

Symmetry breaking in quantum curves and super Chern-Simons matrix models

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ABSTRACT: It was known that quantum curves and super Chern-Simons matrix models correspond to each other. From the viewpoint of symmetry, the algebraic curve of genus one, called the del Pezzo curve, enjoys symmetry of the exceptional algebra, while the super Chern-Simons matrix model is described by the free energy of topological strings on the del Pezzo background with the symmetry broken. We study the symmetry breaking of the quantum cousin of the algebraic curve and reproduce the results in the super Chern-Simons matrix model.

KEYWORDS: Chern-Simons Theories, M-Theory, Matrix Models, Topological Strings

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Contents

1	Introduction	1
2	Superconformal Chern-Simons matrix models	2
3	Quantum curve	6
4	Symmetry breaking	12
5	Degenerate curve	16
6	Conclusion	21
A	Weyl group	22

1 Introduction

In the semi-classical analysis of quantum physics, curves appear as the phase space orbit leading to the semi-classical Bohr-Sommerfeld quantization condition. From the viewpoint of full quantum physics, apparently these curves are technical tools and need to be quantized eventually. Since in algebraic geometry curves are defined by zeros of polynomial rings, in quantization by replacing the polynomial rings by quantum operators, the study of zeros switches smoothly to spectral problems of the quantum operators.

Recently the study of quantization of curves attracts renewed attention due to the important role it plays in understanding the integrability of gauge theories [1–4]. The interplay between curves and gauge theories continues to the three-dimensional M2-brane physics [5]. In [6] it was found that the grand canonical partition function of the M2-branes on the background $\mathbb{C}^4/\mathbb{Z}_k$ can be rewritten as a spectral determinant of a quantum-mechanical operator associated with the geometry $\mathbb{P}^1 \times \mathbb{P}^1$, following preceding works [7–11]. After further studies of the spectral determinant in [12–17] finally it was conjectured [18] that the grand potential of the M2-branes is expressed as the free energy of the topological string theory on the local $\mathbb{P}^1 \times \mathbb{P}^1$ geometry. The rank deformations with the inclusion of fractional M2-branes [19, 20] were studied in [21–24] and found to match the conjectured topological string free energy.

The computation was further generalized to many superconformal Chern-Simons matrix models describing the worldvolume theory of the M2-branes on other orbifold [25–31] or orientifold [32–36] backgrounds and to many spectral determinants associated with other curves [37, 38]. On the matrix model side, we can compute the models with or without rank deformations with similar techniques and find that the results fit in the conjecture with different choices of Kähler parameters and BPS indices on the background geometry, although it is difficult to explain these geometrical data. On the curve side, the general

structure of the correspondence is much clearer, though, besides the difficulty in the interpretation in terms of the M2-branes, it was also difficult to compute directly the kernels of the spectral operators for general parameters of the curves until recently with the important progress in [39, 40]. All of these difficulties on the both sides prevent us from studying the correspondence clearly.

To overcome the difficulties, the viewpoint of symmetry is crucial. On the curve side, among others, the special class of curves of ultimate interest and importance are those of genus one called del Pezzo curves, which are known to enjoy the symmetries of the exceptional algebra. On the matrix model side, by studying rank deformations of matrix models corresponding to curves of genus one, in several cases, the Kähler parameters and the BPS indices were identified [30] and these geometrical data were further interpreted from the symmetry breaking in [31]. Although the symmetry breaking patterns were identified, the explanation of them was missing.

In this paper, we promote the discussions for the classical del Pezzo curves to quantum curves. We define the quantum curves for our setup and study how the symmetry is realized and how the symmetry breaking happens. We find that the breaking patterns are completely consistent with the previous results in [31] from the superconformal Chern-Simons matrix models.

The organization of this paper is as follows. We first review the analysis of the superconformal Chern-Simons matrix models in the next section to explain our motivation. After that, in section 3, we define the quantum curve and study its quantum symmetry using an example of the D_5 curve. In section 4 we identify the superconformal Chern-Simons matrix models in the quantum curve and study its symmetry breaking. We turn to a different example of the E_7 curve in section 5. Finally we conclude with some discussions on future directions. Appendix A is devoted to technical details on the construction of the Weyl group.

2 Superconformal Chern-Simons matrix models

In this section, we review the superconformal Chern-Simons matrix models. The main purpose of this section is to explain our motivation of studying quantum curves.

It was proposed [5, 19, 20] that the $\mathcal{N} = 6$ superconformal Chern-Simons theory with gauge group $U(N_1)_k \times U(N_2)_{-k}$ (with the subscripts $k, -k$ denoting the Chern-Simons levels) and two pairs of bifundamental matters describes the worldvolume theory of $\min(N_1, N_2)$ M2-branes and $|N_2 - N_1|$ fractional M2-branes on the target space $\mathbb{C}^4/\mathbb{Z}_k$. With the localization techniques [41], the partition function, as well as the one-point functions (and hopefully the two-point functions [42]) of the half-BPS Wilson loop in the $\mathcal{N} = 6$ superconformal Chern-Simons theory on S^3 , which is originally defined with the infinite-dimensional path integral, reduces to a finite-dimensional matrix integration.

There are many generalizations for this matrix model. For example, by regarding the quiver diagram of the ABJM theory as the Dynkin diagram of the affine Lie algebra \widehat{A}_1 , there are other generalized theories with quiver diagrams of affine simply-laced Lie algebras

and it is known that they also preserve the $\mathcal{N} = 3$ superconformal symmetries [43].¹ Especially, it was found [46] that, for the \widehat{A}_r quiver diagram with the gauge group $U(N)^{r+1}$, as long as the levels k_a for $a = 1, 2, \dots, r + 1$ are given by

$$k_a = \frac{k}{2}(s_a - s_{a-1}), \quad s_a = \pm 1, \quad (2.1)$$

(with the cyclic identification $s_0 = s_{r+1}$), the superconformal Chern-Simons theory enjoys the supersymmetry enhancement to $\mathcal{N} = 4$. Hence, the $\mathcal{N} = 4$ theories can be characterized by recording $\{s_a\}_{a=1}^{r+1}$ with the order. Following the same localization techniques, the partition functions of these theories are given clearly by associating the vector multiplets (or vertices in the quiver diagram) and the hypermultiplets (or edges in the quiver diagram) respectively with

$$\prod_{m < m'}^N \left(2 \sinh \frac{\lambda_{a,(m)} - \lambda_{a,(m')}}{2} \right)^2, \quad \prod_{m,n=1}^N \left(2 \cosh \frac{\lambda_{a,(m)} - \lambda_{a+1,(n)}}{2} \right)^{-2}, \quad (2.2)$$

and integrating all of the variables $\lambda_{a,(m)}$ with

$$D\lambda_{a,(m)} = \frac{d\lambda_{a,(m)}}{2\pi} e^{\frac{ik_a}{4\pi} \sum_{m=1}^N \lambda_{a,(m)}^2}. \quad (2.3)$$

In [26] the model specified by the ± 1 alignment

$$\{s_a\}_{a=1}^{r+1} = \left(\overbrace{+1, \dots, +1}^{q_1}, \overbrace{-1, \dots, -1}^{p_1}, \overbrace{+1, \dots, +1}^{q_2}, \overbrace{-1, \dots, -1, \dots}^{p_2}, \dots \right), \quad (2.4)$$

was named the $(q_1, p_1, q_2, p_2, \dots)$ model and the grand canonical partition function of the model $\Xi_k(z) = \Xi_k^{(q_1, p_1, q_2, p_2, \dots)}(z)$ defined from the partition function $Z_k(N) = Z_k^{(q_1, p_1, q_2, p_2, \dots)}(N)$ by

$$\Xi_k(z) = \sum_{N=0}^{\infty} z^N Z_k(N), \quad (2.5)$$

was found to be given by

$$\Xi_k(z) = \det(1 + z\widehat{H}^{-1}), \quad (2.6)$$

with $\widehat{H} = \widehat{H}^{(q_1, p_1, q_2, p_2, \dots)}$ of the model given by

$$\widehat{H} = \left(2 \cosh \frac{\widehat{q}}{2} \right)^{q_1} \left(2 \cosh \frac{\widehat{p}}{2} \right)^{p_1} \left(2 \cosh \frac{\widehat{q}}{2} \right)^{q_2} \left(2 \cosh \frac{\widehat{p}}{2} \right)^{p_2} \dots \quad (2.7)$$

Here \widehat{q} and \widehat{p} are canonical operators satisfying the commutation relation $[\widehat{q}, \widehat{p}] = 2\pi ik$.

In a series of works [26–31] following the study of the ABJM matrix model [15–18] it was further found that the reduced grand potential $J_k(\mu) = J_k^{(q_1, p_1, q_2, p_2, \dots)}(\mu)$ of a class of the models (of genus one) defined by

$$\sum_{n=-\infty}^{\infty} e^{J_k(\mu + 2\pi in)} = \Xi_k(e^\mu), \quad (2.8)$$

is split into the perturbative part, the worldsheet instanton part and membrane instanton

¹See [44, 45] for progress in the study of the matrix models for the \widehat{D}_r quiver diagram.

part, $J_k(\mu) = J_k^{\text{pert}}(\mu) + J_k^{\text{WS}}(\mu) + J_k^{\text{MB}}(\mu)$ and if we redefine the chemical potential μ into μ_{eff} [17] and further into the Kähler parameters \mathbf{T} , the instanton parts are described by the free energy of topological strings

$$\begin{aligned}
 J_k^{\text{WS}}(\mu) &= \sum_{j_L, j_R} \sum_{\mathbf{d}} N_{j_L, j_R}^{\mathbf{d}} \sum_{n=1}^{\infty} \frac{(-1)^{(s_L + s_R - 1)n} s_R \sin 2\pi g_s n s_L}{n(2 \sin \pi g_s n)^2 \sin 2\pi g_s n} e^{-n\mathbf{d}\cdot\mathbf{T}}, \\
 J_k^{\text{MB}}(\mu) &= \sum_{j_L, j_R} \sum_{\mathbf{d}} N_{j_L, j_R}^{\mathbf{d}} \sum_{n=1}^{\infty} \frac{\partial}{\partial g_s} \left[g_s \frac{-\sin \frac{\pi n}{g_s} s_L \sin \frac{\pi n}{g_s} s_R}{4\pi n^2 \left(\sin \frac{\pi n}{g_s}\right)^3} e^{-n \frac{\mathbf{d}\cdot\mathbf{T}}{g_s}} \right], \tag{2.9}
 \end{aligned}$$

(with $s_{L/R} = 2j_{L/R} + 1$) on a target space which can be read off from (2.7).

