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Magnetotransport of Weyl semimetals with $\mathbb{Z}_2$ topological charge and chiral anomaly

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ABSTRACT: We calculate the magnetoconductivity of the Weyl semimetal with $\mathbb{Z}_2$ topological charge and chiral anomaly utilizing the recently developed hydrodynamic theory. The system in question will be influenced by magnetic fields connected with ordinary Maxwell and the second U(1)-gauge field, which couples to the anomalous topological charge. The presence of chiral anomaly and $\mathbb{Z}_2$ topological charge endow the system with new transport coefficients. We start with the linear perturbations of the hydrodynamic equations and calculate the magnetoconductivity of this system. The holographic approach in the probe limit is implemented to obtain the explicit dependence of the longitudinal magnetoconductivities on the magnetic fields.

KEYWORDS: AdS-CFT Correspondence, Holography and condensed matter physics (AdS/CMT)

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1 Introduction

The recently observed resurgence of interest in the physics of chiral systems is related to the important discoveries in such fields as dynamics of relativistic quark-gluon plasma [1]–[6] and high energy astrophysical processes in proto-neutron stars [7] on one side and the recent realization of the known and novel relativistic symmetries for electrons in condensed matter systems. Chiral magnetic and vortical effects observed in the high energy nuclear collisions have been reviewed recently [8]. The topological quantum matter is becoming an important play-ground for discoveries of numerous novel phenomena not observed before in particle physics.

Electrons in metals form a system of interacting particles. The strength of the interactions is measured with respect to their kinetic energy and thus depends on the properties of the system under study. The standard paradigm was to describe electrons in metals by the Landau quasiparticle theory. However, there are many very strongly interacting materials which descriptions require approaches going beyond quasiparticle picture. For example, in very clean metals the flow of electrons resembles that of the classical fluid and is described by the laws of classic hydrodynamic theory [9]. In some materials like Dirac or Weyl semimetals [10], the spectrum of electrons in the vicinity of the Fermi level, is linear and thus relativistic despite their velocity v_F being much smaller than the velocity of light. Thus, the condensed matter allows for the relativistic symmetries and opens new possibilities for laboratory studies of exotic particles, which were predicted, but never seen in vacuum. The long sought Majorana fermion may constitute an example of them [11].

The discovery of the aforementioned semimetals of Dirac or Weyl character, has sparked the intense studies of band crossing phenomena in crystals. This resulted in the

discovery of “relativistic” quasiparticles with effective “spin” quantum number different from $1/2$. The crystal space groups hosting such fermions have been identified and the materials realizing them proposed [12]. Moreover, the “spin” $3/2$ electrons were apparently observed [13].

The relativistic spectrum together with the associated chirality of the quantum particles is a source of new phenomena generally known as chiral quantum anomalies, first discovered by Adler [14] and independently by Bell and Jackiw [15]. They have played an important role in understanding the neutral pion decay and recently they are subject of vigorous studies in condensed matter laboratories. In the presence of dense matter, quantum anomalies modify the relativistic hydrodynamics by leading to new phenomena and novel transport coefficients [16], i.e., an anomalous current is generated by an external magnetic field or by vortices in fluid which carries the charge in question [17].

In the realm of condensed matter physics the role of chiral and other anomalies is especially visible under the influence of external magnetic field or a longitudinal temperature gradient. The anomaly also influences the electric DC conductivity. Namely, the longitudinal DC conductivity is amplified by magnetic field [18, 19]. The key prediction was found in a kinetic description at weak coupling, as well as, hydrodynamics and holographic attitude at strong coupling.

The problem of the longitudinal magnetoconductivity, in the system with a chiral anomaly and background magnetic field was examined in [20]. By means of the linear response method in the hydrodynamic limit and holography attitude, it was shown that one needs to have energy, momentum and charge dissipations to obtain a finite DC longitudinal magnetoconductivity. The same problem was studied in [21], where the longitudinal DC conductivity bounded with the Lifshitz like fixed points, in the presence of chiral anomalies, in $(3+1)$ -dimensions was found.

Following the method presented in [22], the generalized expression for DC and Hall conductivities were delivered also for large class of the holographic massive models [23]. Massive gravity models [24]–[28] attract attentions due to the diffeomorphism symmetry breaking which causes the non-conservation of the energy-momentum tensor. The elaborated effect is similar to the dissipation of the momentum.

Among all, the holographic treatment of the system in question reveals that in the intermediate regime of the magnetic field, one gets a negative magnetoresistivity decreasing as an inverse of the magnetic field. The specific angular dependence of this enhancement, in the longitudinal direction along the magnetic field, is the key prediction which arises from the aforementioned descriptions. However, already in the weak coupling limit, it was envisaged that negative magnetoresistance might appear [29].

Similar effect can arise in the hydrodynamical attitude, when the distinctive gradient expansion is taken into account (the non-relativistic constitutive relations can be derived from the most general covariant form of the expression, up to the third order in the field strength) [30]. Moreover, in a relativistic hydrodynamic theory of transport, it depends on the macroscopic model (one can achieve both negative and positive magnetoresistance). In non-Galilean invariant fluids the effect can arise not only due to the presence of a background magnetic field but also depends on the specific structure of the considered hydrodynamics [30].

The other mechanism leading to the negative magnetoresistivity was proposed in [31]–[32], by using two interacting U(1)-gauge fields.

The recent experimental works conducted in Dirac or Weyl semimetals like Na_3Bi , $ZrTe_5$ [33, 34] and $TaAs$, NbP [35]–[37], confirm the evidences of chiral anomaly. In principle there are two classes of Dirac semimetals, in first one the Dirac points appear at the time reversal invariant momenta in the first Brillouin zone. The second class comprises the elements in which the Dirac points take place in pairs at two arbitrary points in the Brillouin zone. They are separated in the momentum space along a rotational axis [38, 39]. Moreover it turns out that they are characterized by a non-trivial $\mathbb{Z}_2$ topological invariant protecting the nodes and leading to the presence of Fermi arc surface states [40]–[43]. It has been argued that the $\mathbb{Z}_2$ anomalous charge affects the transport characteristics of the materials in question [44]. The $\mathbb{Z}_2$ topological charge influence on the transport properties was studied in a relativistic hydrodynamics limit in [45].

The behavior of metals influenced by electromagnetic field is of the great importance for understanding their transport properties. For the most materials under inspection the longitudinal conductivity is a decreasing function of the magnetic field. However, for the Weyl semimetals (materials for which conduction bands intersect at distant points in the Brillouin zone) one has an exception to the rules [20]. In the recent experiments [46] the negative magnetoresistance in the intermediate regime of the magnetic field was confirmed. There were proposed several alternative ways of explaining the above phenomenon. For example in [46]–[47], the weak-antilocalization effect was considered for the explanation of the aforementioned effect.

The main objectives of our work are to examine the influence of $\mathbb{Z}_2$ topological charge and chiral anomaly on conductivities, in the presence of non-zero ingredients of magnetic field components of the U(1)-gauge fields. These considerations constitute the development of the ideas [45], where the hydrodynamical characteristics of the model with chiral anomaly and $\mathbb{Z}_2$ topological charge were presented.

The paper is organized as follows. In section 2 we present the calculations of electric conductivities for both studied U(1)-gauge fields, in the hydrodynamics limit, taking into account chiral anomaly and $\mathbb{Z}_2$ topological charge. We have elaborated the case with background magnetic fields and with all the possible dissipation terms in order to achieve finite value of the DC longitudinal conductivities. Section 3 is devoted to the holographic model of our system, in the probe limit. We discuss the relation between the parameters entering the holographic action and those responsible for anomalies in section 3.1. As a background spacetime we take five-dimensional AdS Schwarzschild black brane. Section 4 is connected with the independent calculation of magnetoconductivities for the studied holographic model with chiral anomaly and $\mathbb{Z}_2$ topological charge in the probe limit, using Kubo formula. The obtained results match the hydrodynamical formula in the appropriate limit. In last section we conclude the main results.

2 Hydrodynamics of the system

The presence of quantum triangle anomalies severely modifies the relativistic hydrodynamic equations. The novel transport coefficients appear and are expressed in terms of anomaly

coefficients (charges) and the systems equation of state [16]. Recently, we have generalized the relativistic hydrodynamic theory to the systems endowed with chiral and topological charges [44]. To describe $\mathbb{Z}_2$ topological charge in chiral Weyl semimetal of the second kind we have used two different electro-magnetic fields.

The motivation standing behind our research is to include the different dissipation terms in studies of longitudinal DC-conductivity. At first we directly find the aforementioned conductivity in the hydrodynamical attitude in the four-dimensional spacetime with two background magnetic fields. One of the fields is connected with the ordinary Maxwell one while the other constitutes the magnetic component of the auxiliary U(1)-gauge field which couples to the anomalous $\mathbb{Z}_2$ topological charge. We restrict our considerations to the linear response level. For the purpose of the future reference and to established the notation we shall briefly present main ideas and results of the mentioned hydrodynamic approach [45].

The inclusion of the dissipation terms [20, 48], is to perturb first a hydrodynamical system in a given equilibrium state and solve the equation of motion with initial values of the perturbations. In the next step, the searched transport coefficients can be found using the response of the electric thermal currents to the initial values of the corresponding perturbations.

2.1 Hydrodynamic equations for system with chiral and $\mathbb{Z}_2$ topological charge

The key set of the studied relations will be the hydrodynamical equations of motion in the presence of $\mathbb{Z}_2$ topological charge and chiral anomaly, provided by [45]

$$\partial_\alpha T^{\alpha\beta}(F, B) = F^{\beta\alpha} j_\alpha(F) + B^{\beta\alpha} j_\alpha(B), \quad (2.1)$$

$$\partial_\alpha j^\alpha(F) = C_1 E_{(F)\alpha} B^{(F)\alpha} + C_2 \tilde{E}_{(B)\alpha} \tilde{B}^{(B)\alpha}, \quad (2.2)$$

$$\partial_\alpha j^\alpha(B) = C_3 \tilde{E}_{(B)\alpha} B^{(F)\alpha} + C_4 E_{(F)\alpha} \tilde{B}^{(B)\alpha}. \quad (2.3)$$

It has to be recalled that $C_2 = C_4$ due to symmetry reasons. In the above equations we have denoted the electric and magnetic fields in the fluid rest frame by

$$E_\alpha^{(F)} = F_{\alpha\beta} u^\beta, \quad B_\alpha^{(F)} = \frac{1}{2} \epsilon_{\alpha\beta\rho\delta} u^\beta F^{\rho\delta}, \quad (2.4)$$

$$\tilde{E}_\alpha^{(B)} = B_{\alpha\beta} u^\beta, \quad \tilde{B}_\alpha^{(B)} = \frac{1}{2} \epsilon_{\alpha\beta\rho\delta} u^\beta B^{\rho\delta}. \quad (2.5)$$

where $F_{\mu\nu} = 2\partial_{[\mu} A_{\nu]}$ stands for the ordinary Maxwell field strength tensor, while the second U(1)-gauge field $B_{\mu\nu}$ is given by $B_{\mu\nu} = 2\partial_{[\mu} B_{\nu]}$. $j_\mu(F)$, $j_\mu(B)$ represent the adequate currents connected with each of the gauge field.

On the other hand, the energy momentum tensor and the adequate currents needed for the hydrodynamic description of the relativistic fluid, imply [49]

$$T^{\mu\nu} = (\epsilon + p) u^\mu u^\nu + p g^{\mu\nu} + \tau^{\mu\nu}, \quad (2.6)$$

$$j^\mu(F) = \rho u^\mu + V_F^\mu, \quad (2.7)$$

$$j^\mu(B) = \rho_d u^\mu + V_B^\mu, \quad (2.8)$$

where $\tau^{\mu\nu}$ and $V_{F(B)}^\mu$ denote corrections, higher order in velocities, responsible for dissipative effects. Using the fact that there is no dissipative force in the rest frame of the liquid element, we obtain $u_\alpha \tau^{\alpha\beta} = u_\alpha V_F^\alpha = u_\alpha V_B^\alpha = 0$. By ϵ we have denoted the energy density, p is connected with pressure and ρ, ρ_d are the U(1) charge densities. The four-vector u^μ , with the normalization $u_\mu u^\mu = -1$, describes the flow of the fluid in the system in question and the general expressions for the dissipative components of the energy — momentum tensor and currents read

$$\tau^{\mu\nu} = -\eta P^{\mu\alpha} P^{\nu\beta} (\partial_\alpha u_\beta + \partial_\beta u_\alpha) - \left(\zeta - \frac{2}{3} \eta \right) P^{\mu\nu} \partial_\alpha u^\alpha, \quad (2.9)$$

$$V_F^\alpha = -\sigma_F \left[T P^{\alpha\beta} \partial_\beta \left(\frac{\mu}{T} \right) - E^{(F)\alpha} \right] - \sigma_{F\tilde{B}} \left[T P^{\alpha\beta} \partial_\beta \left(\frac{\mu_d}{T} \right) - \tilde{E}^{(B)\alpha} \right] + \xi \omega^\alpha \\ + \xi_B B^{(F)\alpha} + \xi_{F\tilde{B}} \tilde{B}^{(B)\alpha}, \quad (2.10)$$

$$V_B^\alpha = -\sigma_B \left[T P^{\alpha\beta} \partial_\beta \left(\frac{\mu_d}{T} \right) - \tilde{E}^{(B)\alpha} \right] - \sigma_{BF} \left[T P^{\alpha\beta} \partial_\beta \left(\frac{\mu}{T} \right) - E^{(F)\alpha} \right] + \xi_d \omega^\alpha \\ + \xi_{\tilde{B}} \tilde{B}^{(B)\alpha} + \xi_{\tilde{B}F} B^{(F)\alpha}. \quad (2.11)$$

where $P^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu$, by $\omega_\mu = 1/2 \epsilon_{\mu\nu\rho\delta} u^\nu \partial^\rho u^\delta$ we have denoted the vorticity. On the other hand, $\xi, \xi_d, \xi_B, \xi_{\tilde{B}}, \xi_{F\tilde{B}}, \xi_{\tilde{B}F}$ are kinetic coefficients being functions of T and μ, μ_d . They are given by the following expressions [45]:

$$\xi = C_1 \mu^2 \left(1 - \frac{2}{3} \frac{\rho \mu}{\epsilon + p} \right) + \mu_d^2 C_2 \left(1 - 2 \frac{\rho \mu}{\epsilon + p} \right) - 4 \frac{\rho T^2}{\epsilon + p} (\mu \tilde{\gamma}_1 + \mu_d \tilde{\gamma}_2) \\ - 2 \frac{\rho T^3}{\epsilon + p} \tilde{\gamma}_3 + 2 \tilde{\gamma}_1 T^2, \quad (2.12)$$

$$\xi_d = -\frac{2}{3} C_1 \frac{\rho_d \mu^3}{\epsilon + p} + 2 C_2 \mu \mu_d \left(1 - \frac{\rho_d \mu_d}{\epsilon + p} \right) - 4 \frac{\rho_d T^2}{\epsilon + p} (\mu \tilde{\gamma}_1 + \mu_d \tilde{\gamma}_2) \\ - 2 \frac{\rho_d T^3}{\epsilon + p} \tilde{\gamma}_3 + 2 \tilde{\gamma}_2 T^2, \quad (2.13)$$

$$\xi_B = C_1 \mu \left(1 - \frac{1}{2} \frac{\rho \mu}{\epsilon + p} \right) - \frac{1}{2} C_3 \frac{\rho \mu_d^2}{\epsilon + p} - \frac{\rho T^2}{\epsilon + p} \tilde{\gamma}_1, \quad (2.14)$$

$$\xi_{\tilde{B}} = C_2 \mu \left(1 - \frac{\rho_d \mu_d}{\epsilon + p} \right) - \frac{\rho_d T^2}{\epsilon + p} \tilde{\gamma}_2. \quad (2.15)$$

$$\xi_{F\tilde{B}} = C_2 \mu_d \left(1 - \frac{\rho \mu}{\epsilon + p} \right) - \frac{\rho T^2}{\epsilon + p} \tilde{\gamma}_2, \quad (2.16)$$

$$\xi_{\tilde{B}F} = C_3 \mu_d \left(1 - \frac{1}{2} \frac{\rho_d \mu_d}{\epsilon + p} \right) - \frac{1}{2} C_1 \frac{\rho_d \mu^2}{\epsilon + p} - \frac{\rho_d T^2}{\epsilon + p} \tilde{\gamma}_1. \quad (2.17)$$

Let us discuss the hydrodynamical regime of the theory in question. Having in mind the relations for V_F^α and V_B^α , suppose that $T \gg \mu, T \gg \mu_d$, as well as, $E^{(F)\alpha}, \tilde{E}^{(B)\alpha}, B^{(F)\alpha}, \tilde{B}^{(B)\alpha} \ll T^2$. These requirements help us to assure that the first derivative expansions constitute the leading order contributions in V_F^α and V_B^α .

As in [20], one also assumes that $|\tilde{c} B^{(F)\alpha}| \ll T^2$ and $|\tilde{d} \tilde{B}^{(F)\alpha}| \ll T^2$, due to the fact that in the first derivative expansions in $j^\mu(F)$ and $j^\mu(B)$, the magnetic components $B^{(F)\alpha}$, $\tilde{B}^{(B)\alpha}$ enter as the contributions of the forms $\tilde{c} B^{(F)\alpha}$, $\tilde{d} \tilde{B}^{(F)\alpha}$, respectively. By $\tilde{c}$, $\tilde{d}$ we have denoted dimensionless numbers, related to the charges C_i which can be either of large or small values.

In our considerations we shall suppose that the system in question is in an equilibrium state in the grand canonical ensemble with chemical potentials μ , μ_d , temperature T and the local velocity will fulfill the condition $u^t = 1$. We also assume that

$$\epsilon + p = Ts + \mu \rho + \mu_d \rho_d, \quad dp = s dT + \rho d\mu + \rho_d d\mu_d. \quad (2.18)$$

2.2 Hydrodynamic magnetotransport of the strongly interacting system

In order to find the magnetic field dependence of the electrical conductivity in the system with chiral anomaly and $\mathbb{Z}_2$ topological charge, we suppose that one deals with a background magnetic fields in z -direction, provided by

$$A_2 = B x, \quad B_2 = \tilde{B}_{\text{add}} x, \quad (2.19)$$

and consider the case when $E^{(F)\mu} = F^{t\mu} = 0$ and $\tilde{E}^{(B)\mu} = B^{t\mu} = 0$. We shall study the response of the currents to the perturbations of $\delta E^{(F)z}$ and $\delta \tilde{E}^{(B)z}$ fields.

In the thermodynamical system under consideration the perturbations of the thermodynamical variables are given by

$$\begin{aligned} \mu(x_\alpha) &= \mu + \delta\mu(x_\alpha), & \mu_d(x_\alpha) &= \mu_d + \delta\mu_d(x_\alpha), \\ T(x_\alpha) &= T + \delta T(x_\alpha), & u^\beta(x_\alpha) &= \left(1, \delta u_j(x_\alpha)\right). \end{aligned} \quad (2.20)$$

Moreover, the perturbations of the other components of the gauge fields are chosen as follows:

$$\delta E^{(F)z} = \delta F^{tz}, \quad \delta \tilde{E}^{(B)z} = \delta B^{tz}, \quad (2.21)$$

$$\delta E^{(F)x} = \delta F^{tx} + B \delta u^y, \quad \delta \tilde{E}^{(B)x} = \delta B^{tx} + \tilde{B}_{\text{add}} \delta u^y, \quad (2.22)$$

$$\delta E^{(F)y} = \delta F^{ty} - B \delta u^x, \quad \delta \tilde{E}^{(B)y} = \delta B^{ty} - \tilde{B}_{\text{add}} \delta u^x. \quad (2.23)$$

Having in mind the above forms of perturbations of the hydrodynamical variables, to the linear order, the perturbations of the conserved quantities fulfill the following relations:

$$\delta T^{00} = \delta\epsilon, \quad (2.24)$$

$$\delta T^{0i} = (\epsilon + p) \delta u^i, \quad (2.25)$$

$$\delta T^{ij} = \delta p g^{ij} - \eta \left(\partial^i \delta u^j + \partial^j \delta u^i - \frac{2}{3} g^{ij} \partial_m \delta u^m \right) - \zeta g^{ij} \partial_m \delta u^m, \quad (2.26)$$

and the modifications of the respective currents connected with U(1)-gauge fields are calculated by combining equations (2.21)–(2.23), having in mind the relations (2.6)–(2.8).

Consequently we obtain

$$\delta j^0(F) = \delta\rho + \xi_B \delta u_z B + \xi_{F\tilde{B}} \delta u_z \tilde{B}_{\text{add}}, \quad (2.27)$$

$$\begin{aligned} \delta j^x(F) = & \rho\delta u^x - \sigma_F T \partial^x \left(\delta \left(\frac{\mu}{T} \right) \right) + \sigma_F (\delta F^{tx} + B\delta u^y) - \sigma_{F\tilde{B}} T \partial^x \left(\delta \left(\frac{\mu_d}{T} \right) \right) \\ & + \sigma_{F\tilde{B}} (\delta B^{tx} + \tilde{B}_{\text{add}}\delta u^y) + \xi \partial_{[y}\delta u_{z]}, \end{aligned} \quad (2.28)$$

$$\begin{aligned} \delta j^y(F) = & \rho\delta u^y - \sigma_F T \partial^y \left(\delta \left(\frac{\mu}{T} \right) \right) + \sigma_F (\delta F^{ty} - B\delta u^x) - \sigma_{F\tilde{B}} T \partial^y \left(\delta \left(\frac{\mu_d}{T} \right) \right) \\ & + \sigma_{F\tilde{B}} (\delta B^{ty} - \tilde{B}_{\text{add}}\delta u^x) + \xi \partial_{[z}\delta u_{x]}, \end{aligned} \quad (2.29)$$

$$\begin{aligned} \delta j^z(F) = & \rho\delta u^z - \sigma_F T \partial^z \left(\delta \left(\frac{\mu}{T} \right) \right) + \sigma_F \delta F^{tz} - \sigma_{F\tilde{B}} T \partial^z \left(\delta \left(\frac{\mu_d}{T} \right) \right) \\ & + \sigma_{F\tilde{B}} \delta B^{tz} + B\delta\xi_B + \tilde{B}_{\text{add}}\delta\xi_{\tilde{B}} + \xi \partial_{[y}\delta u_{z]}. \end{aligned} \quad (2.30)$$

The current perturbations of the additional gauge field imply

$$\delta j^0(B) = \delta\rho_d + \xi_{\tilde{B}} \delta u_z \tilde{B}_{\text{add}} + \xi_{\tilde{B}F} \delta u_z B, \quad (2.31)$$

$$\begin{aligned} \delta j^x(B) = & \rho_d\delta u^x - \sigma_B T \partial^x \left(\delta \left(\frac{\mu_d}{T} \right) \right) + \sigma_B (\delta B^{tx} + \tilde{B}_{\text{add}}\delta u^y) - \sigma_{BF} T \partial^x \left(\delta \left(\frac{\mu}{T} \right) \right) \\ & + \sigma_{BF} (\delta F^{tx} + B\delta u^y) + \xi_d \partial_{[y}\delta u_{z]}, \end{aligned} \quad (2.32)$$

$$\begin{aligned} \delta j^y(B) = & \rho_d\delta u^y - \sigma_B T \partial^y \left(\delta \left(\frac{\mu_d}{T} \right) \right) + \sigma_B (\delta B^{ty} - \tilde{B}_{\text{add}}\delta u^x) - \sigma_{BF} T \partial^y \left(\delta \left(\frac{\mu}{T} \right) \right) \\ & + \sigma_{BF} (\delta F^{ty} - B\delta u^x) + \xi_d \partial_{[z}\delta u_{x]}, \end{aligned} \quad (2.33)$$

$$\begin{aligned} \delta j^z(B) = & \rho_d\delta u^z - \sigma_B T \partial^z \left(\delta \left(\frac{\mu_d}{T} \right) \right) + \sigma_B \delta B^{tz} - \sigma_{BF} T \partial^z \left(\delta \left(\frac{\mu}{T} \right) \right) \\ & + \sigma_{BF} \delta F^{tz} + \tilde{B}_{\text{add}}\delta\xi_{\tilde{B}} + B\delta\xi_{\tilde{B}F} + \xi_d \partial_{[y}\delta u_{z]}, \end{aligned} \quad (2.34)$$

where, as in [20], we suppose that $\delta\epsilon$, δp are entirely impelled by $\delta\mu$, $\delta\mu_d$, δT . Moreover, one neglects the chiral vortical effects because they do not influence the results.

The time evolution of the perturbations is obtained from the conservation equations (2.1)–(2.3). To calculate them, in analogy to [20], we introduce the dissipation terms into the perturbed conservation laws of the considered theory and obtain

$$\partial_\mu \delta T^{\mu 0} = \delta F^{0z} j_z(F) + \delta B^{0z} j_z(B) + \frac{1}{\tau_e} \delta T^{\gamma 0} u_\gamma, \quad (2.35)$$

$$\partial_\mu \delta T^{\mu k} = \rho \delta F^{0k} + \rho_d \delta B^{0k} + F^{k\alpha} \delta j_\alpha(F) + B^{k\alpha} \delta j_\alpha(B) + \frac{1}{\tau_m} \delta T^{\gamma k} u_\gamma, \quad (2.36)$$

$$\partial_\mu \delta j^\mu(F) = C_1 \delta E_{(F)\mu} B^{(F)\mu} + C_2 \delta \tilde{E}_{(B)\mu} \tilde{B}^{(B)\mu} + \frac{1}{\tau_c} \delta j^\gamma(F) u_\gamma, \quad (2.37)$$

$$\partial_\mu \delta j^\mu(B) = C_3 \delta \tilde{E}_{(B)\mu} B^{(F)\mu} + C_4 \delta E_{(F)\mu} \tilde{B}^{(B)\mu} + \frac{1}{\tau_{cd}} \delta j^\gamma(B) u_\gamma, \quad (2.38)$$

where we have denoted by τ_e the energy relaxation time, τ_m stands for the momentum relaxation time, while τ_c and τ_{cd} describe the charge relaxation times for the Maxwell and

the additional U(1)-gauge field. One should have in mind that the relaxation terms affect only the deviations from the equilibrium.

In the next step let us put the perturbations into the energy-momentum and currents perturbation relations, in order to achieve the set of relations for $\delta\mu$, $\delta\mu_d$, δT and δu^α . For the perturbations of the energy-momentum tensor one obtains

$$\begin{aligned} & \left(\partial_t + \frac{1}{\tau_e} \right) \delta\epsilon + \partial_i [(\epsilon + p)\delta u^i] - \delta E^{(F)z} \left(\xi_B B + \xi_{F\tilde{B}} \tilde{B}_{\text{add}} \right) \\ & - \delta \tilde{E}^{(B)z} \left(\xi_{\tilde{B}} \tilde{B}_{\text{add}} + \xi_{\tilde{B}F} B \right) = 0, \end{aligned} \quad (2.39)$$

$$\left(\partial_t + \frac{1}{\tau_m} \right) (\epsilon + p)\delta u^x - \rho\delta F^{0x} - \rho_d\delta B^{0x} + \partial^x\delta p - \eta \left[\partial_i\partial^i\delta u^x + \frac{1}{3}\partial^x\partial_m\delta u^m \right] \quad (2.40)$$

$$\begin{aligned} & -\zeta\partial^x\partial_m\delta u^m = B \left[\rho\delta u_y - \sigma_F T\partial_y \left(\delta \left(\frac{\mu}{T} \right) \right) + \sigma_F (\delta F^{0y} - B\delta u^x) \right. \\ & - \sigma_{F\tilde{B}} T\partial_y \left(\delta \left(\frac{\mu_d}{T} \right) \right) + \sigma_{F\tilde{B}} (\delta B^{0y} - \tilde{B}_{\text{add}}\delta u^x) + \xi\partial_{[z}\delta u^{x]} \left. \right] \\ & + \tilde{B}_{\text{add}} \left[\rho_d\delta u_y - \sigma_B T\partial_y \left(\delta \left(\frac{\mu_d}{T} \right) \right) + \sigma_B (\delta B^{0y} - \tilde{B}_{\text{add}}\delta u^x) \right. \\ & - \sigma_{BF} T\partial_y \left(\delta \left(\frac{\mu}{T} \right) \right) + \sigma_{BF} (\delta F^{0y} - B\delta u^x) + \xi_d\partial_{[z}\delta u^{x]} \left. \right], \end{aligned}$$

