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On the existence of an L∞ structure for the super-Virasoro algebra

Ralph Blumenhagen and Max Brinkmann

Max-Planck-Institut für Physik (Werner-Heisenberg-Institut), Föhringer Ring 6, 80805 München, Germany

E-mail: blumenha@mppmu.mpg.de, mbrinkm@mpp.mpg.de

Abstract: The appearance of L∞ structures for supersymmetric symmetry algebras in two-dimensional conformal field theories is investigated. Looking at the simplest concrete example of the $\mathcal{N}=1$ super-Virasoro algebra in detail, we investigate whether an extension to a super-L∞ algebra is sufficient to capture all appearing signs.

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1 Introduction

L_∞ algebras were introduced to string theory in 1992 when the algebraic structure of bosonic closed string field theory as proposed by Zwiebach [1] was found to encode an L_∞ algebra. These algebras are also called strongly homotopy (sh) Lie algebras in the mathematical literature and are generalizations of Lie algebras. The 2-bracket of an L_∞ algebra may violate the Jacobi identity, however this failure is captured by a homotopical term which defines a 3-bracket. The word "strongly" refers to the fact that this pattern continues, i.e. a generalized Jacobi identity for the n-bracket holds up to homotopical terms defining an $(n+1)$-bracket.

The connection between gauge theories and L_∞ algebras was made by [2] using a geometric formulation of the general BV-formalism. More recently a very tractable procedure for identifying the L_∞ algebra of a gauge theory was formulated [3]. There it was explicitly shown how the L_∞ algebra incorporates the gauge variations, the gauge algebra and the dynamics of the theory. In this way, not only the L_∞ structure of Yang-Mills and Chern-Simons theory but also the L_∞ structure of Double Field Theory and Einstein gravity was derived. In the context of 2D conformal field theories (CFTs), extended non-linear classical conformal algebras, so called classical $\mathcal{W}$ algebras, were also found to exhibit an L_∞ structure [4]. Quantum $\mathcal{W}$ algebras were investigated as well [5], and a quantum L_∞ algebra was proposed. Note that $\mathcal{W}$ algebras are not gauge symmetries but infinitely many global symmetries that can be considered to be holographically dual to higher spin gauge symmetries.

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Following these results, the logic was turned around by the L_∞ bootstrap approach [6]. It was proposed that non-commutative gauge theories can be constructed iteratively by fixing the free theory and the gauge group, then imposing the L_∞ relations order by order while requiring the commutative limit to flow to an ordinary gauge theory. This means that the L_∞ algebra is taken as the guiding principle for obtaining new theories. With this method, derivative and curvature corrections to the equations of motion can be bootstrapped in an algebraic way. As it requires quite simple mathematics in the actual computation (although the equations get exponentially complicated in higher orders), this is an extremely powerful way to obtain general deformations of gauge theories. The question of uniqueness of the bootstrap was subsequently addressed and led to a theorem of equivalence between the two well established concepts of L_∞ quasi-isomorphisms and Seiberg-Witten maps [7].

All of this was developed using purely bosonic theories. A natural question in this context is if there exists an extension of the formalism to supersymmetric theories. While L_∞ algebras and superalgebras have appeared in the same title before [8], the theory considered there does not contain physical fermions. It might be expected that we need to extend L_∞ algebras to super-L_∞ algebras. In this note we investigate this question in the context of supersymmetric extensions of 2D conformal symmetries. To make the first steps we consider a very simple prototype example, namely the $\mathcal{N} = 1$ super-Virasoro algebra.

This letter is structured as follows: first we will briefly introduce the necessary concepts and notations of (super-)L_∞ and super-Virasoro algebras. We then naively follow the calculations presented in [4] to derive L_∞ -like maps from the symmetry variations of the chiral fields. Once the maps are defined, we can discuss the graded vector space that the algebra should be defined on. We will see that with a super-extension of the L_∞ algebra it is possible to capture the extra $\mathbb{Z}_2$ grading present in super-$\mathcal{W}$ algebras. As the main result, we find that the L_∞ formalism for bosonic theories proposed in [3] is also valid for supersymmetric symmetry algebras by simply extending the L_∞ to a super-L_∞ algebra.

2 Preliminaries

In this section we present a brief introduction to L_∞ algebras and the super-Virasoro algebra. We will keep it short and only focus on the material that is compulsory for the following. For further information on L_∞ algebras we refer to [3, 9, 10] and for more on CFTs and the super-Virasoro algebra see e.g. [11–13] or [14] for a textbook introduction.

2.1 Basics of L_∞ algebras

There are multiple equivalent definitions of L_∞ algebras [3, 9, 10]. We will use the so-called ℓ-picture. While this picture has the disadvantage of many more sign factors compared to the b-picture, they allow us to give the first maps a nice interpretation.

Let X be a $\mathbb{Z}$-graded vector space

$$X = \bigoplus_{n\in\mathbb{Z}} X_n \,, \tag{2.1}$$

where elements of X_n have degree n. We will use $x_1, x_2, \ldots$ to represent arbitrary vectors of fixed degree in X, $x_i \in X_n$. Here i is just an arbitrary label to distinguish variables while n is the degree $\deg(x_i) = n$. The degree will appear in sign factors, and we will omit the 'deg' label $(-1)^{\deg(x_1)\deg(x_2)} \equiv (-1)^{x_1 x_2}$ where confusion is impossible.

