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Second order transport coefficients of nonconformal relativistic fluids in various dimensions from Dp-brane

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ABSTRACT: We derive all the dynamical second order transport coefficients for Dp-brane with p from 1 to 6 within the framework of fluid/gravity correspondence in this paper. The D5 and D6-brane do not have dual relativistic fluids; D3-brane corresponds to 4-dimensional conformal relativistic fluid; D1, D2 and D4-brane separately correspond to nonconformal relativistic fluids of dimensions 2, 3 and 5. The Haack-Yarom relation only exists for Dp-branes with p larger than 2 and is also satisfied by them. We also find that the Romatschke and Kleinert-Probst relations need to be generalized in order to be valid for relativistic fluids of dimensions other than 4.

KEYWORDS: Holography and quark-gluon plasmas, AdS-CFT Correspondence, Gauge-gravity correspondence, D-branes

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1 Introduction

Relativistic hydrodynamics is a widely used tool in theoretical physics, but its second-derivative behavior is not well understood. This behavior is mostly studied in two-formalisms that are both developed in 2000s. They are the Minkowskian prescription of AdS/CFT duality [1–6] and the fluid/gravity correspondence [7–12]. Much progress has been made in calculating transport coefficients in both of these formalisms, for various kinds of relativistic fluids. We are mainly focusing on the fluids without conserved vector current in this paper.

To review what is known, we begin by writing down the most general stress-energy tensor for an uncharged, nonconformal, relativistic fluid in d dimensions. This is

$$\begin{aligned}
 T_{\mu\nu} = & \varepsilon u_\mu u_\nu + \mathfrak{p} P_{\mu\nu} - (\eta \cdot 2\sigma_{\mu\nu} + \zeta P_{\mu\nu} \nabla_\rho u^\rho) + \eta \tau_\pi \cdot 2 \left(\langle D \sigma_{\mu\nu} \rangle + \frac{\sigma_{\mu\nu} \nabla_\rho u^\rho}{d-1} \right) \\
 & + \eta \tau_\pi^* \cdot \frac{2\sigma_{\mu\nu} \nabla_\rho u^\rho}{d-1} + \kappa (R_{\langle\mu\nu\rangle} - 2u^\rho u^\sigma R_{\rho\langle\mu\nu\rangle\sigma}) + \kappa^* \cdot 2u^\rho u^\sigma R_{\rho\langle\mu\nu\rangle\sigma} \\
 & + \lambda_1 \cdot 4\sigma_{\langle\mu}{}^\rho \sigma_{\nu\rangle\rho} + \lambda_2 \cdot 2\sigma_{\langle\mu}{}^\rho \Omega_{\nu\rangle\rho} + \lambda_3 \Omega_{\langle\mu}{}^\rho \Omega_{\nu\rangle\rho} + \lambda_4 \nabla_{\langle\mu} \ln s \nabla_{\nu\rangle} \ln s \\
 & + P_{\mu\nu} \left(\zeta \tau_\Pi D(\nabla_\rho u^\rho) + \xi_1 \cdot 4\sigma_{\rho\lambda} \sigma^{\rho\lambda} + \xi_2 (\nabla_\rho u^\rho)^2 + \xi_3 \Omega_{\rho\lambda} \Omega^{\rho\lambda} \right. \\
 & \left. + \xi_4 P^{\rho\lambda} \nabla_\rho \ln s \nabla_\lambda \ln s + \xi_5 R + \xi_6 u^\rho u^\lambda R_{\rho\lambda} \right), \tag{1.1}
 \end{aligned}$$

where ε is the energy density and $\mathfrak{p}$ is the pressure. The metric $g_{\mu\nu}$ gives the Ricci and Riemann tensor $R_{\mu\nu}$ and $R_{\mu\nu\rho\sigma}$ as usual. $P_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu$ is the spatial projection tensor, and $D = u^\mu \nabla_\mu$ is the temporal derivative, and the spatial-projected traceless symmetric tensor is defined as

$$A_{\langle\mu\nu\rangle} = P_\mu^\rho P_\nu^\lambda A_{(\rho\lambda)} - \frac{1}{d-1} P_{\mu\nu} P^{\rho\lambda} A_{\rho\lambda}. \tag{1.2}$$

Matching with our convention in [13], the shear viscous tensor is defined as $\sigma_{\mu\nu} = P_\mu^\rho P_\nu^\sigma \nabla_{(\rho} u_{\sigma)} - \frac{1}{d-1} P_{\mu\nu} \nabla_\rho u^\rho$ where $\nabla_{(\rho} u_{\sigma)} = \frac{1}{2}(\nabla_\rho u_\sigma + \nabla_\sigma u_\rho)$, and the vorticity is defined $\Omega_{\mu\nu} = P_\mu^\rho P_\nu^\lambda \nabla_{[\rho} u_{\lambda]}$ with $\nabla_{[\rho} u_{\lambda]} = \frac{1}{2}(\nabla_\rho u_\lambda - \nabla_\lambda u_\rho)$.

Eq. (1.1) has 2 first order transport coefficients (the shear and bulk viscosities η and ζ) and 15 second order transport coefficients, of which 9 are coefficients of viscous tensors ($\eta, \eta\tau_\pi, \eta\tau_\pi^*, \kappa, \kappa^*, \lambda_{1,2,3,4}$) and 8 are coefficients of viscous scalars ($\zeta, \zeta\tau_\Pi, \xi_{1,2,3,4,5,6}$). A conformal relativistic fluid has only 6 transport coefficients within second order: $\eta, \eta\tau_\pi, \kappa, \lambda_{1,2,3}$. It was shown in [13] that a relativistic fluid living in curved background can have non-zero $\lambda_4, \xi_{4,5,6}$ and κ, κ^* . The other 9 second order transport coefficients i.e., $\eta\tau_\pi, \eta\tau_\pi^*, \lambda_{1,2,3}, \zeta\tau_\Pi, \xi_{1,2,3}$, appear in flat spacetime. The original fluid/gravity correspondence [7] can extract only these 9 transport coefficients.

The study of transport properties within the holographic framework begins with the research on 4d conformal relativistic fluid dual to the AdS_5 black hole that comes from trivial dimensional reduction on a near-extremal black D3-brane. The real-time prescription of AdS/CFT correspondence was proposed in [1], with which Policastro et al. were able to get the shear viscosity [2] and the sound wave dispersion relation [3] for the 4d conformal fluid. The second order transport coefficients were derived several years later in [4–7, 14]. Then the 1st and 2nd order transport coefficients of conformal relativistic fluid in other dimensions [10, 11, 15–18], and the coupling constant correction to the transport coefficients caused by higher order terms in bulk action [19–30] were also studied.

The works on nonconformal relativistic hydrodynamics can be divided into two categories, according to the gravity background of the models. The first class is using scalar field deformed AdS_5 black holes or the Chamblin-Reall background [31–36] and their results are usually numerical [31–35] or approximate [36]. Thereinto, [31–35] use the AdS/CFT calculation to study the scalar deformed AdS_5 black hole and these studies are all numerical, while [36] adopts the method proposed in [37] that the transport coefficients of

nonconformal fluid can be deduced from a higher dimensional conformal one and use the result from [11] together with the relations among the 2nd order coefficients from [38] to derive the 2nd order transport coefficients for a 5d Chamblin-Reall background to first order of $\delta = 1 - 3c_s^2$ corrections. The exact results are given in [35].

The second class of works uses instead brane or supergravity backgrounds from string/M-theory and finds analytic results. Because such backgrounds are 10- or 11-dimensional, they need to be reduced to lower-dimensional manifold in order to use either AdS/CFT calculation or fluid/gravity correspondence to calculate transport coefficients. References of this kind are [13, 16, 37, 39–51], in which [39–42, 47] are about the relativistic fluid that dual to supergravity backgrounds. And [13, 16, 37, 44–51] are about NS5-brane [44], Dp-brane [16, 37, 46–50] and compactified D4-brane [13, 45, 51]. To be more specific, [37, 44–46, 48, 50, 51] study the first order transport coefficients while [13, 16, 47, 49] investigate the second order ones, in which [16, 47, 49] calculate the shear and bulk relaxation time τ_π and τ_Π for Dp-brane and [13] derives all the 7 dynamical second order transport coefficients¹ for compactified D4-brane.

Of the studies reviewed above, only [13, 16, 34–36, 47, 49] are about the 2nd order transport coefficients of nonconformal relativistic fluids. Our knowledge about this subject is still very limited, especially for Dp-brane backgrounds. In this paper, we would like to derive all the 7 dynamical second order transport coefficients for Dp-brane with $p = 1, 2, \dots, 6$, dual to relativistic fluids of dimension $d = p + 1 = 2, 3, \dots, 7$, via fluid/gravity correspondence.

It is said that λ_3 and ξ_3 can be derived from fluid/gravity duality [13]. Since these two coefficients relate with vortical mode of fluid flow, they will only appear in backgrounds which are not isotropic in all spatial directions. This condition can be satisfied in backgrounds with angular momentum [9] or vector charge [12], for the vector perturbations from metric tensor (backgrounds with angular momentum) are involved with those from vector current (backgrounds with vector charge). So λ_3 and ξ_3 can not be extracted from branes with $SO(1, p)$ symmetry in their world-volume directions (p is the spatial dimension of the brane). Thus both of them are 0 for Dp-brane.

The last part of the Introduction is an overview of the paper.

In section 2, we show the procedure of reducing the 10d near-extremal Dp-brane background to a $(p + 2)$ -dimensional one. Then we will solve the first order perturbations in section 3, we find that the tensor perturbations of D5- and D6-brane are non-normalizable in subsection 3.1 thus don't have dual fluid. So we only need to solve the Dp-brane with $1 \leq p \leq 4$ of which the first order perturbations are done in subsection 3.2. Then we calculate the first order stress-energy tensor in section 4 and find the results agree with previous studies.

The second order perturbations are solved in section 5. First we derive the second order expanded metric, the constraint equations of the 2nd order viscous terms and the Navier-Stokes equation in subsection 5.1. Then we redo the calculation for the D3-brane

¹The classification of the second order transport coefficients into dynamical and thermodynamical was proposed in [52], according to whether a viscous term contains either $\sigma_{\mu\nu}$ or $\nabla \cdot u$ which can cause entropy increasing.

with our convention in subsection 5.2. The second order perturbations for D4-, D1- and D2-brane are solved in subsection 5.3, 5.4 and 5.5, respectively.

Section 6 will give the final result of the second order stress-energy tensor for Dp-brane. In subsection 6.1 we formulate the final results of the stress-energy tensor and the 2nd order transport coefficients for D4- eq. (6.5), D1- eq. (6.7) and D2-brane eq. (6.9) both in explicit and unified form eq. (6.11). What's more, the gauge theory language of all the transport coefficients are also given in subsection 6.1. In subsection 6.2 we talk about the relations satisfied by the 2nd order transport coefficients. The most important observation is that the Romatschke and Kleinert-Probst relations need to be generalized when the fluid is not 4-dimensional. The generalized form can be seen in eq. (6.20), eq. (6.25) and eq. (6.23). We give a test on Kanitscheider-Skenderis proposal in subsection 6.3 and calculate the dispersion relations in subsection 6.4.

Section 7 is a summary and some interesting problems for future work will also be mentioned. Appendix A is about the dimensional reduction of bulk metric.

2 Dimensional reduction from 10 to $p + 2$ dimensions

This section will deal with the reduction from 10d supergravity of Dp-brane to a $(p + 2)$ -dimensional effective bulk theory. The 10d action of Dp-brane in Einstein frame is

$$S = \frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{-G} \left[\mathcal{R} - \frac{1}{2}(\nabla_{\hat{M}}\phi)^2 - \frac{g_s^2}{2(8-p)!} e^{\frac{p-3}{2}\phi} F_{8-p}^2 \right] - \frac{1}{\kappa_{10}^2} \int d^9x \sqrt{-H} \mathcal{K} + \frac{1}{\kappa_{10}^2} \int d^9x \sqrt{-H} \frac{9-p}{2L_p} e^{\frac{3-p}{4(7-p)}\phi}. \quad (2.1)$$

The first, second and third part of the above action are separately the bulk term, Gibbons-Hawking surface term and the counter term. The 10d surface gravity is $2\kappa_{10}^2 = (2\pi)^7 g_s^2 l_s^8$ where g_s and l_s are separately the string coupling and the string length. In the bulk term, $G_{\hat{M}\hat{N}}$ is the 10d metric with $\mathcal{R}$ the corresponding Ricci scalar, ϕ is the dilaton with zero VEV and F_{8-p} is the magnetic dual of the Ramond-Ramond field minimally coupled with Dp-brane. The 10d Dp-brane background under the near horizon limit that solves the bulk term of eq. (2.1) is

$$ds^2 = \left(\frac{r}{L_p} \right)^{\frac{(7-p)^2}{8}} \left(-f(r)dt^2 + (dx^i)^2 \right) + \left(\frac{L_p}{r} \right)^{\frac{(p+1)(7-p)}{8}} \left(\frac{dr^2}{f(r)} + r^2 d\Omega_{8-p}^2 \right), \quad (2.2)$$

$$e^\phi = \left(\frac{r}{L_p} \right)^{\frac{(p-3)(7-p)}{4}}, \quad (2.3)$$

$$F_{\theta_1 \dots \theta_{8-p}} = g_s^{-1} Q_p \sqrt{\gamma_{8-p}}, \quad (2.4)$$

where $f(r) = 1 - \frac{r_H^{7-p}}{r^{7-p}}$. The parameter Q_p in eq. (2.4) is defined as $Q_p = (7-p)L_p^{7-p}$ with

$$L_p^{7-p} = \frac{(2\pi l_s)^{7-p} g_s N}{(7-p)\Omega_{8-p}}, \quad (2.5)$$

where $\Omega_{8-p} = 2\pi^{\frac{9-p}{2}}/\Gamma\left(\frac{9-p}{2}\right)$ is the volume of the $(8-p)$ -dimensional unit sphere, and N is the number of the D-branes.

The boundary of eq. (2.2) is at some constant and large value of r with $H_{\hat{M}\hat{N}} = G_{\hat{M}\hat{N}} - \mathbf{n}_{\hat{M}}\mathbf{n}_{\hat{N}}$ the induced metric on it. $\mathbf{n}_{\hat{M}}$ is the unit norm in 10d defined as $\mathbf{n}_{\hat{M}} = \nabla_{\hat{M}}r/\sqrt{G^{\hat{N}\hat{P}}\nabla_{\hat{N}}r\nabla_{\hat{P}}r}$. Then the extrinsic curvature in the Gibbons-Hawking term of eq. (2.1) is $\mathcal{K} = -H^{\hat{M}\hat{N}}\nabla_{\hat{M}}\mathbf{n}_{\hat{N}}$. The counter term for Dp-brane action can refer to e.g., [53], in which L_p appears in the denominator just to balance the dimension.

Now we rewrite eq. (2.2) as

$$ds^2 = e^{2\alpha_1 A} g_{MN} dx^M dx^N + L_p^2 e^{2\alpha_2 A} d\Omega_{8-p}^2 \quad (2.6)$$

with its induced boundary metric as

$$ds^2 = e^{2\alpha_1 A} h_{MN} dx^M dx^N + L_p^2 e^{2\alpha_2 A} d\Omega_{8-p}^2 \quad (2.7)$$

to reduce the 10d theory described by eqs. (2.2), (2.1), (2.3), (2.4) to a $(p+2)$ -dimensional one. The scalar function A can be fixed by comparing eq. (2.6) with eq. (2.2). In order to make the dilaton ϕ not coupled with Ricci scalar in the reduced theory, α_1, α_2 should be set as $\alpha_1 = -\frac{8-p}{p}, \alpha_2 = 1$. g_{MN} and h_{MN} are the $(p+2)$ -dimensional bulk and induced boundary metric, respectively. The coordinate system is $x^{\hat{M}} = \{x^M, \theta^a\} = \{x^\mu, r, \theta^a\}$ with x^μ, r and θ^a independent of each other, thus we have $\nabla_{\theta^a} r = 0$. The 10d unit norm reduces to

$$\mathbf{n}_{\hat{M}} = \frac{\nabla_{\hat{M}}r}{\sqrt{G^{\hat{N}\hat{P}}\nabla_{\hat{N}}r\nabla_{\hat{P}}r}} = \frac{\nabla_M r}{\sqrt{G^{NP}\nabla_N r\nabla_P r}} = e^{-\frac{8-p}{p}A} n_M, \quad (2.8)$$

where $n_M = \frac{\nabla_M r}{\sqrt{g^{NP}\nabla_N r\nabla_P r}}$ is the unit norm in $p+2$ dimensions. Then one also has

$$\mathcal{K} = -H^{\hat{M}\hat{N}}\nabla_{\hat{M}}\mathbf{n}_{\hat{N}} = e^{\frac{2(8-p)}{p}A} h^{MN}\nabla_M \left(e^{-\frac{8-p}{p}A} n_N \right) = e^{\frac{(8-p)}{p}A} K, \quad (2.9)$$

where K is the reduced extrinsic curvature defined as $K = -h^{MN}\nabla_M n_N$. So the full action for $(p+2)$ -dimensional reduced bulk theory is

$$\begin{aligned} S = & \frac{1}{2\kappa_{p+2}^2} \int d^{p+2}x \sqrt{-g} \left[R - \frac{1}{2}(\partial\phi)^2 - \frac{8(8-p)}{p}(\partial A)^2 + V(\phi, A) \right] \\ & - \frac{1}{\kappa_{p+2}^2} \int d^{p+1}x \sqrt{-h} K + \frac{1}{\kappa_{p+2}^2} \int d^{p+1}x \sqrt{-h} \frac{9-p}{2L_p} \left(\frac{r}{L_p} \right)^{-\frac{(p-3)^2}{2p}}, \\ V(\phi, A) = & \frac{(7-p)(8-p)}{L_p^2} e^{-\frac{16}{p}A} - \frac{Q_p^2}{2L_p^{2(8-p)}} e^{\frac{p-3}{2}\phi - \frac{2(p+1)(8-p)}{p}A}, \end{aligned} \quad (2.10)$$

where $\frac{1}{2\kappa_{p+2}^2} = \frac{L_p^{8-p}\Omega_{8-p}}{2\kappa_{10}^2}$ is the surface gravity constant in $(p+2)$ -dimensional reduced bulk theory.

The reduced background which solves eq. (2.10) is

$$\begin{aligned}
 ds^2 &= \left(\frac{r}{L_p}\right)^{\frac{9-p}{p}} (-f(r)dt^2 + (dx^i)^2) + \left(\frac{r}{L_p}\right)^{\frac{p^2-8p+9}{p}} \frac{dr^2}{f(r)}, \\
 e^\phi &= \left(\frac{r}{L_p}\right)^{\frac{(p-3)(7-p)}{4}}, \quad e^A = \left(\frac{r}{L_p}\right)^{\frac{(p-3)^2}{16}}.
 \end{aligned} \tag{2.11}$$

Please note that both of the two scalar fields vanish when $p = 3$, which is the conformal case that has been solved in [4–7]. The equations of motion (EOM) which can be derived from eq. (2.10) are the Einstein equation and the EOM of scalar fields ϕ and A :

$$E_{MN} - T_{MN} = 0, \tag{2.12}$$

$$\nabla^2 \phi - \frac{(p-3)(7-p)^2}{4L_p^2} e^{\frac{p-3}{2}\phi - \frac{2(p+1)(8-p)}{p}A} = 0, \tag{2.13}$$

$$\nabla^2 A - \frac{(7-p)}{L_p^2} e^{-\frac{16}{p}A} + \frac{(p+1)(7-p)^2}{16L_p^2} e^{\frac{p-3}{2}\phi - \frac{2(p+1)(8-p)}{p}A} = 0. \tag{2.14}$$

where $E_{MN} = R_{MN} - \frac{1}{2}g_{MN}R$ is the Einstein tensor and

$$\begin{aligned}
 T_{MN} &= \frac{1}{2} \left(\partial_M \phi \partial_N \phi - \frac{1}{2} g_{MN} (\partial \phi)^2 \right) + \frac{8(8-p)}{p} \left(\partial_M A \partial_N A - \frac{1}{2} g_{MN} (\partial A)^2 \right) \\
 &\quad + \frac{1}{2} g_{MN} V
 \end{aligned} \tag{2.15}$$

is the energy momentum tensor in $(p+2)$ -dimensional spacetime. From the expressions of ϕ and A in eq. (2.11) we have $A = \frac{p-3}{4(7-p)}\phi$, which shows that these two scalar fields are not independent thus we only need to solve one of eq. (2.13) and eq. (2.14) to get the scalar perturbations.