$$\left(\partial_t + \frac{1}{\tau_m} \right) (\epsilon + p)\delta u^y - \rho\delta F^{0y} - \rho_d\delta B^{0y} + \partial^y\delta p - \eta \left[\partial_i\partial^i\delta u^y + \frac{1}{3}\partial^y\partial_m\delta u^m \right] \quad (2.41)$$

$$\begin{aligned} & -\zeta\partial^y\partial_m\delta u^m = -B \left[\rho\delta u_x - \sigma_F T\partial_x \left(\delta \left(\frac{\mu}{T} \right) \right) + \sigma_F (\delta F^{0x} + B\delta u^y) \right. \\ & - \sigma_{F\tilde{B}} T\partial_x \left(\delta \left(\frac{\mu_d}{T} \right) \right) + \sigma_{F\tilde{B}} (\delta B^{0x} + \tilde{B}_{\text{add}}\delta u^y) + \xi\partial^{[y}\delta u_{z]} \left. \right] \\ & - \tilde{B}_{\text{add}} \left[\rho_d\delta u_x - \sigma_B T\partial_x \left(\delta \left(\frac{\mu_d}{T} \right) \right) + \sigma_B (\delta B^{0x} + \tilde{B}_{\text{add}}\delta u^y) \right. \\ & - \sigma_{BF} T\partial_x \left(\delta \left(\frac{\mu}{T} \right) \right) + \sigma_{BF} (\delta F^{0x} + B\delta u^y) + \xi_d\partial^{[y}\delta u_{z]} \left. \right], \end{aligned}$$

$$\left(\partial_t + \frac{1}{\tau_m} \right) (\epsilon + p)\delta u^z - \rho\delta F^{0z} - \rho_d\delta B^{0z} + \partial^z\delta p - \eta \left[\partial_i\partial^i\delta u^z + \frac{1}{3}\partial^z\partial_m\delta u^m \right] \quad (2.42)$$

$$-\zeta\partial^z\partial_m\delta u^m = 0.$$

Consequently, for perturbations of the currents bounded with $F_{\mu\nu}$ and $B_{\mu\nu}$ gauge fields, we arrive at the following relations:

$$\begin{aligned} & \left(\partial_t + \frac{1}{\tau_c} \right) \left(\delta\rho + \xi_B B \delta u_z + \xi_{F\tilde{B}} \tilde{B}_{\text{add}} \delta u_z \right) + \partial_i \left[\rho\delta u^i - \sigma_F T\partial^i \left(\delta \left(\frac{\mu}{T} \right) \right) + \sigma_F \delta F^{0i} \right. \\ & \quad \left. - \sigma_{F\tilde{B}} T\partial^i \left(\delta \left(\frac{\mu_d}{T} \right) \right) + \sigma_{F\tilde{B}} \delta B^{0i} \right] + \partial_i \left(\frac{\xi}{2} \epsilon^{ijk} \partial_j \delta u_k \right) \quad (2.43) \\ & + \partial_x \left(\sigma_F B \delta u^y + \sigma_{F\tilde{B}} \tilde{B}_{\text{add}} \delta u^y \right) - \partial_y \left(\sigma_F B \delta u^x + \sigma_{F\tilde{B}} \tilde{B}_{\text{add}} \delta u^x \right) \\ & + \partial_z \left(B \delta \xi_B + \tilde{B}_{\text{add}} \delta \xi_{F\tilde{B}} \right) - C_1 B \delta E_z^{(F)} - C_2 \tilde{B}_{\text{add}} \delta \tilde{E}_z^{(B)} = 0, \end{aligned}$$

$$\begin{aligned}
 & \left(\partial_t + \frac{1}{\tau_{cd}} \right) \left(\delta \rho_d + \xi_{\tilde{B}} \tilde{B}_{\text{add}} \delta u_z + \xi_{\tilde{B}F} B \delta u_z \right) + \partial_i \left[\rho_d \delta u^i - \sigma_B T \partial^i \left(\delta \left(\frac{\mu_d}{T} \right) \right) + \sigma_B \delta B^{0i} \right. \\
 & \quad \left. - \sigma_{BF} T \partial^i \left(\delta \left(\frac{\mu}{T} \right) \right) + \sigma_{BF} \delta F^{0i} \right] + \partial_i \left(\frac{\xi_d}{2} \epsilon^{ijk} \partial_j \delta u_k \right) \quad (2.44) \\
 & + \partial_x \left(\sigma_B \tilde{B}_{\text{add}} \delta u^y + \sigma_{BF} B \delta u^y \right) - \partial_y \left(\sigma_B \tilde{B}_{\text{add}} \delta u^x + \sigma_{BF} B \delta u^x \right) \\
 & + \partial_z \left(\tilde{B}_{\text{add}} \delta \xi_{\tilde{B}} + B \delta \xi_{\tilde{B}F} \right) - C_3 B \delta \tilde{E}_z^{(B)} - C_4 \tilde{B}_{\text{add}} \delta E_z^{(F)} = 0.
 \end{aligned}$$

Further, one can readily verify that the implementation of the Laplace transformation in t -direction reveals

$$\begin{aligned}
 & \omega_c \delta \epsilon - i \delta \epsilon^{(0)} + i(\epsilon + p) \partial_i \delta u^i - i \delta E^{(F)z} \left(\xi_B B + \xi_{F\tilde{B}} \tilde{B}_{\text{add}} \right) \\
 & - i \delta \tilde{E}^{(B)z} \left(\xi_{\tilde{B}} \tilde{B}_{\text{add}} + \xi_{\tilde{B}F} B \right) = 0, \quad (2.45)
 \end{aligned}$$

$$\begin{aligned}
 & (\epsilon + p) \left[\omega_m \delta u^x - i \delta u^{x(0)} \right] - i \rho \delta F^{0x} - i \rho_d \delta B^{0x} + i \partial^x \delta p - i \eta \left[\partial_i \partial^i \delta u^x + \frac{1}{3} \partial^x \partial_m \delta u^m \right] \\
 & - i \zeta \partial^x \partial_m \delta u^m = i B \left[\rho \delta u_y - \sigma_F T \partial_y \left(\delta \left(\frac{\mu}{T} \right) \right) + \sigma_F (\delta F^{0y} - B \delta u^x) \right. \\
 & - \sigma_{F\tilde{B}} T \partial_y \left(\delta \left(\frac{\mu_d}{T} \right) \right) + \sigma_{F\tilde{B}} (\delta B^{0y} - \tilde{B}_{\text{add}} \delta u^x) + \xi \partial_{[z} \delta u^{x]} \left. \right] \\
 & + i \tilde{B}_{\text{add}} \left[\rho_d \delta u_y - \sigma_B T \partial_y \left(\delta \left(\frac{\mu_d}{T} \right) \right) + \sigma_B (\delta B^{0y} - \tilde{B}_{\text{add}} \delta u^x) \right. \\
 & - \sigma_{BF} T \partial_y \left(\delta \left(\frac{\mu}{T} \right) \right) + \sigma_{BF} (\delta F^{0y} - B \delta u^x) + \xi_d \partial_{[z} \delta u^{x]} \left. \right], \quad (2.46)
 \end{aligned}$$

$$\begin{aligned}
 & (\epsilon + p) \left[\omega_m \delta u^y - i \delta u^{y(0)} \right] - i \rho \delta F^{0y} - i \rho_d \delta B^{0y} + i \partial^y \delta p - i \eta \left[\partial_i \partial^i \delta u^y + \frac{1}{3} \partial^y \partial_m \delta u^m \right] \\
 & - i \zeta \partial^y \partial_m \delta u^m = i B \left[\rho \delta u_x - \sigma_F T \partial_x \left(\delta \left(\frac{\mu}{T} \right) \right) + \sigma_F (\delta F^{0x} + B \delta u^y) \right. \\
 & - \sigma_{F\tilde{B}} T \partial_x \left(\delta \left(\frac{\mu_d}{T} \right) \right) + \sigma_{F\tilde{B}} (\delta B^{0x} + \tilde{B}_{\text{add}} \delta u^y) + \xi \partial_{[y} \delta u^{z]} \left. \right] \\
 & + i \tilde{B}_{\text{add}} \left[\rho_d \delta u_x - \sigma_B T \partial_x \left(\delta \left(\frac{\mu_d}{T} \right) \right) + \sigma_B (\delta B^{0x} + \tilde{B}_{\text{add}} \delta u^y) \right. \\
 & - \sigma_{BF} T \partial_x \left(\delta \left(\frac{\mu}{T} \right) \right) + \sigma_{BF} (\delta F^{0x} + B \delta u^y) + \xi_d \partial_{[y} \delta u^{z]} \left. \right], \quad (2.47)
 \end{aligned}$$

$$\begin{aligned}
 & (\epsilon + P) \left[\omega_m \delta u^z - i \delta u^{z(0)} \right] - i \rho \delta F^{0z} - i \rho_d \delta B^{0z} + i \partial^z \delta p - i \eta \left[\partial_i \partial^i \delta u^z + \frac{1}{3} \partial^z \partial_m \delta u^m \right] \\
 & - i \zeta \partial^z \partial_m \delta u^m = 0, \quad (2.48)
 \end{aligned}$$

$$\begin{aligned}
 & \omega_c \delta \rho - i \delta \rho^{(0)} + \left[\omega_c \delta u_z - i \delta u_z^{(0)} \right] \left(\xi_B B + \xi_{F\tilde{B}} \tilde{B}_{\text{add}} \right) + i \partial_i \left[\rho \delta u^i - \sigma_F T \partial^i \left(\delta \left(\frac{\mu}{T} \right) \right) \right. \\
 & + \sigma_F \delta F^{0i} - \sigma_{F\tilde{B}} T \partial^i \left(\delta \left(\frac{\mu_d}{T} \right) \right) + \sigma_{F\tilde{B}} \delta B^{0i} \left. \right] + i \partial_a \left(\frac{\xi}{2} \epsilon^{ajk} \partial_j \delta u_k \right) \\
 & + i \partial_x \left(\sigma_F B \delta u^y + \sigma_{F\tilde{B}} \tilde{B}_{\text{add}} \delta u^y \right) - i \partial_y \left(\sigma_F B \delta u^x + \sigma_{F\tilde{B}} \tilde{B}_{\text{add}} \delta u^x \right) \\
 & + i \partial_z \left(B \delta \xi_B + \tilde{B}_{\text{add}} \delta \xi_{F\tilde{B}} \right) - i C_1 B \delta E_z^{(F)} - i C_2 \tilde{B}_{\text{add}} \delta \tilde{E}_z^{(B)} = 0, \quad (2.49)
 \end{aligned}$$

$$\begin{aligned}
 & \omega_{c_d} \delta \rho_d - i \delta \rho_d^{(0)} + \left[\omega_{c_d} \delta u_z - i \delta u_z^{(0)} \right] \left(\xi_{\tilde{B}} \tilde{B}_{\text{add}} + \xi_{\tilde{B}F} B \right) + i \partial_i \left[\rho_d \delta u^i - \sigma_B T \partial^i \left(\delta \left(\frac{\mu_d}{T} \right) \right) \right. \\
 & \quad \left. + \sigma_B \delta B^{0i} - \sigma_{BF} T \partial^i \left(\delta \left(\frac{\mu}{T} \right) \right) + \sigma_{BF} \delta F^{0i} \right] + i \partial_a \left(\frac{\xi}{2} \epsilon^{ajk} \partial_j \delta u_k \right) \\
 & \quad + i \partial_x \left(\sigma_B \tilde{B}_{\text{add}} \delta u^y + \sigma_{BF} B \delta u^y \right) - i \partial_y \left(\sigma_B \tilde{B}_{\text{add}} \delta u^x + \sigma_{BF} B \delta u^x \right) \\
 & \quad + i \partial_z \left(\tilde{B}_{\text{add}} \delta \xi_{\tilde{B}} + B \delta \xi_{\tilde{B}F} \right) - i C_3 B \delta \tilde{E}^{(B)z} - i C_2 \tilde{B}_{\text{add}} \delta E^{(F)z} = 0, \tag{2.50}
 \end{aligned}$$

where the symbols with superscript (0) refer to initial values of the perturbations. In the above relations we have denoted

$$\omega_e = \omega + \frac{i}{\tau_e}, \quad \omega_m = \omega + \frac{i}{\tau_m}, \quad \omega_c = \omega + \frac{i}{\tau_c}, \quad \omega_{c_d} = \omega + \frac{i}{\tau_{c_d}}. \tag{2.51}$$

The Fourier transformation in the spatial directions, with the auxiliary condition that $k \rightarrow 0$, leads the following system of equations:

$$\omega_e \delta \epsilon - i \delta \epsilon^{(0)} - i \delta E^{(F)z} \left(\xi_B B + \xi_{F\tilde{B}} \tilde{B}_{\text{add}} \right) - i \delta \tilde{E}^{(B)z} \left(\xi_{\tilde{B}} \tilde{B}_{\text{add}} + \xi_{\tilde{B}F} B \right) = 0, \tag{2.52}$$

$$\begin{aligned}
 (\epsilon + p) \left[\omega_m \delta u^x - i \delta u^{x(0)} \right] - i \rho \delta F^{0x} - i \rho_d \delta B^{0x} &= i B \left[\rho \delta u_y + \sigma_F \left(\delta F^{0y} - B \delta u^x \right) \right. \\
 & \quad \left. + \sigma_{F\tilde{B}} \left(\delta B^{0y} - \tilde{B}_{\text{add}} \delta u^x \right) \right] + i \tilde{B}_{\text{add}} \left[\rho_d \delta u_y + \sigma_B \left(\delta B^{0y} - \tilde{B}_{\text{add}} \delta u^x \right) \right. \\
 & \quad \left. + \sigma_{BF} \left(\delta F^{0y} - B \delta u^x \right) \right], \\
 (\epsilon + p) \left[\omega_m \delta u^y - i \delta u^{y(0)} \right] - i \rho \delta F^{0y} - i \rho_d \delta B^{0y} &= i B \left[\rho \delta u_x + \sigma_F \left(\delta F^{0x} - B \delta u^y \right) \right. \\
 & \quad \left. + \sigma_{F\tilde{B}} \left(\delta B^{0x} - \tilde{B}_{\text{add}} \delta u^y \right) \right] + i \tilde{B}_{\text{add}} \left[\rho_d \delta u_x + \sigma_B \left(\delta B^{0x} - \tilde{B}_{\text{add}} \delta u^y \right) \right. \\
 & \quad \left. + \sigma_{BF} \left(\delta F^{0x} - B \delta u^y \right) \right], \tag{2.53}
 \end{aligned}$$

$$(\epsilon + P) \left[\omega_m \delta u^z - i \delta u^{z(0)} \right] - i \rho \delta E^{(F)z} - i \rho_d \delta \tilde{E}^{(B)z} = 0, \tag{2.54}$$

$$\omega_c \delta \rho - i \delta \rho^{(0)} + \left[\omega_{c_d} \delta u_z - i \delta u_z^{(0)} \right] \left(\xi_B B + \xi_{F\tilde{B}} \tilde{B}_{\text{add}} \right) \tag{2.55}$$

$$-i C_1 B \delta E_z^{(F)} - i C_2 \tilde{B}_{\text{add}} \delta \tilde{E}_z^{(B)} = 0,$$

$$\omega_{c_d} \delta \rho_d - i \delta \rho_d^{(0)} + \left[\omega_{c_d} \delta u_z - i \delta u_z^{(0)} \right] \left(\xi_{\tilde{B}} \tilde{B}_{\text{add}} + \xi_{\tilde{B}F} B \right) \tag{2.56}$$

$$-i C_3 B \delta \tilde{E}^{(B)z} - i C_2 \tilde{B}_{\text{add}} \delta E^{(F)z} = 0,$$