Let $\{\ell_n : X^{\otimes n} \to X\}_{n>0}$ be graded anti-commutative multilinear maps of degree $\deg(\ell_n) = n-2$ acting on elements of homogeneous degree $x_i \in X$:

$$\begin{aligned} \ell_n(x_1, \ldots, x_i, x_j, \ldots, x_n) &= (-1)^{1+x_i x_j} \ell_n(x_1, \ldots, x_j, x_i, \ldots, x_n)\,, \\ \deg(\ell_n(x_1, \ldots, x_n)) &= \sum_{i=1}^{n} \deg(x_i) + n - 2\,. \end{aligned} \tag{2.2}$$

Then $(X, \{\ell_n\})$ is an L_∞ algebra if the maps satisfy the following defining relations for all $n > 0$:

$$\begin{aligned} \mathcal{J}_n := \sum_{\substack{l,k\geq 0\\ l+k=n}} \sum_{\sigma \in \mathrm{Unsh}(k,l)} & (-1)^{kl}\,(-1)^{\sigma}\,\epsilon(\sigma; x) \\ & \times \ell_{l+1}(\ell_k(x_{\sigma(1)}, \ldots, x_{\sigma(k)}), x_{\sigma(k+1)}, \ldots, x_{\sigma(n)}) \;=\; 0 \end{aligned} \tag{2.3}$$

where $(-1)^{\sigma}$ is the sign of the permutation and gives a positive (negative) sign for even (uneven) permutation σ. The sum runs over all unshuffles $\sigma \in \mathrm{Unsh}(k,l)$. Unshuffles correspond to inequivalent partitions of a set with n elements into two subsets of length k and l. Unshuffles are equivalent if the subsets contain the same elements respectively, independent of their order. For example the partition $(1|23)$ is equivalent to $(1|32)$ and inequivalent to $(2|13)$. The Koszul sign $\epsilon(\sigma; x)$ is defined as the sign needed to rearrange a graded commutative algebra $\Lambda(x_1, \ldots, x_n)$:

$$x_{\sigma(1)} \wedge \ldots \wedge x_{\sigma(n)} = \epsilon(\sigma; x) x_1 \wedge \ldots \wedge x_n\,. \tag{2.4}$$

The L_∞ defining relations can be written schematically as

$$0 = \ell_1\ell_1\,, \quad 0 = \ell_1\ell_2 - \ell_2\ell_1\,, \quad 0 = \ell_1\ell_3 + \ell_2\ell_2 + \ell_3\ell_1\,, \quad \ldots\,. \tag{2.5}$$

Explicitly, the L_∞ relations for $n = 1, 2, 3$ read:

$$\begin{aligned} 0 = \mathcal{J}_1 &= \ell_1(\ell_1(x))\,, \\ 0 = \mathcal{J}_2 &= \ell_1(\ell_2(x_1, x_2)) - \ell_2(\ell_1(x_1), x_2)) - (-1)^{x_1}\ell_2(x_1, \ell_1(x_2))\,, \\ 0 = \mathcal{J}_3 &= \ell_1(\ell_3(x_1, x_2, x_3)) + \ell_3(\ell_1(x_1), x_2, x_3) \\ &\quad + (-1)^{x_1}\ell_3(x_1, \ell_1(x_2), x_3) + (-1)^{x_1+x_2}\ell_3(x_1, x_2, \ell_1(x_3)) \\ &\quad + \ell_2(\ell_2(x_1, x_2), x_3) + (-1)^{x_1(x_2+x_3)}\ell_2(\ell_2(x_2, x_3), x_1) \\ &\quad + (-1)^{x_3(x_1+x_2)}\ell_2(\ell_2(x_3, x_1), x_2)\,. \end{aligned} \tag{2.6}$$

The first two equations show that ℓ_1 is nilpotent and acts as a derivation of ℓ_2. The third equation contains two more important features. The first two lines show that ℓ_1 fails to be a derivation of ℓ_3. However the last two lines are the (graded) Jacobi identity for ℓ_2 whose failure to hold is controlled by the first two lines. Mathematically, ℓ_3 is a chain

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homotopy. Thus ℓ_2 satisfies the Jacobi identity up to homotopy exact terms. This pattern continues for higher ℓ-brackets: for n inputs a generalized Jacobi identity of the schematic form $\sum_{i,j>1} \pm\ell_i\ell_j$ will hold up to $\ell_1\ell_n \pm \ell_n\ell_1$ terms [9].

A super-L_∞ algebra is an L_∞ algebra defined on a super vector space [15]. This means one adds an underlying $\mathbb{Z}_2$ grading to the algebra by allowing the vector spaces of homogeneous degree to be a direct sum of a Grassmann even (bosonic) and a Grassmann odd (fermionic) vector space:

$$X = \bigoplus_{n\in\mathbb{Z}} X_n\,, \qquad X_n = X_n^{\mathrm{bos}} \oplus X_n^{\mathrm{ferm}}\,. \tag{2.7}$$

Equivalently, the super-L_∞ algebra is defined on a $\mathbb{Z}\times\mathbb{Z}_2$ graded (Grassmann even) vector space X such that an element of homogeneous degree $x_i \in X$ is equipped with two labels (n_i, s_i) with $n_i \in \mathbb{Z}$ and $s_i \in \{0,1\}$,

$$X = \bigoplus_{\substack{n\in\mathbb{Z}\\ s\in\{0,1\}}} X_{n,s}\,. \tag{2.8}$$

The multilinear maps are then anti-commutative with respect to both n and s,

$$\ell_n(x_1,\dots,x_i,x_j,\dots,x_n) = (-1)^{1+n_in_j+s_is_j}\,\ell_n(x_1,\dots,x_j,x_i,\dots,x_n)\,. \tag{2.9}$$

The degree of x_i is therefore to be given by two integers (n_i, s_i) instead of only one number. The L_∞ defining relations are also modified by taking the Koszul sign with respect to both n- and s-grading. This ensures that both descriptions are equivalent. In the following, we will refer to the fist description as an L_∞ algebra over a super vector space and to the second description as a super-L_∞ algebra, keeping in mind that both are equivalent.