3 The first order perturbations

We will set $L_p = 1$ from now on and restore it when we present our results. Now we switch to the Edington-Finkelstein coordinate by $dt = dv - dr/(r^{\frac{7-p}{2}}f)$, then the metric in eq. (2.11) after coordinate boost becomes

$$ds^2 = -r^{\frac{9-p}{p}} f(r) u_\mu u_\nu dx^\mu dx^\nu + r^{\frac{9-p}{p}} P_{\mu\nu} dx^\mu dx^\nu - 2r^{\frac{(p-3)(p-6)}{2p}} u_\mu dx^\mu dr. \tag{3.1}$$

Following the standard prescription of fluid/gravity correspondence [7], we promote the boost parameters u^μ and r_H to be x^μ dependent and expand them to the first order in ∂_μ like in [51], then we get

$$\begin{aligned}
 ds^2 &= -r^{\frac{9-p}{p}} \left(f - \frac{(7-p)r_H^{6-p}}{r^{7-p}} \delta r_H \right) dv^2 + 2r^{\frac{9-p}{p}} (f-1) \delta \beta_i dx^i dv \\
 &\quad + 2r^{\frac{(p-3)(p-6)}{2p}} dv dr + r^{\frac{9-p}{p}} (dx^i)^2 - 2r^{\frac{(p-3)(p-6)}{2p}} \delta \beta_i dx^i dr.
 \end{aligned} \tag{3.2}$$

The general form of perturbations are set to be

$$ds_{gen. pert.}^2 = r^{\frac{9-p}{p}} k(x, r) u_\mu(x) u_\nu(x) dx^\mu dx^\nu + 2r^{\frac{9-p}{p}} P_\mu^\rho w_\rho(x, r) u_\nu dx^\mu dx^\nu \\ + r^{\frac{9-p}{p}} (\alpha_{\mu\nu}(x, r) + h(x, r) P_{\mu\nu}(x)) dx^\mu dx^\nu - 2r^{\frac{(p-3)(p-6)}{2p}} j(x, r) u_\mu dx^\mu dr. \quad (3.3)$$

Here we use different conventions for setting the perturbations like k and w_μ from that of [13, 51]. After expanded to first order, eq. (3.3) becomes

$$ds_{1st pert.}^2 = r^{\frac{9-p}{p}} k^{(1)} dv^2 - 2r^{\frac{9-p}{p}} w_i^{(1)} dx^i dv + r^{\frac{9-p}{p}} (\alpha_{ij}^{(1)} + h^{(1)} \delta_{ij}) dx^i dx^j \\ + 2r^{\frac{(p-3)(p-6)}{2p}} j^{(1)} dv dr, \quad (3.4)$$

in which all the first order perturbations only depend on r . This is because the expansion actually happens at a specific point on the boundary (which we set by default the original point). It will apply to all x^μ after we write the solved metric back into covariant form, according to [7]. Eq. (3.4) will be solved in this section.

The perturbations in eq. (3.4) can be grouped into three independent parts according to the transformation rules under $SO(p)$. The EOM can be divided into the dynamical and constraint equations by the original terminology in [7]. The dynamical equations for Dp-brane are

$$E_{ij} - \frac{1}{p} \delta_{ij} \delta^{kl} E_{kl} - \left(T_{ij} - \frac{1}{p} \delta_{ij} \delta^{kl} T_{kl} \right) = 0, \quad (3.5)$$

$$E_{ri} - T_{ri} = 0, \quad (3.6)$$

$$E_{rr} - T_{rr} = 0, \quad (3.7)$$

$$\nabla^2 \phi - \frac{(p-3)(7-p)^2}{4} e^{\frac{p-3}{2}\phi - \frac{2(p+1)(8-p)}{p}A} = 0, \quad (3.8)$$

while the constraint equations are from the vector and scalar sectors:

$$g^{r0}(E_{0i} - T_{0i}) + g^{rr}(E_{ri} - T_{ri}) = 0, \quad (3.9)$$

$$g^{rr}(E_{r0} - T_{r0}) + g^{r0}(E_{00} - T_{00}) = 0, \quad (3.10)$$

$$g^{rr}(E_{rr} - T_{rr}) + g^{r0}(E_{r0} - T_{r0}) = 0, \quad (3.11)$$

of which the nature can be specified from their indices. Whether an equation is dynamical or constrained bases on whether it has second derivative order (with respect to r) terms of the perturbations: dynamical equations have while the constraint equations do not. Only the EOM of ϕ eq. (3.8) is an exception because we do not consider perturbation of ϕ in this paper. So eq. (3.8) only contains the first derivative terms of scalar perturbations h , j and k .

Now we put eq. (3.2) plus eq. (3.4) into the EOM from eq. (3.5) to eq. (3.11) and get

$$\partial_r(r^{8-p} f(r) \partial_r \alpha_{ij}(r)) + (9-p)r^{\frac{7-p}{2}} \sigma_{ij} = 0, \quad (3.12)$$

$$\partial_r \left(r^{8-p} \partial_r w_i^{(1)} \right) - \frac{9-p}{2} r^{\frac{7-p}{2}} \partial_0 \beta_i = 0, \quad (3.13)$$

$$2prh'' + p(7-p)h' - 2(9-p)j' = 0, \quad (3.14)$$

$$2(r^{7-p}k)' + 2r^{7-p}fj' + 4(7-p)r^{6-p}j - pr^{7-p}fh' - 2r^{\frac{7-p}{2}}\partial\beta = 0, \quad (3.15)$$

which are the dynamical differential equations and

$$\frac{1}{r_H} \partial_i r_H = -\frac{2}{5-p} \partial_0 \beta_i, \quad (3.16)$$

$$\frac{1}{r_H} \partial_0 r_H = -\frac{2}{9-p} \partial \beta, \quad (3.17)$$

$$(r^{7-p} k)' + 2(7-p)r^{6-p} j - \left(pr^{7-p} - \frac{2p}{9-p} r_H^{7-p} \right) h' - 2r^{\frac{7-p}{2}} \partial \beta = 0, \quad (3.18)$$

which are the constraint equations. The differential equation from the EOM of ϕ eq. (3.15) does not exist for D3-brane. The vector constraint eq. (3.16) and the first scalar constraint eq. (3.17) are only algebraic equations while the second scalar constraint eq. (3.18) is a differential one which will be used to solve the scalar perturbations. The constraint equations of the algebraic type exist because the viscous terms are not independent of each other. In first order, one has the following viscous terms:

$$\sigma_{ij} = \partial_{(i} \beta_{j)} - \frac{1}{p} \delta_{ij} \partial \beta, \quad \partial_0 \beta_i, \quad \partial_i r_H, \quad \partial \beta, \quad \partial_0 r_H. \quad (3.19)$$

In the above, eq. (3.16) and eq. (3.17) tell us that in vector and scalar part of the viscous tensors, $\partial_i r_H$ and $\partial_0 r_H$ are not independent of $\partial_0 \beta_i$ and $\partial \beta$, respectively. We will only use the latter two in this paper. Since the perturbations are all $SO(p)$ invariant, one can set $\alpha_{ij} = F(r) \sigma_{ij}$, $w_i = a(r) \partial_0 \beta_i$, $h = F_h(r) \partial \beta$, $j = F_j(r) \partial \beta$ and $k = F_k(r) \partial \beta$. Using eq. (3.14), eq. (3.15) and eq. (3.18), one can eliminate j, k and get the differential equation for h

$$\partial_r (r^{8-p} f \partial_r h) + \frac{9-p}{p} r^{\frac{7-p}{2}} \partial \beta = 0. \quad (3.20)$$

Comparing eq. (3.20) with eq. (3.12), one gets

$$F_h = \frac{1}{p} F, \quad (3.21)$$

which reflects the fact that the scalar perturbation h is the trace part of the perturbation tensor at the first order.

Next we are going to solve the EOM for all the Dp-brane in the first order. We will divide our discussion into the unphysical D5 and D6-brane case and the Dp-brane cases with $1 \leq p \leq 4$. The D3-brane case is actually dual to the renowned $\mathcal{N} = 4$ super Yang-Mills plasma, of which the results are already known in [4, 6, 7, 11]. The reasons to repeat the calculation for D3-brane are: 1). We will use a different way to solve the perturbations at the first order; 2). At the second order, we will use different conventions to define the viscous terms in order to make them applicable for various dimensions and more convenient to transfer to covariant form. As a result, some of the intermediate results will be different from [7].

3.1 The D5 and D6-brane case

Solving the tensor perturbation equation (3.12) for $p = 5$ and 6 , we get

$$F(r) = C_2 - \ln(r^2 - r_H^2) + \frac{C_1}{2r_H^2} \ln \left(1 - \frac{r_H^2}{r^2} \right), \quad (\text{D5-brane}) \quad (3.22)$$

$$F(r) = C_2 - 4r^{\frac{1}{2}} + 4r_H^{\frac{1}{2}} \operatorname{artanh} \sqrt{\frac{r}{r_H}} + \frac{C_1}{r_H} \ln \left(1 - \frac{r_H}{r} \right), \quad (\text{D6-brane}) \quad (3.23)$$

where $C_{1,2}$ are the integration constants. The function “artanh” is sometimes misnamed as “arctanh”, the “ar” in artanh means area not arc.

Problems arise in eqs. (3.22), (3.23). One can see that both of these two solutions diverge at $r \rightarrow \infty$ thus do not correspond to physical mode. From $F_h = \frac{1}{p}F$ we can conclude that we will not have a physical solution for h , either. This means D5 and D6-brane are not dual to any physical relativistic fluid. Similar conclusions have also been made in [47]. The author finds the formula for deriving the shear relaxation time (originally proposed in [14]) vanishes for D5-brane and diverges for D6-brane. The physical reason behind this is also pointed out in [47] that Dp-branes for $p \geq 5$ have instability: the sound speed vanishes for D5-brane and become negative for D6-brane. We will see this instability by calculating the sound speed in section 4.

The fact that D5 and D6-brane are not dual to any physical relativistic fluid will save us plenty of work, we only need to consider the Dp-brane with $1 \leq p \leq 4$ from now on.

3.2 The Dp-brane case with $1 \leq p \leq 4$

We can solve the tensor perturbations for Dp-brane with $2 \leq p \leq 4$ from eq. (3.12) as

$$F(r) = \frac{2}{5r_H^{3/2}} \left[2 \sin \frac{2\pi}{5} \arctan \frac{2 \sin \frac{\pi}{5} \sqrt{rr_H}}{r - r_H} - 2 \sin \frac{\pi}{5} \arctan \frac{2 \sin \frac{2\pi}{5} \sqrt{rr_H}}{r - r_H} \right. \\ \left. - 2 \cos \frac{2\pi}{5} \operatorname{artanh} \frac{2 \cos \frac{\pi}{5} \sqrt{rr_H}}{r + r_H} - 2 \cos \frac{\pi}{5} \operatorname{artanh} \frac{2 \cos \frac{2\pi}{5} \sqrt{rr_H}}{r + r_H} \right. \\ \left. + \ln \frac{(\sqrt{r} + \sqrt{r_H})^2 (r^4 + r^3 r_H + r^2 r_H^2 + r r_H^3 + r_H^4)}{r^5} \right], \quad (\text{D2-brane}) \quad (3.24)$$

$$F(r) = \frac{1}{2r_H} \left[2 \arctan \frac{r_H}{r} + \ln \frac{(r + r_H)^2 (r^2 + r_H^2)}{r^4} \right], \quad (\text{D3-brane}) \quad (3.25)$$

$$F(r) = \frac{1}{3r_H^{1/2}} \left[2\sqrt{3} \arctan \frac{\sqrt{3rr_H}}{r - r_H} \right. \\ \left. + \ln \frac{(\sqrt{r} + \sqrt{r_H})^4 (r + \sqrt{rr_H} + r_H)^2 (r^2 + rr_H + r_H^2)}{r^6} \right], \quad (\text{D4-brane}) \quad (3.26)$$

Note that D1-brane only has 1 spatial dimension thus can not support tensor perturbation. $F(r)$ for D2-brane is more complicated than the other two cases, yet it can still be cast into a neat form as eq. (3.24). We write the first order tensor perturbation of D3-brane i.e., eq. (3.25) in a different form from that in [7] by using the identity

$$\frac{\pi}{2} - \arctan x = \arctan \frac{1}{x}, \quad (3.27)$$

which is easy to verify.

The D4-brane tensor perturbation eq. (3.26) is the same as in the compactified case [13, 51] which can be shown by first using eq. (3.27) and then the arctangent addition formula. So one can draw the conclusion that compactifying one direction of D4-brane will not effect the coefficients of viscous tensors, i.e., η, τ_π and $\lambda_{1,2}$ will not change compared with the compactified D4-brane case. This can also be seen from the differential equation of second order tensor perturbation for D4-brane later.

The vector perturbations for Dp-brane with $1 \leq p \leq 4$ can be solved from eq. (3.13) as $w_i^{(1)}(r) = a(r)\partial_0\beta_i$ where we define

$$a(r) = -\frac{2}{(5-p)r^{\frac{5-p}{2}}}. \quad (3.28)$$

From the above we can see that the vector perturbation diverges for $p = 5$ and can not be normalized for $p = 6$. Since we use different convention for setting the vector perturbation in eq. (3.3), that's why the above for $p = 4$ is not the same as in [51]. The advantage of this convention is that it can circumvent the appearance of divergent part at the second order, compared with [7, 13, 51]. The vector viscous term $\partial_0\beta_i$ for D1-brane will reduce to a scalar $\partial_0\beta_1$ but we will still call it a vector viscous term throughout the paper.

In solving the first order scalar perturbations of Dp-brane, we have some problems on D3-brane case. As it has been pointed out that the EOM of dilaton eq. (3.8) does not exist in D3-brane case and the scalar perturbations need to be solved under the “background field” gauge [7]. Actually, choosing a gauge condition is just adding one more equation factitiously. We also know that the scalar perturbations of D3-brane are trivial, which indicates that they will not be written in a compact form together with the other Dp-branes. In order to express the scalar perturbation solutions of Dp-brane in a unified form, we will use the condition eq. (3.21) for D3-brane. Thus the 3 scalar perturbations of D3-brane can be solved from eq. (3.14), eq. (3.18) together with eq. (3.21).

The other cases can be solved from eq. (3.14), eq. (3.15) and eq. (3.18). Then the 1st order scalar perturbations of Dp-brane with $1 \leq p \leq 4$ can be nicely formulated as

$$\begin{aligned} F_h &= \frac{1}{p}F, \quad F_j = -\frac{2}{9-p} \frac{r^{\frac{9-p}{2}} - r_H^{\frac{9-p}{2}}}{r^{7-p} - r_H^{7-p}} + \frac{5-p}{2(9-p)}F, \\ F_k &= \frac{4}{(9-p)r^{\frac{5-p}{2}}} - \frac{1}{9-p} \left(5-p + \frac{2r_H^{7-p}}{r^{7-p}} \right) F. \end{aligned} \quad (3.29)$$

In D1-brane case F is just F_h and we only need to keep in mind that for $p = 1$ F has nothing to do with tensor perturbation. Compared with the solution in compactified D4-brane [13, 51], F_j and F_k do not change while F_h does.² This is because h relates with the spatial trace part of the perturbation ansatz, as can be seen from eq. (3.4). And the spatial dimension of compactified D4-brane is 3 while it is 4 for D4-brane case here.

² F_k here is $\frac{1}{r^3}$ times the F_k in [13, 51] due to the convention change on the perturbation ansatz eq. (3.3).

4 The first order stress-energy tensor

The boundary stress-energy tensor is defined as the large r limit of the following Brown-York tensor

$$T_{\mu\nu} = \frac{1}{\kappa_{p+2}^2} \lim_{r \rightarrow \infty} \left(\frac{r}{L_p} \right)^{\frac{(9-p)(p-1)}{2p}} \left(K_{\mu\nu} - h_{\mu\nu} K - \frac{9-p}{2L_p} \left(\frac{r}{L_p} \right)^{-\frac{(p-3)^2}{2p}} h_{\mu\nu} \right), \quad (4.1)$$

from which one can calculate the first order stress-energy tensor for Dp-brane:

$$T_{\mu\nu} = \frac{1}{2\kappa_{p+2}^2} \left[\frac{r_H^{7-p}}{L_p^{8-p}} \left(\frac{9-p}{2} u_\mu u_\nu + \frac{5-p}{2} P_{\mu\nu} \right) - \left(\frac{r_H}{L_p} \right)^{\frac{9-p}{2}} \left(2\sigma_{\mu\nu} + \frac{2(p-3)^2}{p(9-p)} P_{\mu\nu} \partial u \right) \right]. \quad (4.2)$$

One can extract the energy density, pressure and first order transport coefficients from eq. (4.2) as:

$$\begin{aligned} \varepsilon &= \frac{1}{2\kappa_{p+2}^2} \frac{9-p}{2} \frac{r_H^{7-p}}{L_p^{8-p}}, & \mathfrak{p} &= \frac{1}{2\kappa_{p+2}^2} \frac{5-p}{2} \frac{r_H^{7-p}}{L_p^{8-p}}, \\ \eta &= \frac{1}{2\kappa_{p+2}^2} \left(\frac{r_H}{L_p} \right)^{\frac{9-p}{2}}, & \zeta &= \frac{1}{2\kappa_{p+2}^2} \frac{2(p-3)^2}{p(9-p)} \left(\frac{r_H}{L_p} \right)^{\frac{9-p}{2}}. \end{aligned} \quad (4.3)$$

From eq. (2.11) one gets the Hawking temperature

$$T = \frac{(7-p)r_H^{\frac{5-p}{2}}}{4\pi L_p^{\frac{7-p}{2}}}, \quad (4.4)$$

thus the entropy is

$$s = \frac{\varepsilon + \mathfrak{p}}{T} = \frac{1}{2\kappa_{p+2}^2} 4\pi \left(\frac{r_H}{L_p} \right)^{\frac{9-p}{2}}. \quad (4.5)$$

Then we have

$$c_s^2 = \left(\frac{\partial \mathfrak{p}}{\partial \varepsilon} \right)_s = \frac{5-p}{9-p}, \quad \frac{\eta}{s} = \frac{1}{4\pi}, \quad \frac{\zeta}{\eta} = \frac{2(p-3)^2}{p(9-p)}. \quad (4.6)$$

The above results agree with that of [37, 46, 48, 50]. One can see from the sound speed that D5 and D6-brane are indeed unphysical since the sound speed separately become zero and negative for $p = 5$ and 6.