In order to solve the above equations, we write the dependence of $\delta \epsilon$, $\delta \rho$, $\delta \rho_d$, δp on $\delta \mu$, $\delta \mu_d$, δT as

$$\delta \epsilon = e_1 \delta \mu + e_2 \delta T + e_3 \delta \mu_d = \left(\frac{\partial \epsilon}{\partial \mu} \right)_{T, \mu_d} \delta \mu + \left(\frac{\partial \epsilon}{\partial T} \right)_{\mu, \mu_d} \delta T + \left(\frac{\partial \epsilon}{\partial \mu_d} \right)_{T, \mu} \delta \mu_d, \tag{2.57}$$

$$\delta \rho = f_1 \delta \mu + f_2 \delta T + f_3 \delta \mu_d = \left(\frac{\partial \rho}{\partial \mu} \right)_{T, \mu_d} \delta \mu + \left(\frac{\partial \rho}{\partial T} \right)_{\mu, \mu_d} \delta T + \left(\frac{\partial \rho}{\partial \mu_d} \right)_{T, \mu} \delta \mu_d, \tag{2.58}$$

$$\delta\rho_d = g_1 \delta\mu_d + g_2 \delta T + g_3 \delta\mu = \left(\frac{\partial\rho_d}{\partial\mu_d}\right)_{T,\mu} \delta\mu_d + \left(\frac{\partial\rho_d}{\partial T}\right)_{\mu,\mu_d} \delta T + \left(\frac{\partial\rho_d}{\partial\mu}\right)_{T,\mu_d} \delta\mu, \quad (2.59)$$

$$\delta p = \rho \delta\mu + \rho_d \delta\mu_d + s \delta T, \quad (2.60)$$

where the coefficients in the above relations depend on the details corresponding to the system in question.

We solve equations (2.52)–(2.56) in terms of $\delta E^{(F)z}$ and $\delta \tilde{E}^{(B)z}$. Namely, they can be written as follows:

$$\begin{aligned} \delta\mu = & \frac{1}{W} \delta E^{(F)z} \left[\frac{i}{\omega_e} \left(\xi_B B + \xi_{F\tilde{B}} \tilde{B}_{\text{add}} \right) (f_2 g_1 - g_2 f_3) \right. \\ & + \left(-\frac{\rho}{\epsilon + p} \frac{i}{\omega_m} \left(\xi_B B + \xi_{F\tilde{B}} \tilde{B}_{\text{add}} \right) + \frac{i}{\omega_c} C_1 B \right) (g_2 e_3 - e_2 g_1) \\ & + \left(-\frac{\rho}{\epsilon + p} \frac{i}{\omega_m} \left(\xi_{\tilde{B}} \tilde{B}_{\text{add}} + \xi_{\tilde{B}F} B \right) + \frac{i}{\omega_{cd}} C_4 \tilde{B}_{\text{add}} \right) (f_3 e_2 - f_2 e_3) \Big] \\ & + \frac{1}{W} \delta \tilde{E}^{(B)z} \left[\frac{i}{\omega_e} \left(\xi_{\tilde{B}} \tilde{B}_{\text{add}} + \xi_{\tilde{B}F} B \right) (f_2 g_1 - g_2 f_3) \right. \\ & + \left(-\frac{\rho_d}{\epsilon + p} \frac{i}{\omega_m} \left(\xi_B B + \xi_{F\tilde{B}} \tilde{B}_{\text{add}} \right) + \frac{i}{\omega_c} C_2 \tilde{B}_{\text{add}} \right) (g_2 e_3 - e_2 g_1) \\ & + \left(-\frac{\rho_d}{\epsilon + p} \frac{i}{\omega_m} \left(\xi_{\tilde{B}} \tilde{B}_{\text{add}} + \xi_{\tilde{B}F} B \right) + \frac{i}{\omega_{cd}} C_3 B \right) (f_3 e_2 - f_2 e_3) \Big] + \dots \\ & = M_1 \delta E^{(F)z} + M_2 \delta \tilde{E}^{(B)z} + \dots, \end{aligned} \quad (2.61)$$

$$\begin{aligned} \delta\mu_d = & \frac{1}{W} \delta E^{(F)z} \left[\frac{i}{\omega_e} \left(\xi_B B + \xi_{F\tilde{B}} \tilde{B}_{\text{add}} \right) (f_1 g_2 - f_2 g_3) \right. \\ & + \left(-\frac{\rho}{\epsilon + p} \frac{i}{\omega_m} \left(\xi_B B + \xi_{F\tilde{B}} \tilde{B}_{\text{add}} \right) + \frac{i}{\omega_c} C_1 B \right) (e_2 g_3 - g_2 e_1) \\ & + \left(-\frac{\rho}{\epsilon + p} \frac{i}{\omega_m} \left(\xi_{\tilde{B}} \tilde{B}_{\text{add}} + \xi_{\tilde{B}F} B \right) + \frac{i}{\omega_{cd}} C_4 \tilde{B}_{\text{add}} \right) (e_1 f_2 - e_2 f_1) \Big] \\ & + \frac{1}{W} \delta \tilde{E}^{(B)z} \left[\frac{i}{\omega_e} \left(\xi_{\tilde{B}} \tilde{B}_{\text{add}} + \xi_{\tilde{B}F} B \right) (f_1 g_2 - f_2 g_3) \right. \\ & + \left(-\frac{\rho_d}{\epsilon + p} \frac{i}{\omega_m} \left(\xi_B B + \xi_{F\tilde{B}} \tilde{B}_{\text{add}} \right) + \frac{i}{\omega_c} C_2 \tilde{B}_{\text{add}} \right) (g_3 e_2 - e_1 g_2) \\ & + \left(-\frac{\rho_d}{\epsilon + p} \frac{i}{\omega_m} \left(\xi_{\tilde{B}} \tilde{B}_{\text{add}} + \xi_{\tilde{B}F} B \right) + \frac{i}{\omega_{cd}} C_3 B \right) (e_1 f_2 - e_2 f_1) \Big] + \dots \\ & = D_1 \delta E^{(F)z} + D_2 \delta \tilde{E}^{(B)z} + \dots, \end{aligned} \quad (2.62)$$

$$\begin{aligned}
 \delta T &= \frac{1}{W} \delta E^{(F)z} \left[\frac{i}{\omega_e} (\xi_B B + \xi_{F\tilde{B}} \tilde{B}_{\text{add}}) (f_3 g_3 - f_1 g_1) \right. \\
 &\quad + \left(-\frac{\rho}{\epsilon + p} \frac{i}{\omega_m} (\xi_B B + \xi_{F\tilde{B}} \tilde{B}_{\text{add}}) + \frac{i}{\omega_c} C_1 B \right) (e_1 g_1 - g_3 e_3) \\
 &\quad \left. + \left(-\frac{\rho}{\epsilon + p} \frac{i}{\omega_m} (\xi_{\tilde{B}} \tilde{B}_{\text{add}} + \xi_{\tilde{B}F} B) + \frac{i}{\omega_{cd}} C_4 \tilde{B}_{\text{add}} \right) (e_3 f_1 - e_1 f_3) \right] \\
 &\quad + \frac{1}{W} \delta \tilde{E}^{(B)z} \left[\frac{i}{\omega_e} (\xi_{\tilde{B}} \tilde{B}_{\text{add}} + \xi_{\tilde{B}F} B) (f_3 g_3 - f_1 g_1) \right. \\
 &\quad + \left(-\frac{\rho_d}{\epsilon + p} \frac{i}{\omega_m} (\xi_B B + \xi_{F\tilde{B}} \tilde{B}_{\text{add}}) + \frac{i}{\omega_c} C_2 \tilde{B}_{\text{add}} \right) (g_1 e_1 - e_3 g_3) \\
 &\quad \left. + \left(-\frac{\rho_d}{\epsilon + p} \frac{i}{\omega_m} (\xi_{\tilde{B}} \tilde{B}_{\text{add}} + \xi_{\tilde{B}F} B) + \frac{i}{\omega_{cd}} C_3 B \right) (e_3 f_1 - e_1 f_3) \right] + \dots \\
 &= T_1 \delta E^{(F)z} + T_2 \delta \tilde{E}^{(B)z} + \dots, \\
 \delta u^z &= \frac{\rho}{\epsilon + p} \frac{i}{\omega_m} \delta E^z + \frac{\rho_d}{\epsilon + p} \frac{i}{\omega_m} \delta \tilde{E}^z + \dots,
 \end{aligned} \tag{2.64}$$

where we have denoted by W

$$W = \det \begin{pmatrix} e_1 & e_2 & e_3 \\ f_1 & f_2 & f_3 \\ g_3 & g_2 & g_1 \end{pmatrix}. \tag{2.65}$$

By ‘...’ one has in mind the terms which are unrelated to $\delta E^{(F)z}$ and $\delta \tilde{E}^{(B)z}$, which disappear when we choose the initial values of the other perturbations equal to zero. In what follows, for the brevity of notation, one assigns all terms standing in front of δE^z and $\delta \tilde{E}^z$, respectively by

$$\delta \mu = M_1 \delta E^{(F)z} + M_2 \delta \tilde{E}^{(B)z}, \tag{2.66}$$

$$\delta \mu_d = D_1 \delta E^{(F)z} + D_2 \delta \tilde{E}^{(B)z}, \tag{2.67}$$

$$\delta T = T_1 \delta E^{(F)z} + T_2 \delta \tilde{E}^{(B)z}. \tag{2.68}$$

They will be needed for substituting them into the perturbation relations connected with z -directed gauge field currents. On this account we have

$$\begin{aligned}
 \delta j^z(F) &= \rho \delta u^z - \sigma_F T \partial^z \left(\delta \frac{\mu}{T} \right) + \sigma_F \delta E^{(F)z} - \sigma_{F\tilde{B}} T \partial^z \left(\delta \frac{\mu_d}{T} \right) + \sigma_{F\tilde{B}} \delta \tilde{E}^{(B)z} \\
 &\quad + \delta \xi_B B + \delta \xi_{F\tilde{B}} \tilde{B}_{\text{add}} + \xi \partial_{[x} \delta u_{y]},
 \end{aligned} \tag{2.69}$$

and for $B_{\mu\nu}$ field current it yields

$$\begin{aligned}
 \delta j^z(B) &= \rho_d \delta u^z - \sigma_B T \partial^z \left(\delta \frac{\mu_d}{T} \right) + \sigma_B \delta \tilde{E}^{(B)z} - \sigma_{BF} T \partial^z \left(\delta \frac{\mu}{T} \right) + \sigma_{BF} \delta E^{(F)z} \\
 &\quad + \delta \xi_{\tilde{B}} \tilde{B}_{\text{add}} + \delta \xi_{\tilde{B}F} B + \xi \partial_{[x} \delta u_{y]}.
 \end{aligned} \tag{2.70}$$

It can be verified that in the limit of $k \rightarrow 0$ they reduce to

$$\delta j^z(F) = \rho \delta u^z + \sigma_F \delta E^{(F)z} + \sigma_{F\tilde{B}} \delta \tilde{E}^{(B)z} + \delta \xi_B B + \delta \xi_{F\tilde{B}} \tilde{B}_{\text{add}}, \quad (2.71)$$

for the perturbations of the Maxwell field current, and for the additional one we obtain

$$\delta j^z(B) = \rho_d \delta u^z + \sigma_B \delta \tilde{E}^{(B)z} + \sigma_{BF} \delta E^{(F)z} + \delta \xi_{\tilde{B}} \tilde{B}_{\text{add}} + \delta \xi_{\tilde{B}F} B. \quad (2.72)$$