Especially, it turns out that the target spaces for the (1, 1), (2, 2), (1, 1, 1, 1), (2, 1) and (2, 1, 2, 1) models are the del Pezzo curves of genus one, known to be classified by the exceptional Lie algebra E_n . From a careful analysis of the exact values of the partition function in [26, 28, 30], it was found [31] that the (2, 2), (1, 1, 1, 1) and (2, 1) models correspond to the $E_5 = D_5$ del Pezzo curve at the moduli where the Weyl symmetry of the D_5 algebra is broken respectively to those of the subalgebras² D_4 , $(A_1)^3$ and A_3 , while the (2, 1, 2, 1) model corresponds to the E_7 del Pezzo curve at the modulus where the E_7 algebra is broken to the subalgebra $D_5 \times A_1$. Namely, for example for the (2, 2) model and the (1, 1, 1, 1) model, since the total BPS indices $N_{j_L, j_R}^{\mathbf{d}} = \sum_{|\mathbf{d}|=d} N_{j_L, j_R}^{\mathbf{d}}$ at each degree d were computed in [47], our task reduces to identifying the Kähler parameters \mathbf{T} and the split of the total BPS indices at each degree d . This was performed in [30] and it was further found in [31] that, by regarding the BPS indices as representations of the D_5 algebra, the introduction of the Kähler parameters amounts to identifying “the Higgs fields acquiring expectation values” and the split of the total BPS indices corresponds to the decomposition of the representations of the D_5 algebra into those of the unbroken subalgebras.

More concretely, the rank deformations of the (2, 2) model and the (1, 1, 1, 1) model, which are connected by the Hanany-Witten effect, were studied intensively in [30]. For the (2, 2) model with the rank deformations $U(N)_k \times U(N + M_I)_0 \times U(N + 2M_I)_{-k} \times U(N + M_I)_0$, the Kähler parameters and the string coupling constant g_s in the instanton exponents $e^{-\mathbf{d}\cdot\mathbf{T}}$ and $e^{-\mathbf{d}\cdot\mathbf{T}/g_s}$ are

$$T^{\pm} = \frac{\mu_{\text{eff}}}{k} \pm \pi i \left(1 - \frac{M_I}{k} \right), \quad g_s = \frac{1}{k}, \tag{2.10}$$

and the BPS indices forming the representations of the D_5 algebra are broken to representations of D_4 (see table 1 for the split of the BPS indices). Furthermore, for the (2, 2) model with the rank deformations

$$U(N + M_{II})_k \times U(N + M_I)_0 \times U(N + 2M_I + M_{II})_{-k} \times U(N + M_I)_0, \tag{2.11}$$

²As we explain later, the remaining symmetry $(A_1)^4$ for the (1, 1, 1, 1) model identified in [31] should be corrected by $(A_1)^3$.

d	(j_L, j_R)	BPS	$(-1)^{d-1} \sum_{d_I} (\sum_{d_{II}} N_{j_L, j_R}^{(d, d_I, d_{II})})_{d_I}$
1	(0, 0)	16	$8_{+1} + 8_{-1}$
2	$(0, \frac{1}{2})$	10	$1_{+2} + 8_0 + 1_{-2}$
3	(0, 1)	16	$8_{+1} + 8_{-1}$
4	$(0, \frac{1}{2})$	1	1_0
	$(0, \frac{3}{2})$	45	$8_{+2} + 29_0 + 8_{-2}$
	$(\frac{1}{2}, 2)$	1	1_0

Table 1. The split of the BPS indices on the D_5 del Pezzo curve for the (2, 2) model. The split is interpreted as the decomposition of the D_5 representations into the D_4 subalgebra, as in $16 \rightarrow (\mathbf{8}_{s/c})_{+1} + (\mathbf{8}_{s/c})_{-1}$, $10 \rightarrow (\mathbf{1})_{+2} + (\mathbf{8}_v)_0 + (\mathbf{1})_{-2}$ and $45 \rightarrow (\mathbf{8}_v)_{+2} + (\mathbf{28})_0 + (\mathbf{1})_0 + (\mathbf{8}_v)_{-2}$.

which is connected to the (1, 1, 1, 1) model without rank deformations at $(M_I, M_{II}) = (k/2, k/2)$ through the Hanany-Witten effect, the Kähler parameters are

$$\begin{aligned}
 T_1^\pm &= \frac{\mu_{\text{eff}}}{k} \pm \pi i \left(1 - \frac{M_I}{k} - \frac{2M_{II}}{k} \right), \\
 T_2^\pm &= \frac{\mu_{\text{eff}}}{k} \pm \pi i \left(1 - \frac{M_I}{k} \right), \\
 T_3^\pm &= \frac{\mu_{\text{eff}}}{k} \pm \pi i \left(1 - \frac{M_I}{k} + \frac{2M_{II}}{k} \right),
 \end{aligned} \tag{2.12}$$

and the BPS indices in the representations of the D_4 algebra are further split into representations of the $(A_1)^3$ algebra (see table 2 for the further split of the BPS indices). Hence, from table 1 it was found that the symmetry for the (2, 2) model without rank deformations is broken to D_4 while from table 2 the symmetry for the (1, 1, 1, 1) model is further broken to $(A_1)^3$.

Even though the symmetry is broken to $(A_1)^3$ for general rank deformations of the (2, 2) model and the (1, 1, 1, 1) model, we can see an accidental symmetry enhancement for the (1, 1, 1, 1) model without rank deformations. Since the (1, 1, 1, 1) model without rank deformations corresponds to $(M_I, M_{II}) = (k/2, k/2)$, the instanton exponent is given by

$$\mathbf{d} \cdot \mathbf{T} = d \frac{\mu_{\text{eff}}}{k} + \pi i \left(d_I \left(1 - \frac{M_I}{k} \right) - d_{II} \frac{2M_{II}}{k} \right) = d \frac{\mu_{\text{eff}}}{k} + \tilde{d} \frac{\pi i}{2}, \tag{2.13}$$

where we have defined the total $u(1)$ degree and the two Cartan $u(1)$ charges which break the symmetries as

$$\begin{aligned}
 d &= \sum_{i=1}^3 (d_i^+ + d_i^-), \\
 d_I &= (d_1^+ + d_2^+ + d_3^+) - (d_1^- + d_2^- + d_3^-), \\
 d_{II} &= (d_1^+ - d_1^-) - (d_3^+ - d_3^-),
 \end{aligned} \tag{2.14}$$

as well as the special combination of the $u(1)$ charges \tilde{d} characterizing the $(1, 1, 1, 1)$ model without rank deformations and the unbroken $u(1)$ charge \bar{d} as³

$$\tilde{d} = d_I - 2d_{II}, \quad \bar{d} = d_I + d_{II}. \quad (2.15)$$

Then, the accidental symmetry enhancement is observed as follows. The adjoint representation of the D_5 algebra decomposes as

$$\mathbf{45} \rightarrow (\mathbf{8}_v)_{+2} + (\mathbf{28})_0 + (\mathbf{1})_0 + (\mathbf{8}_v)_{-2}, \quad (2.16)$$

in the breaking $D_5 \rightarrow (D_4)_{d_I}$ and further decompositions of various D_4 representations into $(A_1)^4$ are given by

$$\begin{aligned} \mathbf{28} &\rightarrow (\mathbf{3}, \mathbf{1}, \mathbf{1}, \mathbf{1}) + (\mathbf{1}, \mathbf{3}, \mathbf{1}, \mathbf{1}) + (\mathbf{1}, \mathbf{1}, \mathbf{3}, \mathbf{1}) + (\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{3}) + (\mathbf{2}, \mathbf{2}, \mathbf{2}, \mathbf{2}), \\ \mathbf{8}_v &\rightarrow (\mathbf{2}, \mathbf{2}, \mathbf{1}, \mathbf{1}) + (\mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{2}). \end{aligned} \quad (2.17)$$

The last factor of A_1 is broken and the $u(1)$ charge is denoted by d_{II} . After expressing the two $u(1)$ charges d_I and d_{II} in terms of the charges \bar{d} and \tilde{d} , each representation of the third factor of $(A_1)_{\tilde{d}}$ in the unbroken symmetry $(A_1)^3$ combines into the representations of A_2 as

$$\mathbf{8} \rightarrow \mathbf{2}_{+3} + \mathbf{3}_0 + \mathbf{1}_0 + \mathbf{2}_{-3}, \quad \mathbf{3} \rightarrow \mathbf{1}_{+2} + \mathbf{2}_{-1}, \quad \bar{\mathbf{3}} \rightarrow \mathbf{2}_{+1} + \mathbf{1}_{-2}, \quad \mathbf{1} \rightarrow \mathbf{1}_0 \quad (2.18)$$

in $A_2 \rightarrow (A_1)_{\tilde{d}}$. Finally the decomposition of the D_5 adjoint representation into $(A_1 \times A_1 \times A_2)_{\tilde{d}}$ is given by

$$\begin{aligned} \mathbf{45} &\rightarrow (\mathbf{1}, \mathbf{1}, \bar{\mathbf{3}})_{+4} + (\mathbf{2}, \mathbf{2}, \mathbf{3})_{+2} + (\mathbf{3}, \mathbf{1}, \mathbf{1})_0 + (\mathbf{1}, \mathbf{3}, \mathbf{1})_0 + (\mathbf{1}, \mathbf{1}, \mathbf{8})_0 + (\mathbf{1}, \mathbf{1}, \mathbf{1})_0 \\ &\quad + (\mathbf{2}, \mathbf{2}, \bar{\mathbf{3}})_{-2} + (\mathbf{1}, \mathbf{1}, \mathbf{3})_{-4}, \end{aligned} \quad (2.19)$$

which implies that the symmetry $(A_1)^3$ is further enhanced to $(A_1)^2 \times A_2 = A_1 \times A_1 \times A_2$ in the $(1, 1, 1, 1)$ model without rank deformations.

3 Quantum curve

In this section we define carefully what we mean by quantum curves and study the typical example of the D_5 del Pezzo curve.

We define a quantum algebraic curve to be the spectral problem of a polynomial quantum operator \hat{H} generated by $\hat{Q} = e^{\hat{q}}$ and $\hat{P} = e^{\hat{p}}$ where \hat{q} and \hat{p} are the canonical operators of coordinates and momenta satisfying the canonical commutation relation $[\hat{q}, \hat{p}] = i\hbar$. Since the similarity transformation, the adjoint action by \hat{G} ,

$$\hat{H} \sim \hat{G}\hat{H}\hat{G}^{-1}, \quad (3.1)$$

typically does not affect the spectral problem, we define the quantum algebraic curve with the identification of all the similarity transformations. As in the classical algebraic

³The combination of the $u(1)$ charges $\bar{d} = 2(d_1^+ - d_1^-) + (d_2^+ - d_2^-)$ exchanges among the four Kähler parameters $T_1^\pm = \frac{\mu_{\text{eff}}}{k} \mp \frac{\pi i}{2}$ and $T_2^\pm = \frac{\mu_{\text{eff}}}{k} \pm \frac{\pi i}{2}$, while leaving the remaining two $T_3^\pm = \frac{\mu_{\text{eff}}}{k} \pm \frac{3\pi i}{2}$ fixed.

d	(j_L, j_R)	d_I	BPS	$(-1)^{d-1} \sum_{d_{II}} (N_{j_L, j_R}^{(d, d_I, d_{II})})_{d_{II}}$
1	(0, 0)	± 1	8	$2_{+1} + 4_0 + 2_{-1}$
2	$(0, \frac{1}{2})$	0	8	$2_{+1} + 4_0 + 2_{-1}$
		± 2	1	1_0
3	(0, 1)	± 1	8	$2_{+1} + 4_0 + 2_{-1}$
4	$(0, \frac{1}{2})$	0	1	1_0
	$(0, \frac{3}{2})$	0	29	$1_{+2} + 8_{+1} + 11_0 + 8_{-1} + 1_{-2}$
		± 2	8	$2_{+1} + 4_0 + 2_{-1}$
	$(\frac{1}{2}, 2)$	0	1	1_0