2.2 L_∞ algebras and extended conformal symmetries

We will follow the dictionary for L_∞ algebras in gauged theories developed in [3]. When applying this to bosonic extensions of the conformal symmetry, the vector space consists of only two terms

$$X = \underset{\substack{\cup\\ \epsilon}}{X_0} \oplus \underset{\substack{\cup\\ W}}{X_{-1}} \tag{2.10}$$

with chiral fields $W(z)$ and symmetry parameters $\epsilon(z)$. The fields transform under infinitesimal transformations as

$$\begin{aligned}\delta_\epsilon W &= \sum_{n\geq 0}\frac{1}{n!}(-1)^{\frac{n(n-1)}{2}}\ell_{n+1}(\epsilon, W^n)\\ &= \ell_1(\epsilon) + \ell_2(\epsilon, W) - \frac{1}{2}\ell_3(\epsilon, W, W) - \frac{1}{3!}\ell_4(\epsilon, W, W, W) + \dots\end{aligned} \tag{2.11}$$

and the symmetry algebra is given in terms of the L_∞ algebra as

$$\begin{aligned}\left[\delta_{\epsilon_1}, \delta_{\epsilon_2}\right] W &= \delta_{-\mathcal{C}(\epsilon_1,\epsilon_2,W)} W\,,\\ \mathcal{C}(\epsilon_1,\epsilon_2,W) &= \sum_{n\geq 0}\frac{1}{n!}\ell_{n+2}(\epsilon_1,\epsilon_2,W^n)\\ &= \ell_2(\epsilon_1,\epsilon_2) + \ell_3(\epsilon_1,\epsilon_2,W) - \frac{1}{2}\ell_4(\epsilon_1,\epsilon_2,W^2) - \dots\,.\end{aligned} \tag{2.12}$$

2.3 The super-Virasoro algebra

The super-Virasoro algebra we consider is the $\mathcal{N} = 1$ Neveu-Schwarz supersymmetric extension of the Virasoro algebra. The Virasoro algebra contains the generators of infinitesimal conformal transformations of $2D$ conformal field theories and is thus related to the energy-momentum tensor.

The super-Virasoro algebra consists of the energy-momentum tensor $L(z)$ with conformal dimension $(h = 2)$ and a fermionic primary field $G(z)$ of dimension $(h = \frac{3}{2})$ as its superpartner. The mode expansions of the fields are given by

$$L(z) = \sum_{m\in\mathbb{Z}} L_m\, z^{-m-2}\,, \quad G(z) = \sum_{r\in\mathbb{Z}+\frac{1}{2}} G_r\, z^{-r-\frac{3}{2}} \tag{2.13}$$

with the modes of G being half-integers. Note that G as well as G_r are fermionic and thus anticommuting. The super-Virasoro algebra is then given as

$$\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}\,,\\
[L_m, G_r] &= \left(\frac{m}{2} - r\right) G_{m+r}\,,\\
\{G_r, G_s\} &= 2L_{r+s} + \frac{c}{3}\left(r^2 - \frac{1}{4}\right)\delta_{r+s,0}\,,
\end{aligned} \tag{2.14}$$

with the anticommutator appearing because G_r are Grassmann odd [12]. Note that in the here considered classical case, these are actually super Poisson-brackets.

In the following, for computational convenience we will employ the OPE representation of the algebra [13]:

$$\begin{aligned}
L(z)\,L(w) &= \frac{c/2}{(z-w)^4} + \frac{2L(w)}{(z-w)^2} + \frac{\partial_w L(w)}{z-w} + \ldots\,,\\
L(z)\,G(w) &= \frac{\frac{3}{2}G(w)}{(z-w)^2} + \frac{\partial_w G(w)}{z-w} + \ldots\,,\\
G(z)\,G(w) &= \frac{\frac{2}{3}c}{(z-w)^3} + \frac{2L(w)}{z-w} + \ldots\,.
\end{aligned} \tag{2.15}$$

3 The super-Virasoro algebra as an L_∞ algebra

The L_∞ algebra for field theories was motivated by bosonic closed string field theory and was shown to hold also for bosonic gauge theories [3] and bosonic symmetry algebras [4]. It is a priori not clear whether the super-Virasoro algebra also features a (super-)L_∞ structure, and if so whether a super-extension of the L_∞ algebra is strictly necessary. In this section we proceed by carefully deriving L_∞ products from the symmetry variations of the chiral fields and their closed algebra assuming that they fit into the relations (2.11) and (2.12). This provides sufficient information to fix the degree of the vector spaces.