5 The second order perturbations

5.1 General discussions

In this section we are going to solve the second order perturbations for Dp-brane with $1 \leq p \leq 4$. The start line is the first order covariant metric valid on the whole boundary:

$$\begin{aligned}
 ds^2 = & -r^{\frac{9-p}{p}} [f(r_H(x), r) - k(r_H(x), r)] u_\mu(x) u_\nu(x) dx^\mu dx^\nu \\
 & + 2r^{\frac{9-p}{p}} P_\mu^\rho(x, r) w_\rho(x, r) u_\nu(x) dx^\mu dx^\nu \\
 & + r^{\frac{9-p}{p}} [P_{\mu\nu}(x, r) + \alpha_{\mu\nu}(r_H(x), r) + h(r_H(x), r) P_{\mu\nu}(x, r)] dx^\mu dx^\nu \\
 & - 2r^{\frac{(p-3)(p-6)}{2p}} [1 + j(r_H(x), r)] u_\mu(x) dx^\mu dr.
 \end{aligned} \tag{5.1}$$

Then expand $r_H(x)$ and $\beta_i(x)$ to second derivative order like in [13]. If we use $\mathcal{F}$ to denote any of F, F_h, F_j and F_k , then $\mathcal{F}$ should be expanded like

$$\mathcal{F}(r_H(x), r) = \mathcal{F}(r) + \delta\mathcal{F}(r_H(x), r) = \mathcal{F}(r) - \frac{(5-p)\mathcal{F}(r) + 2r\mathcal{F}'(r)}{2r_H} \delta r_H. \tag{5.2}$$

Here $\mathcal{F}(r)$ is $\mathcal{F}(r_H(x), r)$ at the original point $x^\mu = 0$ and $\delta r_H = x^\mu \partial_\mu r_H$. $\delta\mathcal{F}(r_H(x), r)$ will be denoted shortly as $\delta\mathcal{F}$ in the following. Thus eq. (5.1) should be expanded as

$$\begin{aligned}
 ds^2 = & -r^{\frac{9-p}{p}} \left[f + (f-1)\delta\beta_i\delta\beta_i - \frac{(7-p)r_H^{6-p}}{r^{7-p}} \delta r_H - \frac{(7-p)r_H^{6-p}}{2r^{7-p}} \delta^2 r_H - \frac{(7-p)r_H^{6-p}}{r^{7-p}} \delta r_H^{(1)} \right. \\
 & \left. - \frac{(7-p)(6-p)r_H^{5-p}}{2r^{7-p}} (\delta r_H)^2 - (F_k + \delta F_k) \partial\beta - F_k (\delta\partial\beta + \delta\beta_i\partial_0\beta_i) - 2a(r) \delta\beta_i\partial_0\beta_i \right] dv^2 \\
 & + 2r^{\frac{9-p}{p}} \left[(f-1)(\delta\beta_i + \frac{1}{2}\delta^2\beta_i) - a(\partial_0\beta_i + \delta\partial_0\beta_i + \delta\beta_j\partial_j\beta_i) - \frac{(7-p)r_H^{6-p}}{r^{7-p}} \delta r_H \delta\beta_i \right. \\
 & \left. - F_k \partial\beta\delta\beta_i - F\delta\beta_j\partial_{(i}\beta_{j)} \right] dv dx^i \\
 & + 2r^{\frac{(p-3)(p-6)}{2p}} \left[1 + (F_j + \delta F_j) \partial\beta + F_j (\delta\partial\beta + \delta\beta_i\partial_0\beta_i) + \frac{1}{2}\delta\beta_i\delta\beta_i \right] dv dr \\
 & + r^{\frac{9-p}{p}} \left[\delta_{ij} + (1-f)\delta\beta_i\delta\beta_j + 2a\delta\beta_{(i}\partial_{|0|}\beta_{j)} + (F + \delta F) \partial_{(i}\beta_{j)} \right. \\
 & \left. + F (\delta\partial_{(i}\beta_{j)} + \delta\beta_{(i}\partial_{|0|}\beta_{j)}) \right] dx^i dx^j - 2r^{\frac{(p-3)(p-6)}{2p}} \left(\delta\beta_i + \frac{1}{2}\delta^2\beta_i + F_j \partial\beta\delta\beta_i \right) dx^i dr.
 \end{aligned} \tag{5.3}$$

In the above $\delta\beta_i = x^\mu \partial_\mu \beta_i$, similarly as δr_H . We also define $\delta^2\beta_i = x^\mu x^\nu \partial_\mu \partial_\nu \beta_i$ and $\delta\beta_{(i}\partial_{|0|}\beta_{j)} = \frac{1}{2}(\delta\beta_i\partial_0\beta_j + \delta\beta_j\partial_0\beta_i)$. Note after the expansion with respect to the boundary derivative terms, the metric becomes a quadratic function in boundary coordinates x^μ and it is also a complicate function of r via F, F_j and F_k . We will set $r_H = 1$ hereafter and restore it when we give the results of stress-energy tensors.

Since Einstein equation is second order in derivatives, thus one will get differential equations with many second order derivatives of r_H and β_i which can be written as the spatial 2nd order viscous terms. We list them in table 1 where $\Omega_{ij} = \partial_{[i}\beta_{j]}$ is the spatial component of $\Omega_{\mu\nu}$. Some of the conventions for the 2nd order spatial viscous terms in this

Scalars of $SO(p)$	Vectors of $SO(p)$	Tensors of $SO(p)$
$\mathbf{s}_1 = \frac{1}{r_H} \partial_0^2 r_H$	$\mathbf{v}_{1i} = \frac{1}{r_H} \partial_0 \partial_i r_H$	$\mathbf{t}_{1ij} = \frac{1}{r_H} \partial_i \partial_j r_H - \frac{1}{p} \delta_{ij} \mathbf{s}_3$
$\mathbf{s}_2 = \partial_0 \partial_i \beta_i$	$\mathbf{v}_{2i} = \partial_0^2 \beta_i$	$\mathbf{t}_{2ij} = \partial_0 \Omega_{ij}$
$\mathbf{s}_3 = \frac{1}{r_H} \partial_i^2 r_H$	$\mathbf{v}_{4i} = \partial_j \Omega_{ij}$	$\mathbf{t}_{3ij} = \partial_0 \sigma_{ij}$
$\mathfrak{S}_1 = \partial_0 \beta_i \partial_0 \beta_i$	$\mathbf{v}_{5i} = \partial_j \sigma_{ij}$	$\mathfrak{T}_{1ij} = \partial_0 \beta_i \partial_0 \beta_j - \frac{1}{p} \delta_{ij} \mathfrak{S}_1$
$\mathfrak{S}_3 = (\partial_i \beta_i)^2$	$\mathfrak{V}_{1i} = \partial_0 \beta_i \partial \beta$	$\mathfrak{T}_{2ij} = \sigma_{[i}^{ k} \Omega_{j]k}$
$\mathfrak{S}_4 = \Omega_{ij} \Omega_{ij}$	$\mathfrak{V}_{2i} = \partial_0 \beta_j \Omega_{ij}$	$\mathfrak{T}_{3ij} = \Omega_{ij} \partial \beta$
$\mathfrak{S}_5 = \sigma_{ij} \sigma_{ij}$	$\mathfrak{V}_{3i} = \partial_0 \beta_j \sigma_{ij}$	$\mathfrak{T}_{4ij} = \sigma_{ij} \partial \beta$
		$\mathfrak{T}_{5ij} = \Omega_i^{ k} \Omega_{jk} - \frac{1}{p} \delta_{ij} \mathfrak{S}_4$
		$\mathfrak{T}_{6ij} = \sigma_i^{ k} \sigma_{jk} - \frac{1}{p} \delta_{ij} \mathfrak{S}_5$
		$\mathfrak{T}_{7ij} = \sigma_{(i}^{ k} \Omega_{j)k}$

Table 1. A list of second derivative order $SO(p)$ invariant viscous terms for $p \geq 2$. $\mathfrak{T}_{2,5,6}$ are zero when $p = 2$.

paper are different from [7, 13] because we need to make them appropriate for general value of p , not only for the case of $p = 3$. Since the viscous terms of D1-brane are very different from the other cases, thus we will list them separately when we talk about D1-brane later.

We would like to stress the changes of table 1 from that in [7, 13]: first, we rewrite some terms with $l_i = \epsilon_{ijk} \Omega_{ij}$ in terms of Ω_{ij} since l_i only exists for $p = 3$ while Ω_{ij} exists in all cases of $p \geq 2$, e.g. $l_i l_i \rightarrow \Omega_{ij}^2$ and $l_i l_j - \frac{1}{p} \delta_{ij} l_k^2 \rightarrow \Omega_i^{ k} \Omega_{jk} - \frac{1}{p} \delta_{ij} \Omega_{kl}^2$. Second, we remove $\mathfrak{S}_2 = l_i \partial_0 \beta_i$ and $\mathbf{v}_{3i} = \partial_0 l_i$ since they both contain l_i and can not be rewrite as scalar and vector with the role of l_i replaced by Ω_{ij} . In order to be convenient to compare with previous studies, we will not change the ordinal numbers of the other $SO(p)$ invariant scalars and vectors. Third, we redefine $\mathbf{t}_2$ and $\mathfrak{T}_{2,3}$ of $SO(p)$ tensors in terms of Ω_{ij} to express the antisymmetric tensor constraint, i.e. eq. (5.14) for general p . Thus the viscous tensors in table 1 do not only contain the traceless-symmetric tensors but also the antisymmetric tensors. This is different from that in [7, 13]. We will suppress the vector and tensor indices of the spatial viscous terms from now on.

The viscous terms listed in table 1 satisfy 6 constraint equations which are the 2nd order counterparts of eqs. (3.16), (3.17). The usage of them is in deriving the Navier-Stokes equation later. These 6 constraints can be solved from $\partial_\mu \partial^\rho T_{\rho\nu}^{(0)} = 0$ by expanding it to second derivative order, the results can be listed as

$$\frac{9-p}{2} \partial_0^2 r_H + \partial_0 \partial \beta - \frac{2}{9-p} (\partial \beta)^2 - \frac{4}{5-p} \partial_0 \beta_i \partial_0 \beta_i = 0, \quad (5.4)$$

$$\frac{5-p}{2} \partial_i^2 r_H + \partial_0 \partial \beta - \frac{2}{5-p} \partial_0 \beta_i \partial_0 \beta_i - \frac{5-p}{9-p} (\partial \beta)^2 + \partial_i \beta_j \partial_j \beta_i = 0, \quad (5.5)$$

$$\frac{5-p}{2} \partial_0 \partial_i r_H + \partial_0^2 \beta_i - \frac{7-p}{9-p} \partial_0 \beta_i \partial \beta + \partial_0 \beta_j \partial_j \beta_i = 0, \quad (5.6)$$

$$\frac{9-p}{2} \partial_0 \partial_i r_H + \partial_i \partial \beta - \frac{2}{5-p} \partial_0 \beta_i \partial \beta - \frac{4}{5-p} \partial_0 \beta_j \partial_j \beta_i = 0, \quad (5.7)$$

$$\partial_0 \Omega_{ij} - \frac{5-p}{9-p} \Omega_{ij} \partial \beta - \partial_k \beta_{[i} \partial_{j]} \beta_k = 0, \quad (5.8)$$

$$\frac{5-p}{2} \partial_i \partial_j r_H + \partial_0 \partial_{(i} \beta_{j)} - \frac{2}{5-p} \partial_0 \beta_i \partial_0 \beta_j - \frac{5-p}{9-p} \partial_{(i} \beta_{j)} \partial \beta + \partial_k \beta_{(i} \partial_{j)} \beta_k = 0. \quad (5.9)$$

Here $T_{\mu\nu}^{(0)}$ is the zeroth derivative order of eq. (4.2), i.e. the thermodynamic part of the 1st order stress-energy tensor of the relativistic fluid. In terms of the viscous $SO(p)$ invariant terms, the above can be reformulated as

$$\mathbf{s}_1 + \frac{2}{9-p} \mathbf{s}_2 - \frac{8}{(9-p)(5-p)} \mathfrak{S}_1 - \frac{4}{(9-p)^2} \mathfrak{S}_3 = 0, \quad (5.10)$$

$$\mathbf{s}_2 + \frac{5-p}{2} \mathbf{s}_3 - \frac{2}{5-p} \mathfrak{S}_1 + \frac{(p-3)^2}{p(9-p)} \mathfrak{S}_3 - \mathfrak{S}_4 + \mathfrak{S}_5 = 0, \quad (5.11)$$

$$\mathbf{v}_1 + \frac{2}{5-p} \mathbf{v}_2 + \frac{2(p^2 - 8p + 9)}{p(9-p)(5-p)} \mathfrak{V}_1 - \frac{2}{5-p} \mathfrak{V}_2 + \frac{2}{5-p} \mathfrak{V}_3 = 0, \quad (5.12)$$

$$\mathbf{v}_1 + \frac{2p(\mathbf{v}_4 + \mathbf{v}_5)}{(9-p)(p-1)} - \frac{4(p+2)}{p(9-p)(5-p)} \mathfrak{V}_1 - \frac{8(\mathfrak{V}_2 + \mathfrak{V}_3)}{(9-p)(5-p)} = 0, \quad (5.13)$$

$$\mathbf{t}_2 - 2\mathfrak{T}_2 + \frac{p^2 - 7p + 18}{p(9-p)} \mathfrak{T}_3 = 0, \quad (5.14)$$

$$\mathbf{t}_1 + \frac{2}{5-p} \mathbf{t}_3 - \frac{4}{(5-p)^2} \mathfrak{T}_1 + \frac{2(p^2 - 7p + 18)}{p(9-p)(5-p)} \mathfrak{T}_4 - \frac{2}{5-p} \mathfrak{T}_5 + \frac{2}{5-p} \mathfrak{T}_6 = 0. \quad (5.15)$$

The above 6 constraints from eq. (5.10) to eq. (5.15) are completely appropriate for D3 and D4-brane. Since all the components of $\mathfrak{T}_2$, $\mathfrak{T}_5$ and $\mathfrak{T}_6$ are 0 for $p = 2$ thus we only need to remove them from eq. (5.14) and eq. (5.15) to get the constraints for D2-brane. The D1-brane needs more changes: first, it does not have tensor part so there should be no tensorial constraints; next, we need to remove the terms $\mathfrak{V}_{2,3}$ from the vector constraint and $\mathfrak{S}_{4,5}$ from scalar constraint because these terms consist of σ_{ij} and Ω_{ij} ; the last is to use $\mathbf{v}_4 + \mathbf{v}_5 = \frac{p-1}{p} \partial_i \partial \beta$ to rewrite eq. (5.13) since in this way there is no $p-1$ in the denominator which will blow up as $p = 1$. The second order constraints for D1-brane will be given later.

It is interesting to compare the above constraints with that of the compactified D4-brane [13] and the D3-brane [7]. Since the convention for the $SO(p)$ invariant viscous terms are different from that of [7], thus the coefficients of the viscous terms here will also change for D3-brane. Also note that the antisymmetric part of the tensor constraint eq. (5.14) is tensorial here but it is of axial vector type in the cases of D3-brane [7] and compactified D4-brane [13]. Since the spatial dimension for both of these two cases are 3 in which one can define the axial vector l_i out of Ω_{ij} , but this can not be done in other dimensions. Thus we write the antisymmetric tensor constraint in terms of three newly defined antisymmetric viscous tensors:³ $\mathbf{t}_2 = \partial_0 \Omega_{ij}$, $\mathfrak{T}_2 = \sigma_{[i}^k \Omega_{j]k}$ and $\mathfrak{T}_3 = \Omega_{ij} \partial \beta$.

There are also two other important equations which can be derived from the full first order stress-energy tensor eq. (4.2), which we denote as $T_{\mu\nu}^{(0+1)}$. They are the Navier-Stokes

³The old definitions of $\mathbf{t}_2$ and $\mathfrak{T}_{2,3}$ contain l_i which are not used in calculation in [7, 13, 51].

equations $\partial^\mu T_{\mu\nu}^{(0+1)} = 0$ with the index ν being set to 0 and i :

$$\frac{1}{r_H^{(p-3)/2}} \partial_0 r_H^{(1)} = \frac{4(p-3)^2}{p(9-p)^2(7-p)} \mathfrak{S}_3 + \frac{4}{(9-p)(7-p)} \mathfrak{S}_5, \quad (5.16)$$

$$\begin{aligned} \frac{1}{r_H^{(p-3)/2}} \partial_i r_H^{(1)} &= \frac{4(p-3)^2 \mathbf{v}_4 + 16p \mathbf{v}_5}{(9-p)(7-p)(5-p)(p-1)} + \frac{2(p-1)(p^2 - 22p + 77)}{p(9-p)(7-p)(5-p)^2} \mathfrak{V}_1 \\ &\quad - \frac{2(19-3p)}{(9-p)(7-p)(5-p)} \mathfrak{V}_2 - \frac{2(p^2 - 14p + 77)}{(9-p)(7-p)(5-p)^2} \mathfrak{V}_3. \end{aligned} \quad (5.17)$$

The above two equations are suitable for $2 \leq p \leq 4$. As what we have specified for the 2nd order constraints of D1-brane case, we need to remove $\mathfrak{S}_5$ and $\mathfrak{V}_{2,3}$ first and then recast the terms with $\mathbf{v}_{4,5}$ into $\partial_1^2 \beta_1$. The explicit form of the Navier-Stokes equations for $p = 1$ will be offered later. As we have said that in deriving these two equations, we need to use the constraints from eq. (5.4) to eq. (5.9). The Navier-Stokes equations will be used in deriving the differential equations for the 2nd order perturbations. They can also be got from the bulk vector and scalar constraint equations at second order, which will be specified later.

Following [7, 13], the second order perturbations can be solved by integrations as

$$\alpha_{ij}^{(2)} \text{ or } h^{(2)} = \int_r^\infty \frac{-1}{x^{8-p} f} dx \int_1^x S_{\alpha,h}(y) dy, \quad (5.18)$$

$$j^{(2)} \text{ or } k^{(2)} = - \int_r^\infty S_{j,k}(x) dx, \quad (5.19)$$

$$w_i^{(2)} = \int_r^\infty \frac{1}{x^{8-p}} dx \int_x^\infty S_w(y) dy, \quad (5.20)$$

where $S_\#$ with $\# = \{\alpha, w, h, j, k\}$ are the source terms which can be easily understood as the right hand side of the perturbation equations. The above integrations are suitable for the nonconformal branes, i.e., $p = 1, 2, 4$. For $p = 3$, $h^{(2)}$ needs to be integrated in the same way as $w_i^{(2)}$ but not like $\alpha_{ij}^{(2)}$ since the scalar part of D3-brane is trivial. While the other perturbations of D3-brane are still integrated in the same way as the other Dp-branes.

5.2 The D3-brane case

Although the second order transport coefficients of D3-brane has been studied in [7], we still would like to redo the calculation here. The reason is that we use different conventions, thus the intermediate steps of the calculation will also change. Further more, the D3-brane calculation can help us to understand the nonconformal cases.