Table 2. The split of the BPS indices on the D_5 del Pezzo curve for the $(1, 1, 1, 1)$ model. The split is interpreted as the decomposition of the D_4 representations into the $(A_1)^4$ subalgebra, as in $\mathbf{8}_v \rightarrow (\mathbf{2}, \mathbf{2}, \mathbf{1}, \mathbf{1}) + (\mathbf{1}, \mathbf{1}, \mathbf{2}, \mathbf{2})$, $\mathbf{8}_s \rightarrow (\mathbf{2}, \mathbf{1}, \mathbf{2}, \mathbf{1}) + (\mathbf{1}, \mathbf{2}, \mathbf{1}, \mathbf{2})$, $\mathbf{8}_c \rightarrow (\mathbf{2}, \mathbf{1}, \mathbf{1}, \mathbf{2}) + (\mathbf{1}, \mathbf{2}, \mathbf{2}, \mathbf{1})$ and $\mathbf{28} \rightarrow (\mathbf{3}, \mathbf{1}, \mathbf{1}, \mathbf{1}) + (\mathbf{1}, \mathbf{3}, \mathbf{1}, \mathbf{1}) + (\mathbf{1}, \mathbf{1}, \mathbf{3}, \mathbf{1}) + (\mathbf{1}, \mathbf{1}, \mathbf{1}, \mathbf{3}) + (\mathbf{2}, \mathbf{2}, \mathbf{2}, \mathbf{2})$, where the last A_1 factor contributes as the $u(1)$ charge in the subscript (and hence is broken).

curve, the curve is studied within a linear combination of a certain class of the independent operators $\widehat{Q}^m \widehat{P}^n$ with $(m, n) \in \mathbb{Z}^2$. Note that in the quantization, the order of the operators is important and we adopt the normal ordering such that \widehat{Q} is in the left and \widehat{P} is in the right. The set of (m, n) with non-vanishing coefficients is often referred to as the Newton polygon.

The classical algebraic curve of genus one is called the del Pezzo curve and is known to be classified by the exceptional algebra E_n . As one of simple and abundant cases, here we mainly consider the quantization of the $E_5 = D_5$ curve, where the quantum curve or the quantum Hamiltonian is a linear combination of the independent operators

$$\widehat{Q}^m \widehat{P}^n, \quad m = -1, 0, 1, \quad n = -1, 0, 1, \tag{3.2}$$

(see figure 1 for the Newton polygon of the D_5 del Pezzo curve). Instead of fixing the coefficients, as in the classical case [48] (see sections 8.2.5 and 8.4.4), it is often convenient to fix the asymptotic values of the curve

$$(\infty, e_1^{-1}), (\infty, e_2^{-1}), (e_3, \infty), (e_4, \infty), (0, h_2^{-1} e_5), (0, h_2^{-1} e_6), (h_1 e_7^{-1}, 0), (h_1 e_8^{-1}, 0), \tag{3.3}$$

(see figure 2 for the asymptotic values). Each two points out of the eight points are the solutions to the quadratic equations obtained by setting $\widehat{Q} \rightarrow \infty, \widehat{P} \rightarrow \infty, \widehat{Q} \rightarrow 0$ and $\widehat{P} \rightarrow 0$ respectively. In other words, the eight values are the asymptotic values of the dual graph of the Newton polygon. Due to the Vieta's formulas on products of roots, the eight points are not independent and should be subject to the constraint

$$\prod_{i=1}^8 e_i = h_1^2 h_2^2. \tag{3.4}$$

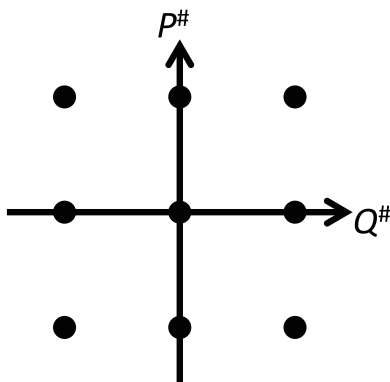


Figure 1. The Newton polygon of the D_5 del Pezzo curve.

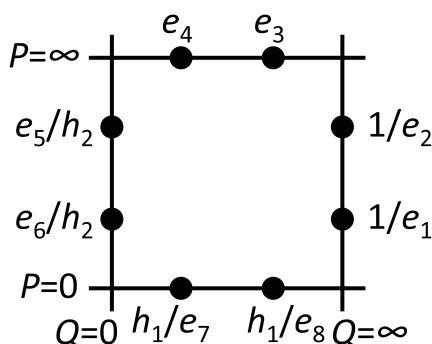


Figure 2. The asymptotic values of the D_5 del Pezzo curve, (∞, e_1^{-1}) , (∞, e_2^{-1}) , (e_3, ∞) , (e_4, ∞) , $(0, h_2^{-1}e_5)$, $(0, h_2^{-1}e_6)$, $(h_1e_7^{-1}, 0)$, $(h_1e_8^{-1}, 0)$. The four lines denote “lines at infinity” $\widehat{Q} = \infty$, $\widehat{P} = \infty$, $\widehat{Q} = 0$ and $\widehat{P} = 0$ respectively.

Then, our quantum curve is given by

$$\widehat{H}/\alpha = \begin{matrix} \widehat{Q}\widehat{P} & -(e_3 + e_4)\widehat{P} & +e_3e_4\widehat{Q}^{-1}\widehat{P} \\ -(e_1^{-1} + e_2^{-1})\widehat{Q} & +E/\alpha & -h_2^{-1}e_3e_4(e_5 + e_6)\widehat{Q}^{-1} \\ +(e_1e_2)^{-1}\widehat{Q}\widehat{P}^{-1} & -h_1(e_1e_2)^{-1}(e_7^{-1} + e_8^{-1})\widehat{P}^{-1} & +h_1^2(e_1e_2e_7e_8)^{-1}\widehat{Q}^{-1}\widehat{P}^{-1}, \end{matrix} \quad (3.5)$$

where the coefficient of the last term $\widehat{Q}^{-1}\widehat{P}^{-1}$ can be alternatively expressed as $h_1^2(e_1e_2e_7e_8)^{-1} = h_2^{-2}e_3e_4e_5e_6$ due to (3.4). Note that the classical algebraic curve is defined from the zeros of the curve and characterized by their asymptotic zeros. For the quantum case, the asymptotic zeros are obtained only after the normal ordering.

This curve enjoys a lot of symmetries. Especially our labelling of the curve is redundant and the same curve can be realized by different choices of the parameters. For example, we use ten parameters h_1, h_2 and e_1, \dots, e_8 to describe the eight asymptotic values and apparently two degrees of freedom can be fixed arbitrarily. Also, by the similarity transformation (3.1) generated by $\widehat{G} = e^{\frac{ia}{\hbar}\widehat{P}}$ or $\widehat{G} = e^{-\frac{ib}{\hbar}\widehat{Q}}$, it is clear that a curve and the same curve with \widehat{Q}, \widehat{P} rescaled as

$$(\widehat{Q}, \widehat{P}) \rightarrow (A\widehat{Q}, \widehat{P}), \quad (\widehat{Q}, \widehat{P}) \rightarrow (\widehat{Q}, B\widehat{P}), \quad (3.6)$$

(with $A = e^a$ and $B = e^b$) should be identified. Using these two rescalings we can further fix two degrees of freedom. After the identification, aside from the parameters α and E , we have ten parameters subject to four continuous symmetries and one constraint (3.4), which leaves only five parameters.

After identifying these continuous gauge symmetries, there also remain discrete gauge symmetries, which should be clarified. The analysis for the classical algebraic curve is well-known and explained carefully for example in [48]. Here we study the same problem for the quantum curve.

Classically the D_5 del Pezzo curve enjoys the Weyl symmetry of D_5 , which is basically generated by s_1, s_2, s_3, s_4, s_5 exchanging the asymptotical points [48]

$$\begin{aligned}
 s_1 &: h_1 e_7^{-1} \leftrightarrow h_1 e_8^{-1}, \\
 s_2 &: e_3 \leftrightarrow e_4, \\
 s_3 &: e_3 \leftrightarrow h_1 e_7^{-1}, \\
 s_4 &: e_1^{-1} \leftrightarrow h_2^{-1} e_5, \\
 s_5 &: e_1^{-1} \leftrightarrow e_2^{-1},
 \end{aligned}
 \tag{3.7}$$

and has $2^4 \times 5! = 1920$ elements in total (see figure 3 for the Dynkin diagram of D_5 and the numbering of the roots). Though in our setup the affine root does not appear, we can introduce the lowest root

$$s_0 : h_2^{-1} e_5 \leftrightarrow h_2^{-1} e_6, \tag{3.8}$$

to complete the affine Dynkin diagram. Of course, this is not necessary because the lowest root is generated by the simple roots as $s_0 = s_4 s_3 s_2 s_5 s_4 s_3 s_1 s_3 s_4 s_5 s_2 s_3 s_4$.

To provide the Weyl symmetry explicitly, in the following we adopt the gauge fixing condition

$$e_2 = e_4 = e_6 = e_8 = 1, \tag{3.9}$$

using the two degrees of freedom in the redundant description of the eight points with ten parameters and the other two degrees of freedom in the continuous rescaling as explained in (3.6). Then, the constraint (3.4) becomes

$$h_1^2 h_2^2 = e_1 e_3 e_5 e_7. \tag{3.10}$$

We often drop e_7 with

$$e_7 = h_1^2 h_2^2 (e_1 e_3 e_5)^{-1}, \tag{3.11}$$

to display the transformations unambiguously.

After the gauge fixing, it is not difficult to realize that the exchanges of s_1, s_2, s_5 and s_0 are given as

$$\begin{aligned}
 s_1 &: (h_1, h_2, e_1, e_3, e_5, e_7; \alpha) \mapsto (h_1 e_7^{-1}, h_2, e_1, e_3, e_5, e_7^{-1}; \alpha), \\
 s_2 &: (h_1, h_2, e_1, e_3, e_5, e_7; \alpha) \mapsto (h_1 e_3^{-1}, h_2, e_1, e_3^{-1}, e_5, e_7; e_3 \alpha), \\
 s_5 &: (h_1, h_2, e_1, e_3, e_5, e_7; \alpha) \mapsto (h_1, h_2 e_1^{-1}, e_1^{-1}, e_3, e_5, e_7; e_1^{-1} \alpha), \\
 s_0 &: (h_1, h_2, e_1, e_3, e_5, e_7; \alpha) \mapsto (h_1, h_2 e_5^{-1}, e_1, e_3, e_5^{-1}, e_7; \alpha),
 \end{aligned}
 \tag{3.12}$$

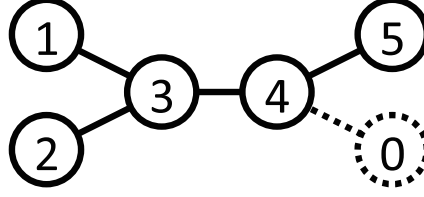


Figure 3. The Dynkin diagram of the D_5 algebra.