3.1 The L_∞ products

The variations of the fields $X \in \{L, G\}$ with respect to the symmetry generated by the modes of $Y \in \{L, G\}$ are given by

$$\delta_{\epsilon^Y} X(w) = \frac{1}{2\pi i} \oint_{\mathcal{C}(0)} dz\, \epsilon^Y(z)[Y(z), X(w)]_\pm\,. \tag{3.1}$$

Here $[X, Y]_\pm$ denotes the anticommutator if both X and Y are fermionic, and the commutator in all other cases. Using radial ordering one can rewrite this as an integral over the OPE around w

$$\delta_{\epsilon^Y} X(w) = \frac{1}{2\pi i} \oint_{\mathcal{C}(w)} dz\, \epsilon^Y(z)\Big(Y(z)X(w)\Big)_{\mathrm{OPE}} \tag{3.2}$$

where the contour integral extracts the singular part of the OPE. In this way we find the four symmetry variations

$$\begin{aligned}
\delta_{\epsilon^L} L &= \frac{c}{12}\partial^3\epsilon^L + 2L\,\partial\epsilon^L + \epsilon^L\,\partial L\,,\\
\delta_{\epsilon^G} G &= \frac{c}{3}\partial^2\epsilon^G + 2\epsilon^G\,L\,,\\
\delta_{\epsilon^L} G &= \frac{3}{2}G\,\partial\epsilon^L + \partial G\,\epsilon^L\,,\\
\delta_{\epsilon^G} L &= -\frac{3}{2}\partial\epsilon^G\,G - \frac{1}{2}\epsilon^G\,\partial G\,.
\end{aligned} \tag{3.3}$$

For the following it is important to note that, as the variation of a bosonic (fermionic) field should also be bosonic (fermionic), ϵ^L must be Grassmann even and ϵ^G Grassmann odd. One must therefore be careful with extra signs when computing the symmetry algebra and the L_∞ relations.

Using (2.11), one can directly read off the nontrivial products with one symmetry parameter

$$\begin{aligned}
\ell_1^L(\epsilon^L) &= \frac{c}{12}\partial^3\epsilon^L\,,\\
\ell_1^G(\epsilon^G) &= \frac{c}{3}\partial^2\epsilon^G\,,\\
\ell_2^L(\epsilon^L, L) &= 2\partial\epsilon^L L + \epsilon^L\partial L\,,\\
\ell_2^G(\epsilon^L, G) &= \frac{3}{2}\partial\epsilon^L G + \epsilon^L\partial G\,,\\
\ell_2^G(\epsilon^G, L) &= 2\epsilon^G L\,,\\
\ell_2^L(\epsilon^G, G) &= -\frac{3}{2}\partial\epsilon^G G - \frac{1}{2}\epsilon^G\partial G\,.
\end{aligned} \tag{3.4}$$

The superscript of the products indicates the type of field the products map to. Next we have 6 equations to calculate the commutator of two symmetry transformations

$$\big[\delta_{\epsilon^X}, \delta_{\tilde{\epsilon}^Y}\big] Z = \delta_{-\epsilon_{(XY)}} Z\,, \qquad X, Y, Z \in \{L, G\}\,. \tag{3.5}$$

However we expect the algebra to close, i.e. the commutator of two transformations is again a transformation with some new parameter. This means it should not matter on which

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field we act, the commutator should give the same new parameter. This is indeed the case and one obtains

$$\begin{aligned}
\epsilon^L_{(LL)} &= \epsilon^L \partial \tilde{\epsilon}^L - \partial \epsilon^L \tilde{\epsilon}^L \,, \\
\epsilon^L_{(GG)} &= -2\epsilon^G \tilde{\epsilon}^G \,, \\
\epsilon^G_{(LG)} &= \epsilon^L \partial \epsilon^G - \frac{1}{2} \partial \epsilon^L \epsilon^G \,,
\end{aligned} \tag{3.6}$$

where the superscript denotes the kind of transformation the resulting symmetry parameter is associated to. As expected, both the commutator of two bosonic as well as of two fermionic transformations give a bosonic transformation, while the commutator of a bosonic and a fermionic transformation gives a fermionic transformation. Using (2.12) we can directly identify the L_∞ products:

$$\begin{aligned}
\ell_2^{\epsilon^L}(\epsilon^L, \tilde{\epsilon}^L) &= \epsilon^L \, \partial \tilde{\epsilon}^L - \partial \epsilon^L \, \tilde{\epsilon}^L \,, \\
\ell_2^{\epsilon^L}(\epsilon^G, \tilde{\epsilon}^G) &= -2\epsilon^G \, \tilde{\epsilon}^G \,, \\
\ell_2^{\epsilon^G}(\epsilon^L, \epsilon^G) &= \epsilon^L \, \partial \epsilon^G - \frac{1}{2} \partial \epsilon^L \, \epsilon^G \,.
\end{aligned} \tag{3.7}$$

We have now read off all the L_∞ products of the super-Virasoro algebra. In the next section we will find out what vector space actually underlies them.

3.2 The graded vector space

Having two distinct fields L, G as well as two symmetry parameters ϵ^L, ϵ^G in the theory, we can treat the space as effectively four-dimensional. Assuming the degrees are mapped correctly, we have an equation for the relative degree of the vector spaces for each map defined in (3.4) and (3.7):

$$\ell_n^X(A, B, \ldots) \neq 0 \quad \Rightarrow \quad |X| = |A| + |B| + \ldots + n - 2 \,. \tag{3.8}$$

With this data we can fix the graded vector space. Consider for example the following set of equations:

$$\begin{aligned}
\ell_1^L(\epsilon^L): &\quad |L| = |\epsilon^L| - 1 \,, \\
\ell_1^G(\epsilon^G): &\quad |G| = |\epsilon^G| - 1 \,, \\
\ell_2^L(\epsilon^L, L): &\quad |L| = |\epsilon^L| + |L| \,, \\
\ell_2^{\epsilon^L}(\epsilon^G, \tilde{\epsilon}^G): &\quad |\epsilon^L| = |\epsilon^G| \times 2 \,.
\end{aligned} \tag{3.9}$$