At the second order there is no simple relation between h and α_{ij} like eq. (3.21). Since in D3-brane case we do not have the dilaton equation and one only has eq. (3.7) and eq. (3.11) at hand to solve h , j and k , which are not enough. Thus we need a gauge condition to eliminate one of the degrees of freedom. Here we use the “background field” gauge $g^{(0)MN} g_{MN}^{(n)} = 0$ from [7], which gives $j^{(n)} = -\frac{3}{2} h^{(n)}$, where (n) specifies the order of each perturbation. With this we can solve h, j, k for D3-brane at first order as

$$h^{(1)} = j^{(1)} = 0, \quad k^{(1)} = \frac{2}{3r} \partial \beta. \quad (5.21)$$

Since $F_{h,j,k}$ are trivial for D3-brane under the background field gauge, the second order expanded metric of D3-brane can not be got by simply setting $p = 3$ in eq. (5.3), but needs to be derived separately. The result is:

$$\begin{aligned}
 ds^2 = & -r^2 \left[f + (f-1)\delta\beta_i\delta\beta_i - \frac{4r_H^3}{r^4}\delta r_H - \frac{2r_H^3}{r^4}\delta^2 r_H - \frac{4r_H^3}{r^4}\delta r_H^{(1)} - \frac{6r_H^2}{r^4}(\delta r_H)^2 \right. \\
 & \left. - F_k\partial\beta - F_k(\delta\partial\beta + \delta\beta_i\partial_0\beta_i) - 2a\delta\beta_i\partial_0\beta_i \right] dv^2 \\
 & + 2r^2 \left[(f-1)(\delta\beta_i + \frac{1}{2}\delta^2\beta_i) - a(\partial_0\beta_i + \delta\partial_0\beta_i + \delta\beta_j\partial_j\beta_i) - \frac{4r_H^3}{r^4}\delta r_H\delta\beta_i \right. \\
 & \left. - F_k\partial\beta\delta\beta_i - F\delta\beta_j\partial_{(i}\beta_{j)} + \frac{1}{3}F\partial\beta\delta\beta_i \right] dv dx^i + 2 \left(1 + \frac{1}{2}\delta\beta_i\delta\beta_i \right) dv dr \\
 & + r^2 \left[\left(1 - \frac{1}{3}(F + \delta F)\partial\beta - \frac{1}{3}F(\delta\partial\beta + \delta\beta_k\partial_0\beta_k) \right) \delta_{ij} + (1-f)\delta\beta_i\delta\beta_j \right. \\
 & \left. + 2a\delta\beta_{(i}\partial_{|0|}\beta_{j)} + (F + \delta F)\partial_{(i}\beta_{j)} + F(\delta\partial_{(i}\beta_{j)} + \delta\beta_{(i}\partial_{|0|}\beta_{j)}) \right] dx^i dx^j \\
 & - 2 \left(\delta\beta_i + \frac{1}{2}\delta^2\beta_i \right) dx^i dr
 \end{aligned} \tag{5.22}$$

The metric of the perturbative ansatz at 2nd order has the same form as eq. (3.4) with the first order perturbations changing to second order ones. By substituting eq. (5.22) and the 2nd order perturbations into the EOM eq. (3.5) to eq. (3.7) and the constraint equations (3.9), (3.10), (3.11), we can get⁴

$$\begin{aligned}
 \partial_r \left(r^5 f \partial_r \alpha_{ij}^{(2)} \right) = & (2r - 3r^2 F - 2r^3 F') \left(\mathfrak{t}_3 + \mathfrak{T}_1 + \frac{\mathfrak{T}_4}{3} \right) - \left(\frac{4}{r^3} + 4r \right) \mathfrak{T}_5 \\
 & + (4r - 6r^2 F + r^5 f F'^2) \mathfrak{T}_6 - (4r + 6r^2 F + 4r^3 F') \mathfrak{T}_7,
 \end{aligned} \tag{5.23}$$

$$\begin{aligned}
 \partial_r \left(r^5 \partial_r w_i^{(2)} \right) = & -2r \mathbf{v}_4 + r^3 F' \mathbf{v}_5 + \left(\frac{2}{3}r + \frac{5}{9}r^3 F' \right) \mathfrak{V}_1 + \left(r - \frac{5}{6}r^3 F' \right) \mathfrak{V}_2 \\
 & - \left(r + \frac{11}{6}r^3 F' \right) \mathfrak{V}_3,
 \end{aligned} \tag{5.24}$$

$$r h_{(2)}'' + 2h_{(2)}' - 2j_{(2)}' = \frac{2}{3r^3} \mathfrak{S}_4 + \left(\frac{1}{6}r F'^2 + \frac{2}{3}F F' + \frac{1}{3}r F F'' \right) \mathfrak{S}_5 \tag{5.25}$$

and

$$\partial_i r_H^{(1)} = (2r^3 + r^5 f F') \left(\frac{1}{4} \mathbf{v}_5 + \frac{5}{36} \mathfrak{V}_1 - \frac{5}{24} \mathfrak{V}_2 - \frac{11}{24} \mathfrak{V}_3 \right), \tag{5.26}$$

$$\partial_0 r_H^{(1)} = \frac{1}{3r} (\mathbf{s}_2 + \mathbf{s}_3 - \mathfrak{S}_1 - \mathfrak{S}_4) + \left(\frac{1}{3r} + \frac{1}{6}r^3 + \frac{1}{12}r^5 f F' \right) \mathfrak{S}_5, \tag{5.27}$$

$$\begin{aligned}
 (r^4 k_{(2)})' + 8r^3 j_{(2)} - (3r^4 - 1)h_{(2)}' = & -\frac{4}{3}r \mathbf{s}_2 + \frac{2}{3}r \mathfrak{S}_1 - \frac{2}{9}r \mathfrak{S}_3 + \left(\frac{1}{3r^3} + 2r \right) \mathfrak{S}_4 \\
 & - \left[\frac{5}{3}r + \frac{1}{3}(3r^4 - 1)F F' + \frac{1}{12}r^5 f F'^2 \right] \mathfrak{S}_5.
 \end{aligned} \tag{5.28}$$

⁴Please note that there is no EOM for dilaton in D3-brane case.

During the procedure of casting the above differential equations, it needs to use the differential equations of first order perturbations like eqs. (3.12), (3.13), (3.14), (3.15), (3.18), together with the Navier-Stokes equations (5.16), (5.17) to eliminate the x^μ dependent terms.

Now we can use eqs. (5.18), (5.20) separately to get the solutions for the differential equations of tensor eq. (5.23) and vector eq. (5.24) perturbations:

$$\alpha_{ij}^{(2)} = \frac{2 - \ln 2}{4r^4} \left(\mathfrak{t}_3 + \mathfrak{T}_1 + \frac{\mathfrak{T}_4}{3} \right) + \frac{1}{r^2} \mathfrak{T}_5 + \left(\frac{1}{r^2} + \frac{1}{2r^4} \right) \mathfrak{T}_6 + \left(\frac{2}{r^2} - \frac{\ln 2}{2r^4} \right) \mathfrak{T}_7, \quad (5.29)$$

$$w_i^{(2)} = \frac{1}{2r^2} (\mathbf{v}_4 + \mathbf{v}_5) + \frac{1}{9r^2} \mathfrak{V}_1 - \frac{2}{3r^2} (\mathfrak{V}_2 + \mathfrak{V}_3). \quad (5.30)$$

As for the scalar part, first we should use eq. (5.25) together with the gauge condition to get the differential equation for $h^{(2)}$, then solve it via

$$h^{(2)} = \int_r^\infty \frac{1}{x^5} dx \int_x^\infty S_h(y) dy. \quad (5.31)$$

As one can see from the above that the way to get $h^{(2)}$ for D3-brane is the same as w_i eq. (5.20), the other Dp-branes will follow the way of eq. (5.18). Then one can get $j^{(2)}$ and $k^{(2)}$ easily from eq. (5.19), the results of the scalar perturbations for $p = 3$ are

$$\begin{aligned} h^{(2)} &= -\frac{1}{6r^2} \mathfrak{S}_4 - \frac{1}{6r^2} \mathfrak{S}_5, \\ j^{(2)} &= -\frac{3}{2} h^{(2)} = \frac{1}{4r^2} \mathfrak{S}_4 + \frac{1}{4r^2} \mathfrak{S}_5, \\ k^{(2)} &= -\frac{2}{3r^2} \mathfrak{s}_2 + \frac{1}{3r^2} \mathfrak{S}_1 - \frac{1}{9r^2} \mathfrak{S}_3 + \frac{1}{2r^2} \mathfrak{S}_4 + \frac{1}{2r^2} \mathfrak{S}_5. \end{aligned} \quad (5.32)$$

We only retain the leading order in the large r expansion in presenting the solutions of the vector and scalar perturbations. Because both of these two parts are trivial and only the leading order is relevant in calculating the stress-energy tensor of the dual fluid. The first scalar constraint eq. (5.27) and the vector constraint eq. (5.26) will reproduce the Navier-Stokes equations (5.16), (5.17) after we expand $F(r)$.

5.3 The D4-brane case

Solving the D4-brane is almost the same like the compactified D4-brane case [13], we just need to keep in mind that the spatial dimension is 4 but not 3 for D4-brane case here. The differential equations for tensor perturbation is

$$\begin{aligned} \partial_r(r^4 f \partial_r \alpha_{ij}^{(2)}) &= \left(6r - \frac{5}{2} r^{\frac{3}{2}} F - 2r^{\frac{5}{2}} F' \right) (\mathfrak{t}_3 + \mathfrak{T}_1) + \left[2r + 2r^{\frac{3}{2}} F - \frac{21}{5} r^{\frac{5}{2}} F' \right. \\ &\quad - \frac{4}{5} r^{\frac{7}{2}} F'' + (4r^3 - 1) F F' + r^4 f F F'' + 5r^{\frac{3}{2}} F_j - 15r^2 F F_j \\ &\quad + 2(4r^3 - 1) F' F_j - (5r^3 - 2) F F'_j + r^4 f F' F'_j + 2r^4 f F'' F_j \\ &\quad \left. - \frac{15}{2} r^2 F F_k + 4r^3 F' F_k - \frac{13}{2} r^3 F F'_k + r^4 F' F'_k + r^4 F'' F_k - r^4 F F''_k \right] \mathfrak{T}_4 \\ &\quad - \left(\frac{4}{r^2} + 8r \right) \mathfrak{T}_5 + \left(8r - 5r^{\frac{3}{2}} F + r^4 f F'^2 \right) \mathfrak{T}_6 \\ &\quad - \left(4r + 5r^{\frac{3}{2}} F + 4r^{\frac{5}{2}} F' \right) \mathfrak{T}_7, \end{aligned} \quad (5.33)$$

Compared with the compactified D4-brane in [13], the above has the same coefficient functions for viscous tensors $\mathbf{t}_3, \mathfrak{T}_{1,5,6,7}$ (the definitions for $\mathfrak{T}_{5,6,7}$ are different from [13]). Only the coefficient function of $\mathfrak{T}_4$ is different. Since coefficient functions determine the transport coefficients. Thus confirms that $\tau_\pi, \lambda_{1,2}$ of D4-brane are the same as the compactified D4-brane but τ_π^* is different. The solution of eq. (5.33) can be got from eq. (5.18) as

$$\begin{aligned} \alpha_{ij}^{(2)} = & \left(\frac{4}{3} - \frac{\pi}{9\sqrt{3}} - \frac{\ln 3}{3} \right) \frac{1}{r^3} (\mathbf{t}_3 + \mathfrak{T}_1) + \left[\frac{2}{r} + \left(\frac{2}{5} - \frac{\pi}{45\sqrt{3}} - \frac{\ln 3}{15} \right) \frac{1}{r^3} \right] \mathfrak{T}_4 \\ & + \frac{4}{r} \mathfrak{T}_5 + \left(\frac{4}{r} + \frac{4}{3r^3} \right) \mathfrak{T}_6 + \left[\frac{8}{r} - \left(\frac{2\pi}{9\sqrt{3}} + \frac{2\ln 3}{3} \right) \frac{1}{r^3} \right] \mathfrak{T}_7. \end{aligned} \quad (5.34)$$

The differential equation for the 2nd order vector perturbation of D4-brane is

$$\begin{aligned} \partial_r(r^4 \partial_r w_i^{(2)}) = & \left(-2r - r^{\frac{5}{2}} F' + \frac{10}{3} r^{\frac{3}{2}} F_j - \frac{4}{3} r^{\frac{5}{2}} F_j' \right) \mathbf{v}_4 + \left(\frac{10}{3} r^{\frac{3}{2}} F_j - \frac{4}{3} r^{\frac{5}{2}} F_j' \right) \mathbf{v}_5 \\ & + \left(\frac{3}{4} r - \frac{29}{8} r^{\frac{5}{2}} F' - 2r^{\frac{7}{2}} F'' + \frac{45}{8} r^{\frac{3}{2}} F_j + \frac{11}{4} r^{\frac{5}{2}} F_j' - 2r^{\frac{7}{2}} F_j'' \right) \mathfrak{V}_1 \\ & + \left(5r - \frac{3}{2} r^{\frac{5}{2}} F' + \frac{5}{2} r^{\frac{3}{2}} F_j - r^{\frac{5}{2}} F_j' \right) \mathfrak{V}_2 \\ & + \left(-r - \frac{9}{2} r^{\frac{5}{2}} F' + \frac{5}{2} r^{\frac{3}{2}} F_j - r^{\frac{5}{2}} F_j' \right) \mathfrak{V}_3, \end{aligned} \quad (5.35)$$

whose solution can be got from eq. (5.20) as

$$w_i^{(2)} = -\frac{1}{r} \mathfrak{V}_1 - \frac{4}{r} \mathfrak{V}_2 - \frac{4}{r} \mathfrak{V}_3 + \mathcal{O}\left(\frac{1}{r^{7/2}}\right). \quad (5.36)$$

Here we only keep the leading order for the vector perturbation, the same as D3-brane. The D1 and D2-brane will be treated similarly. The vector constraint is

$$\begin{aligned} \partial_i r_H^{(1)} = & \left[-\frac{4}{3} r^{\frac{5}{2}} - \frac{2}{3} r^4 f F' + \frac{8}{9} (5r^3 - 2) F_j + \frac{20}{9} r^3 F_k + \frac{8}{9} r^4 F_k' \right] \mathbf{v}_4 \\ & + \left[\frac{8}{9} (5r^3 - 2) F_j + \frac{20}{9} r^3 F_k + \frac{8}{9} r^4 F_k' \right] \mathbf{v}_5 \\ & + \left[\frac{35}{6} r^{\frac{5}{2}} - \frac{11}{12} r F' + \frac{35}{12} r^4 F' - \frac{5}{3} F_j - \frac{95}{6} r^3 F_j - \frac{95}{12} r^3 F_k - \frac{5}{2} r^4 F_k' \right] \mathfrak{V}_1 \\ & + \left[-2r^{\frac{5}{2}} - r^4 f F' + \frac{2}{3} (5r^3 - 2) F_j + \frac{5}{3} r^3 F_k + \frac{2}{3} r^4 F_k' \right] \mathfrak{V}_2 \\ & + \left[-6r^{\frac{5}{2}} - 3r^4 f F' + \frac{2}{3} (5r^3 - 2) F_j + \frac{5}{3} r^3 F_k + \frac{2}{3} r^4 F_k' \right] \mathfrak{V}_3. \end{aligned} \quad (5.37)$$

After expanded, eq. (5.37) becomes

$$\partial_i r_H^{(1)} = \frac{4}{45} \mathbf{v}_4 + \frac{64}{45} \mathbf{v}_5 + \frac{1}{2} \mathfrak{V}_1 - \frac{14}{15} \mathfrak{V}_2 - \frac{74}{15} \mathfrak{V}_3, \quad (5.38)$$

which is just eq. (5.17) with $p = 4$.

In the scalar sector, the first scalar perturbation

$$\begin{aligned}
 \partial_0 r_H^{(1)} = & \left(\frac{4}{5r^{\frac{1}{2}}} + \frac{4}{15}r^{\frac{5}{2}} - \frac{1}{5}F + \frac{2}{15}r^4 fF' - \frac{2}{3}r^3 fF_j - \frac{1}{3}r^3 F_k \right) \mathbf{s}_2 + \frac{2}{5r^{\frac{1}{2}}} \mathbf{s}_3 \\
 & + \left(-\frac{8}{5r^{\frac{1}{2}}} + \frac{4}{15}r^{\frac{5}{2}} - \frac{1}{5}F + \frac{2}{15}r^4 fF' - \frac{2}{3}r^3 fF_j - \frac{1}{3}r^3 F_k \right) \mathfrak{S}_1 \\
 & + \left(\frac{1}{25r^{\frac{1}{2}}} + \frac{1}{15}r^{\frac{5}{2}} - \frac{1}{25}F + \frac{17}{150}r^4 fF' + \frac{4}{75}r^5 fF'' - \frac{2}{15}r^3 fF_j - \frac{4}{15}r^4 fF'_j \right. \\
 & \quad \left. - \frac{1}{15}r^3 F_k \right) \mathfrak{S}_3 - \frac{4}{5r^{\frac{1}{2}}} \mathfrak{S}_4 + \left(\frac{4}{5r^{\frac{1}{2}}} + \frac{4}{15}r^{\frac{5}{2}} + \frac{2}{15}r^4 fF' \right) \mathfrak{S}_5
 \end{aligned} \tag{5.39}$$

will reproduce the scalar component of Navier-Stokes equations eq. (5.16) with $p = 4$. To get this, one needs to expand F, F_j and F_k in eq. (5.39) and has:

$$\partial_0 r_H^{(1)} = \frac{1}{75} \mathfrak{S}_3 + \frac{4}{15} \mathfrak{S}_5 + \frac{4}{5r^{1/2}} \left(\mathbf{s}_2 + \frac{1}{2} \mathbf{s}_3 - 2\mathfrak{S}_1 + \frac{1}{20} \mathfrak{S}_3 - \mathfrak{S}_4 + \mathfrak{S}_5 \right). \tag{5.40}$$

The part inside the parenthesis is just the constraints eq. (5.11). Thus we reproduce eq. (5.16) for the case of $p = 4$.

In order to solve h, j, k at the second order, we use the second scalar perturbation eq. (3.11):

$$\begin{aligned}
 5(r^3 k_{(2)})' + 30r^2 j_{(2)} - 4(5r^3 - 2)h'_{(2)} = & \left(-16r + 5r^{\frac{3}{2}}F + 2r^{\frac{5}{2}}F' \right) \mathbf{s}_2 \\
 & + \left(24r + 5r^{\frac{3}{2}}F + 2r^{\frac{5}{2}}F' \right) \mathfrak{S}_1 + \left[-\frac{5}{2}r + r^{\frac{5}{2}}F + \frac{26}{5}r^{\frac{5}{2}}F' + \frac{4}{5}r^{\frac{7}{2}}F'' + \frac{3}{8}r^4 fF'^2 \right. \\
 & - \frac{1}{4}(5r^3 - 2)FF' - 10r^{\frac{2}{3}}F_j + 45r^2 F_j^2 - 2(5r^3 - 2)F'F_j - 5r^3 F'F_k - r^4 F'F'_k \\
 & \left. + 30r^2 F_j F_k + 10r^3 F_j F'_k \right] \mathfrak{S}_3 + \left(\frac{2}{r^2} + 20r \right) \mathfrak{S}_4 \\
 & - \left[18r + (5r^3 - 2)FF' + \frac{1}{2}r^4 fF'^2 \right] \mathfrak{S}_5,
 \end{aligned} \tag{5.41}$$

the (rr) component of Einstein equation (3.7):

$$\begin{aligned}
 4rh''_{(2)} + 6h'_{(2)} - 5j'_{(2)} = & \left(\frac{1}{8}rF'^2 + \frac{3}{8}FF' + \frac{1}{4}rFF'' - 5F_j F'_j + rF'F'_j \right) \mathfrak{S}_3 \\
 & + \frac{2}{r^2} \mathfrak{S}_4 + \left(\frac{1}{2}rF' + \frac{3}{2}FF' + rFF'' \right) \mathfrak{S}_5,
 \end{aligned} \tag{5.42}$$

and the dilaton's EOM:

$$\begin{aligned}
 (r^3 k_{(2)})' + r^3 f j'_{(2)} + 6r^2 j_{(2)} - 2r^3 f h'_{(2)} = & \left(-2r + \frac{1}{2}r^{\frac{3}{2}}F \right) \mathbf{s}_2 + \left(6r + \frac{1}{2}r^{\frac{3}{2}}F \right) \mathfrak{S}_1 \\
 & + \left[-\frac{1}{2}r + \frac{1}{10}r^{\frac{3}{2}}F + \frac{1}{5}r^{\frac{5}{2}}F' - \frac{1}{8}r^3 fFF' - r^{\frac{3}{2}}F_j + 9r^2 F_j^2 + 3r^3 fF_j F'_j - r^3 fF'F_j \right. \\
 & \left. - \frac{1}{2}r^3 F'F_k + 6r^2 F_j F_k + r^3 F'_j F_k + 2r^3 F_j F'_k \right] \mathfrak{S}_3 + 2r \mathfrak{S}_4 - \left(2r + \frac{1}{2}r^3 fFF' \right) \mathfrak{S}_5
 \end{aligned} \tag{5.43}$$

to eliminate j and k and get the different equation for $h_{(2)}$, then $h_{(2)}$ can be solved by using eq. (5.18). The final results for the 2nd order scalar perturbations of D4-brane are

$$\begin{aligned}
 h^{(2)} &= \left(\frac{1}{3} - \frac{\pi}{36\sqrt{3}} - \frac{\ln 3}{12} \right) \frac{1}{r^3} (s_2 + \mathfrak{S}_1) + \left[\frac{1}{4r} + \left(\frac{1}{60} - \frac{\pi}{180\sqrt{3}} - \frac{\ln 3}{60} \right) \frac{1}{r^3} \right] \mathfrak{S}_3 \\
 &\quad + \frac{1}{r} \mathfrak{S}_4 + \left(\frac{1}{r} + \frac{1}{3r^3} \right) \mathfrak{S}_5, \\
 j^{(2)} &= - \left(\frac{2}{3} - \frac{\pi}{18\sqrt{3}} - \frac{\ln 3}{6} \right) \frac{1}{r^3} (s_2 + \mathfrak{S}_1) - \left(\frac{1}{30} - \frac{\pi}{90\sqrt{3}} - \frac{\ln 3}{30} \right) \frac{1}{r^3} \mathfrak{S}_3 \\
 &\quad - \frac{2}{3r^3} \mathfrak{S}_5, \\
 r^3 k^{(2)} &= \left[- \left(\frac{4}{15} - \frac{\pi}{45\sqrt{3}} - \frac{\ln 3}{15} \right) + \frac{1}{10r} - \left(\frac{2}{3} - \frac{\pi}{18\sqrt{3}} - \frac{\ln 3}{6} \right) \frac{1}{r^3} \right] s_2 \\
 &\quad + \left[4r^2 - \left(\frac{4}{15} - \frac{\pi}{45\sqrt{3}} - \frac{\ln 3}{15} \right) + \frac{1}{10r} - \left(\frac{2}{3} - \frac{\pi}{18\sqrt{3}} - \frac{\ln 3}{6} \right) \frac{1}{r^3} \right] \mathfrak{S}_1 \\
 &\quad + \left[- \left(\frac{1}{75} - \frac{\pi}{225\sqrt{3}} - \frac{\ln 3}{75} \right) + \frac{2}{35r^{1/2}} + \frac{163}{100r} + \left(\frac{1}{90} + \frac{\pi}{90\sqrt{3}} + \frac{\ln 3}{30} \right) \frac{1}{r^3} \right] \mathfrak{S}_3 \\
 &\quad - \frac{2}{r} \mathfrak{S}_4 - \left(\frac{4}{15} - \frac{8}{7r^{1/2}} - \frac{1}{r} + \frac{2}{3r^3} \right) \mathfrak{S}_5. \tag{5.44}
 \end{aligned}$$

Here the constants at the right hand side of the solution of $k^{(2)}$ are integration constants, which are fixed by requiring the boundary stress-energy tensor is in Landau frame. We do not need to fix the integration constants for D3-brane because the scalar perturbations are trivial and do not contribute to the stress-energy tensor.