Inserting the explicit values for $\delta \xi_B$, $\delta \xi_{F\tilde{B}}$ and $\delta \xi_{\tilde{B}}$, $\delta \xi_{\tilde{B}F}$, obtained from (2.14)–(2.15) with help of (2.57)–(2.60), one arrives at the relation

$$\begin{aligned} \delta j_z(F) = \delta E_z^{(F)} & \left[\sigma_F + \frac{\rho^2}{\epsilon + p} \frac{i}{\omega_m} + B \left(M_1 H_1 + D_1 H_2 + T_1 H_3 \right) \right. \\ & \left. + \tilde{B}_{\text{add}} \left(D_1 G_1 + M_1 G_2 + T_1 G_3 \right) \right] \\ & + \delta \tilde{E}_z^{(B)} \left[\sigma_{F\tilde{B}} + \frac{\rho \rho_d}{\epsilon + p} \frac{i}{\omega_m} + B \left(M_2 H_1 + D_2 H_2 + T_2 H_3 \right) \right. \\ & \left. + \tilde{B}_{\text{add}} \left(D_2 G_1 + M_2 G_2 + T_2 G_3 \right) \right], \end{aligned} \quad (2.73)$$

where we have defined the following quantities connected with the terms multiplied by B -magnetic field:

$$H_1 = C_1 \left(1 - \frac{\rho \mu}{\epsilon + p} \right) + \frac{x_1}{2(\epsilon + p)} \left(-f_1 + \frac{\rho}{\epsilon + p} (e_1 + \rho) \right), \quad (2.74)$$

$$H_2 = -C_3 \frac{\rho \mu_d}{\epsilon + p} + \frac{x_1}{2(\epsilon + p)} \left(-f_3 + \frac{\rho}{\epsilon + p} (e_3 + \rho_d) \right), \quad (2.75)$$

$$H_3 = -\frac{2\rho T}{\epsilon + p} \tilde{\gamma}_1 + \frac{x_1}{2(\epsilon + p)} \left(-f_2 + \frac{\rho}{\epsilon + p} (e_2 + s) \right), \quad (2.76)$$

$$x_1 = C_3 \mu_d^2 + C_1 \mu^2 + 2 \tilde{\gamma}_1 T^2, \quad (2.77)$$

and with the terms multiplied by $\tilde{B}_{\text{add}}$

$$G_1 = C_2 \left(1 - \frac{\rho \mu}{\epsilon + p} \right) + \frac{x_2}{(\epsilon + p)^2} (\rho(\rho_d + e_3) - f_3(\epsilon + p)), \quad (2.78)$$

$$G_2 = -C_2 \frac{\rho \mu_d}{\epsilon + p} + \frac{x_2}{(\epsilon + p)^2} (\rho(\rho + e_1) - f_1(\epsilon + p)), \quad (2.79)$$

$$G_3 = -\frac{2\rho T}{\epsilon + p} \tilde{\gamma}_2 + \frac{x_2}{(\epsilon + p)^2} (\rho(s + e_2) - f_3(\epsilon + p)), \quad (2.80)$$

$$x_2 = C_2 \mu \mu_d + \tilde{\gamma}_2 T^2. \quad (2.81)$$

On the other hand, for the second gauge field current one obtains

$$\begin{aligned} \delta j_z(B) = & \delta E_z^{(F)} \left[\sigma_{BF} + \frac{\rho \rho_d}{\epsilon + p} \frac{i}{\omega_m} + \tilde{B}_{\text{add}} (M_1 I_1 + D_1 I_2 + T_1 I_3) \right. \\ & \left. + B (M_1 J_1 + D_1 J_2 + T_1 J_3) \right] \\ & + \delta \tilde{E}_z^{(B)} \left[\sigma_B + \frac{\rho_d^2}{\epsilon + p} \frac{i}{\omega_m} + \tilde{B}_{\text{add}} (M_2 I_1 + D_2 I_2 + T_2 I_3) \right. \\ & \left. + B (M_2 J_1 + D_2 J_2 + T_2 J_3) \right], \end{aligned} \quad (2.82)$$

where we have defined the following quantities:

$$I_1 = C_2 \left(1 - \frac{\rho_d \mu_d}{\epsilon + p} \right) + \frac{x_2}{(\epsilon + p)^2} (\rho_d(\rho + e_1) - g_3(\epsilon + p)), \quad (2.83)$$

$$I_2 = -C_2 \frac{\rho \mu_d}{\epsilon + p} + \frac{x_2}{(\epsilon + p)^2} (\rho_d(\rho_d + e_3) - g_1(\epsilon + p)), \quad (2.84)$$

$$I_3 = -\frac{2\rho_d T}{\epsilon + p} \tilde{\gamma}_2 + \frac{x_2}{(\epsilon + p)^2} (\rho_d(s + e_2) - g_2(\epsilon + p)), \quad (2.85)$$

and

$$J_1 = -C_1 \frac{\rho_d \mu}{\epsilon + p} + \frac{x_1}{2(\epsilon + p)} \left(-g_3 + \frac{\rho_d}{\epsilon + p} (e_1 + \rho) \right), \quad (2.86)$$

$$J_2 = C_3 \left(1 - \frac{\rho_d \mu}{\epsilon + p} \right) + \frac{x_1}{2(\epsilon + p)} \left(-g_1 + \frac{\rho_d}{\epsilon + p} (e_3 + \rho_d) \right), \quad (2.87)$$

$$J_3 = -\frac{2\rho_d T}{\epsilon + p} \tilde{\gamma}_1 + \frac{x_1}{2(\epsilon + p)} \left(-g_2 + \frac{\rho_d}{\epsilon + p} (e_2 + s) \right). \quad (2.88)$$

Consequently, we can write them in a more compact forms, given by

$$\delta j_z(F) = \tilde{\sigma}_F \delta E_z^{(F)} + \tilde{\sigma}_{FB} \delta \tilde{E}_z^{(B)}, \quad (2.89)$$

$$\delta j_z(B) = \tilde{\sigma}_B \delta \tilde{E}_z^{(B)} + \tilde{\sigma}_{BF} \delta E_z^{(F)}. \quad (2.90)$$

By virtue of the equations (2.89)–(2.90), we can read off the explicit forms of the adequate conductivities

$$\tilde{\sigma}_F = \sigma_F + \frac{\rho^2}{\epsilon + p} \frac{i}{\omega_m} + B \left(M_1 H_1 + D_1 H_2 + T_1 H_3 \right) \quad (2.91)$$

$$+ \tilde{B}_{\text{add}} \left(D_1 G_1 + M_1 G_2 + T_1 G_3 \right),$$

$$\tilde{\sigma}_{FB} = \sigma_{FB} + \frac{\rho \rho_d}{\epsilon + p} \frac{i}{\omega_m} + B \left(M_2 H_1 + D_2 H_2 + T_2 H_3 \right) \quad (2.92)$$

$$+ \tilde{B}_{\text{add}} \left(D_2 G_1 + M_2 G_2 + T_2 G_3 \right),$$

$$\begin{aligned}\tilde{\sigma}_B &= \sigma_B + \frac{\rho_d^2}{\epsilon + p} \frac{i}{\omega_m} + \tilde{B}_{\text{add}} \left(M_2 I_1 + D_2 I_2 + T_2 I_3 \right) \\ &\quad + B \left(M_2 J_1 + D_2 J_2 + T_2 J_3 \right),\end{aligned}\tag{2.93}$$

$$\begin{aligned}\tilde{\sigma}_{BF} &= \sigma_{BF} + \frac{\rho \rho_d}{\epsilon + p} \frac{i}{\omega_m} + \tilde{B}_{\text{add}} \left(M_1 I_1 + D_1 I_2 + T_1 I_3 \right) \\ &\quad + B \left(M_1 J_1 + D_1 J_2 + T_1 J_3 \right).\end{aligned}\tag{2.94}$$

These results constitute the expressions for the longitudinal electric conductivities of a fluid with chiral anomaly and $\mathbb{Z}_2$ topological charge, in the background of the adequate magnetic fields B and $\tilde{B}_{\text{add}}$, connected with Maxwell and auxiliary U(1)-gauge fields.

2.3 The limits

In this subsection we shall discuss physical consequences of the model. It is important to note that DC conductivity corresponding to $\omega = 0$ limit is finite, if all of the relaxation times τ_i , i.e., energy for $i = e$, momentum ($i = m$) and two charge relaxations ($i = c, cd$), are finite. This agrees with the earlier observation [20] for the model without $\mathbb{Z}_2$ topological charge. For the finite values of the relaxation times, the conductivities have both real $\sigma'(\omega)$ and imaginary $\sigma''(\omega)$ parts. The imaginary one characterizes by an antisymmetric contribution in frequency, resulting from the terms like $i\omega/(\omega^2 + 1/\tau_i^2)$. The real parts of the conductivities are governed by the corresponding relaxation times and contain terms with factors $(1/\tau_i)/(\omega^2 + 1/\tau_i^2)$, which in the limit of large relaxation times (i.e., $1/\tau \rightarrow 0$) envisage the zero frequency, $\delta(\omega)$, contribution only.

If we consider the case without relaxation terms, i.e., $\omega_e, \omega_m, \omega_c, \omega_{cd} \rightarrow \omega$ should be replaced in (2.91)–(2.94). The conductivities will have poles in the imaginary part at $\omega = 0$. Consequently, one will encounter $\delta(\omega)$ in the real part of the DC-conductivities in question. For a system without $\mathbb{Z}_2$ topological charge there exist only a single non-zero parameter $C_1 = C$ and our general formula for the conductivity matrix reduces to the element $\tilde{\sigma}_F$, which agrees with that found earlier [20].

Because of the complexity of the obtained results we pay attention to some simpler limit cases. Namely, our analysis will be addressed to the case when $\rho = \mu = 0$, which implies also that $f_2 = f_3 = 0$. In the case under consideration, one obtains the following forms of the conductivities:

$$\begin{aligned}\tilde{\sigma}_F(\rho = \mu = 0) &= \sigma_F + B \left[\frac{i}{\omega_c} \left(\frac{C_1^2 B}{f_1} - \frac{C_1 B}{2(\epsilon + p)} (C_3 \mu_d^2 + 2\tilde{\gamma}_1 T^2) \right) \right] \\ &\quad - \tilde{B}_{\text{add}} \frac{i}{\omega_c} \frac{C_1 B \tilde{\gamma}_2 T^2}{\epsilon + p},\end{aligned}\tag{2.95}$$

$$\tilde{\sigma}_{FB}(\rho = \mu = 0) = \sigma_{F\tilde{B}} + B M_2^{(L)} H_1^{(L)} + \tilde{B}_{\text{add}} \left(D_2^{(L)} G_1^{(L)} + M_2^{(L)} C_2 \right),\tag{2.96}$$

$$\begin{aligned}\tilde{\sigma}_{BF}(\rho = \mu = 0) &= \sigma_{BF} + \tilde{B}_{\text{add}} \left[\frac{i}{\omega_e} \frac{C_1 B}{f_1} I_1^{(L)} + D_1^{(L)} I_2^{(L)} + T_1^{(L)} I_3^{(L)} \right] \\ &\quad + B \left[\frac{i}{\omega_e} \frac{C_1 B}{f_1} J_1^{(L)} + D_1^{(L)} J_2^{(L)} + T_1^{(L)} I_2^{(L)} + T_1^{(L)} I_3^{(L)} \right],\end{aligned}\tag{2.97}$$

$$\begin{aligned} \tilde{\sigma}_B(\rho = \mu = 0) = & \sigma_B + \frac{\rho_d^2}{\epsilon + p} \frac{i}{\omega_e} + \tilde{B}_{\text{add}} \left[M_2^{(L)} I_1^{(L)} + D_2^{(L)} I_2^{(L)} + T_2^{(L)} I_3^{(L)} \right] \\ & + B \left[M_2^{(L)} J_1^{(L)} + D_2^{(L)} J_2^{(L)} + T_2^{(L)} J_3^{(L)} \right], \end{aligned} \quad (2.98)$$

where for the brevity of the subsequent notion we have denoted the following quantities:

$$M_2^{(L)} = -\frac{\rho_d}{\epsilon + p} \frac{i}{\omega_e} \frac{\tilde{B}_{\text{add}} C_2 \mu_d}{f_1} + \frac{i}{\omega_c} \frac{\tilde{B}_{\text{add}} C_2 \mu_d}{f_1}, \quad (2.99)$$

$$D_1^{(L)} = \frac{i}{\omega_e} \frac{\tilde{B}_{\text{add}} C_2 \mu_d g_2}{g_2 e_3 - e_2 g_1} + \frac{i}{\omega_c} \frac{C_1 B (g_3 e_2 - g_2 e_1)}{f_1 (g_2 e_3 - e_2 g_1)} - \frac{i}{\omega_{cd}} \frac{C_4 \tilde{B}_{\text{add}} e_2}{g_2 e_3 - e_2 g_1}, \quad (2.100)$$

$$\begin{aligned} D_2^{(L)} = & \frac{i}{\omega_e} \frac{\tilde{B}_{\text{add}} C_2 \mu_d g_2}{g_2 e_3 - e_2 g_1} - \frac{i}{\omega_m} \frac{\rho_d}{\epsilon + p} \frac{C_2 \tilde{B}_{\text{add}} \mu_d (g_3 e_2 - g_2 e_1)}{f_1 (g_2 e_3 - e_2 g_1)} - \frac{i}{\omega_{cd}} \frac{C_3 \tilde{B}_{\text{add}} e_2}{g_2 e_3 - e_2 g_1} \\ & + \frac{i}{\omega_m} \frac{\rho_d}{\epsilon + p} \frac{C_2 \tilde{B}_{\text{add}} \mu_d e_2}{f_1 (g_2 e_3 - e_2 g_1)}, \end{aligned} \quad (2.101)$$

$$T_1^{(L)} = -\frac{i}{\omega_e} \frac{\tilde{B}_{\text{add}} C_2 \mu_d g_1}{g_2 e_3 - e_2 g_1} + \frac{i}{\omega_c} \frac{C_1 B (e_1 g_1 - e_3 g_3)}{f_1 (g_2 e_3 - e_2 g_1)}, \quad (2.102)$$

$$\begin{aligned} T_2^{(L)} = & -\frac{i}{\omega_e} \frac{\tilde{B}_{\text{add}} C_2 \mu_d g_1}{g_2 e_3 - e_2 g_1} - \frac{i}{\omega_m} \frac{\rho_d}{\epsilon + p} \frac{\tilde{B}_{\text{add}} C_2 \mu_d (e_1 g_1 - e_3 g_3)}{f_1 (g_2 e_3 - e_2 g_1)} \\ & - \frac{i}{\omega_{cd}} \frac{C_3 \tilde{B}_{\text{add}} e_3}{g_2 e_3 - e_2 g_1} - \frac{i}{\omega_m} \frac{\rho_d}{\epsilon + p} \frac{\tilde{B}_{\text{add}} C_2 \mu_d e_3}{g_2 e_3 - e_2 g_1}, \end{aligned} \quad (2.103)$$

$$H_1^{(L)} = C_1 - \frac{C_3 \mu_d^2 + 2\tilde{\gamma}_1 T^2}{2(\epsilon + p)} f_1, \quad (2.104)$$

$$G_2^{(L)} = -f_1 \frac{\tilde{\gamma}_2 T^2}{\epsilon + p}, \quad (2.105)$$

$$I_1^{(L)} = C_1 \left(1 - \frac{\rho_d \mu_d}{\epsilon + p} \right) + \frac{\tilde{\gamma}_2 T^2}{(\epsilon + p)^2} (e_1 \rho_d - g_3 (\epsilon + p)), \quad (2.106)$$

$$I_2^{(L)} = \frac{\tilde{\gamma}_2 T^2}{(\epsilon + p)^2} (\rho_d (\rho_d + e_3) - g_1 (\epsilon + p)), \quad (2.107)$$

$$I_3^{(L)} = -2 \frac{\rho_d \tilde{\gamma}_2 T}{\epsilon + p} + \frac{\tilde{\gamma}_2 T^2}{(\epsilon + p)^2} (\rho_d (s + e_2) - g_2 (\epsilon + p)), \quad (2.108)$$

$$J_1^{(L)} = \frac{C_3 \mu_d^2 + 2\tilde{\gamma}_1 T^2}{2(\epsilon + p)} \left(-g_3 + \frac{\rho_d e_1}{\epsilon + p} \right), \quad (2.109)$$

$$J_2^{(L)} = C_3 + \frac{C_3 \mu_d^2 + 2\tilde{\gamma}_1 T^2}{2(\epsilon + p)} \left(-g_1 + \frac{\rho_d (e_3 + \rho_d)}{\epsilon + p} \right), \quad (2.110)$$

$$J_3^{(L)} = -2 \frac{\rho_d \tilde{\gamma}_1 T}{\epsilon + p} + \frac{C_3 \mu_d^2 + 2\tilde{\gamma}_1 T^2}{2(\epsilon + p)} \left(-g_2 + \frac{\rho_d (e_2 + s)}{\epsilon + p} \right). \quad (2.111)$$