From the first and third equation, clearly the bosonic components have the expected degree $|L| = -1$ and $|\epsilon^L| = 0$. The last equation fixes $|\epsilon^G| = 0$ and with the second equation the final degree is also fixed. All in all we end up with

$$|L| = |G| = -1 \,, \quad |\epsilon^L| = |\epsilon^G| = 0 \,. \tag{3.10}$$

Thus we can write the total vector space as a graded vector space such that the homogeneous graded subspaces are again a direct sum of a bosonic and a fermionic space

$$X = X_0 \oplus X_{-1} \,, \quad X_{i \in \{0,-1\}} = X_i^{\mathrm{bos}} \oplus X_i^{\mathrm{ferm}} \,, \tag{3.11}$$

with elements

$$
\begin{aligned}
L &\in X_{-1}^{\text{bos}}, & G &\in X_{-1}^{\text{ferm}}, \\
\epsilon^L &\in X_0^{\text{bos}}, & \epsilon^G &\in X_0^{\text{ferm}}.
\end{aligned} \tag{3.12}
$$

This is exactly of the form (2.7). We can also explicitly compute the symmetry properties of the ℓ_2-products

$$
\begin{aligned}
\ell_2^{\epsilon^L}(\epsilon^L, \tilde{\epsilon}^L) &= \epsilon^L\,\partial\tilde{\epsilon}^L - \partial\epsilon^L\,\tilde{\epsilon}^L = -\left(\tilde{\epsilon}^L\,\partial\epsilon^L - \partial\tilde{\epsilon}^L\,\epsilon^L\right) = -\ell_2^{\epsilon^L}(\tilde{\epsilon}^L, \epsilon^L)\,, \\
\ell_2^{\epsilon^L}(\epsilon^G, \tilde{\epsilon}^G) &= -2\epsilon^G\,\tilde{\epsilon}^G = 2\tilde{\epsilon}^G\,\epsilon^G = -\ell_2^{\epsilon^L}(\tilde{\epsilon}^G, \epsilon^G)\,,
\end{aligned} \tag{3.13}
$$

where in the second line we have used that ϵ^G is Grassmann odd. Both ℓ_2 products are antisymmetric which is just the symmetry property already contained in the L_∞ algebra without introducing any further $\mathbb{Z}_2$ grading to the maps. This indicates that we have in fact found an L_∞ algebra over a super vector space that describes the super-Virasoro algebra, assuming of course the L_∞ relations are found to hold.

We will proceed with checking this in the next section.

3.3 Checking the L_∞ relations

Since we only have fields with degree 0 and -1 and no additional $\mathbb{Z}_2$ grading on the maps, we expect the usual L_∞ relations (2.3) to hold. There is no X_{-2} space, so the $\mathcal{J}_1$ relation $\ell_1\ell_1 = 0$ is trivially satisfied. As per the argument in [4], we only need to consider $\mathcal{J}_2$ and $\mathcal{J}_3$ with 2 or 3 gauge parameters as inputs.[1]

The computations are lengthy but elementary. We start with $\mathcal{J}_2$, which we need to compute on combinations of two symmetry parameters.

$$
\begin{aligned}
\mathcal{J}_2(\epsilon_1^L, \epsilon_2^L) &= \ell_1\left(\epsilon_1^L\partial\epsilon_2^L - \partial\epsilon_1^L\epsilon_2^L\right) - \ell_2\left(\frac{c}{12}\partial^3\epsilon_1^L, \epsilon_2^L\right) - \ell_2\left(\epsilon_1^L, \frac{c}{12}\partial^3\epsilon_2^L\right) \\
&= \frac{c}{12}\left(\partial^3\left(\epsilon_1^L\partial\epsilon_2^L - \partial\epsilon_1^L\epsilon_2^L\right) + 2\partial^3\epsilon_1^L\partial\epsilon_2^L + \epsilon_2^L\partial^4\epsilon_1^L - 2\partial^3\epsilon_2^L\partial\epsilon_1^L - \epsilon_1^L\partial^4\epsilon_2^L\right) \\
&= 0.
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{J}_2(\epsilon_1^G, \epsilon_2^G) &= \ell_1^L\left(\frac{1}{2}\epsilon_1^G\epsilon_2^G\right) - \ell_2^L\left(\frac{c}{12}\partial^2\epsilon_1^G, \epsilon_2^G\right) - \ell_2^L\left(\epsilon_1^G, \frac{c}{12}\partial^2\epsilon_2^G\right) \\
&= \frac{c}{24}\left(\partial^3\left(\epsilon_1^G\epsilon_2^G\right) + \partial\epsilon_2^G\partial^2\epsilon_1^G + \epsilon_2^G\partial^3\epsilon_1^G - \partial\epsilon_1^G\partial^2\epsilon_2^G - \epsilon_1^G\partial^3\epsilon_2^G\right) \\
&= 0\,.
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{J}_2(\epsilon^L, \epsilon^G) &= \ell_1^G\left(\epsilon^L\partial\epsilon^G - \frac{1}{2}\epsilon^G\partial\epsilon^L\right) - \ell_2^G\left(\frac{c}{12}\partial^3\epsilon^L, \epsilon^G\right) - \ell_2^G\left(\epsilon^L, \frac{c}{12}\partial^2\epsilon^G\right) \\
&= \frac{c}{12}\left(\partial^2\left(\epsilon^L\partial\epsilon^G - \frac{1}{2}\epsilon^G\partial\epsilon^L\right) + \frac{1}{2}\epsilon^G\partial^3\epsilon^L - \frac{3}{2}\partial^2\epsilon^G\partial\epsilon^L - \epsilon^L\partial^3\epsilon^G\right) \\
&= 0\,.
\end{aligned}
$$