5.4 The D1-brane case

The D1-brane case is very different from the other cases for no viscous tensors can exist in this case. We list the viscous scalars and vectors for $p = 1$ in table 2. Note we define a new viscous vector $\mathbf{v}_3 = \partial_1^2 \beta_1$ for D1-brane out of a linear combination of $\mathbf{v}_4$ and $\mathbf{v}_5$ since both of them can not exist for $p = 1$. The viscous terms for D1-brane case satisfy the following 4 constraints:

$$s_1 + \frac{1}{4}s_2 - \frac{1}{4}\mathfrak{S}_1 - \frac{1}{16}\mathfrak{S}_3 = 0, \tag{5.45}$$

$$s_2 + 2s_3 - \frac{1}{2}\mathfrak{S}_1 + \frac{1}{2}\mathfrak{S}_3 = 0, \tag{5.46}$$

$$\mathbf{v}_1 + \frac{1}{2}\mathbf{v}_2 + \frac{1}{8}\mathfrak{V}_1 = 0, \tag{5.47}$$

$$\mathbf{v}_1 + \frac{1}{4}\mathbf{v}_3 - \frac{3}{8}\mathfrak{V}_1 = 0, \tag{5.48}$$

which can be got from the general form of the constraints eqs. (5.10), (5.12), (5.13), (5.11) by specifying $p = 1$.

Scalars of SO(1)	Vectors of SO(1)
$\mathbf{s}_1 = \frac{1}{r_H} \partial_0^2 r_H$	$\mathbf{v}_1 = \frac{1}{r_H} \partial_0 \partial_1 r_H$
$\mathbf{s}_2 = \partial_0 \partial_1 \beta_1$	$\mathbf{v}_2 = \partial_0^2 \beta_1$
$\mathbf{s}_3 = \frac{1}{r_H} \partial_1^2 r_H$	$\mathbf{v}_3 = \partial_1^2 \beta_1$
$\mathfrak{S}_1 = \partial_0 \beta_1 \partial_0 \beta_1$	$\mathfrak{V}_1 = \partial_0 \beta_1 \partial_1 \beta_1$
$\mathfrak{S}_3 = (\partial_1 \beta_1)^2$	

Table 2. The list of all the second order viscous terms for D1-brane. The vector viscous terms are reduced to scalars in this case but we still call them vectors.

There is no tensor perturbation for D1-brane, thus we start from the vector perturbation. The vector constraint for D1-brane is

$$\begin{aligned} \partial_1 r_H^{(1)} = & \left[-\frac{1}{6} F_j + \frac{2}{3} r^6 F_j + \frac{1}{3} r^6 F_k + \frac{1}{12} r^7 F'_k \right] \mathbf{v}_3 \\ & + \left[\frac{1}{6} r^4 + \frac{1}{24} r F' + \frac{1}{12} r^7 F' - \frac{1}{3} (2r^6 + 1) F_j - \frac{1}{3} r^6 F_k \right] \mathfrak{V}_1. \end{aligned} \quad (5.49)$$

After expansion the above becomes

$$\partial_1 r_H^{(1)} = \frac{1}{12} \mathbf{v}_3, \quad (5.50)$$

which is just the spatial component of the Navier-Stokes equation (5.17). The vector dynamical equation is

$$\partial_r (r^7 \partial_r w_1^{(2)}) = (4r^3 F_j - r^4 F'_j) \mathbf{v}_3 + \left(-\frac{5}{2} r^4 F' - \frac{1}{2} r^5 F'' + 12r^3 F_j + \frac{1}{2} r^4 F'_j - \frac{1}{2} r^5 F''_j \right) \mathfrak{V}_1 \quad (5.51)$$

from which we get

$$w_1^{(2)} = -\frac{1}{4r^4} \mathfrak{V}_1. \quad (5.52)$$

The scalar constraint for D1-brane is

$$\begin{aligned} \partial_0 r_H^{(1)} = & \left(\frac{1}{8r^2} + \frac{1}{12} r^4 - \frac{1}{8} F + \frac{1}{24} r^7 f F' - \frac{1}{3} r^6 f F_j - \frac{1}{6} r^6 F_k \right) \mathbf{s}_2 + \frac{1}{4r^2} \mathbf{s}_3 \\ & + \left(-\frac{1}{16r^2} + \frac{1}{12} r^4 - \frac{1}{8} F + \frac{1}{24} r^7 f F' - \frac{1}{3} r^6 f F_j - \frac{1}{6} r^6 F_k \right) \mathfrak{S}_1 \\ & + \left(\frac{1}{16r^2} + \frac{1}{12} r^4 - \frac{1}{16} F + \frac{7}{96} r^7 f F' + \frac{1}{96} r^8 f F'' - \frac{1}{6} r^6 f F_j - \frac{1}{12} r^7 f F'_j \right. \\ & \quad \left. - \frac{1}{12} r^6 F_k \right) \mathfrak{S}_3 \end{aligned} \quad (5.53)$$

under the large r expansion one has

$$\partial_0 r_H^{(1)} = \frac{1}{24} \mathfrak{S}_3 + \frac{1}{8r^2} \left(\mathbf{s}_2 + 2\mathbf{s}_3 - \frac{1}{2} \mathfrak{S}_1 + \frac{1}{2} \mathfrak{S}_3 \right) \quad (5.54)$$

The terms inside the parenthesis in the above equation is just the 2nd order constraint of D1-brane eq. (5.46) thus gives no contribution. So we get the scalar component of the Navier-Stokes equation for D1-brane as

$$\partial_0 r_H^{(1)} = \frac{1}{24} \mathfrak{S}_3, \quad (5.55)$$

which can be got from eq. (5.16) by setting $p = 1$ and remove $\mathfrak{S}_5$ (there is no shear tensor for D1-brane). The dynamical equation for the scalar perturbations are the second scalar constraint eq. (3.11)

$$\begin{aligned} & 4(r^6 k_{(2)})' + 48r^5 j_{(2)} - (4r^6 - 1)h_{(2)}' \\ &= (-2r + 4r^3 F + r^4 F') \mathfrak{s}_2 + (4r^3 F + r^4 F') \mathfrak{S}_1 + \left[-2r + 2r^3 F + \frac{11}{4} r^4 F' + \frac{1}{4} r^5 F'' \right. \\ &\quad - (4r^6 - 1) F F' - 8r^3 F_j + 72r^5 F_j^2 - 2(4r^6 - 1) F' F_j - 4r^6 F' F_k - \frac{1}{2} r^7 F' F'_k \\ &\quad \left. + 48r^5 F_j F_k + 8r^6 F_j F'_k \right] \mathfrak{S}_3, \end{aligned} \quad (5.56)$$

the (rr) component of Einstein equation (3.7)

$$r h_{(2)}'' + 3h_{(2)}' - 8j_{(2)}' = \left(\frac{1}{2} r F'^2 + 3F F' + r F F'' - 8F_j F'_j + r F' F'_j \right) \mathfrak{S}_3, \quad (5.57)$$

and the EOM of ϕ eq. (3.8)

$$\begin{aligned} & 2(r^6 k_{(2)})' + 2r^6 f j_{(2)}' + 24r^5 j_{(2)} - r^6 f h_{(2)}' = (-r + r^3 F) \mathfrak{s}_2 + r^3 F \mathfrak{S}_1 \\ &+ \left[-r + \frac{1}{2} r^3 F + \frac{1}{4} r^4 F' - r^6 f F F' - 2r^3 F_j + 36r^5 F_j^2 + 6r^6 f F_j F'_j - 2r^6 f F' F_j \right. \\ &\quad \left. - r^6 F' F_k + 24r^5 F_j F_k + 2r^6 F'_j F_k + 4r^6 F_j F'_k \right] \mathfrak{S}_3, \end{aligned} \quad (5.58)$$

from which we can solve all the scalar perturbations as

$$h^{(2)} = \left(\frac{1}{6} + \frac{\pi}{36\sqrt{3}} - \frac{\ln 3}{12} \right) \frac{1}{r^6} (\mathfrak{s}_2 + \mathfrak{S}_1) + \left[\frac{1}{4r^4} + \left(\frac{1}{12} + \frac{\pi}{72\sqrt{3}} - \frac{\ln 3}{24} \right) \frac{1}{r^6} \right] \mathfrak{S}_3, \quad (5.59)$$

$$j^{(2)} = - \left(\frac{1}{12} + \frac{\pi}{72\sqrt{3}} - \frac{\ln 3}{24} \right) \frac{1}{r^6} (\mathfrak{s}_2 + \mathfrak{S}_1) - \left(\frac{1}{24} + \frac{\pi}{144\sqrt{3}} - \frac{\ln 3}{48} \right) \frac{1}{r^6} \mathfrak{S}_3, \quad (5.60)$$

$$\begin{aligned} r^6 k^{(2)} = & - \left[\left(\frac{1}{12} + \frac{\pi}{72\sqrt{3}} - \frac{\ln 3}{24} \right) - \frac{1}{40r^4} + \left(\frac{1}{12} + \frac{\pi}{72\sqrt{3}} - \frac{\ln 3}{24} \right) \frac{1}{r^6} \right] \mathfrak{s}_2 \\ & + \left[\frac{1}{4} r^2 - \left(\frac{1}{12} + \frac{\pi}{72\sqrt{3}} - \frac{\ln 3}{24} \right) + \frac{1}{40r^4} - \left(\frac{1}{12} + \frac{\pi}{72\sqrt{3}} - \frac{\ln 3}{24} \right) \frac{1}{r^6} \right] \mathfrak{S}_1 \\ & - \left[\left(\frac{1}{24} + \frac{\pi}{144\sqrt{3}} - \frac{\ln 3}{48} \right) - \frac{1}{16r^2} - \frac{1}{16r^4} \right. \\ & \left. + \left(\frac{1}{72} + \frac{\pi}{144\sqrt{3}} - \frac{\ln 3}{48} \right) \frac{1}{r^6} \right] \mathfrak{S}_3. \end{aligned} \quad (5.61)$$

5.5 The D2-brane case

The D2-brane case is the most complicated one and that's why we leave it to the last. The differential equation for the tensor perturbation of D2-brane is

$$\begin{aligned}
 \partial_r \left(r^6 f \partial_r \alpha_{ij}^{(2)} \right) = & \left(\frac{2}{3} r - \frac{7}{2} r^{\frac{5}{2}} F - 2 r^{\frac{7}{2}} F' \right) (t_3 + \mathfrak{T}_1) + \left[\frac{2}{3} r - \frac{3}{2} r^{\frac{5}{2}} F - \frac{31}{7} r^{\frac{7}{2}} F' \right. \\
 & - \frac{4}{7} r^{\frac{9}{2}} F'' + \frac{1}{2} r^6 f F'^2 + (6r^5 - 1) F F' + r^6 f F F'' + 7 r^{\frac{5}{2}} F_j - 35 r^4 F F_j \\
 & + 2(6r^5 - 1) F' F_j + 2 r^6 f F'' F_j - (7r^5 - 2) F F'_j + r^6 f F' F'_j - \frac{35}{2} r^4 F F_k \\
 & + 6 r^5 F' F_k + r^6 F'' F_k - \frac{19}{2} r^5 F F'_k + r^6 F' F'_k - r^6 F F''_k \left. \right] \mathfrak{T}_4 \\
 & - \left(4r + 7 r^{\frac{5}{2}} F + 4 r^{\frac{7}{2}} F' \right) \mathfrak{T}_7.
 \end{aligned} \tag{5.62}$$

Note in the source term there is no $\mathfrak{T}_{5,6}$ at present since both of them are zero for $p = 2$. Using the relations between $F_{j,k}$ and F eq. (3.29) and the differential equation for F eq. (3.12) at the case $p = 2$ will simplify the source term and make it easy to be integrated. The solution can be got from eq. (5.18) as

$$\begin{aligned}
 \alpha_{ij}^{(2)}(r) = & \left(\frac{4}{15} + \frac{\pi}{25} \sqrt{1 - \frac{2}{\sqrt{5}}} + \frac{1}{5\sqrt{5}} \operatorname{arccoth} \sqrt{5} - \frac{\ln 5}{10} \right) \frac{1}{r^5} (t_3 + \mathfrak{T}_1) \\
 & + \left[\frac{4}{9r^3} + \left(\frac{16}{105} + \frac{3\pi}{175} \sqrt{1 - \frac{2}{\sqrt{5}}} + \frac{3}{35\sqrt{5}} \operatorname{arccoth} \sqrt{5} - \frac{3}{70} \ln 5 \right) \frac{1}{r^5} \right] \mathfrak{T}_4 \\
 & + \left[\frac{8}{9r^3} + \left(\frac{2\pi}{25} \sqrt{1 - \frac{2}{\sqrt{5}}} + \frac{2}{5\sqrt{5}} \operatorname{arccoth} \sqrt{5} - \frac{\ln 5}{5} \right) \frac{1}{r^5} \right] \mathfrak{T}_7.
 \end{aligned} \tag{5.63}$$

As one can see from the above that the solution of D2-brane is indeed more complicated than the other nonconformal situations: except the terms proportional to π and logarithm, there appears another arccoth term.

The 2nd order vector constraint for D2-brane is

$$\begin{aligned}
 \partial_i r_H^{(1)} = & \left[-\frac{4}{15} r^{\frac{7}{2}} - \frac{2}{15} r^6 f F' + \frac{4}{15} (7r^5 - 2) F_j + \frac{14}{15} r^5 F_k + \frac{4}{15} r^6 F'_k \right] \mathbf{v}_4 \\
 & + \left[\frac{4}{15} (7r^5 - 2) F_j + \frac{14}{15} r^5 F_k + \frac{4}{15} r^6 F'_k \right] \mathbf{v}_5 \\
 & + \left[\frac{23}{45} r^{\frac{7}{2}} - \frac{1}{30} r F' + \frac{23}{90} r^6 F' - \frac{2}{5} F_j - \frac{77}{45} r^5 F_j - \frac{77}{90} r^5 F_k - \frac{1}{9} r^6 F'_k \right] \mathfrak{V}_1 \\
 & + \left[-\frac{2}{5} r^{\frac{7}{2}} - \frac{1}{5} r^6 f F' + \frac{2}{15} (7r^5 - 2) F_j + \frac{7}{15} r^5 F_k + \frac{2}{15} r^6 F'_k \right] \mathfrak{V}_2 \\
 & + \left[-\frac{22}{45} r^{\frac{7}{2}} - \frac{11}{45} r^6 f F' + \frac{2}{15} (7r^5 - 2) F_j + \frac{7}{15} r^5 F_k + \frac{2}{15} r^6 F'_k \right] \mathfrak{V}_3.
 \end{aligned} \tag{5.64}$$

After large r expansion, the above becomes

$$\partial_i r_H^{(1)} = \frac{4}{105} \mathbf{v}_4 + \frac{32}{105} \mathbf{v}_5 + \frac{37}{315} \mathfrak{V}_1 - \frac{26}{105} \mathfrak{V}_2 - \frac{106}{315} \mathfrak{V}_3. \tag{5.65}$$

This is just the vector component of Navier-Stokes equation (5.17) when $p = 2$. The vector dynamical equation is

$$\begin{aligned} \partial_r(r^6 \partial_r w_i^{(2)}) = & \left(-2r - r^{\frac{7}{2}} F' + 7r^{\frac{5}{2}} F_j - 2r^{\frac{7}{2}} F'_j\right) \mathbf{v}_4 + \left(7r^{\frac{5}{2}} F_j - 2r^{\frac{7}{2}} F'_j\right) \mathbf{v}_5 \\ & + \left(\frac{1}{2}r - \frac{25}{12}r^{\frac{7}{2}} F' - \frac{2}{3}r^{\frac{9}{2}} F'' + \frac{35}{4}r^{\frac{5}{2}} F_j + \frac{7}{6}r^{\frac{7}{2}} F'_j - \frac{2}{3}r^{\frac{9}{2}} F''_j\right) \mathfrak{V}_1 \\ & + \left(-\frac{1}{3}r - \frac{3}{2}r^{\frac{7}{2}} F' + \frac{7}{2}r^{\frac{5}{2}} F_j - r^{\frac{7}{2}} F'_j\right) \mathfrak{V}_2 \\ & + \left(-r - \frac{11}{6}r^{\frac{7}{2}} F' + \frac{7}{2}r^{\frac{5}{2}} F_j - r^{\frac{7}{2}} F'_j\right) \mathfrak{V}_3, \end{aligned} \quad (5.66)$$

whose leading order solution is

$$w_i^{(2)} = -\frac{2}{9r^3} \mathfrak{V}_1 - \frac{4}{9r^3} \mathfrak{V}_2 - \frac{4}{9r^3} \mathfrak{V}_3. \quad (5.67)$$

The first scalar constraint for $p = 2$ is

$$\begin{aligned} \partial_0 r_H^{(1)} = & \left(\frac{4}{21r^{3/2}} + \frac{4}{35}r^{\frac{7}{2}} - \frac{1}{7}F + \frac{2}{35}r^6 f F' - \frac{2}{5}r^5 f F_j - \frac{1}{5}r^5 F_k\right) \mathbf{s}_2 + \frac{2}{7r^{3/2}} \mathbf{s}_3 \\ & + \left(-\frac{8}{63r^{3/2}} + \frac{4}{35}r^{\frac{7}{2}} - \frac{1}{7}F + \frac{2}{35}r^6 f F' - \frac{2}{5}r^5 f F_j - \frac{1}{5}r^5 F_k\right) \mathfrak{S}_1 \\ & + \left(\frac{2}{147r^{3/2}} + \frac{2}{35}r^{\frac{7}{2}} - \frac{3}{49}F + \frac{17}{245}r^6 f F' + \frac{4}{245}r^7 f F'' - \frac{6}{35}r^5 f F_j - \frac{4}{35}r^6 f F'_j \right. \\ & \quad \left. - \frac{3}{35}r^5 F_k\right) \mathfrak{S}_3 - \frac{4}{21r^{3/2}} \mathfrak{S}_4 + \left(\frac{4}{21r^{3/2}} + \frac{4}{35}r^{\frac{7}{2}} + \frac{2}{35}r^6 f F'\right) \mathfrak{S}_5. \end{aligned} \quad (5.68)$$

After expansion, the above will reproduce the scalar component of the Navier-Stokes equation (5.16) for $p = 2$:

$$\partial_0 r_H^{(1)} = \frac{2}{245} \mathfrak{S}_3 + \frac{4}{35} \mathfrak{S}_5 + \frac{4}{21r^{\frac{3}{2}}} \left(\mathbf{s}_2 + \frac{3}{2} \mathbf{s}_3 - \frac{2}{3} \mathfrak{S}_1 + \frac{1}{14} \mathfrak{S}_3 - \mathfrak{S}_4 + \mathfrak{S}_5\right). \quad (5.69)$$

The terms in the parenthesis of the above is just the second order constraint eq. (5.11) at $p = 2$ thus gives 0. So we finally have

$$\partial_0 r_H^{(1)} = \frac{2}{245} \mathfrak{S}_3 + \frac{4}{35} \mathfrak{S}_5. \quad (5.70)$$

The above equation is just the scalar component of Navier-Stokes equation at $p = 2$.