The above relations simplify to great extend, when one assumes that the magnetic fields

are equal to zero. Consequently this assumption leads to

$$\tilde{\sigma}_F(\rho = \mu = 0) = \sigma_F, \quad (2.112)$$

$$\tilde{\sigma}_{FB}(\rho = \mu = 0) = \sigma_{F\tilde{B}}, \quad (2.113)$$

$$\tilde{\sigma}_{BF}(\rho = \mu = 0) = \sigma_{BF}, \quad (2.114)$$

$$\tilde{\sigma}_B(\rho = \mu = 0) = \sigma_B + \frac{i}{\omega_m} \frac{\rho_d^2}{\epsilon + p}. \quad (2.115)$$

One can see that in the considered case the additional U(1)-gauge field density ρ_d gives the finite conductivity.

On the other hand, in the limit when $\rho_d = \mu_d = 0$, which implies that $g_2 = g_3 = 0$, the relations for the conductivities yield

$$\begin{aligned} \tilde{\sigma}_F(\rho_d = \mu_d = 0) &= \sigma_F + \frac{\rho^2}{\epsilon + p} \frac{i}{\omega_m} + B \left[M_1^{(D)} H_1^{(D)} + D_1^{(D)} H_2^{(D)} + T_1^{(D)} H_3^{(D)} \right] \\ &= \tilde{B}_{\text{add}} \left[D_1^{(D)} G_1^{(D)} + M_1^{(D)} G_2^{(D)} + T_1^{(D)} G_3^{(D)} \right], \end{aligned} \quad (2.116)$$

$$\tilde{\sigma}_{FB}(\rho_d = \mu_d = 0) = \sigma_{F\tilde{B}} + B \left[M_2^{(D)} H_1^{(D)} + \frac{i}{\omega_{cd}} \frac{C_3 B}{g_1} H_2^{(D)} + T_2^{(D)} H_3^{(D)} \right] \quad (2.117)$$

$$+ \tilde{B}_{\text{add}} \left[\frac{i}{\omega_{cd}} \frac{C_3 B}{g_1} G_1^{(D)} + M_2^{(D)} G_2^{(D)} + T_2^{(D)} G_3^{(D)} \right], \quad (2.118)$$

$$\tilde{\sigma}_{BF}(\rho_d = \mu_d = 0) = \sigma_{BF} + \tilde{B}_{\text{add}} \left[M_1^{(D)} C_2 - D_1^{(D)} \frac{\tilde{\gamma}_2 T^2 g_1}{\epsilon + p} \right] \quad (2.119)$$

$$+ B D_1^{(D)} \left(C_3 - \frac{C_1 \mu^2 + 2\tilde{\gamma}_1 T^2}{2(\epsilon + p)} g_1 \right), \quad (2.120)$$

$$\begin{aligned} \tilde{\sigma}_B(\rho_d = \mu_d = 0) &= \sigma_B + \tilde{B}_{\text{add}} \left[M_2^{(D)} C_2 - \frac{i}{\omega_m} \frac{C_3 B}{g_1} \frac{\tilde{\gamma}_2 T^2 g_1}{\epsilon + p} \right] \\ &+ B \frac{i}{\omega_{cd}} \frac{C_3 B}{g_1} \left(C_3 - \frac{C_1 \mu^2 + 2\tilde{\gamma}_1 T^2}{2(\epsilon + p)} g_1 \right). \end{aligned} \quad (2.121)$$

In the above relations we set the quantities

$$M_1^{(D)} = \left(\frac{i}{\omega_e} f_2 + \frac{\rho}{\epsilon + p} \frac{i}{\omega_m} \right) \frac{1}{e_1 f_2 - e_2 f_1} \left(1 - \frac{1}{g_1} \right) \left[C_1 \mu B \left(1 - \frac{1}{2} \frac{\rho \mu}{\epsilon + p} \right) \right. \quad (2.122)$$

$$\left. - \frac{\rho \tilde{\gamma}_1 T^2 B}{e + p} - \frac{\rho \tilde{\gamma}_2 T^2 \tilde{B}_{\text{add}}}{\epsilon + p} \right] - \frac{i}{\omega_c} \frac{C_1 B e_2}{e_1 f_2 - e_2 f_1} + \frac{i}{\omega_{cd}} \frac{C_4 \tilde{B}_{\text{add}} (e_2 f_3 - e_3 f_2)}{g_1 (e_1 f_2 - e_2 f_1)},$$

$$M_2^{(D)} = \frac{1}{e_1 f_2 - e_2 f_1} \left[\frac{i}{\omega_e} \tilde{B}_{\text{add}} C_2 \mu f_2 - \frac{i}{\omega_c} C_2 \tilde{B}_{\text{add}} e_2 + \frac{i}{\omega_{cd}} C_3 B (e_2 f_3 - e_3 f_2) \right], \quad (2.123)$$

$$D_1^{(D)} = -\frac{i}{\omega_m} \frac{\rho}{\epsilon + p} \frac{\tilde{B}_{\text{add}} C_2 \mu}{g_1} + \frac{i}{\omega_{cd}} \frac{C_4 \tilde{B}_{\text{add}}}{g_1}, \quad (2.124)$$

$$T_1^{(D)} = \left[C_1 \mu B \left(1 - \frac{1}{2} \frac{\rho \mu}{\epsilon + p} \right) - \frac{\rho \tilde{\gamma}_1 T^2 B}{e + p} - \frac{\rho \tilde{\gamma}_2 T^2 \tilde{B}_{\text{add}}}{\epsilon + p} \right] \frac{1}{e_1 f_2 - e_2 f_1} \quad (2.125)$$

$$\left(-\frac{i}{\omega_e} f_1 - \frac{i}{\omega_m} \frac{\rho}{\epsilon + p} \left(e_1 + \frac{f_1 e_3 - f_3 e_1}{g_1} \right) \right) \quad (2.126)$$

$$+ \frac{1}{e_1 f_2 - e_2 f_1} \left[\frac{i}{\omega_c} C_1 B e_1 + \frac{i}{\omega_{cd}} C_4 \tilde{B}_{\text{add}} \frac{f_1 e_3 - f_3 e_1}{g_1} \right], \quad (2.127)$$

$$T_2^{(D)} = \frac{1}{e_1 f_2 - e_2 f_1} \left[-\frac{i}{\omega_e} \tilde{B}_{\text{add}} C_2 \mu f_1 + -\frac{i}{\omega_c} C_2 \tilde{B}_{\text{add}} e_1 + -\frac{i}{\omega_{cd}} C_3 B \frac{f_1 e_3 - f_3 e_1}{g_1} \right], \quad (2.128)$$

$$H_1^{(D)} = C_1 \left(1 - \frac{\rho \mu}{\epsilon + p} \right) + \frac{C_1 \mu + 2 \tilde{\gamma}_1 T^2}{2(\epsilon + p)} \left(-f_1 + \frac{\rho}{\epsilon + p} (e_1 + \rho) \right), \quad (2.129)$$

$$H_2^{(D)} = \frac{C_1 \mu^2 + 2 \tilde{\gamma}_1 T^2}{2(\epsilon + p)} \left(-f_3 + \frac{\rho e_3}{\epsilon + p} \right), \quad (2.130)$$

$$H_3^{(D)} = -2 \frac{\rho \tilde{\gamma}_1 T}{\epsilon + p} + \frac{C_1 \mu + 2 \tilde{\gamma}_1 T^2}{2(\epsilon + p)} \left(-f_2 + \frac{\rho}{\epsilon + p} (e_2 + s) \right), \quad (2.131)$$

$$G_1^{(D)} = C_2 \left(1 - \frac{\rho \mu}{\epsilon + p} \right) + \frac{\tilde{\gamma}_2 T^2}{(\epsilon + p)^2} (e_3 \rho - f_3 (\epsilon + p)), \quad (2.132)$$

$$G_2^{(D)} = \frac{\tilde{\gamma}_2 T^2}{(\epsilon + p)^2} (\rho^2 - f_1 (\epsilon + p)), \quad (2.133)$$

$$G_3^{(D)} = -2 \frac{\rho \tilde{\gamma}_2 T^2}{\epsilon + p} + \frac{\tilde{\gamma}_2 T^2}{(\epsilon + p)^2} (s \rho - f_3 (\epsilon + p)). \quad (2.134)$$

As in the latter case, let us suppose that the magnetic fields are equal to zero. It implies the following:

$$\tilde{\sigma}_F(\rho_d = \mu_d = 0) = \sigma_F + \frac{i}{\omega_m} \frac{\rho^2}{\epsilon + p}, \quad (2.135)$$

$$\tilde{\sigma}_{FB}(\rho_d = \mu_d = 0) = \sigma_{F\tilde{B}}, \quad (2.136)$$

$$\tilde{\sigma}_{BF}(\rho_d = \mu_d = 0) = \sigma_{BF}, \quad (2.137)$$

$$\tilde{\sigma}_B(\rho_d = \mu_d = 0) = \sigma_B. \quad (2.138)$$

As was previously stated, the additional gauge field density, now ρ , gives the finite conductivity.

In the case when $\rho = \mu = \rho_d = \mu_d = 0$, and $f_2 = f_3 = g_2 = g_3 = 0$, one receives the relations provided by

$$\tilde{\sigma}_F(\rho_m = \mu_m = 0) = \sigma_F + \frac{i}{\omega_c} \frac{C_1^2 B^2}{f_1} + \frac{i}{\omega_{cd}} \frac{C_2 C_4 \tilde{B}_{\text{add}}}{g_1} - \frac{i}{\omega_c} \frac{C_1 B}{\epsilon + p} T^2 (\tilde{\gamma}_1 B + \tilde{\gamma}_2 \tilde{B}_{\text{add}}), \quad (2.139)$$

$$\begin{aligned} \tilde{\sigma}_{FB}(\rho_m = \mu_m = 0) &= \sigma_{F\tilde{B}} + \frac{i}{\omega_c} \frac{C_1 C_2 B \tilde{B}_{\text{add}}}{f_1} + \frac{i}{\omega_{cd}} \frac{C_2 C_3 B \tilde{B}_{\text{add}}}{g_1} \\ &\quad - \frac{i}{\omega_c} \frac{C_2 \tilde{B}_{\text{add}}}{\epsilon + p} T^2 (\tilde{\gamma}_1 B + \tilde{\gamma}_2 \tilde{B}_{\text{add}}), \end{aligned} \quad (2.140)$$

$$\begin{aligned} \tilde{\sigma}_{BF}(\rho_m = \mu_m = 0) &= \sigma_{BF} + \frac{i}{\omega_c} \frac{C_1 C_2 B \tilde{B}_{\text{add}}}{f_1} + \frac{i}{\omega_{cd}} \frac{C_3 C_4 B \tilde{B}_{\text{add}}}{g_1} \\ &\quad - \frac{i}{\omega_{cd}} \frac{C_4 \tilde{B}_{\text{add}}}{\epsilon + p} T^2 (\tilde{\gamma}_1 B + \tilde{\gamma}_2 \tilde{B}_{\text{add}}), \end{aligned} \quad (2.141)$$

$$\begin{aligned} \tilde{\sigma}_B(\rho_m = \mu_m = 0) &= \sigma_B + \frac{i}{\omega_c} \frac{C_2^2 B \tilde{B}_{\text{add}}}{f_1} + \frac{i}{\omega_{cd}} \frac{C_3^2 B^2}{g_1} \\ &\quad - \frac{i}{\omega_{cd}} \frac{C_3 B}{\epsilon + p} T^2 (\tilde{\gamma}_1 B + \tilde{\gamma}_2 \tilde{B}_{\text{add}}), \end{aligned} \quad (2.142)$$

where for the brevity of the notion we introduce index ‘m’ denoting both cases, i.e., the case of the ordinary Maxwell field and the additional U(1)-gauge one.

Neglecting the presence of $\mathbb{Z}_2$ topological charge, which means that $\tilde{B}_{\text{add}} = 0$, and assuming $\tilde{\gamma}_1 = 0$, we get

$$\tilde{\sigma}_F(\rho_m = \mu_m = 0) = \sigma_F + \frac{i}{\omega_c} \frac{C_1^2 B^2}{\left(\frac{\partial \rho}{\partial \mu}\right)_{|T, \mu_d}}, \quad (2.143)$$

$$\tilde{\sigma}_{FB}(\rho_m = \mu_m = 0) = \sigma_{F\tilde{B}}, \quad (2.144)$$

$$\tilde{\sigma}_{BF}(\rho_m = \mu_m = 0) = \sigma_{BF}, \quad (2.145)$$

$$\tilde{\sigma}_B(\rho_m = \mu_m = 0) = \sigma_B + \frac{i}{\omega_{cd}} \frac{C_3^2 B^2}{\left(\frac{\partial \rho_d}{\partial \mu_d}\right)_{|T, \mu}}. \quad (2.146)$$

From the above relations one can draw a conclusion that even at zero densities we obtain still finite DC-conductivities. The terms in question can only be dissipated by the charge dissipations, ω_c and ω_{cd} , respectively. On the other hand, the form of $\tilde{\sigma}_F$ conductivity is the same as in [20] (see the next section), but in our case ρ is also dependent on the auxiliary gauge field components.