[1]Although the L_∞ relations are defined on the homogeneous graded spaces X_i with $X_i = X_i^{\text{bos}} \oplus X_i^{\text{ferm}}$, one can choose a basis with homogeneously Grassmann even and odd vectors. Then with linearity of the L_∞ products it suffices to show that the L_∞ relations hold for any combination of Grassmann even/odd elements.

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With this all $\mathcal{J}_2$ relations are satisfied. In the second equation we have used that ϵ^G fields anticommute. Since there are no ℓ_3 products, the $\mathcal{J}_3$ relations reduce to graded Jacobi identities

$$\ell_2(\ell_2(x,y),z)+(-1)^{x(y+z)}\ell_2(\ell_2(y,z),x)+(-1)^{(x+y)z}\ell_2(\ell_2(z,x),y). \tag{3.14}$$

Next we compute the $\mathcal{J}_3$ relations with two gauge parameters.

$$\begin{aligned}
\mathcal{J}_3(\epsilon_1^L,\epsilon_2^L,L) &= 2L\partial\left(\epsilon_1^L\partial\epsilon_2^L-\partial\epsilon_1^L\epsilon_2^L\right)+\left(\epsilon_1^L\partial\epsilon_2^L-\partial\epsilon_1^L\epsilon_2^L\right)\partial L\\
&\quad+2\left(2L\partial\epsilon_1^L+\epsilon_1^L\partial L\right)\partial\epsilon_2^L+\epsilon_2^L\partial\left(2L\partial\epsilon_1^L+\epsilon_1^L\partial L\right)\\
&\quad-2\left(2L\partial\epsilon_2^L+\epsilon_2^L\partial L\right)\partial\epsilon_1^L-\epsilon_1^L\partial\left(2L\partial\epsilon_2^L+\epsilon_2^L\partial L\right)\\
&=0.\\
\mathcal{J}_3(\epsilon_1^L,\epsilon_2^L,G) &= \frac{3}{2}G\partial\left(\epsilon_1^L\partial\epsilon_2^L-\partial\epsilon_1^L\epsilon_2^L\right)+\left(\epsilon_1^L\partial\epsilon_2^L-\partial\epsilon_1^L\epsilon_2^L\right)\partial G\\
&\quad+\frac{3}{2}\left(\frac{3}{2}G\partial\epsilon_1^L+\epsilon_1^L\partial G\right)\partial\epsilon_2^L+\epsilon_2^L\partial\left(\frac{3}{2}G\partial\epsilon_1^L+\epsilon_1^L\partial G\right)\\
&\quad-\frac{3}{2}\left(\frac{3}{2}G\partial\epsilon_2^L+\epsilon_2^L\partial G\right)\partial\epsilon_1^L-\epsilon_1^L\partial\left(\frac{3}{2}G\partial\epsilon_2^L+\epsilon_2^L\partial G\right)\\
&=0.\\
\mathcal{J}_3(\epsilon_1^G,\epsilon_2^G,L) &= L\partial\left(\epsilon_1^G\epsilon_2^G\right)+\frac{1}{2}\epsilon_1^G\epsilon_2^G\partial L-\frac{3}{4}\partial\epsilon_1^G\epsilon_2^G L\\
&\quad-\frac{1}{4}\epsilon_1^G\partial\left(\epsilon_2^G L\right)+\frac{3}{4}\partial\epsilon_2^G\epsilon_1^G L+\frac{1}{4}\epsilon_2^G\partial\left(\epsilon_1^G L\right)\\
&=0.\\
\mathcal{J}_3(\epsilon_1^G,\epsilon_2^G,G) &= \frac{3}{4}G\partial\left(\epsilon_1^G\epsilon_2^G\right)+\frac{1}{2}\epsilon_1^G\epsilon_2^G\partial G\\
&\quad-\frac{1}{4}\epsilon_1^G\left(3\partial\epsilon_2^G G+\epsilon_2^G\partial G\right)+\frac{1}{4}\epsilon_2^G\left(3\partial\epsilon_1^G G+\epsilon_1^G\partial G\right)\\
&=0.\\
\mathcal{J}_3(\epsilon^L,\epsilon^G,L) &= \frac{1}{2}\left(\epsilon^L\partial\epsilon^G-\frac{1}{2}\epsilon^G\partial\epsilon^L\right)L-\frac{3}{4}\epsilon^G L\partial\epsilon^L\\
&\quad-\frac{1}{2}\epsilon^L\partial\left(\epsilon^G L\right)+\epsilon^G\left(L\partial\epsilon^L+\frac{1}{2}\epsilon^L\partial L\right)\\
&=0.\\
\mathcal{J}_3(\epsilon^L,\epsilon^G,G) &= \frac{3}{2}\partial\left(\epsilon^L\partial\epsilon^G-\frac{1}{2}\epsilon^G\partial\epsilon^L\right)G+\frac{1}{2}\left(\epsilon^L\partial\epsilon^G-\frac{1}{2}\epsilon^G\partial\epsilon^L\right)\partial G\\
&\quad-\left(3\partial\epsilon^G G+\epsilon^G\partial G\right)\partial\epsilon^L-\epsilon^L\partial\left(\frac{3}{2}\partial\epsilon^G G+\frac{1}{2}\epsilon^G\partial G\right)\\
&\quad+\frac{3}{2}\partial\epsilon^G\left(\frac{3}{2}G\partial\epsilon^L+\epsilon^L\partial G\right)+\frac{1}{2}\epsilon^G\partial\left(\frac{3}{2}G\partial\epsilon^L+\epsilon^L\partial G\right)\\
&=0.
\end{aligned}$$