The scalar dynamical equations are still the second scalar constraint eq. (3.11)

$$\begin{aligned} & 7(r^5 k_{(2)})' + 70r^4 j_{(2)} - 2(7r^5 - 2)h'_{(2)} \\ = & \left(-\frac{16}{3}r + 7r^{\frac{5}{2}} F + 2r^{\frac{7}{2}} F'\right) \mathbf{s}_2 + \left(\frac{8}{9}r + 7r^{\frac{5}{2}} F + 2r^{\frac{7}{2}} F'\right) \mathfrak{S}_1 + \left[-\frac{5}{3}r + 3r^{\frac{5}{2}} F \right. \\ & \quad + \frac{38}{7}r^{\frac{7}{2}} F' + \frac{4}{7}r^{\frac{9}{2}} F'' + \frac{1}{4}r^6 f F'^2 - \frac{1}{2}(7r^5 - 2)FF' - 14r^{\frac{5}{2}} F_j + 105r^4 F_j^2 \\ & \quad \left. - 2(7r^5 - 2)F'F_j - 7r^5 F'F_k - r^6 F'F'_k + 70r^4 F_j F_k + 14r^5 F_j F'_k\right] \mathfrak{S}_3 \\ & + \left(\frac{2}{r^4} + \frac{28}{3}r\right) \mathfrak{S}_4 - \left[\frac{22}{3}r + \frac{1}{2}r^6 f F'^2 + (7r^5 - 2)FF'\right] \mathfrak{S}_5, \end{aligned} \quad (5.71)$$

the (rr) component of the Einstein equation (3.7)

$$2rh_{(2)}'' + 5h_{(2)}' - 7j_{(2)}' = \left(\frac{1}{4}rF'^2 + \frac{5}{4}FF' + \frac{1}{2}rFF'' - 7F_jF_j' + rF'F_j' \right) \mathfrak{S}_3 \\ + \frac{2}{r^4} \mathfrak{S}_4 + \left(\frac{1}{2}rF'^2 + \frac{5}{2}FF' + rFF'' \right) \mathfrak{S}_5, \quad (5.72)$$

and the EOM of dilaton eq. (3.8)

$$(r^5k_{(2)})' + r^5fj_{(2)}' + 10r^4j_{(2)} - r^5fh_{(2)}' = \left(-\frac{2}{3}r + \frac{1}{2}r^{\frac{5}{2}}F \right) \mathfrak{s}_2 \\ + \left(\frac{2}{9}r + \frac{1}{2}r^{\frac{5}{2}}F \right) \mathfrak{S}_1 + \left[-\frac{1}{3}r + \frac{3}{14}r^{\frac{5}{2}}F + \frac{1}{7}r^{\frac{7}{2}}F' - \frac{1}{4}r^5fFF' - r^{\frac{5}{2}}F_j + 15r^4F_j^2 \right. \\ \left. - r^5fF'F_j + 3r^5fF_jF_j' - \frac{1}{2}r^5F'F_k + 10r^4F_jF_k + r^5F_j'F_k + 2r^5F_jF_k' \right] \mathfrak{S}_3 \\ + \frac{2}{3}r\mathfrak{S}_4 - \left(\frac{2}{3}r + \frac{1}{2}r^5fFF' \right) \mathfrak{S}_5. \quad (5.73)$$

From which we can solve the scalar perturbations for $p = 2$ as

$$h^{(2)} = \left(\frac{2}{15} + \frac{\pi}{50}\sqrt{1 - \frac{2}{\sqrt{5}}} + \frac{1}{10\sqrt{5}} \operatorname{arccoth} \sqrt{5} - \frac{\ln 5}{20} \right) \frac{1}{r^5} (\mathfrak{s}_2 + \mathfrak{S}_1) \\ + \left[\frac{1}{9r^3} + \left(\frac{1}{105} + \frac{3\pi}{350}\sqrt{1 - \frac{2}{\sqrt{5}}} + \frac{3}{70\sqrt{5}} \operatorname{arccoth} \sqrt{5} - \frac{3}{140} \ln 5 \right) \frac{1}{r^5} \right] \mathfrak{S}_3 \\ + \frac{2}{9r^3} \mathfrak{S}_4 + \left(\frac{2}{9r^3} + \frac{2}{15r^5} \right) \mathfrak{S}_5, \quad (5.74)$$

$$j^{(2)} = - \left(\frac{2}{15} + \frac{\pi}{50}\sqrt{1 - \frac{2}{\sqrt{5}}} + \frac{1}{10\sqrt{5}} \operatorname{arccoth} \sqrt{5} - \frac{\ln 5}{20} \right) \frac{1}{r^5} (\mathfrak{s}_2 + \mathfrak{S}_1) \\ - \left(\frac{1}{105} + \frac{3\pi}{350}\sqrt{1 - \frac{2}{\sqrt{5}}} + \frac{3}{70\sqrt{5}} \operatorname{arccoth} \sqrt{5} - \frac{\ln 5}{140} \right) \frac{1}{r^5} \mathfrak{S}_3 - \frac{2}{15r^5} \mathfrak{S}_5, \quad (5.75)$$

$$r^5k^{(2)} = \left[- \left(\frac{4}{35} + \frac{3\pi}{175}\sqrt{1 - \frac{2}{\sqrt{5}}} + \frac{3}{35\sqrt{5}} \operatorname{arccoth} \sqrt{5} - \frac{3}{70} \ln 5 \right) + \frac{1}{28r^3} \right. \\ \left. - \left(\frac{2}{15} + \frac{\pi}{50}\sqrt{1 - \frac{2}{\sqrt{5}}} + \frac{1}{10\sqrt{5}} \operatorname{arccoth} \sqrt{5} - \frac{\ln 5}{20} \right) \frac{1}{r^5} \right] \mathfrak{s}_2 \\ + \left[\frac{4}{9}r^2 - \left(\frac{4}{35} + \frac{3\pi}{175}\sqrt{1 - \frac{2}{\sqrt{5}}} + \frac{3}{35\sqrt{5}} \operatorname{arccoth} \sqrt{5} - \frac{3}{70} \ln 5 \right) + \frac{1}{28r^3} \right. \\ \left. - \left(\frac{2}{15} + \frac{\pi}{50}\sqrt{1 - \frac{2}{\sqrt{5}}} + \frac{1}{10\sqrt{5}} \operatorname{arccoth} \sqrt{5} - \frac{\ln 5}{20} \right) \frac{1}{r^5} \right] \mathfrak{S}_1 \\ + \left[- \left(\frac{2}{245} + \frac{9\pi}{1225}\sqrt{1 - \frac{2}{\sqrt{5}}} + \frac{9}{245\sqrt{5}} \operatorname{arccoth} \sqrt{5} - \frac{9}{490} \ln 5 \right) + \frac{4}{273r^{\frac{3}{2}}} \right]$$

$$\begin{aligned}
 & + \frac{17}{147r^3} + \left(\frac{13}{525} - \frac{3\pi}{350} \sqrt{1 - \frac{2}{\sqrt{5}}} - \frac{3}{70\sqrt{5}} \operatorname{arccoth} \sqrt{5} + \frac{3}{140} \ln 5 \right) \frac{1}{r^5} \Big] \mathfrak{S}_3 \\
 & - \frac{2}{9r^3} \mathfrak{S}_4 + \left(-\frac{4}{35} + \frac{8}{39r^{\frac{3}{2}}} + \frac{1}{18r^3} - \frac{2}{15r^5} \right) \mathfrak{S}_5.
 \end{aligned} \tag{5.76}$$

6 The second order stress-energy tensor

6.1 Formulation of the result

Now, using the second order expanded metric eq. (5.3) and eq. (5.22) together with the already solved second order perturbations of Dp-brane, we can calculate the boundary stress-energy tensor via eq. (4.1). We will only offer the results here, the readers who are interested in the details can refer to [13].

In order to get the covariant form of the stress-energy tensor, we will need the following rules to substitute the 2nd order spatial viscous tensors:

$$\begin{aligned}
 \mathfrak{t}_{3ij} &= \partial_0 \sigma_{ij} \rightarrow \langle D \partial_\mu u_\nu \rangle, & \mathfrak{T}_{1ij} &= \partial_0 \beta_i \partial_0 \beta_j - \frac{1}{p} \delta_{ij} \mathfrak{s}_1 \rightarrow Du_{\langle \mu} Du_{\nu \rangle}, \\
 \mathfrak{T}_{4ij} &= \sigma_{ij} \partial \beta \rightarrow \sigma_{\mu\nu} \partial_\rho u^\rho, & \mathfrak{T}_{6ij} &= \sigma_{ik} \sigma_{kj} - \frac{1}{p} \delta_{ij} \mathfrak{s}_5 \rightarrow \sigma_{\langle \mu}{}^\rho \sigma_{\nu \rangle \rho}, \\
 \mathfrak{T}_{7ij} &= \sigma_{(i}{}^k \Omega_{j)k} \rightarrow \sigma_{\langle \mu}{}^\rho \Omega_{\nu \rangle \rho}
 \end{aligned} \tag{6.1}$$

Then the boundary stress-energy tensor for D3-brane under full consideration of dimension can be calculated as

$$\begin{aligned}
 T_{\mu\nu} &= \frac{1}{2\kappa_5^2} \left\{ \frac{r_H^4}{L_3^5} (3u_\mu u_\nu + P_{\mu\nu}) - \left(\frac{r_H}{L_3} \right)^3 \cdot 2\sigma_{\mu\nu} \right. \\
 &\quad \left. + \frac{r_H^2}{L_3} \left[\frac{2 - \ln 2}{2} \cdot 2 \left(\langle D \sigma_{\mu\nu} \rangle + \frac{1}{3} \sigma_{\mu\nu} \partial u \right) + \frac{1}{2} \cdot 4 \sigma_{\langle \mu}{}^\rho \sigma_{\nu \rangle \rho} - \ln 2 \cdot 2 \sigma_{\langle \mu}{}^\rho \Omega_{\nu \rangle \rho} \right] \right\}.
 \end{aligned} \tag{6.2}$$

Compare the above with eq. (1.1) one can get the 2nd order transport coefficients for D3-brane

$$\eta\tau_\pi = \frac{1}{2\kappa_5^2} \frac{2 - \ln 2}{2} \frac{r_H^2}{L_3}, \quad \lambda_1 = \frac{1}{2\kappa_5^2} \frac{1}{2} \frac{r_H^2}{L_3}, \quad \lambda_2 = -\frac{1}{2\kappa_5^2} \ln 2 \frac{r_H^2}{L_3}. \tag{6.3}$$

We have restored the parameters r_H , L_p and κ_{p+2}^2 which has length dimension in eqs. (6.2), (6.3). The results in eq. (6.3) have already been derived in e.g. [4, 6, 7].

As for the D4-brane, we need further substitution rules for the scalar viscous terms

$$\mathfrak{s}_2 + \mathfrak{S}_1 = \partial_0 \partial \beta + \partial_0 \beta_i \partial_0 \beta_i \rightarrow D(\partial u), \quad \mathfrak{S}_3 = (\partial \beta)^2 \rightarrow (\partial u)^2, \quad \mathfrak{S}_5 = \sigma_{ij}^2 \rightarrow \sigma_{\mu\nu}^2. \tag{6.4}$$

The boundary stress-energy tensor for D4-brane can then be formulated as

$$\begin{aligned}
 T_{\mu\nu} = & \frac{1}{2\kappa_6^2} \left\{ \frac{r_H^3}{L_4^4} \left(\frac{5}{2} u_\mu u_\nu + \frac{1}{2} P_{\mu\nu} \right) - \left(\frac{r_H}{L_4} \right)^{\frac{5}{2}} \left(2\sigma_{\mu\nu} + \frac{1}{10} \partial_\rho u^\rho P_{\mu\nu} \right) \right. \\
 & + \frac{r_H^2}{L_4} \left[\left(2 - \frac{\pi}{6\sqrt{3}} - \frac{\ln 3}{2} \right) \cdot 2 \left(\langle D \sigma_{\mu\nu} \rangle + \frac{1}{4} \sigma_{\mu\nu} \partial u \right) + \left(\frac{2}{5} + \frac{\pi}{30\sqrt{3}} + \frac{\ln 3}{10} \right) \frac{2\sigma_{\mu\nu} \partial u}{4} \right. \\
 & + 4\sigma_{\langle \mu}{}^\rho \sigma_{\nu \rangle \rho} - \left(\frac{\pi}{3\sqrt{3}} + \ln 3 \right) \cdot 2\sigma_{\langle \mu}{}^\rho \Omega_{\nu \rangle \rho} \left. \right] + \frac{r_H^2}{L_4} P_{\mu\nu} \left[\left(\frac{1}{5} - \frac{\pi}{60\sqrt{3}} - \frac{\ln 3}{20} \right) D(\partial u) \right. \\
 & \left. \left. + \left(\frac{1}{100} - \frac{\pi}{300\sqrt{3}} - \frac{\ln 3}{100} \right) (\partial u)^2 + \frac{1}{20} \cdot 4\sigma_{\alpha\beta}^2 \right] \right\}, \quad (6.5)
 \end{aligned}$$

from which we can read all the second order transport coefficients as

$$\begin{aligned}
 \eta\tau_\pi &= \frac{1}{2\kappa_6^2} \left(2 - \frac{\pi}{6\sqrt{3}} - \frac{\ln 3}{2} \right) \frac{r_H^2}{L_4}, \quad \eta\tau_\pi^* = \frac{1}{2\kappa_6^2} \left(\frac{2}{5} + \frac{\pi}{30\sqrt{3}} + \frac{\ln 3}{10} \right) \frac{r_H^2}{L_4}, \quad \lambda_1 = \frac{1}{2\kappa_6^2} \frac{r_H^2}{L_4}, \\
 \lambda_2 &= -\frac{1}{2\kappa_6^2} \left(\frac{\pi}{3\sqrt{3}} + \ln 3 \right) \frac{r_H^2}{L_4}, \quad \zeta\tau_\Pi = \frac{1}{2\kappa_6^2} \left(\frac{1}{5} - \frac{\pi}{60\sqrt{3}} - \frac{\ln 3}{20} \right) \frac{r_H^2}{L_4}, \quad \xi_1 = \frac{1}{2\kappa_6^2} \frac{1}{20} \frac{r_H^2}{L_4}, \\
 \xi_2 &= \frac{1}{2\kappa_6^2} \left(\frac{1}{100} - \frac{\pi}{300\sqrt{3}} - \frac{\ln 3}{100} \right) \frac{r_H^2}{L_4}. \quad (6.6)
 \end{aligned}$$

Compared with the results for compactified D4-brane [13], one can see that $\eta\tau_\pi$ and $\lambda_{1,2}$ which are the transport coefficients of viscous tensors do indeed not change, just as what we have discussed in subsections 3.2, 5.3. But the other 4 coefficients i.e., $\eta\tau_\pi^*$, $\zeta\tau_\Pi$ and $\xi_{1,2}$ which associate with nonconformal viscous terms are different from that in [13]. This may lie in the difference of spatial dimensions. Further more, $\eta\tau_\pi^*$ and ξ_1 are separately half and $\frac{3}{8}$ of that for compactified D4-brane, while for $\zeta\tau_\Pi$ and ξ_2 there are no simple ratios.

As one can check, the shear and bulk relaxation time of D4-brane eq. (6.6) that we get here satisfies the relation in equation (4.7) of [16] after using the Hawking temperature eq. (4.4) with $p = 4$.

The relativistic fluid dual to D1-brane has the stress-energy tensor

$$\begin{aligned}
 T_{\mu\nu} = & \frac{1}{2\kappa_3^2} \left\{ \frac{r_H^6}{L_1^7} (4u_\mu u_\nu + 2P_{\mu\nu}) - \left(\frac{r_H}{L_1} \right)^4 P_{\mu\nu} \partial u \right. \\
 & \left. + \frac{r_H^2}{L_1} P_{\mu\nu} \left[\left(\frac{1}{2} + \frac{\pi}{12\sqrt{3}} - \frac{\ln 3}{4} \right) D(\partial u) + \left(\frac{1}{4} + \frac{\pi}{24\sqrt{3}} - \frac{\ln 3}{8} \right) (\partial u)^2 \right] \right\}. \quad (6.7)
 \end{aligned}$$

It has only two 2nd order transport coefficients from the scalar viscous terms

$$\zeta\tau_\Pi = \frac{1}{2\kappa_3^2} \left(\frac{1}{2} + \frac{\pi}{12\sqrt{3}} - \frac{\ln 3}{4} \right) \frac{r_H^2}{L_1}, \quad \xi_2 = \frac{1}{2\kappa_3^2} \left(\frac{1}{4} + \frac{\pi}{24\sqrt{3}} - \frac{\ln 3}{8} \right) \frac{r_H^2}{L_1}. \quad (6.8)$$

One can check that our results of the 2nd order transport coefficients of D1-brane eq. (6.8) agree with equation (4.6) of [16] where the author derived the bulk relaxation time τ_Π for D1-brane.

The stress-energy tensor of D2-brane is

$$\begin{aligned}
 T_{\mu\nu} = & \frac{1}{2\kappa_4^2} \left\{ \frac{r_H^5}{L_2^6} \left(\frac{7}{2} u_\mu u_\nu + \frac{3}{2} P_{\mu\nu} \right) - \left(\frac{r_H}{L_2} \right)^{\frac{7}{2}} \left(2\sigma_{\mu\nu} + \frac{1}{7} P_{\mu\nu} \partial u \right) \right. \\
 & + \frac{r_H^2}{L_2} \left[\left(\frac{2}{3} + \frac{\pi}{10} \sqrt{1 - \frac{2}{\sqrt{5}}} + \frac{1}{2\sqrt{5}} \operatorname{arccoth} \sqrt{5} - \frac{\ln 5}{4} \right) 2 \left(\langle D \sigma_{\mu\nu} \rangle + \frac{1}{2} \sigma_{\mu\nu} \partial_\rho u^\rho \right) \right. \\
 & + \left(\frac{2}{21} - \frac{\pi}{70} \sqrt{1 - \frac{2}{\sqrt{5}}} - \frac{1}{14\sqrt{5}} \operatorname{arccoth} \sqrt{5} + \frac{\ln 5}{28} \right) 2 \cdot \frac{\sigma_{\mu\nu} \partial_\rho u^\rho}{2} \\
 & + \left. \left(\frac{\pi}{5} \sqrt{1 - \frac{2}{\sqrt{5}}} + \frac{1}{\sqrt{5}} \operatorname{arccoth} \sqrt{5} - \frac{\ln 5}{2} \right) 2 \sigma_{\langle \mu}{}^\rho \Omega_{\nu \rangle \rho} \right] \\
 & + \frac{r_H^2}{L_2} P_{\mu\nu} \left[\left(\frac{2}{21} + \frac{\pi}{70} \sqrt{1 - \frac{2}{\sqrt{5}}} + \frac{1}{14\sqrt{5}} \operatorname{arccoth} \sqrt{5} - \frac{\ln 5}{28} \right) D \partial u \right. \\
 & + \left. \left(\frac{1}{147} + \frac{3\pi}{490} \sqrt{1 - \frac{2}{\sqrt{5}}} + \frac{3}{98\sqrt{5}} \operatorname{arccoth} \sqrt{5} - \frac{3 \ln 5}{196} \right) (\partial u)^2 + \frac{1}{42} \cdot 4 \sigma_{\alpha\beta}^2 \right] \left. \right\}. \tag{6.9}
 \end{aligned}$$

The 2nd order transport coefficients can be read from the above as

$$\begin{aligned}
 \eta \tau_\pi &= \frac{1}{2\kappa_4^2} \left(\frac{2}{3} + \frac{\pi}{10} \sqrt{1 - \frac{2}{\sqrt{5}}} + \frac{1}{2\sqrt{5}} \operatorname{arccoth} \sqrt{5} - \frac{\ln 5}{4} \right) \frac{r_H^2}{L_2}, \\
 \eta \tau_\pi^* &= \frac{1}{2\kappa_4^2} \left(\frac{2}{21} - \frac{\pi}{70} \sqrt{1 - \frac{2}{\sqrt{5}}} - \frac{1}{14\sqrt{5}} \operatorname{arccoth} \sqrt{5} + \frac{\ln 5}{28} \right) \frac{r_H^2}{L_2}, \\
 \lambda_2 &= \frac{1}{2\kappa_4^2} \left(\frac{\pi}{5} \sqrt{1 - \frac{2}{\sqrt{5}}} + \frac{1}{\sqrt{5}} \operatorname{arccoth} \sqrt{5} - \frac{\ln 5}{2} \right) \frac{r_H^2}{L_2}, \\
 \zeta \tau_\Pi &= \frac{1}{2\kappa_4^2} \left(\frac{2}{21} + \frac{\pi}{70} \sqrt{1 - \frac{2}{\sqrt{5}}} + \frac{1}{14\sqrt{5}} \operatorname{arccoth} \sqrt{5} - \frac{\ln 5}{28} \right) \frac{r_H^2}{L_2}, \quad \xi_1 = \frac{1}{2\kappa_4^2} \frac{1}{42} \frac{r_H^2}{L_2}, \\
 \xi_2 &= \frac{1}{2\kappa_4^2} \left(\frac{1}{147} + \frac{3\pi}{490} \sqrt{1 - \frac{2}{\sqrt{5}}} + \frac{3}{98\sqrt{5}} \operatorname{arccoth} \sqrt{5} - \frac{3 \ln 5}{196} \right) \frac{r_H^2}{L_2}. \tag{6.10}
 \end{aligned}$$

Compared with the 2nd order coefficients of D4-brane, the D2-brane does not have λ_1 . This is due to the fact that $\sigma_{\langle \mu}{}^\rho \sigma_{\nu \rangle \rho}$ can not exist in 3d relativistic hydrodynamics. In the literature, [47] also gives the explicit result of the shear relaxation time of D2-brane (as in equation (4.6) of [47]). But by comparing, we find that the shear relaxation time of D2-brane of our results eq. (6.10) does not agree with that in [47]. We think that the result offered in [47] should be wrong which will be specified later.