Finally, we assume that $B = T = 0$. This limit is beyond the hydrodynamical calculations which require $B/T^2 \ll 1$, as was mentioned in subsection 2.1, but for the completeness of the considerations we present these relations

$$\tilde{\sigma}_F(\rho_m = \mu_m = 0) = \sigma_F + \frac{i}{\omega_{cd}} \frac{C_2 C_4 \tilde{B}_{\text{add}}^2}{\left(\frac{\partial \rho_d}{\partial \mu_d}\right)_{|T, \mu}}, \quad (2.147)$$

$$\tilde{\sigma}_{FB}(\rho_m = \mu_m = 0) = \sigma_{F\tilde{B}}, \quad (2.148)$$

$$\tilde{\sigma}_{BF}(\rho_m = \mu_m = 0) = \sigma_{BF}, \quad (2.149)$$

$$\tilde{\sigma}_B(\rho_m = \mu_m = 0) = \sigma_B + \frac{i}{\omega_c} \frac{C_2^2 \tilde{B}_{\text{add}}^2}{\left(\frac{\partial \rho}{\partial \mu}\right)_{|T, \mu_d}}. \quad (2.150)$$

The terms in question are only dissipated by the charge dissipative terms.

3 Magneto-transport of the holographic system

In this section we shall elaborate the holographic model of the system, i.e., Dirac semimetals with $\mathbb{Z}_2$ topological charge and chiral anomaly. Our goal is to use holographic model of the system in question and calculate its thermodynamic properties, in particular the relations among the charge densities and chemical potentials. We shall restrict our attention to the analysis of the simplest cases as given by the equations (2.143)–(2.146). It has to be stressed that hydrodynamic approach does not fix the values of the Boltzmann conductivities σ_F , $\sigma_{F\tilde{B}}$, σ_{BF} and σ_B in the above expressions. It provides the general conditions [45] stemming from the positivity of the entropy production. However, hydrodynamics completely determines the anomaly related kinetic coefficients.

In the probe limit, we are interested in, one considers the AdS Schwarzschild black brane background, in five-dimensional Einstein-Chern Simons gravity with negative cosmological constant with two U(1)-gauge fields. The gauge Chern-Simons terms in gravitational action constitute the possible interactions of the aforementioned fields, which play the crucial role from the point of view of the chiral anomaly and $\mathbb{Z}_2$ topological charge in the studied system. The validity of the probe limit approximation requires $B/T^2 \ll 1$, $\tilde{B}_{\text{add}}/T^2 \ll 1$.

The bulk action provided by the holographic model is composed of terms responsible for chiral anomaly and $\mathbb{Z}_2$ topological charge and reads

$$S = \int dx^5 \sqrt{-g} \left(R + \frac{12}{L^2} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu} + \frac{\alpha_1}{3} \epsilon^{\mu\nu\rho\delta\tau} A_\mu F_{\nu\rho} F_{\delta\tau} \right. \\ \left. + \frac{\alpha_2}{3} \epsilon^{\mu\nu\rho\delta\tau} B_\mu B_{\nu\rho} B_{\delta\tau} + \frac{\alpha_3}{3} \epsilon^{\mu\nu\rho\delta\tau} A_\mu F_{\nu\rho} B_{\delta\tau} + \frac{\alpha_4}{3} \epsilon^{\mu\nu\rho\delta\tau} B_\mu F_{\nu\rho} B_{\delta\tau} \right). \quad (3.1)$$

In what follows, our convention is $\epsilon^{txyzt} = 1$. The equation of motion for the U(1)-gauge fields can be written as

$$\nabla_\alpha F^{\alpha\beta} + \alpha_1 \epsilon^{\beta\mu\nu\rho\delta} F_{\mu\nu} F_{\rho\delta} + \frac{2}{3} \alpha_3 \epsilon^{\beta\mu\nu\rho\delta} F_{\mu\nu} B_{\rho\delta} + \frac{\alpha_4}{3} \epsilon^{\beta\mu\nu\rho\delta} B_{\mu\nu} B_{\rho\delta} = 0, \quad (3.2)$$

and for the auxiliary $B_{\mu\nu}$ field it has the form as

$$\nabla_\alpha B^{\alpha\beta} + \alpha_2 \epsilon^{\beta\mu\nu\rho\delta} B_{\mu\nu} B_{\rho\delta} + \frac{2}{3} \alpha_4 \epsilon^{\beta\mu\nu\rho\delta} B_{\mu\nu} F_{\rho\delta} + \frac{\alpha_3}{3} \epsilon^{\beta\mu\nu\rho\delta} F_{\mu\nu} F_{\rho\delta} = 0, \quad (3.3)$$

The components of Maxwell and the additional gauge field are provided by

$$A_\mu = (\phi(r), 0, Bx, A_z(r), 0), \quad B_\mu = (\psi(r), 0, \tilde{B}_{\text{add}}x, B_z(r), 0). \quad (3.4)$$

In our consideration, as the background metric we take the line element of AdS-Schwarzschild five-dimensional black brane

$$ds^2 = r^2 \left(-f(r) dt^2 + dx^2 + dy^2 + dz^2 \right) + \frac{dr^2}{r^2 f(r)}, \quad (3.5)$$

where $f(r) = 1 - \frac{r_0^4}{r^4}$ and r_0 is the radius of the event horizon. The energy density, entropy density and the Hawking temperature for the black brane are given, respectively by

$$\epsilon = 3 r_0^4, \quad s = 4\pi r_0^3, \quad T = \frac{r_0}{\pi}. \quad (3.6)$$

The equations of motion for the gauge fields in the aforementioned background yield

$$\phi''(r) + \frac{3}{r}\phi'(r) + \frac{8\alpha_1}{r^3}BA'_z(r) + \frac{8\alpha_3}{3r^3}\left(A'_z(r)\tilde{B}_{\text{add}} + B'_z(r)B\right) + \frac{8\alpha_4}{3r^3}B'_z(r)\tilde{B}_{\text{add}} = 0, \quad (3.7)$$

$$\psi''(r) + \frac{3}{r}\psi'(r) + \frac{8\alpha_2}{r^3}\tilde{B}_{\text{add}}B'_z(r) + \frac{8\alpha_4}{3r^3}\left(B'_z(r)B + A'_z(r)\tilde{B}_{\text{add}}\right) + \frac{8\alpha_4}{3r^3}A'_z(r)B = 0, \quad (3.8)$$

$$\begin{aligned} A''_z(r) + A'_z(r)\left(\frac{3}{r} + \frac{f'(r)}{f(r)}\right) + \frac{8\alpha_1}{r^3f(r)}B\phi'(r) + \frac{8\alpha_3}{3r^3f(r)}\left(\phi'(r)\tilde{B}_{\text{add}} + \psi'(r)B\right) \\ + \frac{8\alpha_4}{3r^3f(r)}\psi'(r)\tilde{B}_{\text{add}} = 0, \end{aligned} \quad (3.9)$$

$$\begin{aligned} B''_z(r) + B'_z(r)\left(\frac{3}{r} + \frac{f'(r)}{f(r)}\right) + \frac{8\alpha_2}{r^3f(r)}\tilde{B}_{\text{add}}\psi'(r) + \frac{8\alpha_4}{3r^3f(r)}\left(\psi'(r)B + \phi'(r)\tilde{B}_{\text{add}}\right) \\ + \frac{8\alpha_3}{3r^3f(r)}\phi'(r)B = 0, \end{aligned} \quad (3.10)$$

where the prime denotes the derivative with respect to r -coordinate.

The above set of differential equation can be simplified to the following forms:

$$\left[r^3 \phi'(r) + A_z(r) a_1 + B_z c\right]' = 0, \quad (3.11)$$

$$\left[r^3 f(r) A'_z(r) + \phi(r) a_1 + \psi(r) c\right]' = 0, \quad (3.12)$$

$$\left[r^3 \psi'(r) + B_z(r) b_1 + A_z c\right]' = 0, \quad (3.13)$$

$$\left[r^3 f(r) B'_z(r) + \psi(r) b_1 + \phi(r) c\right]' = 0, \quad (3.14)$$

where for the brevity of the notation we have introduced the quantities defined as follows:

$$a_1 = 8 \left(\alpha_1 B + \frac{\alpha_3}{3} \tilde{B}_{\text{add}} \right), \quad (3.15)$$

$$b_1 = 8 \left(\alpha_2 \tilde{B}_{\text{add}} + \frac{\alpha_4}{3} B \right), \quad (3.16)$$

$$c = \frac{8}{3} \left(\alpha_4 \tilde{B}_{\text{add}} + \alpha_3 B \right). \quad (3.17)$$

Solving the above set of equations for ϕ and ψ , with the boundary conditions that at the event horizon of the considered black brane one has $\phi(r_0) = \psi(r_0) = 0$, after changing the coordinates given by the relation

$$u = \frac{r_0^2}{r^2}, \quad (3.18)$$

one arrives at the following set of the differential equations:

$$\frac{d^2\phi(u)}{du^2} = \frac{1}{4r_0^4(1-u^2)} \left(a_1 b_1 - c^2 \right) \frac{a_1}{b_1} \phi(u), \quad (3.19)$$

$$\frac{d^2\psi(u)}{du^2} = \frac{1}{4r_0^4(1-u^2)} \left(a_1 b_1 - c^2 \right) \psi(u). \quad (3.20)$$

The analytical solution near the black brane event horizon may be written as

$$\phi_i(u) = G_1 {}_2F_1 \left[\frac{1}{4} \left(-\sqrt{1-K_i} - 1 \right); \frac{1}{4} \left(\sqrt{1-K_i} - 1 \right); \frac{1}{2}; u^2 \right] \quad (3.21)$$

$$+ G_2 u {}_2F_1 \left[\frac{1}{4} \left(1 - \sqrt{1-K_i} \right); \frac{1}{4} \left(1 + \sqrt{1-K_i} \right); \frac{1}{2}; u^2 \right], \quad (3.22)$$

where $i = \phi, \psi$ and K_i are given by

$$K_\psi = \frac{(a_1 b_1 - c^2)}{r_0^4}, \quad K_\phi = K_\psi \frac{a_1}{b_1}. \quad (3.23)$$

G_1 and G_2 stand for constants. Having in mind that near $u \rightarrow 0$ the solutions for $\phi(r)$ and $\psi(r)$ behave like

$$\phi(r) = \mu - \frac{\rho}{2r_0^2} u + \dots, \quad \psi(r) = \mu_d - \frac{\rho_d}{2r_0^2} u + \dots, \quad (3.24)$$

one obtains the dual charge density for the Maxwell field

$$\rho = 4 \mu r_0^2 \frac{\Gamma \left[\frac{5-\sqrt{1-K_\phi}}{4} \right] \Gamma \left[\frac{5+\sqrt{1-K_\phi}}{4} \right]}{\Gamma \left[\frac{3-\sqrt{1-K_\phi}}{4} \right] \Gamma \left[\frac{3+\sqrt{1-K_\phi}}{4} \right]}. \quad (3.25)$$

For the additional U(1)-gauge field, it implies

$$\rho_d = 4 \mu_d r_0^2 \frac{\Gamma \left[\frac{5-\sqrt{1-K_\psi}}{4} \right] \Gamma \left[\frac{5+\sqrt{1-K_\psi}}{4} \right]}{\Gamma \left[\frac{3-\sqrt{1-K_\psi}}{4} \right] \Gamma \left[\frac{3+\sqrt{1-K_\psi}}{4} \right]}. \quad (3.26)$$

Let us now apply the holographic description for the system in zero density limits, i.e., $\rho = \rho_d \rightarrow 0$, $\omega_c = \omega_{cd} \rightarrow \omega$ and the additional requirement of vanishing the adequate magnetic field. For example, by virtue of the equations (2.143) and (2.146) we can find the explicit forms of the conductivities, when $\tilde{B}_{\text{add}} = 0$. They are provided by

$$\tilde{\sigma}_F = \sigma_F + \frac{i}{\omega} \frac{C_1^2 B^2}{4 \pi^2 T^2} \frac{\Gamma \left[\frac{3-\sqrt{1-K_\phi}}{4} \right] \Gamma \left[\frac{3+\sqrt{1-K_\phi}}{4} \right]}{\Gamma \left[\frac{5-\sqrt{1-K_\phi}}{4} \right] \Gamma \left[\frac{5+\sqrt{1-K_\phi}}{4} \right]}, \quad (3.27)$$

$$\tilde{\sigma}_B = \sigma_B + \frac{i}{\omega} \frac{C_3^2 B^2}{4 \pi^2 T^2} \frac{\Gamma \left[\frac{3-\sqrt{1-K_\psi}}{4} \right] \Gamma \left[\frac{3+\sqrt{1-K_\psi}}{4} \right]}{\Gamma \left[\frac{5-\sqrt{1-K_\psi}}{4} \right] \Gamma \left[\frac{5+\sqrt{1-K_\psi}}{4} \right]}. \quad (3.28)$$

Figure 1. The magnetic field dependence of the real part of conductivity $\tilde{\sigma}_F$ calculated from the equation (2.143) for a system described by (3.25). The upper curve is obtained for all $C_i \neq 0$, while the lower one for $C_1 = C = 1$ and $C_3 = 0$. The existence of $\mathbb{Z}_2$ topological charge leads to narrowing of the magneto-conductivity. For actual calculations we have used $T = 1$, $\tau_e = 0.1$, $\omega = 0$.

Similar calculations can be conducted for the case when $B = 0$, described by the equations (2.147) and (2.150).