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Finally there are four $\mathcal{J}_3$ relations with three symmetry parameters left to consider.

$$\begin{aligned}
\mathcal{J}_3(\epsilon_1^L,\epsilon_2^L,\epsilon_3^L) &= \left(\epsilon_1^L\partial\epsilon_2^L - \partial\epsilon_1^L\epsilon_2^L\right)\partial\epsilon_3^L - \partial\left(\epsilon_1^L\partial\epsilon_2^L - \partial\epsilon_1^L\epsilon_2^L\right)\epsilon_3^L + \text{cyclic}\\
&= 0.\\
\mathcal{J}_3(\epsilon_1^L,\epsilon_2^L,\epsilon^G) &= \left(\epsilon_1^L\partial\epsilon_2^L - \partial\epsilon_1^L\epsilon_2^L\right)\partial\epsilon^G - \frac{1}{2}\epsilon^G\partial\left(\epsilon_1^L\partial\epsilon_2^L - \partial\epsilon_1^L\epsilon_2^L\right)\\
&\quad - \epsilon_1^L\partial\left(\epsilon_2^L\partial\epsilon^G - \frac{1}{2}\epsilon^G\partial\epsilon_2^L\right) + \frac{1}{2}\left(\epsilon_2^L\partial\epsilon^G - \frac{1}{2}\epsilon^G\partial\epsilon_2^L\right)\partial\epsilon_1^L\\
&\quad + \epsilon_2^L\partial\left(\epsilon_1^L\partial\epsilon^G - \frac{1}{2}\epsilon^G\partial\epsilon_1^L\right) - \frac{1}{2}\left(\epsilon_1^L\partial\epsilon^G - \frac{1}{2}\epsilon^G\partial\epsilon_1^L\right)\partial\epsilon_2^L\\
&= 0.\\
\mathcal{J}_3(\epsilon^L,\epsilon_1^G,\epsilon_2^G) &= \frac{1}{2}\left(\epsilon^L\partial\epsilon_1^G - \frac{1}{2}\epsilon_1^G\partial\epsilon^L\right)\epsilon_2^G - \frac{1}{2}\left(\epsilon^L\partial\epsilon_2^G - \frac{1}{2}\epsilon_2^G\partial\epsilon^L\right)\epsilon_1^G\\
&\quad + \frac{1}{2}\epsilon_1^G\epsilon_2^G\partial\epsilon^L - \partial\left(\frac{1}{2}\epsilon_1^G\epsilon_2^G\right)\epsilon^L\\
&= 0.\\
\mathcal{J}_3(\epsilon_1^G,\epsilon_2^G,\epsilon_3^G) &= \frac{1}{2}\epsilon_1^G\epsilon_2^G\partial\epsilon_3^G - \frac{1}{4}\partial\left(\epsilon_1^G\epsilon_2^G\right)\epsilon_3^G + \text{ cyclic}\\
&= 0.
\end{aligned}$$

3.4 The dual super-L_∞

As we have seen in section 2.1, there are two equivalent forms of the super-L_∞ algebra. Since we have found a simple nontrivial example for such an algebra, it is an interesting exercise to work out the dual description in detail. In the previous sections we have found the algebra to be a regular L_∞ algebra over a super vector space. The dual picture is simply obtained by exchanging the internal $\mathbb{Z}_2$ Grassmann parity for the external $\mathbb{Z}_2$ s-grading and modifying the symmetry of the products as well as the L_∞ relations accordingly. In practice this means that the following, previously anticommuting products become commuting according to (2.9):

$$\begin{aligned}
\ell_2^{\epsilon^L}(\epsilon_1^G,\epsilon_2^G) &= \ell_2^{\epsilon^L}(\epsilon_2^G,\epsilon_1^G) = -2\epsilon_1^G\,\epsilon_2^G\,,\\
\ell_2^L(\epsilon^G,G) &= \ell_2^L(G,\epsilon^G) = -\frac{3}{2}\partial\epsilon^G G - \frac{1}{2}\epsilon^G\partial G\,,
\end{aligned} \tag{3.15}$$

with $(n,s)=(0,1)$ and $(-1,1)$ the degree of the now Grassmann even ϵ^G and G. The degree of the bosonic fields ϵ^L and L is $(0,0)$ and $(-1,0)$. Here it also becomes clear how the two descriptions are equivalent. In the previous discussion both ϵ^G and G were Grassmann odd, and their ℓ_2 maps were anti-commutative. By turning the Grassmann parity into an additional $\mathbb{Z}_2$ grading that only affects the symmetry of the L_∞ maps, the right hand side becomes commuting. The L_∞ maps get an additional sign factor when exchanging two previously Grassmann odd inputs, which turns the left hand side commutative as well.