The 2nd order stress-energy tensor for Dp-brane with $1 \leq p \leq 4$ can be nicely written in the following unified form in terms of Harmonic number:

$$T_{\mu\nu} = \frac{1}{2\kappa_{p+2}^2} \left\{ \frac{r_H^{7-p}}{L_p^{8-p}} \left(\frac{9-p}{2} u_\mu u_\nu + \frac{5-p}{2} P_{\mu\nu} \right) - \left(\frac{r_H}{L_p} \right)^{\frac{9-p}{2}} \left(2\sigma_{\mu\nu} + \frac{2(p-3)^2}{p(9-p)} P_{\mu\nu} \partial u \right) \right\}$$

$$\begin{aligned}
 & + \frac{r_H^2}{L_p} \left[\left(\frac{1}{5-p} + \frac{1}{7-p} H_{\frac{5-p}{7-p}} \right) \cdot 2 \left(\langle D \sigma_{\mu\nu} \rangle + \frac{1}{p} \sigma_{\mu\nu} \partial u \right) \right. \\
 & \quad + \left(\frac{3(p-3)^2}{(5-p)(9-p)} - \frac{(p-3)^2}{(7-p)(9-p)} H_{\frac{5-p}{7-p}} \right) \frac{2\sigma_{\mu\nu} \partial u}{p} + \frac{1}{5-p} \cdot 4\sigma_{\langle\mu}^{\rho} \sigma_{\nu\rangle\rho} \\
 & \quad \left. + \left(-\frac{2}{5-p} + \frac{2}{7-p} H_{\frac{5-p}{7-p}} \right) \cdot 2\sigma_{\langle\mu}^{\rho} \Omega_{\nu\rangle\rho} \right] \\
 & + \frac{r_H^2}{L_p} P_{\mu\nu} \left[\left(\frac{2(p-3)^2}{p(5-p)(9-p)} + \frac{2(p-3)^2}{p(7-p)(9-p)} H_{\frac{5-p}{7-p}} \right) D(\partial u) \right. \\
 & \quad + \left(\frac{2(p-3)^2(3p^2-17p+18)}{p^2(5-p)(9-p)^2} + \frac{2(p-3)^2(5-p)}{p(7-p)(9-p)^2} H_{\frac{5-p}{7-p}} \right) (\partial u)^2 \\
 & \quad \left. + \frac{(p-3)^2}{p(5-p)(9-p)} \cdot 4\sigma_{\alpha\beta}^2 \right] \Bigg\}. \tag{6.11}
 \end{aligned}$$

Note the surface gravity in the above results for Dp-brane is κ_{p+2}^2 which has the mass dimension $-p$. This is consistent with the mass dimension of the stress-energy tensor of Dp-brane i.e. $p+1$. Then one can read the general form of all the 2nd order transport coefficients as

$$\begin{aligned}
 \eta\tau_{\pi} &= \frac{1}{2\kappa_{p+2}^2} \left(\frac{1}{5-p} + \frac{1}{7-p} H_{\frac{5-p}{7-p}} \right) \frac{r_H^2}{L_p}, \\
 \eta\tau_{\pi}^* &= \frac{1}{2\kappa_{p+2}^2} \left(\frac{3(p-3)^2}{(5-p)(9-p)} - \frac{(p-3)^2}{(7-p)(9-p)} H_{\frac{5-p}{7-p}} \right) \frac{r_H^2}{L_p}, \\
 \lambda_1 &= \frac{1}{2\kappa_{p+2}^2} \frac{1}{5-p} \frac{r_H^2}{L_p}, \quad \lambda_2 = \frac{1}{2\kappa_{p+2}^2} \left(-\frac{2}{5-p} + \frac{2}{7-p} H_{\frac{5-p}{7-p}} \right) \frac{r_H^2}{L_p}, \\
 \zeta\tau_{\Pi} &= \frac{1}{2\kappa_{p+2}^2} \left(\frac{2(p-3)^2}{p(5-p)(9-p)} + \frac{2(p-3)^2}{p(7-p)(9-p)} H_{\frac{5-p}{7-p}} \right) \frac{r_H^2}{L_p}, \\
 \xi_1 &= \frac{1}{2\kappa_{p+2}^2} \frac{(p-3)^2}{p(5-p)(9-p)} \frac{r_H^2}{L_p}, \\
 \xi_2 &= \frac{1}{2\kappa_{p+2}^2} \left(\frac{2(p-3)^2(3p^2-17p+18)}{p^2(5-p)(9-p)^2} + \frac{2(p-3)^2(5-p)}{p(7-p)(9-p)^2} H_{\frac{5-p}{7-p}} \right) \frac{r_H^2}{L_p}. \tag{6.12}
 \end{aligned}$$

Here $H_{\frac{5-p}{7-p}}$ is the Harmonic number which is defined as

$$H_{\frac{a}{b}} = \frac{b}{a} + 2 \sum_{n=1}^{\lfloor \frac{b-1}{2} \rfloor} \cos \left(\frac{2\pi na}{b} \right) \ln \sin \left(\frac{n\pi}{b} \right) - \frac{\pi}{2} \cot \left(\frac{\pi a}{b} \right) - \ln(2b), \tag{6.13}$$

where a, b are positive integers with $a < b$. For the special cases of $1 \leq p \leq 4$, one has

$$H_{\frac{2}{3}} = \frac{3}{2} + \frac{\pi}{2\sqrt{3}} - \frac{3}{2} \ln 3, \tag{6.14} \quad (p=1)$$

$$H_{\frac{3}{5}} = \frac{5}{3} + \frac{\pi}{2} \sqrt{1 - \frac{2}{\sqrt{5}}} + \frac{5}{2\sqrt{5}} \operatorname{arccoth} \sqrt{5} - \frac{5}{4} \ln 5, \tag{6.15} \quad (p=2)$$

$$H_{\frac{1}{2}} = 2 - 2 \ln 2, \quad (p = 3) \quad (6.16)$$

$$H_{\frac{1}{3}} = 3 - \frac{\pi}{2\sqrt{3}} - \frac{3}{2} \ln 3. \quad (p = 4) \quad (6.17)$$

Please remember that $\eta\tau_\pi, \eta\tau_\pi^*, \lambda_2, \xi_1$ are not appropriate for $p = 1$ and λ_1 is not suitable for both $p = 1, 2$ in eq. (6.12).

We find previous studies [49] and [47] also give general expressions on Dp-brane relaxation time. The author of [49] calculates the sound mode dispersion relation for Dp-branes and finds a relation between the shear and bulk relaxation time as in its equation (79). One can check that our results eq. (6.12) satisfy that relation. In equation (4.5) of [47], the author offers a general formula for the shear relaxation time of Dp-brane. If one compares the result of τ_π in eq. (6.12) with that from [47], one can see that the result in [47] misses a term of $\frac{1}{5-p}$. That is how τ_π of D2-brane in [47] disagrees with our result in eq. (6.10).

The way to express all the results for Dp-brane i.e. the energy density, pressure and the first order transport coefficients in eq. (4.3), as well as the second order transport coefficients listed in eq. (6.12) are all expressed via geometric parameters $2\kappa_{p+2}^2, r_H$ and L_p . One can also reformulate the expressions in field theory language via 't Hooft coupling of the gauge theory on Dp-brane world-volume λ_{p+1} , the number of Dp-branes N and the temperature of the boundary fluid T . λ_{p+1} here is defined as $\lambda_{p+1} = g_{p+1}^2 N \Lambda^{p-3}$, where $g_{p+1}^2 = (2\pi)^{p-2} g_s l_s^{p-3}$ is the gauge coupling of the effective field theory on Dp-brane world-volume and $\Lambda = r/\alpha'$ is the characteristic UV cutoff scale under the field theory limit. With the definitions of $\lambda_{p+1}, g_{p+1}^2, L_p$ eq. (2.5) and Hawking temperature eq. (4.4), we can translate the geometric language of expressing the results into field theory language. The results are listed in table 3.

6.2 Relations between second order transport coefficients

Now we would like to discuss the identities satisfied by the 2nd order transport coefficients calculated in this paper.

First is the Haack-Yarom (HY) relation $4\lambda_1 + \lambda_2 = 2\eta\tau_\pi$. As it has been discussed in [13] that the HY relation has been verified to be valid for a wide range of cases except for the second order λ_{GB} correction [28–30]. By the 2nd order stress-energy tensor of D1 eq. (6.7) and D2-brane eq. (6.9) we can see that the HY relation does not exist for these two cases. Because D1-brane does not support any tensor viscous term and D2-brane does not have the term $\sigma_{\langle\mu}{}^\rho \sigma_{\nu\rangle\rho}$ which relates λ_1 . Similar observation has also been made in [54] for U(1) charged conformal fluid. But the HY relation holds for D3 and D4-brane which can be seen from their 2nd order transport coefficients i.e. eqs. (6.3), (6.6). Thus we can conclude that the HY relation exists for Dp-brane with only $p > 2$ and is also satisfied by those cases. If one does not consider D5 and D6-brane that do not have dual relativistic fluid, we find D4-brane as a new example that confirms the robustness of HY relation in nonconformal regime.

ε	$\frac{9-p}{2} \left(\frac{4\pi}{7-p} \right)^2 \frac{2^{\frac{2(7-p)}{5-p}} \pi^{\frac{3-p}{5-p}}}{(7-p)^{\frac{11-p}{5-p}}} \Gamma\left(\frac{9-p}{2}\right)^{\frac{2}{5-p}} \lambda_{p+1}^{\frac{p-3}{5-p}} N^2 \frac{T^{\frac{2(7-p)}{5-p}}}{\Lambda^{\frac{(p-3)^2}{5-p}}}$
$\mathbf{p}$	$\frac{5-p}{2} \left(\frac{4\pi}{7-p} \right)^2 \frac{2^{\frac{2(7-p)}{5-p}} \pi^{\frac{3-p}{5-p}}}{(7-p)^{\frac{11-p}{5-p}}} \Gamma\left(\frac{9-p}{2}\right)^{\frac{2}{5-p}} \lambda_{p+1}^{\frac{p-3}{5-p}} N^2 \frac{T^{\frac{2(7-p)}{5-p}}}{\Lambda^{\frac{(p-3)^2}{5-p}}}$
η	$\left(\frac{4\pi}{7-p} \right)^2 \frac{2^{\frac{2(7-p)}{5-p}} \pi^{\frac{3-p}{5-p}}}{(7-p)^{\frac{11-p}{5-p}}} \Gamma\left(\frac{9-p}{2}\right)^{\frac{2}{5-p}} \lambda_{p+1}^{\frac{p-3}{5-p}} N^2 \frac{T^{\frac{9-p}{5-p}}}{\Lambda^{\frac{(p-3)^2}{5-p}}}$
ζ	$\frac{2(p-3)^2}{p(9-p)} \left(\frac{4\pi}{7-p} \right)^2 \frac{2^{\frac{2(7-p)}{5-p}} \pi^{\frac{3-p}{5-p}}}{(7-p)^{\frac{11-p}{5-p}}} \Gamma\left(\frac{9-p}{2}\right)^{\frac{2}{5-p}} \lambda_{p+1}^{\frac{p-3}{5-p}} N^2 \frac{T^{\frac{9-p}{5-p}}}{\Lambda^{\frac{(p-3)^2}{5-p}}}$
$\eta\tau_\pi$	$\left(\frac{1}{5-p} + \frac{1}{7-p} H_{\frac{5-p}{7-p}} \right) \frac{2^{\frac{2(7-p)}{5-p}} \pi^{\frac{3-p}{5-p}}}{(7-p)^{\frac{11-p}{5-p}}} \Gamma\left(\frac{9-p}{2}\right)^{\frac{2}{5-p}} \lambda_{p+1}^{\frac{p-3}{5-p}} N^2 \frac{T^{\frac{4}{5-p}}}{\Lambda^{\frac{(p-3)^2}{5-p}}}$
$\eta\tau_\pi^*$	$\left(\frac{3(p-3)^2}{(5-p)(9-p)} - \frac{(p-3)^2}{(7-p)(9-p)} H_{\frac{5-p}{7-p}} \right) \frac{2^{\frac{2(7-p)}{5-p}} \pi^{\frac{3-p}{5-p}}}{(7-p)^{\frac{11-p}{5-p}}} \Gamma\left(\frac{9-p}{2}\right)^{\frac{2}{5-p}} \lambda_{p+1}^{\frac{p-3}{5-p}} N^2 \frac{T^{\frac{4}{5-p}}}{\Lambda^{\frac{(p-3)^2}{5-p}}}$
λ_1	$\frac{1}{5-p} \frac{2^{\frac{2(7-p)}{5-p}} \pi^{\frac{3-p}{5-p}}}{(7-p)^{\frac{11-p}{5-p}}} \Gamma\left(\frac{9-p}{2}\right)^{\frac{2}{5-p}} \lambda_{p+1}^{\frac{p-3}{5-p}} N^2 \frac{T^{\frac{4}{5-p}}}{\Lambda^{\frac{(p-3)^2}{5-p}}}$
λ_2	$\left(-\frac{2}{5-p} + \frac{2}{7-p} H_{\frac{5-p}{7-p}} \right) \frac{2^{\frac{2(7-p)}{5-p}} \pi^{\frac{3-p}{5-p}}}{(7-p)^{\frac{11-p}{5-p}}} \Gamma\left(\frac{9-p}{2}\right)^{\frac{2}{5-p}} \lambda_{p+1}^{\frac{p-3}{5-p}} N^2 \frac{T^{\frac{4}{5-p}}}{\Lambda^{\frac{(p-3)^2}{5-p}}}$
$\zeta\tau_\Pi$	$\left(\frac{2(p-3)^2}{p(5-p)(9-p)} + \frac{2(p-3)^2}{p(7-p)(9-p)} H_{\frac{5-p}{7-p}} \right) \frac{2^{\frac{2(7-p)}{5-p}} \pi^{\frac{3-p}{5-p}}}{(7-p)^{\frac{11-p}{5-p}}} \Gamma\left(\frac{9-p}{2}\right)^{\frac{2}{5-p}} \lambda_{p+1}^{\frac{p-3}{5-p}} N^2 \frac{T^{\frac{4}{5-p}}}{\Lambda^{\frac{(p-3)^2}{5-p}}}$
ξ_1	$\frac{(p-3)^2}{p(5-p)(9-p)} \frac{2^{\frac{2(7-p)}{5-p}} \pi^{\frac{3-p}{5-p}}}{(7-p)^{\frac{11-p}{5-p}}} \Gamma\left(\frac{9-p}{2}\right)^{\frac{2}{5-p}} \lambda_{p+1}^{\frac{p-3}{5-p}} N^2 \frac{T^{\frac{4}{5-p}}}{\Lambda^{\frac{(p-3)^2}{5-p}}}$
ξ_2	$\left(\frac{2(p-3)^2(3p^2-17p+18)}{p^2(5-p)(9-p)^2} + \frac{2(p-3)^2(5-p)}{p(7-p)(9-p)^2} H_{\frac{5-p}{7-p}} \right) \frac{2^{\frac{2(7-p)}{5-p}} \pi^{\frac{3-p}{5-p}}}{(7-p)^{\frac{11-p}{5-p}}} \Gamma\left(\frac{9-p}{2}\right)^{\frac{2}{5-p}} \lambda_{p+1}^{\frac{p-3}{5-p}} N^2 \frac{T^{\frac{4}{5-p}}}{\Lambda^{\frac{(p-3)^2}{5-p}}}$

Table 3. Reformulation of the results in field theory language. Remember that λ_3 and ξ_3 are both 0.

The next are two relations that were first proposed by Romatschke in [55]:

$$\tau_\pi = \tau_\Pi, \quad (6.18)$$

$$\xi_1 = \frac{1-3c_s^2}{3} \lambda_1. \quad (6.19)$$

It has been shown that the above are satisfied by the 2nd order transport coefficients of compactified D4-brane in [13]. What we want to point out first is eq. (6.18) is not suitable for D1-brane and eq. (6.19) is only valid for $p > 2$. Then if we check the above two relations using eq. (6.12), we can find eq. (6.18) still holds while eq. (6.19) is not. This is because Romatschke only considered the hydrodynamics of 4d in [55]. The second relation has to be generalized to arbitrary dimensions as

$$\xi_1 = \frac{1}{d-1} (1 - (d-1)c_s^2) \lambda_1, \quad (d > 3) \quad (6.20)$$

where $d = p + 1$ is the dimension of the relativistic fluid. As one can check that eq. (6.12) satisfies eq. (6.20).

Besides the two relations discussed in the last paragraph, [55] also proposed many other relations between the 2nd order coefficients, e.g.

$$\tau_\pi^* = -(1 - 3c_s^2)\tau_\pi, \quad \xi_2 = \frac{1 - 3c_s^2}{3} 2c_s^2 \eta \tau_\pi. \quad (6.21)$$

But it has been pointed out in [35] by Kleinert and Probst that these two relations may miss the term of λ_1 , and the correct form should be

$$\eta \tau_\pi^* = (1 - 3c_s^2)(4\lambda_1 - \eta \tau_\pi), \quad \xi_2 = \frac{2}{9}(1 - 3c_s^2)[3c_s^2 \eta \tau_\pi + (1 - 6c_s^2)2\lambda_1]. \quad (6.22)$$

Kleinert and Probst have also checked that the above two relations are satisfied by the compactified D4-brane [13, 45, 51]. But if we are in dimensions other than 4 like the case in this paper, the above Kleinert-Probst (KP) relations eq. (6.22) should be generalized to

$$\begin{aligned} \eta \tau_\pi^* &= (1 - (d-1)c_s^2)(4\lambda_1 - \eta \tau_\pi), & (d > 3) \\ \xi_2 &= \frac{2}{(d-1)^2} (1 - (d-1)c_s^2) [(1 - 2(d-1)c_s^2)2\lambda_1 + (d-1)c_s^2 \eta \tau_\pi]. & (d > 3) \end{aligned} \quad (6.23)$$

As one can check, the above two generalized KP relations are indeed satisfied by the 2nd order transport coefficients of Dp-brane eq. (6.12) in this paper. Since the above two relations also have λ_1 , so they are only appropriate for the cases with $p > 2$ (or $d > 3$), like the HY relation and the generalized Romatschke relation eq. (6.20).

There are some other nontrivial relations among the 2nd order thermodynamical transport coefficients⁵ proposed in [55], which are

$$\begin{aligned} \kappa^* &= -\frac{1 - 3c_s^2}{2c_s^2} \kappa, & \xi_3 &= \frac{1 - 3c_s^2}{3} \lambda_3, \\ \xi_5 &= \frac{1 - 3c_s^2}{3} \kappa, & \xi_6 &= \frac{1 - 3c_s^2}{3c_s^2} \kappa. \end{aligned} \quad (6.24)$$

Kleinert and Probst check the above relations to be true for the 5d Chamblin-Reall background using the method of [37]. According to the generalization from eq. (6.22) to eq. (6.23), we think the above relations may also need to be generalized in arbitrary dimensions to:

$$\begin{aligned} \kappa^* &= -\frac{1 - (d-1)c_s^2}{2c_s^2} \kappa, & \xi_3 &= \frac{1 - (d-1)c_s^2}{d-1} \lambda_3, \\ \xi_5 &= \frac{1 - (d-1)c_s^2}{d-1} \kappa, & \xi_6 &= \frac{1 - (d-1)c_s^2}{(d-1)c_s^2} \kappa. \end{aligned} \quad (6.25)$$

Although we do not know for now whether eq. (6.25) is completely correct since it needs to be checked by strict calculation in nonconformal backgrounds. We hope future studies will give us a satisfied answer.