as in the classical case. For s_3 and s_4 the situation is more complicated. For s_3 we apply the canonical transformation,

$$\widehat{Q}' = \widehat{Q}, \quad \widehat{P}' = (\widehat{Q} - e_3)\widehat{P}(\widehat{Q} - h_1e_7^{-1})^{-1}, \quad (3.13)$$

which can be obtained by applying the similarity transformation (3.1) generated by

$$\widehat{G} = e^{F_3(\widehat{q}) - F_7(\widehat{q})}, \quad (3.14)$$

with $F_3(q)$ and $F_7(q)$ defined by

$$e^{F_3(q) - F_3(q - i\hbar)} = e^q - e_3, \quad e^{F_7(q + i\hbar) - F_7(q)} = e^q - h_1e_7^{-1}. \quad (3.15)$$

Indeed, by using the formula $e^{\mp\widehat{P}}f(\widehat{q})e^{\pm\widehat{P}} = f(\widehat{q} \pm i\hbar)$ repeatedly one can show

$$\begin{aligned} \widehat{G}\widehat{P}\widehat{G}^{-1} &= e^{F_3(\widehat{q}) - F_7(\widehat{q})}e^{\widehat{P}}e^{-F_3(\widehat{q}) + F_7(\widehat{q})} \\ &= e^{F_3(\widehat{q}) - F_3(\widehat{q} - i\hbar)}e^{\widehat{P}}e^{F_7(\widehat{q}) - F_7(\widehat{q} + i\hbar)} = (\widehat{Q} - e_3)\widehat{P}(\widehat{Q} - h_1e_7^{-1})^{-1}. \end{aligned} \quad (3.16)$$

Then, after the normal ordering, we find that the terms in \widehat{H}/α proportional to \widehat{P} and those proportional to \widehat{P}^{-1} are respectively given by ($q = e^{i\hbar}$)

$$\begin{aligned} \widehat{Q}^{-1}(\widehat{Q} - e_3)(\widehat{Q} - 1)\widehat{P} &= \widehat{Q}'^{-1}(\widehat{Q}' - qh_1e_7^{-1})(\widehat{Q}' - 1)(q^{-1}\widehat{P}'), \\ e_1^{-1}\widehat{Q}^{-1}(\widehat{Q} - h_1e_7^{-1})(\widehat{Q} - h_1)\widehat{P}^{-1} &= e_1^{-1}\widehat{Q}'^{-1}(\widehat{Q}' - q^{-1}e_3)(\widehat{Q}' - h_1)(q^{-1}\widehat{P}')^{-1}. \end{aligned} \quad (3.17)$$

Similarly for s_4 we apply the similarity transformation

$$\widehat{Q}' = (\widehat{P} - h_2^{-1}e_5)^{-1}\widehat{Q}(\widehat{P} - e_1^{-1}), \quad \widehat{P}' = \widehat{P}, \quad (3.18)$$

and perform a similar normal ordering. These transformations imply that

$$\begin{aligned} s_3 : (h_1, h_2, e_1, e_3, e_5, e_7; \alpha) &\mapsto (h_1, qh_1h_2(e_3e_7)^{-1}, e_1, qh_1e_7^{-1}, e_5, qh_1e_3^{-1}; \alpha), \\ s_4 : (h_1, h_2, e_1, e_3, e_5, e_7; \alpha) &\mapsto (h_1h_2(qe_1e_5)^{-1}, h_2, h_2(qe_5)^{-1}, e_3, h_2(qe_1)^{-1}, e_7; \alpha). \end{aligned} \quad (3.19)$$

Using the constraint (3.11), we find

$$\begin{aligned} s_1 : (h_1, h_2, e_1, e_3, e_5; \alpha) &\mapsto \left(\frac{e_1e_3e_5}{h_1h_2^2}, h_2, e_1, e_3, e_5; \alpha \right), \\ s_2 : (h_1, h_2, e_1, e_3, e_5; \alpha) &\mapsto \left(\frac{h_1}{e_3}, h_2, e_1, \frac{1}{e_3}, e_5; e_3\alpha \right), \end{aligned}$$

$$\begin{aligned}
 s_3 : (h_1, h_2, e_1, e_3, e_5; \alpha) &\mapsto \left(h_1, \frac{qe_1e_5}{h_1h_2}, e_1, \frac{qe_1e_3e_5}{h_1h_2^2}, e_5; \alpha \right), \\
 s_4 : (h_1, h_2, e_1, e_3, e_5; \alpha) &\mapsto \left(\frac{h_1h_2}{qe_1e_5}, h_2, \frac{h_2}{qe_5}, e_3, \frac{h_2}{qe_1}; \alpha \right), \\
 s_5 : (h_1, h_2, e_1, e_3, e_5; \alpha) &\mapsto \left(h_1, \frac{h_2}{e_1}, \frac{1}{e_1}, e_3, e_5; \frac{\alpha}{e_1} \right), \\
 s_0 : (h_1, h_2, e_1, e_3, e_5; \alpha) &\mapsto \left(h_1, \frac{h_2}{e_5}, e_1, e_3, \frac{1}{e_5}; \alpha \right).
 \end{aligned} \tag{3.20}$$

It is not difficult to see the algebraic relations

$$\begin{aligned}
 s_1^2 = s_2^2 = s_3^2 = s_4^2 = s_5^2 &= 1, \\
 (s_1s_2)^2 = (s_1s_4)^2 = (s_1s_5)^2 = (s_2s_4)^2 &= (s_2s_5)^2 = (s_3s_5)^2 = 1, \\
 (s_1s_3)^3 = (s_2s_3)^3 = (s_3s_4)^3 = (s_4s_5)^3 &= 1.
 \end{aligned} \tag{3.21}$$

By comparing with the general relations of the Weyl group $(s_i s_j)^{r+2} = 1$ for two different simple roots connected by r edges, the relations (3.21) indicate that the transformations generate the Weyl group of D_5 in figure 3.

Apparently, in the transformations (3.20) only s_3 and s_4 contain the quantum deformation parameter q explicitly and, by setting $q = 1$, the transformations reproduce those for the classical curves. It is, then, natural to ask whether the transformations for the quantum curves essentially change from the classical ones. To answer this question, let us redefine h_1 and h_2 by

$$\bar{h}_1 = qh_1, \quad \bar{h}_2 = q^{-1}h_2. \tag{3.22}$$

After the redefinition, the transformations s_3 and s_4 become

$$\begin{aligned}
 s_3 : (\bar{h}_1, \bar{h}_2, e_1, e_3, e_5; \alpha) &\mapsto \left(\bar{h}_1, \frac{e_1e_5}{\bar{h}_1\bar{h}_2}, e_1, \frac{e_1e_3e_5}{\bar{h}_1\bar{h}_2^2}, e_5; \alpha \right), \\
 s_4 : (\bar{h}_1, \bar{h}_2, e_1, e_3, e_5; \alpha) &\mapsto \left(\frac{\bar{h}_1\bar{h}_2}{e_1e_5}, \bar{h}_2, \frac{\bar{h}_2}{e_5}, e_3, \frac{\bar{h}_2}{e_1}; \alpha \right),
 \end{aligned} \tag{3.23}$$

and the other transformations are unaffected by the change of variables. Hence we conclude that the only change from the transformations for the classical curves is the shift of the parameters (3.22).

This fact implies that we can regard the parameter space of the curve as the root or weight space and identify these transformations as the standard Weyl actions, reflections by the simple root vectors α ,

$$s_\alpha(v) = v - \alpha \frac{2(\alpha, v)}{(\alpha, \alpha)}, \tag{3.24}$$

where v is an element of a five-dimensional space and (\cdot, \cdot) is a bilinear form in the space. To identify the simple root vectors in the parameter space of the curve, we also prepare the fundamental weight vectors ω_i ($1 \leq i \leq 5$), which are defined as the dual basis of

the coroot vectors, $(\omega_i, \alpha_j^\vee) = \delta_{ij}$, with the coroot vectors being $\alpha_i^\vee = 2\alpha_i/(\alpha_i, \alpha_i)$. Then, we find that the root vectors are expanded by the fundamental weight vectors with the coefficients given by the Cartan matrix $A_{ij} = (\alpha_i, \alpha_j^\vee)$,

$$\alpha_i = A_{ij}\omega_j, \tag{3.25}$$

and that transformation s_{α_i} acts on ω_j as

$$s_{\alpha_i}(\omega_j) = \omega_j - \delta_{ij}\alpha_i, \tag{3.26}$$

with no sum over i . Now, it turns out that our task for finding simple roots and fundamental weights is to solve (3.25) and (3.26) simultaneously under the identification $s_{\alpha_i}(v) = s_i(v)$ along with the explicit form of the Cartan matrix of D_5

$$A = \begin{pmatrix} 2 & 0 & -1 & 0 & 0 \\ 0 & 2 & -1 & 0 & 0 \\ -1 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{pmatrix}. \tag{3.27}$$

Then, we find that the final results of the identification are given as

$$\begin{aligned} \alpha_1 &= (1, 0, 0, 0, 0), & \omega_1 &= (1, -1, 0, 0, -1), \\ \alpha_2 &= (1, 0, 0, 2, 0), & \omega_2 &= (1, -1, 0, 1, -1), \\ \alpha_3 &= (0, -1, 0, -1, 0), & \omega_3 &= (1, -2, 0, 0, -2), \\ \alpha_4 &= (-1, 0, -1, 0, -1), & \omega_4 &= (0, -1, 0, 0, -2), \\ \alpha_5 &= (0, 1, 2, 0, 0), & \omega_5 &= (0, 0, 1, 0, -1), \end{aligned} \tag{3.28}$$

where we have represented the parameters of curves by $(\log \bar{h}_1, \log \bar{h}_2, \log e_1, \log e_3, \log e_5)$. Note that in this expression, our symmetries of the algebraic curve s_i (3.20), (3.23) reduce to the standard Weyl action s_{α_i} (3.24).

4 Symmetry breaking

After establishing the Weyl symmetries of the quantum D_5 del Pezzo curve, we start our study of the symmetry breaking. For the (2, 2) model and the (1, 1, 1, 1) model with the expressions of the quantum operator

$$\begin{aligned} \widehat{H}^{(2,2)} &= (\widehat{Q}^{\frac{1}{2}} + \widehat{Q}^{-\frac{1}{2}})^2(\widehat{P}^{\frac{1}{2}} + \widehat{P}^{-\frac{1}{2}})^2, \\ \widehat{H}^{(1,1,1,1)} &= (\widehat{Q}^{\frac{1}{2}} + \widehat{Q}^{-\frac{1}{2}})(\widehat{P}^{\frac{1}{2}} + \widehat{P}^{-\frac{1}{2}})(\widehat{Q}^{\frac{1}{2}} + \widehat{Q}^{-\frac{1}{2}})(\widehat{P}^{\frac{1}{2}} + \widehat{P}^{-\frac{1}{2}}), \end{aligned} \tag{4.1}$$

after the shift $\widehat{q} \rightarrow \widehat{q} + \pi i$ and $\widehat{p} \rightarrow \widehat{p} - \pi i$, generated by the similarity transformations (3.6), we can easily identify the parameters

$$\begin{aligned} (h_1, h_2, e_1, e_3, e_5, e_7; \alpha)^{(2,2)} &= (1, 1, 1, 1, 1, 1; 1), \\ (h_1, h_2, e_1, e_3, e_5, e_7; \alpha)^{(1,1,1,1)} &= (1, 1, q^{-\frac{1}{2}}, q^{\frac{1}{2}}, q^{-\frac{1}{2}}, q^{\frac{1}{2}}; q^{-\frac{1}{4}}). \end{aligned} \tag{4.2}$$

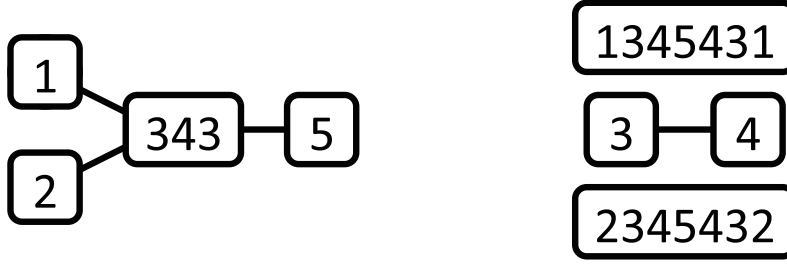


Figure 4. The Dynkin diagram of the D_4 subalgebra within the original D_5 algebra which preserves the $(2, 2)$ model without rank deformations (Left) and that of the $(A_1)^2 \times A_2$ subalgebra which preserves the $(1, 1, 1, 1)$ model without rank deformations (Right).