Similarly, the super-L_∞ relations differ from the L_∞ relations if they involve two or more previously fermionic fields. There are six relations that must be checked again to verify the algebra, $\mathcal{J}_2(\epsilon_1^G,\epsilon_2^G)$, $\mathcal{J}_3(\epsilon^G,\epsilon^L,G)$, and $\mathcal{J}_3(\epsilon_1^G,\epsilon_2^G,x)$ with $x\in\{L,G,\epsilon^L,\epsilon^G\}$. Note that the relations now include sign factors for the additional $\mathbb{Z}_2$ grading, i.e.

$$\mathcal{J}_2=\ell_1(\ell_2(x_1,x_2))-\ell_2(\ell_1(x_1),x_2)+(-1)^{n_1n_2+s_1s_2}\ell_2(\ell_1(x_2),x_1) \tag{3.16}$$

and analogously for the signs in $\mathcal{J}_3$. In exchange one can freely commute all fields during calculations.

$$\begin{aligned}
\mathcal{J}_2(\epsilon_1^G,\epsilon_2^G)&=\frac{c}{2}\partial^2\epsilon_1^G\partial\epsilon_2^G+\frac{c}{2}\partial\epsilon_1^G\partial^2\epsilon_2^G+\frac{c}{6}\partial^3\epsilon_1^G\epsilon_2^G+\frac{c}{6}\epsilon_1^G\partial^3\epsilon_2^G-\frac{c}{6}\partial^3\left(\epsilon_1^G\epsilon_2^G\right)\\
&=0.\\
\mathcal{J}_3(\epsilon^G,\epsilon^L,G)&=-\frac{3}{2}\partial\left(\frac{1}{2}\partial\epsilon^L\epsilon^G-\epsilon^L\partial\epsilon^G\right)G-\frac{1}{2}\left(\frac{1}{2}\partial\epsilon^L\epsilon^G-\epsilon^L\partial\epsilon^G\right)\partial G\\
&\quad-\partial\epsilon^L\left(3\partial\epsilon^GG+\epsilon^G\partial G\right)-\frac{1}{2}\epsilon^L\partial\left(3\partial\epsilon^GG+\epsilon^G\partial G\right)\\
&\quad+\frac{3}{2}\partial\epsilon^G\left(\frac{3}{2}\partial\epsilon^LG+\epsilon^L\partial G\right)+\frac{1}{2}\epsilon^G\partial\left(\frac{3}{2}\partial\epsilon^LG+\epsilon^L\partial G\right)\\
&=0.\\
\mathcal{J}_3(\epsilon_1^G,\epsilon_2^G,L)&=-4\partial\left(\epsilon_1^G\epsilon_2^G\right)L-2\epsilon_1^G\epsilon_2^G\partial L+3\partial\epsilon_2^G\epsilon_1^GL\\
&\quad+\epsilon_2^G\partial\left(\epsilon_1^GL\right)+3\partial\epsilon_1^G\epsilon_2^GL+\epsilon_1^G\partial\left(\epsilon_2^GL\right)\\
&=0.\\
\mathcal{J}_3(\epsilon_1^G,\epsilon_2^G,G)&=-3\partial\left(\epsilon_1^G\epsilon_2^G\right)G-2\epsilon_1^G\epsilon_2^G\partial G+3\partial\epsilon_1^G\epsilon_2^GG\\
&\quad+\epsilon_1^G\epsilon_2^G\partial G+3\epsilon_1^G\partial\epsilon_2^GG+\epsilon_1^G\epsilon_2^G\partial G\\
&=0.\\
\mathcal{J}_3(\epsilon_1^G,\epsilon_2^G,\epsilon^L)&=-2\left(\epsilon_1^G\epsilon_2^G\partial\epsilon^L-\partial\left(\epsilon_1^G\epsilon_2^G\right)\epsilon^L\right)-2\left(\epsilon^L\partial\epsilon_1^G-\frac{1}{2}\partial\epsilon^L\epsilon_1^G\right)\epsilon_2^G\\
&\quad-2\left(\epsilon^L\partial\epsilon_2^G-\frac{1}{2}\partial\epsilon^L\epsilon_1^G\right)\epsilon_1^G\\
&=0.\\
\mathcal{J}_3(\epsilon_1^G,\epsilon_2^G,\epsilon_3^G)&=\partial\epsilon_1^G\epsilon_2^G\epsilon_3^G+\epsilon_1^G\partial\epsilon_2^G\epsilon_3^G-2\epsilon_1^G\epsilon_2^G\partial\epsilon_3^G+\text{ cyclic}\\
&=0.
\end{aligned}$$

As expected, the products that were read off from the symmetry algebra indeed satisfy both definitions of super-L_∞ algebras.

4 Conclusion

In this letter, in the context of two-dimensional conformal field theories, we explored the possibility of extending the L_∞ structure to super-$\mathcal{W}$ algebras. The question was whether such a generalization exists, and if so whether it does require the introduction of super-L_∞ algebras. As a simple example to clarify this issue we employed the super-Virasoro algebra but expect that a more involved analysis of genuine nonlinear super-$\mathcal{W}$ algebras will confirm our findings.

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We found that both the bosonic and fermionic symmetry parameters and the bosonic and fermionic fields carry the same degree in each case. The symmetry variations of the fields and their closure algebra are correctly described by an ordinary L_∞ algebra over a super vector space. We have seen how the description is equivalent to a super-L_∞ algebra with only Grassmann even fields and modified symmetries and algebra relations. This is in many ways similar to the superspace approach to supersymmetric field theories and might simplify calculations in more complicated cases.

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