⁵As it has been classified in [52] that there are 8 thermodynamical transport coefficients out of the 15 second order coefficients which do not cause the increase of entropy.

6.3 Test of Kanitscheider-Skenderis proposal

Kanitscheider and Skenderis proposed a way of generating the second order part of non-conformal stress-energy tensor from that of a higher dimensional conformal relativistic fluid [37]. With the help of the results in this paper, we can test whether their method is correct.

First we need the result of stress-energy tensor of conformal relativistic fluid in various dimensions. The following stress-energy tensor summarizes all the results from [2–7, 10, 11, 15–18] and offers the stress-energy tensor of d -dimensional conformal relativistic fluid that corresponds to AdS_{d+1} black hole:

$$T_{\mu\nu} = \frac{1}{2\kappa_{d+1}^2} \left\{ \frac{r_H^d}{L^{d+1}} [(d-1)u_\mu u_\nu + P_{\mu\nu}] - \left(\frac{r_H}{L}\right)^{d-1} 2\sigma_{\mu\nu} \right. \\ \left. + \frac{r_H^{d-2}}{L^{d-3}} \left[\left(\frac{1}{2} + \frac{1}{d}H_{\frac{2}{d}}\right) 2 \left(\langle D \sigma_{\mu\nu} \rangle + \frac{1}{d-1} \sigma_{\mu\nu} \partial u \right) + \frac{1}{2} \cdot 4\sigma_{\langle\mu}{}^\rho \sigma_{\nu\rangle\rho} \right. \right. \\ \left. \left. + \left(\frac{2}{d}H_{\frac{2}{d}} - 1\right) 2\sigma_{\langle\mu}{}^\rho \Omega_{\nu\rangle\rho} + \frac{2}{d-2} (R_{\langle\mu\nu\rangle} - 2u^\rho u^\sigma R_{\rho\langle\mu\nu\rangle\sigma}) \right] \right\}, \quad (6.26)$$

where $\frac{1}{2\kappa_{d+1}^2}$, L and r_H are separately the surface gravity constant, the curvature scale and the location of horizon of AdS_{d+1} black hole background. The energy density, pressure and the transport coefficients can then be read as⁶

$$\begin{aligned} \varepsilon &= \frac{1}{2\kappa_{d+1}^2} (d-1) \frac{r_H^d}{L^{d+1}}, & \mathfrak{p} &= \frac{1}{2\kappa_{d+1}^2} \frac{r_H^d}{L^{d+1}}, & \eta &= \frac{1}{2\kappa_{d+1}^2} \frac{r_H^{d-1}}{L^{d-1}}, \\ \eta\tau_\pi &= \frac{1}{2\kappa_{d+1}^2} \left(\frac{1}{2} + \frac{1}{d}H_{\frac{2}{d}}\right) \frac{r_H^{d-2}}{L^{d-3}}, & \kappa &= \frac{1}{2\kappa_{d+1}^2} \frac{2}{d-2} \frac{r_H^{d-2}}{L^{d-3}}, \\ \lambda_1 &= \frac{1}{2\kappa_{d+1}^2} \frac{1}{2} \frac{r_H^{d-2}}{L^{d-3}}, & \lambda_2 &= \frac{1}{2\kappa_{d+1}^2} \left(\frac{2}{d}H_{\frac{2}{d}} - 1\right) \frac{r_H^{d-2}}{L^{d-3}}, & \lambda_3 &= 0. \end{aligned} \quad (6.27)$$

The Hawking temperature, entropy density and shear to entropy ratio are

$$T = \frac{d}{4\pi} \frac{r_H}{L^2}, \quad s = \frac{1}{2\kappa_{d+1}^2} 4\pi \frac{r_H^{d-1}}{L^{d-1}}, \quad \frac{\eta}{s} = \frac{1}{4\pi}. \quad (6.28)$$

Following the method that proposed by Kanitscheider and Skenderis in [37], the 2nd order viscous terms in eq. (6.26) will transform like

$$\begin{aligned} \langle D \sigma_{\mu\nu} \rangle &\rightarrow \langle D \sigma_{\mu\nu} \rangle + \chi P_{\mu\nu} D(\partial u), & \sigma_{\mu\nu} \partial u &\rightarrow \sigma_{\mu\nu} \partial u + \chi P_{\mu\nu} (\partial u)^2, \\ \sigma_{\langle\mu}{}^\rho \sigma_{\nu\rangle\rho} &\rightarrow \sigma_{\langle\mu}{}^\rho \sigma_{\nu\rangle\rho} + \chi P_{\mu\nu} \sigma_{\alpha\beta}^2, & \sigma_{\langle\mu}{}^\rho \Omega_{\nu\rangle\rho} &\rightarrow \sigma_{\langle\mu}{}^\rho \Omega_{\nu\rangle\rho}. \end{aligned} \quad (6.29)$$

Here χ is defined as

$$\chi = \frac{\tilde{d} - d}{(\tilde{d} - 1)(d - 1)}. \quad (6.30)$$

⁶The result in [11] is expressed via $H_{\frac{2}{d}-1}$, which can be related to $H_{\frac{2}{d}}$ by the recurrence relation of Harmonic number: $H_\alpha = H_{\alpha-1} + \frac{1}{\alpha}$.

We denote $\tilde{d}$ as the dimension of conformal fluid that is dual to $\text{AdS}_{\tilde{d}+1}$ black hole background, and $d = p + 1$ as the dimension of the nonconformal fluid which correspond to Dp-brane, with $\tilde{d} > d$. Now we want to see with the transformation rule eq. (6.29), whether we can get the 2nd order part of stress-energy tensor for nonconformal d -dimensional fluid from that of a $\tilde{d}$ -dimensional conformal fluid. The 2nd order part stress-energy tensor of $\tilde{d}$ -dimensional conformal fluid is

$$\widetilde{\eta\tau_\pi} \cdot 2 \left(\langle D \sigma_{\mu\nu} \rangle + \frac{1}{\tilde{d}-1} \sigma_{\mu\nu} \partial u \right) + \tilde{\lambda}_1 \cdot 4 \sigma_{\langle \mu}{}^\rho \sigma_{\nu \rangle \rho} + \tilde{\lambda}_2 \cdot 2 \sigma_{\langle \mu}{}^\rho \Omega_{\nu \rangle \rho}, \quad (6.31)$$

where we also write the 2nd order coefficients of $\text{AdS}_{\tilde{d}+1}$ black hole background with a tilde. Also note that here we do not keep quantities with dimension like r_H and L , since the Kanitscheider-Skenderis proposal only involves with numerical value of the 2nd order transport coefficients. After using the transformation rule eq. (6.29), the above will become

$$\begin{aligned} & \widetilde{\eta\tau_\pi} \cdot 2 \left(\langle D \sigma_{\mu\nu} \rangle + \frac{1}{d-1} \sigma_{\mu\nu} \partial u \right) - (d-1) \chi \widetilde{\eta\tau_\pi} \frac{2 \sigma_{\mu\nu} \partial u}{d-1} + \tilde{\lambda}_1 \cdot 4 \sigma_{\langle \mu}{}^\rho \sigma_{\nu \rangle \rho} \\ & + \tilde{\lambda}_2 \cdot 2 \sigma_{\langle \mu}{}^\rho \Omega_{\nu \rangle \rho} + P_{\mu\nu} \left(2 \chi \widetilde{\eta\tau_\pi} D(\partial u) + \frac{2 \chi \widetilde{\eta\tau_\pi}}{\tilde{d}-1} (\partial u)^2 + \chi \tilde{\lambda}_1 \cdot 4 \sigma_{\alpha\beta}^2 \right). \end{aligned} \quad (6.32)$$

Thus we get the relations between the 2nd order transport coefficients of nonconformal fluid and that of the conformal fluid:

$$\begin{aligned} \eta\tau_\pi &= \widetilde{\eta\tau_\pi}, & \eta\tau_\pi^* &= -(d-1) \chi \widetilde{\eta\tau_\pi}, & \lambda_1 &= \tilde{\lambda}_1, & \lambda_2 &= \tilde{\lambda}_2, \\ \zeta\tau_\Pi &= 2 \chi \widetilde{\eta\tau_\pi}, & \xi_1 &= \chi \tilde{\lambda}_1, & \xi_2 &= \frac{2 \chi}{\tilde{d}-1} \widetilde{\eta\tau_\pi}. \end{aligned} \quad (6.33)$$

Now we will use the results of D1, D2 and D4-brane to test whether the relations eq. (6.33) are reliable. Since the D4-brane result can be got from AdS_7 black hole background, so we have $\tilde{d} = 6$, $d = p + 1 = 5$ and $\chi = 1/20$. The 2nd order transport coefficients for AdS_7 black hole are

$$\widetilde{\eta\tau_\pi} = \frac{1}{2} + \frac{1}{6} H_{\frac{1}{3}}, \quad \tilde{\lambda}_1 = \frac{1}{2}, \quad \tilde{\lambda}_2 = -1 + \frac{1}{3} H_{\frac{1}{3}}. \quad (6.34)$$

Thus we can get from eq. (6.33) that

$$\begin{aligned} \eta\tau_\pi &= \frac{1}{2} + \frac{1}{6} H_{\frac{1}{3}}, & \eta\tau_\pi^* &= -\frac{1}{10} - \frac{1}{30} H_{\frac{1}{3}}, & \lambda_1 &= \frac{1}{2}, & \lambda_2 &= -1 + \frac{1}{3} H_{\frac{1}{3}}, \\ \zeta\tau_\Pi &= \frac{1}{20} + \frac{1}{60} H_{\frac{1}{3}}, & \xi_1 &= \frac{1}{40}, & \xi_2 &= \frac{1}{100} + \frac{1}{300} H_{\frac{1}{3}}. \end{aligned} \quad (6.35)$$

Since a global scaling factor for the above results do not matter, we can multiply a scaling factor 2 to the above results and get

$$\begin{aligned} \eta\tau_\pi &= 1 + \frac{1}{3} H_{\frac{1}{3}}, & \eta\tau_\pi^* &= -\frac{1}{5} - \frac{1}{15} H_{\frac{1}{3}}, & \lambda_1 &= 1, & \lambda_2 &= -2 + \frac{2}{3} H_{\frac{1}{3}}, \\ \zeta\tau_\Pi &= \frac{1}{10} + \frac{1}{30} H_{\frac{1}{3}}, & \xi_1 &= \frac{1}{20}, & \xi_2 &= \frac{1}{50} + \frac{1}{150} H_{\frac{1}{3}}. \end{aligned} \quad (6.36)$$

Compare the above with eq. (6.12) for the case $p = 4$, we can see that $\eta\tau_\pi, \zeta\tau_\Pi, \lambda_{1,2}, \xi_1$ are the same as the results of D4-brane, but $\eta\tau_\pi^*$ and ξ_2 do not match.

By the same token, one can check the situations of D1 and D2-brane. The reduced bulk gravity of D1-brane relates to AdS_4 black hole background, after using the relations eq. (6.33) and multiplying a scaling factor 2 one can reproduce exactly the second order transport coefficients for D1-brane. The second order transport coefficients of D2-brane should be got from that of AdS_6 black hole by using eq. (6.33).⁷ But one will see that the results got this way can not match any transport coefficient of D2-brane in eq. (6.12).

The authors of [35] derive out the exact analytical results of the second order transport coefficients for 5d Chamblin-Reall background and use the leading nonconformal corrections of the results to compare with [36]. It is interesting to note that they also find $\eta\tau_\pi^*$ and ξ_2 in [36] are wrong. The reason they think is that [36] uses eq. (6.21) (first proposed in [55]) to get $\eta\tau_\pi^*$ and ξ_2 whose correct form should be eq. (6.22). Considering that [55] derived the relations eq. (6.21) also using the method of [37], we may draw the conclusion that the method of getting the 2nd order transport coefficients of nonconformal Dp-brane from AdS black hole background as suggested in [37] is applicable only for some special cases. It is better to use the standard methods like the Minkowskian AdS/CFT prescription or the fluid/gravity correspondence to calculate the 2nd order transport coefficients for nonconformal backgrounds.

6.4 Dispersion relations

At last, we would like to offer the dispersion relations for the dual relativistic fluid of Dp-brane with $1 \leq p \leq 4$. One needs to expand $r_H(x)$ and $\beta_i(x)$ as $r_H(x) = r_H(0) + \delta r_H e^{ik_\mu x^\mu}$ and $\beta_i(x) = \delta\beta_i e^{ik_\mu x^\mu}$, respectively. Then put them into the conservation equation of the stress-energy tensor of Dp-brane $\partial^\mu T_{\mu\nu} = 0$ and find the coefficient matrix of the vector $(\delta r_H, \delta\beta_i)^T$ where the subscript “T” stands for taking the transpose. Then the determinant of that coefficient matrix equate to 0 will give two equations from which we can solve the transverse and the longitudinal modes of the dispersion relations (denoted separately as ω_T and ω_L). The result is

$$\omega_T = -i \frac{1}{7-p} \frac{L_p^{\frac{7-p}{2}}}{r_H^{\frac{5-p}{2}}} k^2 - i \frac{(7-p) + (5-p) H_{\frac{5-p}{7-p}}}{(5-p)(7-p)^3} \frac{L_p^{\frac{3(7-p)}{2}}}{r_H^{\frac{3(5-p)}{2}}} k^4, \quad (p \neq 1) \quad (6.37)$$

$$\begin{aligned} \omega_L = & \pm \sqrt{\frac{5-p}{9-p}} k - i \frac{4}{(7-p)(9-p)} \frac{L_p^{\frac{7-p}{2}}}{r_H^{\frac{5-p}{2}}} k^2 \pm \frac{4(5-p)^{\frac{1}{2}} \left(1 + H_{\frac{5-p}{7-p}}\right)}{(7-p)^2(9-p)^{\frac{3}{2}}} \frac{L_p^{7-p}}{r_H^{5-p}} k^3 \\ & - i \frac{32 \left(7-p + (5-p) H_{\frac{5-p}{7-p}}\right)}{(5-p)(7-p)^3(9-p)^2} \frac{L_p^{\frac{3(7-p)}{2}}}{r_H^{\frac{3(5-p)}{2}}} k^4. \end{aligned} \quad (6.38)$$

Here $k = |\vec{k}|$ and also note that p can not be 1 for the transverse mode.

⁷One will need the reflection relation of Harmonic number $H_{1-\alpha} = H_\alpha + \pi \cot \pi\alpha + \frac{1}{1-\alpha} - \frac{1}{\alpha}$ in comparing the results got from eq. (6.33) with the second order transport coefficients of D2-brane in eq. (6.12).

7 Summary and outlook

In this paper, we have derived all the dynamical second order transport coefficients for Dp-brane using fluid/gravity correspondence. D5 and D6-brane do not have physical dual relativistic fluid; D3-brane is dual to 4d conformal fluid of which the results are already well known in the literature; thus the new results are mainly for D1, D2 and D4-brane, which correspond to nonconformal fluid of dimension 2, 3 and 5.

The validity of the identities among 2nd order transport coefficients has exceptions: $\tau_\pi = \tau_\Pi$ is not valid for D1-brane; the HY relation, the generalized Romatschke eq. (6.20) and KP eq. (6.23) relations are appropriate for Dp-brane with $p > 2$, or for relativistic fluid of dimension $d > 3$. The general form of the 2nd order transport coefficients of Dp-brane eq. (6.12) satisfies the HY, Romatschke and KP relations. We also think that the constraint relations between the 2nd order transport coefficients, which were first proposed in [56] and rewritten in a more useful form in [52], need to be generalized if the relativistic fluid is not in 4d. So one should be careful when using them in dimensions other than 4.

This work leaves us some interesting problems to explore. First, we can continue to study the thermodynamical 2nd order transport coefficients for Dp-brane via the real-time AdS/CFT correspondence. Second, D5 and D6-brane are not dual to any relativistic fluid. So we may need to use the NS5-brane if we want to study the relativistic fluid of dimension larger than 5. NS5-brane, if it has, will be dual to a nonconformal relativistic fluid of 6d. Finally, we can tell from the results of D4-brane and compactified D4-brane that the compactification on the brane will change the spatial dimension of the fluid thus the 2nd order transport coefficients relating viscous scalars, i.e., $\eta\tau_\pi^*$, $\zeta\tau_\Pi$ and $\xi_{1,2}$. So we can compactify more directions on D4-brane to get nonconformal fluids of dimension 2 and 3, and one can also compactify the other Dp-branes. It will be interesting to see whether D3-brane will correspond to a nonconformal fluid after compactification.

Through [13, 51] and this work, we have successfully generalized the original discussion of fluid/gravity correspondence to nonconformal regime for various dimensions. By the stress-energy tensor of general dimension eq. (6.11), we actually offers a group of nonconformal relativistic fluid models. We hope these models can be used to get the analytic solutions by using the framework of [57–63] thus will give us more hints on the phenomena of relativistic heavy ion collider.

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A Dimensional reduction of bulk metric

We will do the dimensional reduction on the following 10d metric

$$ds^2 = e^{2\alpha_1 A} g_{MN} dx^M dx^N + L_p^2 e^{2\alpha_2 A} d\Omega_{8-p}^2, \quad (\text{A.1})$$

where the dimension of the $\{x^M\}$ part is $(p+2)$ -dimensional, $A = A(x^M)$ only depend on coordinates of the first $p+2$ dimensions, the second part is a $(8-p)$ -dimensional unit sphere with the metric γ_{ab} . $\alpha_{1,2}$, L_p are all constants. The components of connection are

$$\hat{\Gamma}_{NP}^M = \Gamma_{NP}^M + \alpha_1 (\delta_N^M \partial_P A + \delta_P^M \partial_N A - g_{NP} \nabla^M A), \quad (\text{A.2})$$

$$\hat{\Gamma}_{ab}^M = -\alpha_2 L_p^2 e^{(-2\alpha_1 + 2\alpha_2)A} \nabla^M A \gamma_{ab}, \quad (\text{A.3})$$

$$\hat{\Gamma}_{Mb}^a = \alpha_2 \partial_M A \delta_b^a, \quad (\text{A.4})$$

$$\hat{\Gamma}_{bc}^a = \Gamma_{bc}^a. \quad (\text{A.5})$$

The Ricci tensor is

$$\begin{aligned} \hat{R}_{MN} = R_{MN} - [p\alpha_1 + (8-p)\alpha_2] \nabla_M \nabla_N A - \alpha_1 g_{MN} \nabla^2 A + [p\alpha_1^2 + 2(8-p)\alpha_1\alpha_2 \\ - (8-p)\alpha_2^2] \nabla_M A \nabla_N A - [p\alpha_1^2 + (8-p)\alpha_1\alpha_2] g_{MN} (\nabla A)^2, \end{aligned} \quad (\text{A.6})$$

$$\hat{R}_{ab} = (7-p)\gamma_{ab} - L_p^2 e^{(-2\alpha_1 + 2\alpha_2)A} [\alpha_2 \nabla^2 A + (p\alpha_1\alpha_2 + (8-p)\alpha_2^2) (\nabla A)^2] \gamma_{ab}. \quad (\text{A.7})$$

The Ricci scalar is

$$\begin{aligned} \hat{R} = \frac{(7-p)(8-p)}{L_p^2} e^{-2\alpha_2 A} + e^{-2\alpha_1 A} [R - ((2p+2)\alpha_1 + (16-2p)\alpha_2) \nabla^2 A \\ - ((p^2+p)\alpha_1^2 + (16p-2p^2)\alpha_1\alpha_2 + (8-p)(9-p)\alpha_2^2) (\nabla A)^2] \end{aligned} \quad (\text{A.8})$$

The quantities with a hat like $\hat{\Gamma}$ and $\hat{R}$ are 10-dimensional quantities while those without like R are $(p+2)$ -dimensional ones.

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