For the study of the $(2, 1)$ model we need to rescale \hat{p} by 2,

$$\hat{H}^{(2,1)} = (\hat{Q}^{\frac{1}{2}} + \hat{Q}^{-\frac{1}{2}})^2 (\hat{P} + \hat{P}^{-1}). \quad (4.3)$$

Then, the parameters are

$$(h_1, h_2, e_1, e_3, e_5, e_7; \alpha)^{(2,1)} = (1, 1, -1, 1, -1, 1; 1). \quad (4.4)$$

Hence, the main question is, out of the 1920 elements of the D_5 Weyl group, which elements leave these parameters including α invariant and what group these elements form. We can easily generate the 1920 elements with a computer by subsequently acting the transformations s_1, s_2, s_3, s_4, s_5 and choosing only the transformations which do not appear previously. We can then act these 1920 transformations on the parameters of the various models (4.2), (4.4) and pick up the invariant transformations.

Let us search for the transformations leaving the parameters for these models invariant. For the $(2, 2)$ model, out of the 1920 elements, we find that there are 192 elements leaving the parameter invariant. If we look closely, we further find that, among them, the four transformations

$$s_1, \quad s_2, \quad s_3 s_4 s_3, \quad s_5, \quad (4.5)$$

satisfy the relations

$$(s_1 s_3 s_4 s_3)^3 = (s_2 s_3 s_4 s_3)^3 = (s_5 s_3 s_4 s_3)^3 = 1, \quad (4.6)$$

and generate $2^3 \times 4! = 192$ different elements. Hence we conclude that the invariant subgroup leaving the $(2, 2)$ model is the D_4 Weyl group (see figure 4 for the Dynkin diagram). The result matches with the studies of the superconformal Chern-Simons matrix models as in table 1.

We can further ask which subspace in the parameter space $(h_1, h_2, e_1, e_3, e_5)$ enjoys the same D_4 symmetry as the $(2, 2)$ model without rank deformations for arbitrary values of α . Due to the expression of the transformation $s_3 s_4 s_3$

$$s_3 s_4 s_3 : (h_1, h_2, e_1, e_3, e_5; \alpha) \mapsto \left(\frac{1}{h_2}, \frac{1}{h_1}, \frac{e_1}{h_1 h_2}, \frac{e_3}{h_1 h_2}, \frac{e_5}{h_1 h_2}; \alpha \right), \quad (4.7)$$

cosets	$(\bar{h}_1, \bar{h}_2, e_1, e_3, e_5, e_7)$	D_4 symmetry	invariant subspace
$1, s_3 s_1 s_2 s_3$	$(q^{\pm 1}, q^{\mp 1}, 1, 1, 1, 1)$	$\langle s_3 s_4 s_3; s_1, s_2, s_5 \rangle$	$(h^{-1}, h, 1, 1, 1, 1)$
$s_3, s_1 s_2 s_3$	$(q^{\pm 1}, 1, 1, q^{\pm 1}, 1, q^{\pm 1})$	$\langle s_4; s_1 s_3 s_1, s_2 s_3 s_2, s_5 \rangle$	$(h^{-1}, 1, 1, h^{-1}, 1, h^{-1})$
$s_4, s_4 s_3 s_1 s_2 s_3$	$(1, q^{\mp 1}, q^{\mp 1}, 1, q^{\mp 1}, 1)$	$\langle s_3; s_1, s_2, s_4 s_5 s_4 \rangle$	$(1, h, h, 1, h, 1)$
$s_1 s_3, s_2 s_3$	$(1, 1, 1, q^{\pm 1}, 1, q^{\mp 1})$	$\langle s_4; s_3, s_1 s_2 s_3 s_2 s_1, s_5 \rangle$	$(1, 1, 1, h^{-1}, 1, h)$
$s_5 s_4, s_5 s_4 s_3 s_1 s_2 s_3$	$(1, 1, q^{\pm 1}, 1, q^{\mp 1}, 1)$	$\langle s_3; s_1, s_2, s_4 \rangle$	$(1, 1, h^{-1}, 1, h, 1)$

Table 3. Different representative choices of the parameters for the (2, 2) model without rank deformations. The parameters $(\bar{h}_1, \bar{h}_2, e_1, e_3, e_5, e_7)$ are obtained by acting the 10 cosets of the D_5 Weyl group by the invariant D_4 Weyl group (where the upper/lower double-sign corresponds to the first/second coset respectively). For the D_4 symmetry we first denote the root corresponding to the adjoint representation and then the other three corresponding to the three **8** representations.

the condition $h_1 h_2 = 1$ is required. Along with the actions of s_1, s_2 and s_5 in (3.20), we further find the conditions $e_1 = e_3 = e_5 = 1$. Namely, the subspace in the parameter space $(h_1, h_2, e_1, e_3, e_5, e_7)$ enjoying the same D_4 symmetry as the (2, 2) model without rank deformations is parametrized by

$$\{(\bar{h}_1, \bar{h}_2, e_1, e_3, e_5, e_7) = (h^{-1}, h, 1, 1, 1, 1)\}. \tag{4.8}$$

We find that, instead of the original parameter of $(h_1, h_2, e_1, e_3, e_5, e_7)$, in discussing the subspace with the symmetry enhancement, it is convenient to use the redefined parameter $(\bar{h}_1, \bar{h}_2, e_1, e_3, e_5, e_7)$ introduced in (3.22).

Note that the symmetry breaking does not mean that the D_5 Weyl symmetry disappears completely. Even though the broken symmetries do not leave the parameters invariant, since the transformations come from the similarity transformations, the new parameters share the same spectrum. In the analogy of the spontaneous symmetry breaking, the broken symmetry is realized “non-linearly”. More concretely, with the broken symmetries, the vacuum expectation value is mapped to other equivalent values which share the same symmetry breaking. In fact since the order of the D_5 Weyl group is 1920 and the order of the invariant D_4 Weyl group is 192, we find $1920/192 = 10$ cosets. Using these cosets we can map the original parameter $(\bar{h}_1, \bar{h}_2, e_1, e_3, e_5, e_7)^{(2,2)}$ (4.2) into other equivalent parameters. In table 3 we display the 10 parameters mapped by the cosets, the generators of the invariant D_4 Weyl groups and the one-dimensional subspaces invariant under these generators.

Similarly, we work for the (1, 1, 1, 1) model. This time we find that, out of the 1920 elements, there are 24 elements that leave the parameter invariant. We find that the whole 24 elements are generated from the following four transformations

$$s_3, \quad s_4, \quad s_1 s_3 s_4 s_5 s_4 s_3 s_1, \quad s_2 s_3 s_4 s_5 s_4 s_3 s_2. \tag{4.9}$$

Since the latter two commute with the others it is clear that the invariant subgroup leaving the (1, 1, 1, 1) model is $A_2 \times (A_1)^2$ (see figure 4 for the Dynkin diagram). Compared with the analysis from the superconformal Chern-Simons matrix models where the invariant

subgroup was originally found to be $(A_1)^3$ as reviewed in table 2, the results do not coincide. The reason is that in the study of the matrix models the invariant subgroup was found as the further deformation from the $(2, 2)$ model and the subgroup was considered within the invariant subgroup of the $(2, 2)$ model. In fact, if we investigate the intersection of the 192 elements of the invariant subgroup D_4 for the $(2, 2)$ model and the 24 elements of the invariant subgroup $A_2 \times (A_1)^2$, we find only 8 elements containing the commuting elements

$$s_3s_4s_3, \quad s_1s_3s_4s_5s_4s_3s_1, \quad s_2s_3s_4s_5s_4s_3s_2. \quad (4.10)$$

Hence, the invariant subgroup is reduced to $(A_1)^3$. Alternatively, in (2.19) we have seen that the invariant subgroup of the $(1, 1, 1, 1)$ model without rank deformations is enhanced to $A_2 \times (A_1)^2$ accidentally.

Besides the A_1 symmetry $s_3s_4s_3$ requiring $h_1h_2 = 1$, since the actions of the extra two generators $s_1s_3s_4s_5s_4s_3s_1$ and $s_2s_3s_4s_5s_4s_3s_2$ are given by

$$\begin{aligned} s_1s_3s_4s_5s_4s_3s_1 : (h_1, h_2, e_1, e_3, e_5; \alpha) &\mapsto \left(h_1, h_2, \frac{e_1e_3e_5}{h_1h_2}, \frac{h_1h_2}{e_5}, \frac{h_1h_2}{e_3}; \frac{e_3e_5}{h_1h_2}\alpha \right), \\ s_2s_3s_4s_5s_4s_3s_2 : (h_1, h_2, e_1, e_3, e_5; \alpha) &\mapsto \left(h_1, h_2, \frac{h_1h_2}{e_3}, \frac{h_1h_2}{e_1}, \frac{e_1e_3e_5}{h_1h_2}; \alpha \right) \end{aligned} \quad (4.11)$$

the conditions $e_1 = e_3^{-1} = e_5$ are further required. Hence, the subspace enjoying the $(A_1)^3$ symmetry is

$$\{(\bar{h}_1, \bar{h}_2, e_1, e_3, e_5, e_7) = (h^{-1}e, he^{-1}, e^{-1}, e, e^{-1}, e)\}. \quad (4.12)$$

As previously, we can use representatives of the cosets to map the parameter for the $(1, 1, 1, 1)$ model without rank deformations into other parameters. Since the order of the $A_2 \times (A_1)^2$ invariant subgroup is 24, we have $1920/24 = 80$ cosets and hence 80 equivalent parameters for the same model.

Since it was known that the $(2, 2)$ model and the $(1, 1, 1, 1)$ model are connected through rank deformations as studied carefully in [30], it is natural to expect that they are also connected in the parameter space. Intending to understand better “the moduli space of the M2-branes”, we concentrate on the subspace (4.12) and study the invariant subgroup at each point. We find that the symmetry enhances at certain linear subspaces as depicted in figure 5.

