \otimes \sigma^I \\ \tau^4 &= \sigma^2 \otimes 1 \\ \tau^5 &= -\sigma^3 \otimes 1 \end{aligned}$$

such that we have $\gamma^5 = \gamma^{1234}$. Then we have

$$C' = \sigma^3 \otimes \epsilon$$

and we realize

$$(\gamma^A)^T = C' \gamma^A C'^{-1}$$

It is important to note that when we use 5d language, we completely dispose of the chirality matrix, and we use 5d spinors which are not Weyl. Then we define in 5d language

$$\bar{\epsilon} = \epsilon^T C \otimes C'$$

with ϵ being the 5d Dirac spinor, whereas in 6d we use the Weyl spinor and define

$$\bar{\epsilon} = \epsilon^T c \otimes C \otimes C'$$

The difference is crucial, since $C_{11d} = c \otimes C \otimes C'$ is antisymmetric, while $C_{5d} = C \times C'$ is symmetric. So for example, we use this fact to show that

$$\bar{\epsilon} \tau^{AB} \epsilon = (\epsilon^T C_{5d} \tau^{AB} \epsilon)^T = \epsilon^T (-C \tau^{AB} C^{-1}) C_{5d}^T \epsilon = -\bar{\epsilon} \tau^{AB} \epsilon$$

is vanishing.

F Fierz identities

Using 11d gamma matrices, we have for commuting ϵ subject to the 6d Weyl projection $\Gamma \epsilon = -\epsilon$

$$\epsilon \bar{\epsilon} = \frac{1}{16} \left(\bar{\epsilon} \Gamma_M \epsilon \Gamma^M - \bar{\epsilon} \Gamma_{MA} \epsilon \Gamma^{MA} + \frac{1}{12} \Gamma_{MNPAB} \epsilon \Gamma^{MNPAB} \right) P_+$$

where $P_+ = (1 + \Gamma)/2$. Using 6d and 5d gamma matrices that we relate to the 11d gamma matrices through

$$\begin{aligned} \Gamma^M &= \Gamma^M \otimes 1 \\ \Gamma^A &= \Gamma \otimes \tau^A \end{aligned}$$

where we recycle the same letter Γ^M to refer to the 6d gamma matrices, we get

$$\epsilon \bar{\epsilon} = \frac{1}{16} \left(\bar{\epsilon} \Gamma_M \epsilon \Gamma^M + \bar{\epsilon} \Gamma_M \tau^A \epsilon \Gamma^M \tau^A + \frac{1}{12} \bar{\epsilon} \Gamma_{MNP} \tau^{AB} \epsilon \Gamma^{MNP} \tau^{AB} \right) P_+$$

We also have

$$\begin{aligned} \epsilon \bar{\eta} &= \frac{1}{16} \left(\bar{\eta} \epsilon + \bar{\eta} \tau^A \epsilon \tau^A - \frac{1}{2} \bar{\eta} \tau^{AB} \epsilon \tau^{AB} \right) P_- \\ &+ \frac{1}{16} \left(-\frac{1}{2} \bar{\eta} \Gamma^{MN} \epsilon \Gamma_{MN} - \frac{1}{2} \bar{\eta} \Gamma^{MN} \tau^A \epsilon \Gamma_{MN} \tau^A + \frac{1}{4} \bar{\eta} \Gamma^{MN} \tau^{AB} \Gamma_{MN} \tau^{AB} \right) P_- \end{aligned}$$

and

$$\begin{aligned}\eta\bar{\varepsilon} = & -\frac{1}{16} \left(\bar{\eta}\varepsilon + \bar{\eta}\tau^A\varepsilon\tau^A + \frac{1}{2}\bar{\eta}\tau^{AB}\varepsilon\tau^{AB} \right) P_+ \\ & -\frac{1}{16} \left(\frac{1}{2}\bar{\eta}\Gamma^{MN}\varepsilon\Gamma_{MN} + \frac{1}{2}\bar{\eta}\Gamma^{MN}\tau^A\varepsilon\Gamma_{MN}\tau^A + \frac{1}{4}\bar{\eta}\Gamma^{MN}\tau^{AB}\varepsilon\Gamma_{MN}\tau^{AB} \right) P_+\end{aligned}$$

We have gamma matrix identities

$$\begin{aligned}\{\Gamma_{MN}, \Gamma^{RST}\} &= 12\delta_{MN}^{RS}\Gamma^T + 2\Gamma_{MN}^{RST} \\ \Gamma^{MNP}\Gamma^R\Gamma_{NP} &= -4\Gamma^{MR} - 20g^{MR} \\ \Gamma^{MNP}\Gamma^{RST}\Gamma_{NP} &= 4\Gamma^{MRST} + 12g^{MR}\Gamma^{ST} \\ \Gamma^{MNP}\Gamma^{RS}\Gamma_{MNP} &= 24\Gamma^{RS} \\ \tau^A\tau^B\tau^A &= -3\tau^B \\ \tau^A\tau^{BC}\tau^A &= \tau^{BC}\end{aligned}$$