We can identify the one-dimensional subspace (4.8) spanned by h as the M_I deformation space of the $(2, 2)$ model and the two-dimensional subspace (4.12) as the (M_I, M_{II}) deformation space because of the correspondence of the symmetries. Furthermore, if we take the fact into account that the $(2, 2)$ model with $(M_I, M_{II}) = (k/2, k/2)$ is equal to the $(1, 1, 1, 1)$ model without rank deformations as explained above (2.13), we can tentatively identify the correspondence between parameters of the curve and parameters of the $(2, 2)$ model as

$$h = e^{2\pi i(M_I - k)}, \quad e = e^{2\pi i M_{II}}. \quad (4.13)$$

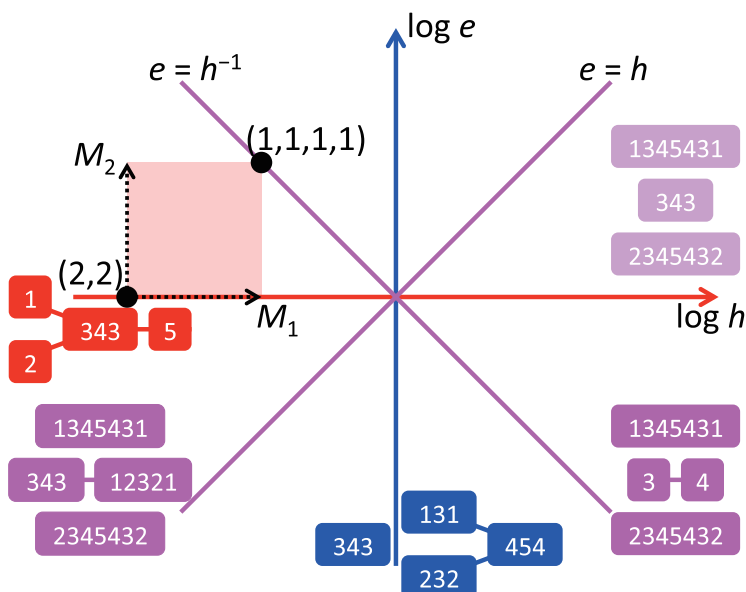


Figure 5. The patterns of the symmetry breaking in the $(\log h, \log e)$ plane (4.12).

The result of the rank deformations in [30] seems consistent with our current analysis. To really understand the rank deformations, however, we need some further clarifications which is beyond the scope of the present work [49].

Finally let us turn to the $(2, 1)$ model. There are 24 elements containing

$$s_1, \quad s_2, \quad s_3 s_4 s_3, \tag{4.14}$$

which leave (4.4) invariant. These three elements are part of (4.5), so the invariant subgroup for the $(2, 1)$ model is A_3 , which again accords with the analysis on the matrix model side.

5 Degenerate curve

Let us turn to the E_7 del Pezzo curve. Classically, the E_7 del Pezzo curve is realized as a linear combination of

$$Q^m P^n, \quad m = -2, -1, 0, 1, 2, \quad n = -1, 0, 1. \tag{5.1}$$

See figure 6 for the Newton polygon. This realization of the E_7 del Pezzo curve appeared in [50] following the proposal of utilizing the degenerate genus in [51]. Note that the coefficients of these operators are not all independent, otherwise the number of inner points in the Newton polygon indicating that the genus is three. To reduce the genus to one, classically we require the curve to be singular at $(Q, P) = (0, h_1)$ and $(Q, P) = (\infty, h_2)$. Since the singular point of an algebraic curve $H(Q, P) = 0$ at (Q_0, P_0) is defined by

$$H(Q_0, P_0) = 0, \quad \frac{\partial H}{\partial Q}(Q_0, P_0) = 0, \quad \frac{\partial H}{\partial P}(Q_0, P_0) = 0, \tag{5.2}$$

the requirements we have imposed become the conditions that the quadratic polynomials of P at Q^2 and Q^1 respectively have a double root and a single root at h_2 and those at Q^{-2}

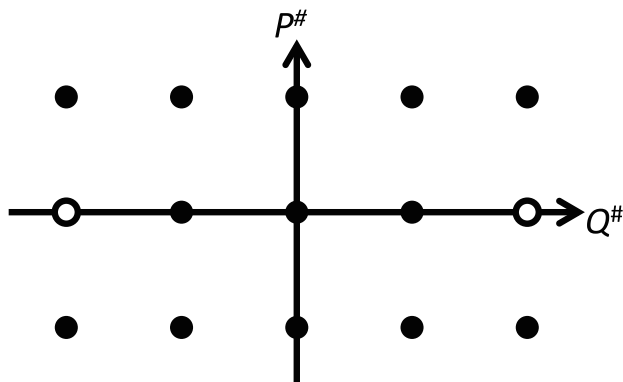


Figure 6. The Newton polygon of the E_7 del Pezzo curve.

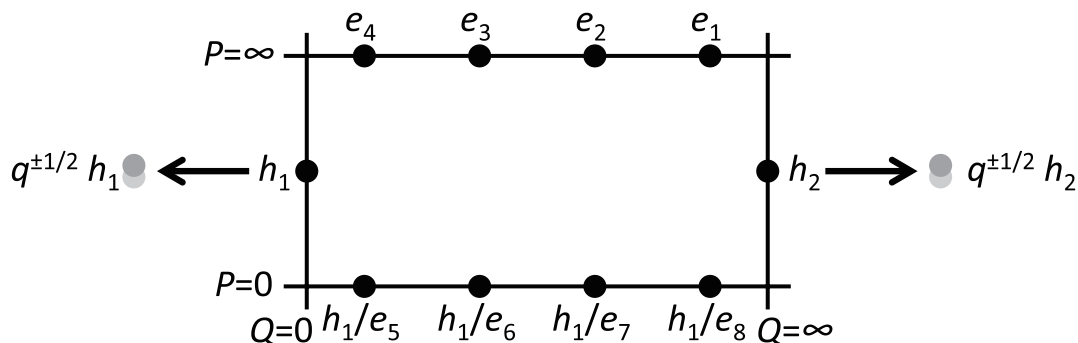


Figure 7. The asymptotic values of the E_7 del Pezzo curve.

and Q^{-1} respectively have a double root and a single root at h_1 . For other asymptotical values, we set

$$(e_1, \infty), (e_2, \infty), (e_3, \infty), (e_4, \infty), (h_1 e_5^{-1}, 0), (h_1 e_6^{-1}, 0), (h_1 e_7^{-1}, 0), (h_1 e_8^{-1}, 0). \quad (5.3)$$

See figure 7 for the asymptotical values. Again from the Vieta's formulas on products of roots, the parameters satisfy

$$\prod_{i=1}^8 e_i = h_1^2 h_2^2. \quad (5.4)$$

Then we find that if we define s_1 to be the exchange of the two singular asymptotical points

$$s_1 : h_1 \leftrightarrow h_2, \quad (5.5)$$

generated by the canonical transformation

$$Q' = (P - h_1)^{-1} Q (P - h_2), \quad P' = P, \quad (5.6)$$

along with

$$\begin{aligned} s_2 : e_4 \leftrightarrow e_3, & & s_3 : e_3 \leftrightarrow e_2, & & s_4 : e_2 \leftrightarrow e_1, & & s_5 : e_1 \leftrightarrow h_1 e_5^{-1}, \\ s_6 : h_1 e_5^{-1} \leftrightarrow h_1 e_6^{-1}, & & s_7 : h_1 e_6^{-1} \leftrightarrow h_1 e_7^{-1}, & & & & \end{aligned} \quad (5.7)$$

these actions generate the whole E_7 Weyl group whose Dynkin diagram is given in figure 8.

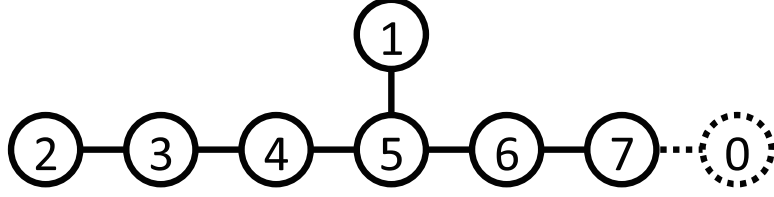


Figure 8. The Dynkin diagram of the E_7 algebra.

Now let us turn to quantum curves. Our working hypothesis for the condition of the degeneracy (5.2) for the quantum curves is that the relative coefficients are determined so that the quantum deformation of the transformation (5.6) is again the symmetry of the curve. Then, for the transformation in the positive or negative quadratic terms to work we need to split the asymptotic double roots h_1 and h_2 respectively into $q^{\pm\frac{1}{2}}h_1$ and $q^{\pm\frac{1}{2}}h_2$ and consider the curve specified by

$$\begin{aligned}
\widehat{H}/\alpha &= \widehat{Q}^2(\widehat{P}-q^{-\frac{1}{2}}h_2)(\widehat{P}-q^{\frac{1}{2}}h_2)\widehat{P}^{-1} \\
&\quad - \widehat{Q}(\widehat{P}-q^{-\frac{1}{2}}h_2)((e_1+e_2+e_3+e_4)\widehat{P}-q^{\frac{1}{2}}h_1h_2(e_5^{-1}+e_6^{-1}+e_7^{-1}+e_8^{-1}))\widehat{P}^{-1} \\
&\quad + \left\{ (e_1e_2+e_1e_3+e_1e_4+e_2e_3+e_2e_4+e_3e_4)\widehat{P}+E/\alpha \right. \\
&\quad \quad \left. + h_1^2h_2^2(e_5^{-1}e_6^{-1}+e_5^{-1}e_7^{-1}+e_5^{-1}e_8^{-1}+e_6^{-1}e_7^{-1}+e_6^{-1}e_8^{-1}+e_7^{-1}e_8^{-1})\widehat{P}^{-1} \right\} \\
&\quad - e_1e_2e_3e_4Q^{-1}(P-q^{\frac{1}{2}}h_1)((e_1^{-1}+e_2^{-1}+e_3^{-1}+e_4^{-1})\widehat{P}-q^{-\frac{1}{2}}(e_5+e_6+e_7+e_8))\widehat{P}^{-1} \\
&\quad + e_1e_2e_3e_4\widehat{Q}^{-2}(\widehat{P}-q^{\frac{1}{2}}h_1)(\widehat{P}-q^{-\frac{1}{2}}h_1)\widehat{P}^{-1}. \tag{5.8}
\end{aligned}$$

Instead we can display the same curve by listing each order of \widehat{P} , as

$$\begin{aligned}
\widehat{H}/\alpha &= (\widehat{Q}-e_1)(\widehat{Q}-e_2)(\widehat{Q}-e_3)(\widehat{Q}-e_4)\widehat{Q}^{-2}\widehat{P} \\
&\quad + \left\{ -(q^{\frac{1}{2}}+q^{-\frac{1}{2}})h_2\widehat{Q}^2+h_2[q^{-\frac{1}{2}}(e_1+e_2+e_3+e_4)+q^{\frac{1}{2}}h_1(e_5^{-1}+e_6^{-1}+e_7^{-1}+e_8^{-1})]\widehat{Q} \right. \\
&\quad \quad \left. + E/\alpha+e_1e_2e_3e_4[q^{-\frac{1}{2}}(e_5+e_6+e_7+e_8)+q^{\frac{1}{2}}h_1(e_1^{-1}+e_2^{-1}+e_3^{-1}+e_4^{-1})]\widehat{Q}^{-1} \right. \\
&\quad \quad \left. -(q^{\frac{1}{2}}+q^{-\frac{1}{2}})h_1e_1e_2e_3e_4\widehat{Q}^{-2} \right\} \\
&\quad + h_2^2(\widehat{Q}-h_1e_5^{-1})(\widehat{Q}-h_1e_6^{-1})(\widehat{Q}-h_1e_7^{-1})(\widehat{Q}-h_1e_8^{-1})\widehat{Q}^{-2}\widehat{P}^{-1}. \tag{5.9}
\end{aligned}$$

Let us turn to the Weyl symmetry of the curve. For s_1 , we consider the similarity transformation

$$\widehat{Q}' = (\widehat{P}-q^{\frac{1}{2}}h_1)^{-1}\widehat{Q}(\widehat{P}-q^{-\frac{1}{2}}h_2), \quad \widehat{P}' = \widehat{P}, \tag{5.10}$$

which also implies

$$\widehat{Q}'^2 = (q\widehat{P}-q^{\frac{1}{2}}h_1)^{-1}(\widehat{P}-q^{\frac{1}{2}}h_1)^{-1}\widehat{Q}^2(\widehat{P}-q^{-\frac{1}{2}}h_2)(q^{-1}\widehat{P}-q^{-\frac{1}{2}}h_2). \tag{5.11}$$

After combining with the rescaling

$$\widehat{Q}'' = q^{-1}\widehat{Q}', \quad \widehat{P}'' = \widehat{P}' \tag{5.12}$$

we find that s_1 transforms the parameters as

$$s_1 : (h_1, h_2, e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8; \alpha) \mapsto (q^{-2}h_2, q^2h_1, e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8; \alpha) \quad (5.13)$$

The other transformations are parallel to the previous studies in the D_5 case.

As in the D_5 case, if we use the degrees of freedom of the rescaling of \widehat{Q} and \widehat{P} (3.6) to fix the gauge

$$e_4 = e_8 = 1, \quad (5.14)$$

and drop e_7 by using the constraint

$$e_7 = \frac{h_1^2 h_2^2}{e_1 e_2 e_3 e_5 e_6}, \quad (5.15)$$

we find that the transformations are given by

$$\begin{aligned} s_1 &: (h_1, h_2, e_1, e_2, e_3, e_5, e_6; \alpha) \mapsto \left(\frac{h_2}{q^2}, q^2 h_1, e_1, e_2, e_3, e_5, e_6; \alpha \right) \\ s_2 &: (h_1, h_2, e_1, e_2, e_3, e_5, e_6; \alpha) \mapsto \left(\frac{h_1}{e_3}, \frac{h_2}{e_3}, \frac{e_1}{e_3}, \frac{e_2}{e_3}, \frac{1}{e_3}, e_5, e_6; e_3^3 \alpha \right), \\ s_3 &: (h_1, h_2, e_1, e_2, e_3, e_5, e_6; \alpha) \mapsto (h_1, h_2, e_1, e_3, e_2, e_5, e_6; \alpha), \\ s_4 &: (h_1, h_2, e_1, e_2, e_3, e_5, e_6; \alpha) \mapsto (h_1, h_2, e_2, e_1, e_3, e_5, e_6; \alpha), \\ s_5 &: (h_1, h_2, e_1, e_2, e_3, e_5, e_6; \alpha) \mapsto \left(h_1, \frac{qh_1 h_2}{e_1 e_5}, \frac{qh_1}{e_5}, e_2, e_3, \frac{qh_1}{e_1}, e_6; \frac{e_1 e_5}{qh_1} \alpha \right), \\ s_6 &: (h_1, h_2, e_1, e_2, e_3, e_5, e_6; \alpha) \mapsto (h_1, h_2, e_1, e_2, e_3, e_6, e_5; \alpha) \\ s_7 &: (h_1, h_2, e_1, e_2, e_3, e_5, e_6; \alpha) \mapsto \left(h_1, h_2, e_1, e_2, e_3, e_5, \frac{h_1^2 h_2^2}{e_1 e_2 e_3 e_5 e_6}; \alpha \right). \end{aligned} \quad (5.16)$$

Note again that, if we introduce \bar{h}_1 and \bar{h}_2 as in (3.22),

$$\bar{h}_1 = qh_1, \quad \bar{h}_2 = q^{-1}h_2, \quad (5.17)$$

we can absorb the quantum deformation parameter q completely in (5.16).

As in the D_5 case, we find that the Weyl symmetries are realized as standard Weyl actions. In this case, root vectors and fundamental weight vectors are

$$\begin{aligned} \alpha_1 &= (1, -1, 0, 0, 0, 0, 0), & \omega_1 &= (2, 1, 0, 0, 0, 2, 2), \\ \alpha_2 &= (-1, -1, -1, -1, -2, 0, 0), & \omega_2 &= (0, 0, -1, -1, -1, 1, 1), \\ \alpha_3 &= (0, 0, 0, -1, 1, 0, 0), & \omega_3 &= (1, 1, -1, -1, 0, 2, 2), \\ \alpha_4 &= (0, 0, -1, 1, 0, 0, 0), & \omega_4 &= (2, 2, -1, 0, 0, 3, 3), \\ \alpha_5 &= (0, 1, 1, 0, 0, 1, 0), & \omega_5 &= (3, 3, 0, 0, 0, 4, 4), \\ \alpha_6 &= (0, 0, 0, 0, 0, -1, 1), & \omega_6 &= (2, 2, 0, 0, 0, 2, 3), \\ \alpha_7 &= (0, 0, 0, 0, 0, 0, -1), & \omega_7 &= (1, 1, 0, 0, 0, 1, 1), \end{aligned} \quad (5.18)$$

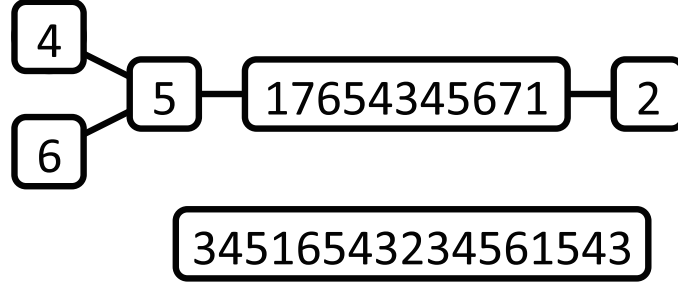


Figure 9. The Dynkin diagram of the $D_5 \times A_1$ subalgebra within the original E_7 algebra, which preserves the $(2, 1, 2, 1)$ model without rank deformations.

where we have represented the parameters by $(\log \bar{h}_1, \log \bar{h}_2, \log e_1, \log e_2, \log e_3, \log e_5, \log e_6)$ as in the D_5 case.

In the study of the $(2, 1, 2, 1)$ model without rank deformations, it was found that the model falls into the class of the E_7 del Pezzo curve with the symmetry broken to $D_5 \times A_1$. As in the previous case we can identify the parameters of the $(2, 1, 2, 1)$ model

$$\begin{aligned}
\hat{H} &= (\hat{Q}^{\frac{1}{2}} + \hat{Q}^{-\frac{1}{2}})^2 (\hat{P}^{\frac{1}{2}} + \hat{P}^{-\frac{1}{2}}) (\hat{Q}^{\frac{1}{2}} + \hat{Q}^{-\frac{1}{2}})^2 (\hat{P}^{\frac{1}{2}} + \hat{P}^{-\frac{1}{2}}) \\
&= q^{-\frac{1}{2}} \hat{Q}^2 \hat{P} + 2(1 + q^{-\frac{1}{2}}) \hat{Q} \hat{P} + (q^{\frac{1}{2}} + 4 + q^{-\frac{1}{2}}) \hat{P} + 2(q^{\frac{1}{2}} + 1) \hat{Q}^{-1} \hat{P} + q^{\frac{1}{2}} \hat{Q}^{-2} \hat{P} \\
&\quad + (q^{\frac{1}{2}} + q^{-\frac{1}{2}}) \hat{Q}^2 + 2(q^{\frac{1}{2}} + 2 + q^{-\frac{1}{2}}) \hat{Q} + 2(q^{\frac{1}{2}} + 4 + q^{-\frac{1}{2}}) \\
&\quad\quad + 2(q^{\frac{1}{2}} + 2 + q^{-\frac{1}{2}}) \hat{Q}^{-1} + (q^{\frac{1}{2}} + q^{-\frac{1}{2}}) \hat{Q}^{-2} \\
&\quad + q^{\frac{1}{2}} \hat{Q}^2 \hat{P}^{-1} + 2(q^{\frac{1}{2}} + 1) \hat{Q} \hat{P}^{-1} + (q^{\frac{1}{2}} + 4 + q^{-\frac{1}{2}}) \hat{P}^{-1} \\
&\quad\quad + 2(1 + q^{-\frac{1}{2}}) \hat{Q}^{-1} \hat{P}^{-1} + q^{-\frac{1}{2}} \hat{Q}^{-2} \hat{P}^{-1}, \tag{5.19}
\end{aligned}$$

as

$$(h_1, h_2, e_1, e_2, e_3, e_5, e_6, e_7)^{(2,1,2,1)} = (1, q, q^{\frac{1}{2}}, q^{\frac{1}{2}}, 1, q^{\frac{1}{2}}, q^{\frac{1}{2}}, 1). \tag{5.20}$$

Then we can ask again which elements of the E_7 Weyl group generated by $s_1, s_2, s_3, \dots, s_7$ preserve this parameter and what group those elements form. We again generate all elements of the E_7 Weyl group by using a computer, then we find that the answer is 3840 elements generated by

$$s_4, \quad s_5, \quad s_6, \quad s_1 s_7 s_6 s_5 s_4 s_3 s_4 s_5 s_6 s_7 s_1, \quad s_2, \tag{5.21}$$

along with the commuting element $s_3 s_4 s_5 s_1 s_6 s_5 s_4 s_3 s_2 s_3 s_4 s_5 s_6 s_1 s_5 s_4 s_3$. See figure 9 for the Dynkin diagram for these elements. The former five elements generate the Weyl group of D_5 while the latter is A_1 . The result matches again with the study from the superconformal Chern-Simons matrix model.

Since the E_7 Weyl group has 2903040 elements in total, it is not easy to generate the elements without strategies. We shall explain how we have generated them in appendix A.

6 Conclusion

In this paper we have studied the symmetry breaking of the quantum curves. We first find that the symmetry for the classical algebraic curve given in the Weyl group of the exceptional algebra is promoted to that for the quantum curve in our definition. We then fix the values of the parameters to those of the superconformal Chern-Simons matrix models and study the symmetry breaking patterns for these values.

The main motivation of our work is to reproduce the symmetry breaking patterns we found previously for the superconformal Chern-Simons matrix models. We find that we can reproduce the results perfectly and at the same time detect an unexpected symmetry enhancement. After the reproduction, we continue to study the breaking patterns of other values, partially expecting that this gives “the moduli space of the M2-branes”. We find that the moduli space does not change from the classical limit $q \rightarrow 1$ and conjecture that this is the case as well for other curves. It is interesting to find that the whole moduli space of the M2-branes enjoys a generalization of the Weyl group of the exceptional algebra.

Previously the correspondence between the superconformal Chern-Simons matrix models and the algebraic curves was mainly studied from the analytical viewpoint. We believe that, along with [31], our computation in terms of the symmetry breaking of the Weyl group has opened up a new avenue to understand better the correspondence. We shall list several further directions.