These Fierz identities can be used to obtain the closure relation for the abelian $(2,0)$ 6d tensor multiplet on the fermion. From (3.1) we first get

$$\begin{aligned}\delta^2\psi = & i \left(\frac{1}{4}\Gamma^{MNP}\varepsilon\bar{\varepsilon}\Gamma_{NP} - \Gamma^M\tau^A\varepsilon\bar{\varepsilon}\tau^A \right) D_M\psi \\ & + i \left(-\frac{1}{4}\Gamma^{MNP}\varepsilon\bar{\eta}\Gamma_{MNP} + \Gamma^M\tau^A\varepsilon\bar{\eta}\tau^A\Gamma_M - 4\tau^A\eta\bar{\varepsilon}\tau^A \right) \psi\end{aligned}$$

We get from the first line

$$\begin{aligned}(\delta^2\psi)_{1st} = & -iS^M D_M\psi \\ & + \frac{3i}{8}S_R\Gamma^R\Gamma^M D_M\psi \\ & - \frac{1}{8}\bar{\varepsilon}\Gamma_R\tau^A\varepsilon\tau^A\Gamma^R\Gamma^M D_M\psi\end{aligned}$$

This is the complete result, the contribution from $\Gamma^{RST}\tau^{CD}$ completely cancels out by some lucky circumstances. Let us now move on to the second line,

$$\begin{aligned}(\delta^2\psi)_{2nd} = & \frac{i}{2}\bar{\eta}\Gamma_{MN}\varepsilon\Gamma^{MN}\psi \\ = & -\frac{i}{4}D_MS_N\Gamma^{MN}\psi\end{aligned}$$

This term combines with a term from the first line into a full Lie derivative

$$-iS^M D_M\psi - \frac{i}{4}D_MS_N\Gamma^{MN}\psi =: -i\mathcal{L}_S\psi$$

and so we get one of the closure relations presented in the main text.

Reduction to $(1,0)$ supersymmetry is done by imposing the condition

$$\tau^5\varepsilon = -\varepsilon$$

and then we obtain the Fierz identity

$$\varepsilon \bar{\varepsilon} = \frac{1}{8} (\bar{\varepsilon} \varepsilon + \bar{\varepsilon} \gamma_m \varepsilon \gamma^m) - \frac{1}{64} \bar{\varepsilon} \gamma_{mn} \tau^{ij} \varepsilon \gamma^{mn} \tau^{ij}$$

We also have made use of the following 5d gamma matrix identities

$$\begin{aligned} \gamma^m \gamma^p \gamma_m &= -3\gamma^p \\ \gamma^m \gamma^{pq} \gamma_m &= \gamma^{pq} \\ \gamma^{mn} \gamma^p \gamma_{mn} &= -4\gamma^p \\ \gamma^{mn} \gamma^{pq} \gamma_{mn} &= 4\gamma^{pq} \end{aligned}$$

G Selfduality from a Lagrangian

Here we review the main result of [1, 10]. For abelian case, we have the selfduality equation

$$\frac{1}{r} \mathcal{F}_{mn} = -\frac{1}{6} \mathcal{E}_{mn}{}^{qrs} H_{qrs} + \frac{1}{2} \mathcal{E}_{mn}{}^{qrs} \mathcal{F}_{qr} \kappa_s$$

where

$$\begin{aligned} \mathcal{F}_{mn} &= F_{mn} + \partial_0 B_{mn} \\ F_{mn} &= 2\partial_m A_n \end{aligned}$$

By using the Bianchi identity for H_{qrs} we get the Maxwell equation

$$D^m \left(\frac{1}{r} \mathcal{F}_{mn} \right) - \frac{1}{2} \mathcal{E}_{mn}{}^{qrs} D^m (\mathcal{F}_{qr} \kappa_s) = 0$$

which follows from an action

$$\mathcal{L} = \frac{1}{4r} \mathcal{F}^{mn} \mathcal{F}_{mn} - \frac{1}{8} \mathcal{E}^{mnpqr} \mathcal{F}_{mn} \mathcal{F}_{pq} \kappa_r + \frac{1}{24} \mathcal{E}^{mnpqr} B_{mn} \partial_0 H_{rpq}$$

by varying A_m . But now we can also vary B_{mn} and then we get

$$\partial_0 \left(\frac{1}{2r} \mathcal{F}_{mn} - \frac{1}{4} \mathcal{E}_{mn}{}^{pqr} \mathcal{F}_{pq} \kappa_r - \frac{1}{12} \mathcal{E}_{mn}{}^{pqr} H_{pqr} \right) = 0$$

The last term can be replaced by

$$-\frac{1}{24} \mathcal{E}^{mnpqr} B_{mn} \partial_0 H_{rpq} = \frac{1}{8} \mathcal{E}^{mnpqr} \mathcal{F}_{mn} \partial_0^{-1} \partial_r \mathcal{F}_{pq}$$

where we note that $\partial_0 H_{rpq} = 3\partial_r \mathcal{F}_{pq}$. Thus we can write

$$\mathcal{L} = \frac{1}{4r} \mathcal{F}^{mn} \mathcal{F}_{mn} + \frac{1}{8} \mathcal{E}^{mnpqr} \mathcal{F}_{mn} \partial_0^{-1} (\partial_r - \kappa_r \partial_0) \mathcal{F}_{pq}$$

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