First, our method is applicable to many generalizations and the study in these directions may lead to many clarifications of the correspondence. As well as the grand canonical ensembles of the matrix models constructed from other $\mathcal{N} = 4$ $U(N)^{r+1}$ superconformal Chern-Simons theories for the \hat{A}_r quiver diagram with (2.1), those constructed from the $\mathcal{N} = 3$ theories with the same field contents ((2.1) with $\{s_a\}_{a=1}^{r+1}$ being arbitrary integers) also take the form of the Fredholm determinant (2.6). In these cases \hat{H} is given as a product of $2 \cosh(\hat{q} - s_a \hat{p})/2$, and hence is expanded by finite terms of $\hat{Q}^m \hat{P}^n$ with $(m, n) \in \mathbb{Z}^2$. In general, the curve is a higher-genus generalization of the del Pezzo curves we have considered in this paper. It is interesting to study the higher-genus generalizations and compare with the results in [52].

Secondly, so far in this paper we have mainly restricted our studies to the case without rank deformations. To fully understand the moduli space of the M2-branes we need to proceed to the rank deformations. For the rank deformations, however, we encounter several interesting new points to be clarified which we would like to study more carefully and report in our future work [49].

Thirdly, the rank deformations of the $(2, 1, 2, 1)$ model have another interesting aspect. If we consider the $(2, 1, 2, 1)$ model with general choice of six ranks, the whole moduli space would also include the $(4, 2)$ model, whose curve is not degenerate according to the definition of degenerate quantum curve in section 5. It would be interesting to study the rank deformations of the $(2, 1, 2, 1)$ model and identify the special class which keeps the degeneracy of the E_7 curve.

Fourthly, it would be nice to establish the quantum notion of degenerate curves for general type of singularities. In the case of the classical E_7 curve, there is a pair of

singularities at $Q = 0$ and $Q = \infty$, which are exchanged by the symmetry s_1 (5.6). Hence we can define the degeneracy condition for the quantum curve by requiring that s_1 , now the similarity transformation of the quantum operators (5.10), remains to be the symmetry of the quantum curves. This strategy does not work for general un-paired singularities. Nevertheless, we notice that, if we introduce the q -derivative as $d_x^q f(x) = (f(qx) - f(x))/(qx - x)$, the E_7 curve (5.8) satisfies $H(0, q^{-1/2}h_2) = d_Q^q H(0, q^{-1/2}h_2) = d_P^q H(0, q^{-1/2}h_2) = 0$ at the singularity on $Q = 0$ (and the same condition for $Q = \infty$). This is analogous to the degeneracy condition for the classical curves (5.2), hence might be a good starting point. Once we know the definition of quantum degeneracy, it would be possible to study the models corresponding to the E_6 curve or the E_8 curve as well, which can be obtained by starting from the higher genus rectangular curve and tuning the parameters so that the curve is singular and all but a single genus are degenerate, similar to the case of the E_7 curve [50].

Fifthly, the relation to the q -deformed Painlevé equation is another interesting direction. It was shown in [53] that the grand canonical ensemble of the ABJM matrix model satisfies the q -deformed Painlevé equation. On the other hand, the Weyl symmetries we have studied in this paper are also known to be the symmetries of the q -deformed Painlevé equations [48]. Hence, our studies of the relation between the superconformal Chern-Simons matrix models and the Weyl symmetries should be connected via the q -deformed Painlevé equations. The integrable structure of the superconformal Chern-Simons matrix models would be made clearer from the studies. Recent developments between matrix models and Painlevé equations [54, 55] may be helpful in studying the generalizations.

Finally, it is also interesting to study the connection to five-dimensional gauge theories. Historically the Newton polygons and the dual asymptotic values played an important role in studying five-dimensional gauge theories constructed from the (p, q) 5-brane webs [56]. The superconformal Chern-Simons matrix models studied in this paper can also be realized by the type IIB brane setups consisting of D3-branes spanning between NS5-branes and $(1, k)$ 5-branes, where the rank of the gauge group is given by the number of D3-branes on each segment. We hope to clarify the relation between the five-dimensional gauge theories and the three-dimensional gauge theories.

A Weyl group

In this appendix we comment on the computation of the Weyl group of the exceptional algebra, when the order, the number of the elements, is large such as E_6, E_7, E_8 . See table 4 for the order of the Weyl group for the exceptional algebra $W(E_n)$ where we denote the Weyl group of the Lie algebra G by $W(G)$.

A first trial would be to collect all of different elements of the Weyl group from the simple roots by multiplying them one by one to see whether the transformation is new or not. Namely, we prepare a set of elements obtained so far and try to generate a new element of the Weyl group by multiplying the simple roots to those in the original set. If the transformation is new, we add the new element to the original set. Otherwise we forget it and proceed to the next multiplication. This is valid for the D_5 Weyl group with only

$E_3 = A_2 \times A_1$	$E_4 = A_4$	$E_5 = D_5$	E_6	E_7	E_8
12	192	1920	51840	2903040	696729600

Table 4. The order of the Weyl group for each exceptional algebra $\#W(E_n)$.

$A_4/(A_2 \times A_1)$	D_5/A_4	E_6/D_5	E_7/E_6	E_8/E_7
16	10	27	56	240

Table 5. The number of cosets of two Weyl groups $\#(W(E_n)/W(E_{n-1}))$.

1920 elements in total, though for the E_7 Weyl group with 2903040 elements this method is very time-consuming. The article [57] is helpful for us to improve this situation.

The main time-consuming process would be to judge whether the generated element is new or not. In fact, for the final step of the E_7 Weyl group, we need to multiply 7 elements to the existing $2903040 - 1$ elements in the set to find out only one, the longest element, by comparing the generated $(2903040 - 1) \times 7$ elements with each of the $2903040 - 1$ elements. To improve the situation, it is nice to consider cosets. Namely, if we need to study the Weyl group of G , we can start from the Weyl group of a subalgebra $H(\subset G)$ and consider only the cosets in $W(G)/W(H)$. When we generate the E_6 Weyl group from the D_5 Weyl group the number of the cosets $W(E_6)/W(D_5)$ is 27, while when we generate the E_7 Weyl group from the E_6 Weyl group the number of the cosets $W(E_7)/W(E_6)$ is 56. See table 5 for the number of the cosets of two Weyl groups for the subsequent exceptional algebras. Then, each time we find out a new coset in $W(G)/W(H)$, as a bonus, we obtain $\#W(H)$ new elements in the Weyl group $W(G)$. We have followed this process to find out the E_7 Weyl group. Namely, starting from the D_5 Weyl group generated by s_1, s_4, s_5, s_6, s_7 , to find out the E_6 Weyl group generated by $s_1, s_3, s_4, s_5, s_6, s_7$ all we have to do is to find out the 27 cosets. (The numbering of the simple roots is the same as in figure 8.) Each time we find out one element in the coset, we can generate 1920 elements in the original E_6 Weyl group. See table 6 for the representatives of the cosets $W(E_6)/W(D_5)$. After generating the E_6 Weyl group, to find out the E_7 Weyl group, we only need to find out the 56 cosets, which can be done similarly. See table 7 for the representatives of the cosets $W(E_7)/W(E_6)$. This process saves a lot of time, though the computation still takes several hours on a decent laptop computer.

We can further improve the computation. Namely, although we have reduced the computation by considering the coset, in the final step for E_7 we still need to generate $(56 - 1) \times 7$ elements and compare them with the existing $2903040 - 51840$ elements. It is nice if we can compare the generated elements only with those in the cosets $W(G)/W(H)$.

The main idea in [57] is to restrict the consideration to a Weyl chamber of $W(H)$. Namely, by changing our transformations into the standard Weyl actions, we can choose the representatives of the cosets in the Weyl chamber. Since we only consider those representatives, if the generated element is not located in the Weyl chamber we can simply discard it without comparing with the existing representatives and proceed to the next element. In this sense we do not need to compare with the whole set of the Weyl group

step	representatives
0	1
1	s_3
2	s_3s_4
3	$s_3s_4s_5$
4	$s_3s_4s_5s_1, s_3s_4s_5s_6$
5	$s_3s_4s_5s_1s_6, s_3s_4s_5s_6s_7$
6	$s_3s_4s_5s_1s_6s_5, s_3s_4s_5s_1s_6s_7$
7	$s_3s_4s_5s_1s_6s_5s_4, s_3s_4s_5s_1s_6s_5s_7$
8	$s_3s_4s_5s_1s_6s_5s_4s_3, s_3s_4s_5s_1s_6s_5s_4s_7, s_3s_4s_5s_1s_6s_5s_7s_6$
9	$s_3s_4s_5s_1s_6s_5s_4s_3s_7, s_3s_4s_5s_1s_6s_5s_4s_7s_6$
10	$s_3s_4s_5s_1s_6s_5s_4s_3s_7s_6, s_3s_4s_5s_1s_6s_5s_4s_7s_6s_5$
11	$s_3s_4s_5s_1s_6s_5s_4s_3s_7s_6s_5, s_3s_4s_5s_1s_6s_5s_4s_7s_6s_5s_1$
12	$s_3s_4s_5s_1s_6s_5s_4s_3s_7s_6s_5s_1, s_3s_4s_5s_1s_6s_5s_4s_3s_7s_6s_5s_4$
13	$s_3s_4s_5s_1s_6s_5s_4s_3s_7s_6s_5s_1s_4$
14	$s_3s_4s_5s_1s_6s_5s_4s_3s_7s_6s_5s_1s_4s_5$
15	$s_3s_4s_5s_1s_6s_5s_4s_3s_7s_6s_5s_1s_4s_5s_6$
16	$s_3s_4s_5s_1s_6s_5s_4s_3s_7s_6s_5s_1s_4s_5s_6s_7$

Table 6. The 27 cosets of the D_5 Weyl group (generated by s_1, s_4, s_5, s_6, s_7) in the E_6 Weyl group (generated by $s_1, s_3, s_4, s_5, s_6, s_7$).

but instead only with the representatives. After the simplifications are taken into account, finally we can generate the E_7 Weyl group within a few minutes.

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step	representatives
1	2
2	23
3	234
4	2345
5	23451, 23456
6	234516, 234567
7	2345165, 2345167
8	23451654, 23451657
9	234516543, 234516547, 234516576
10	2345165432, 2345165437, 2345165476
11	23451654327, 23451654376, 23451654765
12	234516543276, 234516543765, 234516547651
13	2345165432765, 2345165437651, 2345165437654
14	23451654327651, 23451654327654, 23451654376514
15	234516543276514, 234516543276543, 234516543765145
16	2345165432765143, 2345165432765145, 2345165437651456
17	23451654327651435, 23451654327651456, 23451654376514567
18	234516543276514354, 234516543276514356, 234516543276514567
19	2345165432765143546, 2345165432765143567
20	23451654327651435465, 23451654327651435467
21	234516543276514354651, 234516543276514354657
22	2345165432765143546517, 2345165432765143546576
23	23451654327651435465176
24	234516543276514354651765
25	2345165432765143546517654
26	23451654327651435465176543
27	234516543276514354651765432

Table 7. The 56 cosets of the E_6 Weyl group (generated by $s_1, s_3, s_4, s_5, s_6, s_7$) in the E_7 Weyl group (generated by $s_1, s_2, s_3, s_4, s_5, s_6, s_7$). We abbreviate $s_1, s_2, s_3, s_4, s_5, s_6, s_7$ as 1, 2, 3, 4, 5, 6, 7 for simplicity.

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