In the hydrodynamical limit, we assume that $B \ll T^2$ and $\tilde{B}_{\text{add}} \ll T^2$. It implies that the quantities $K_\phi \ll 1$ and $K_\psi \ll 1$. Thus, in the limit under inspection, we receive that the dual densities are provided by

$$\rho = 2 \mu r_0^2 \left(1 + \mathcal{O} \left(\frac{B^2}{T^4}, \frac{\tilde{B}_{\text{add}}^2}{T^4} \right) \right), \quad \rho_d = 2 \mu_d r_0^2 \left(1 + \mathcal{O} \left(\frac{B^2}{T^2}, \frac{\tilde{B}_{\text{add}}^2}{T^4} \right) \right). \quad (3.29)$$

Consequently, the adequate conductivities are given by expressions

$$\tilde{\sigma}_F = \sigma_F + \frac{i}{\omega} \frac{C_1^2 B^2}{4 \pi^2 T^2} + \mathcal{O} \left(\frac{B^2}{T^2}, \frac{\tilde{B}_{\text{add}}^2}{T^4} \right), \quad (3.30)$$

$$\tilde{\sigma}_B = \sigma_B + \frac{i}{\omega} \frac{C_3^2 B^2}{4 \pi^2 T^2} + \mathcal{O} \left(\frac{B^2}{T^2}, \frac{\tilde{B}_{\text{add}}^2}{T^4} \right). \quad (3.31)$$

It can be remarked that in the case when $\tilde{B}_{\text{add}} = 0$ and respectively $C_3 = 0$, we arrive at the results presented in [20], where the only one gauge field was considered and anomaly was bounded with electric and magnetic components of the ordinary Maxwell field.

In figure 1 we plot the magnetic field dependence of the relative conductivity $(\tilde{\sigma}_F - \sigma_F)/(\pi^2 T^2)$ for a system with the density given by the formula (3.25) and assuming that B_{add} vanishes. This assumption is valid for materials like Na_3Bi (or Cd_2As_3), in which the spin projection is related to $\mathbb{Z}_2$ topological charge. In such systems one does not expect spin analog of the magnetic field and this justifies our assumption. Two curves in the

figure correspond to the systems with and without $\mathbb{Z}_2$ topological charge. The lower curve, calculated for $C_1 = C = 1, C_3 = 0 = C_2 = C_4$ describes the system with chiral anomaly but without $\mathbb{Z}_2$ topological charge. The modifications due to the $\mathbb{Z}_2$ charge are depicted in the upper curve. It occurs that the presence of the $\mathbb{Z}_2$ topological charge leads to the narrowing of the magneto-conductivity curve. This conclusion seem to agree with the one based on the kinetic equation approach to Na₃Bi material [44].

The plot has been obtained for $T = 1$, $\tau_e = 0.1$, and $\omega = 0$. In the next section, we calculate the conductivity $\tilde{\sigma}_F$ using the holographic approach to the problem and we shall obtain σ_F part of the conductivity, as well as, its magnetic field dependence. The effect of $\mathbb{Z}_2$ topological charge shows up as a narrowing of the magneto-conductivity line. However, the overall dependence on the $CB/(\pi^2 T^2)$ for the system with additional topological charge is similar to that with the chiral symmetry only.

3.1 Relation between hydrodynamic and holographic parameters

In order to connect the hydrodynamical and holographic descriptions we shall find the relation between C_i and α_i constants. In order to obtain them we expand the action (3.1), to the second order in perturbations of both gauge fields and the metric

$$A_\mu \rightarrow A_\mu + a_\mu, \quad B_\mu \rightarrow B_\mu + b_\mu, \quad g_{\mu\nu} \rightarrow g_{\mu\nu} + h_{\mu\nu}. \quad (3.32)$$

We focus on the part of the action responsible to the first order perturbations connected with currents of gauge fields. Namely, to the first order in gauge field fluctuations, the adequate part of the action reduces to a boundary term of the following form:

$$\begin{aligned} \delta S^{(1)} = \int d^4x \left[\sqrt{-g} \left(F^{\beta r} + \frac{4\alpha_1}{3} \epsilon^{\beta\mu\nu\rho} A_\mu F_{\nu\rho} + \frac{2\alpha_3}{3} \epsilon^{\beta\mu\nu\rho} A_\mu B_{\nu\rho} + \frac{2\alpha_4}{3} \epsilon^{\beta\mu\nu\rho} B_\mu B_{\nu\rho} \right) a_\beta \right. \\ \left. + \sqrt{-g} \left(B^{\beta r} + \frac{4\alpha_2}{3} \epsilon^{\beta\mu\nu\rho} B_\mu B_{\nu\rho} + \frac{2\alpha_3}{3} \epsilon^{\beta\mu\nu\rho} A_\mu F_{\nu\rho} + \frac{2\alpha_4}{3} \epsilon^{\beta\mu\nu\rho} B_\mu F_{\nu\rho} \right) b_\beta \right] \Big|_{r \rightarrow \infty}. \end{aligned} \quad (3.33)$$

Having in mind the above relation, we can read off the forms of the boundary currents

$$J_{(F)}^\beta = \frac{\delta S^{(1)}}{\delta A_\mu} \Big|_{r \rightarrow \infty}, \quad J_{(B)}^\beta = \frac{\delta S^{(1)}}{\delta B_\mu} \Big|_{r \rightarrow \infty}. \quad (3.34)$$

Using the equations of motion (3.2) and (3.3) and the divergence of $\nabla_\beta (J_{(F)}^\beta + J_{(B)}^\beta)$, we arrive at the relation

$$C_i = \frac{8}{3} \alpha_i, \quad (3.35)$$

where $i = 1, \dots, 4$.

4 Holographic calculation of conductivities in the probe limit

This section is devoted to the direct calculations of the DC-conductivities for the studied system. As in the later section we shall elaborate the probe limit of the holographic model,

starting from small perturbations in the AdS-Schwarzschild background and computing the longitudinal conductivities from the perturbations.

In order to find the longitudinal holographic conductivity in the model in question, we assume that the fluctuations of the vector potentials for both gauge fields are provided by

$$A_\mu = (\delta\phi(r)e^{-i\omega t}, 0, 0, B x, \delta A_z(r)e^{-i\omega t}), \quad (4.1)$$

$$B_\mu = (\delta\psi(r)e^{-i\omega t}, 0, 0, \tilde{B}_{\text{add}} x, \delta B_z(r)e^{-i\omega t}). \quad (4.2)$$

The equations of motion for the above perturbations imply

$$\delta\phi'(r) + \frac{\delta A_z}{r^3} a_1 + \frac{\delta B_z(r)}{r^3} c = 0, \quad (4.3)$$

$$\delta\psi'(r) + \frac{\delta B_z}{r^3} b_1 + \frac{\delta A_z(r)}{r^3} c = 0, \quad (4.4)$$

$$\delta A_z''(r) + \left(\frac{3}{r} + \frac{f'(r)}{f(r)}\right) \delta A_z'(r) + \frac{\omega^2}{r^4 f(r)^2} \delta A_z(r) + \frac{\delta\phi'(r)}{r^3 f(r)} a_1 + \frac{\delta\psi'(r)}{r^3 f(r)} c = 0, \quad (4.5)$$

$$\delta B_z''(r) + \left(\frac{3}{r} + \frac{f'(r)}{f(r)}\right) \delta B_z'(r) + \frac{\omega^2}{r^4 f(r)^2} \delta B_z(r) + \frac{\delta\phi'(r)}{r^3 f(r)} c + \frac{\delta\psi'(r)}{r^3 f(r)} b_1 = 0. \quad (4.6)$$

By virtue of the above, one has the following relations for $\delta A_z(r)$ and $\delta B_z(r)$

$$\delta A_z''(r) + \left(\frac{3}{r} + \frac{f'(r)}{f(r)}\right) \delta A_z'(r) + \left(\frac{\omega^2}{r^4 f(r)^2} - \frac{\tilde{A}}{r^6 f(r)}\right) \delta A_z(r) = 0, \quad (4.7)$$

$$\delta B_z''(r) + \left(\frac{3}{r} + \frac{f'(r)}{f(r)}\right) \delta B_z'(r) + \left(\frac{\omega^2}{r^4 f(r)^2} - \frac{\tilde{B}}{r^6 f(r)}\right) \delta B_z(r) = 0, \quad (4.8)$$

where we have introduced the following abbreviations

$$\tilde{A} = a_1 (a_1 + c) + c (b_1 + c), \quad \tilde{B} = b_1 (b_1 + c) + c (a_1 + c). \quad (4.9)$$

In $u = r_0^2/r^2$ coordinate the relations (4.7)–(4.8) reduce to the forms

$$\delta A_z''(u) - \frac{2u}{1-u^2} \delta A_z'(r) + \left(\frac{\omega^2}{4 r_0^2 u(1-u^2)^2} - \frac{\tilde{A}}{4 r_0^4 (1-u^2)}\right) \delta A_z(r) = 0, \quad (4.10)$$

$$\delta B_z''(u) - \frac{2u}{1-u^2} \delta B_z'(r) + \left(\frac{\omega^2}{4 r_0^2 u(1-u^2)^2} - \frac{\tilde{B}}{4 r_0^4 (1-u^2)}\right) \delta B_z(r) = 0, \quad (4.11)$$

where now the prime denotes derivatives with respect to u -coordinate.

For the near horizon limit, i.e. $u \rightarrow 1$, the above equations can be rewritten as

$$\delta A_z''(u) - \frac{1}{1-u} \delta A_z'(r) + \left(\frac{\omega^2}{16 r_0^2 u(1-u)^2} - \frac{\tilde{A}}{8 r_0^4 (1-u)}\right) \delta A_z(r) = 0, \quad (4.12)$$

$$\delta B_z''(u) - \frac{1}{1-u} \delta B_z'(r) + \left(\frac{\omega^2}{16 r_0^2 u(1-u)^2} - \frac{\tilde{B}}{8 r_0^4 (1-u)}\right) \delta B_z(r) = 0, \quad (4.13)$$

with the solution given in terms of the modified Bessel functions

$$\delta A_z(u) = E_1(-1)^{-\frac{i\omega}{4r_0}} I_{-\frac{i\omega}{2r_0}} \left[\sqrt{\frac{\tilde{A}}{2}} \frac{(1-u)^{\frac{1}{2}}}{r_0^2} \right] + E_2(-1)^{\frac{i\omega}{4r_0}} I_{\frac{i\omega}{2r_0}} \left[\sqrt{\frac{\tilde{A}}{2}} \frac{(1-u)^{\frac{1}{2}}}{r_0^2} \right], \quad (4.14)$$

$$\delta B_z(u) = D_1(-1)^{-\frac{i\omega}{4r_0}} I_{-\frac{i\omega}{2r_0}} \left[\sqrt{\frac{\tilde{B}}{2}} \frac{(1-u)^{\frac{1}{2}}}{r_0^2} \right] + D_2(-1)^{\frac{i\omega}{4r_0}} I_{\frac{i\omega}{2r_0}} \left[\sqrt{\frac{\tilde{B}}{2}} \frac{(1-u)^{\frac{1}{2}}}{r_0^2} \right], \quad (4.15)$$

where E_i , D_i are integration constants. The in-falling boundary conditions for u -coordinate correspond to disappearing of E_2 and D_2 .

In the far away region $1-u \gg \omega/r_0$, the above relations take forms as

$$\delta A_z''(u) - \frac{2u}{1-u^2} \delta A_z'(r) - \frac{\tilde{A}}{4 r_0^4 (1-u^2)} \delta A_z(r) = 0, \quad (4.16)$$

$$\delta B_z''(u) - \frac{2u}{1-u^2} \delta B_z'(r) - \frac{\tilde{B}}{4 r_0^4 (1-u^2)} \delta B_z(r) = 0, \quad (4.17)$$

with the solution in terms of the Legendre functions

$$\delta A_z(u) = \tilde{E}_1 P_{\frac{1}{2}} \left[\sqrt{1 - \frac{\tilde{A}}{r_0^4}} - 1 \right] (u) + \tilde{E}_2 Q_{\frac{1}{2}} \left[\sqrt{1 - \frac{\tilde{A}}{r_0^4}} - 1 \right] (u), \quad (4.18)$$

$$\delta B_z(u) = \tilde{D}_1 P_{\frac{1}{2}} \left[\sqrt{1 - \frac{\tilde{B}}{r_0^4}} - 1 \right] (u) + \tilde{D}_2 Q_{\frac{1}{2}} \left[\sqrt{1 - \frac{\tilde{B}}{r_0^4}} - 1 \right] (u), \quad (4.19)$$

where $\tilde{E}_i$, $\tilde{D}_i$ are constants.

In order to find the integration constants one should match the near-horizon solution with the far away one, in some intermediate, matching, region $\omega/r_0 \ll 1-u \ll 1$. It can be done exactly as in [20]. Using the same reasoning it can be revealed that the integration constants fulfill

$$\frac{\tilde{E}_1}{E_1} = 1 + \mathcal{O}(\omega), \quad \frac{\tilde{D}_1}{D_1} = 1 + \mathcal{O}(\omega), \quad (4.20)$$

$$\frac{\tilde{E}_2}{E_1} = \frac{i\omega}{2r_0} (1 + \mathcal{O}(\omega)), \quad \frac{\tilde{D}_2}{D_1} = \frac{i\omega}{2r_0} (1 + \mathcal{O}(\omega)). \quad (4.21)$$

To proceed further let us find the value of the Legendre functions at the boundary $u \rightarrow 0$. Namely, one has

$$P_{\frac{1}{2}} \left[\sqrt{1 - \frac{\tilde{A}}{r_0^4}} - 1 \right] (u \rightarrow 0) = p_1(\tilde{A}_i) + p_2(\tilde{A}_i)u + \mathcal{O}(u^2), \quad (4.22)$$

$$Q_{\frac{1}{2}} \left[\sqrt{1 - \frac{\tilde{A}}{r_0^4}} - 1 \right] (u \rightarrow 0) = q_1(\tilde{A}_i) + q_2(\tilde{A}_i)u + \mathcal{O}(u^2), \quad (4.23)$$

where we have defined

$$p_1(\tilde{A}_i) = \frac{\sqrt{\pi}}{\Gamma\left[\frac{3-\sqrt{1-\frac{\tilde{A}_i}{r_0^4}}}{4}\right] \Gamma\left[\frac{3+\sqrt{1-\frac{\tilde{A}_i}{r_0^4}}}{4}\right]}, \quad p_2(\tilde{A}_i) = -\frac{\sqrt{\pi} \tilde{A}_i}{8r_0^4 \Gamma\left[\frac{5-\sqrt{1-\frac{\tilde{A}_i}{r_0^4}}}{4}\right] \Gamma\left[\frac{5+\sqrt{1-\frac{\tilde{A}_i}{r_0^4}}}{4}\right]}, \quad (4.24)$$

$$q_1(\tilde{A}_i) = -\frac{\sqrt{\pi} \sin\left(\frac{\pi}{4} \left(\sqrt{1-\frac{\tilde{A}_i}{r_0^4}} - 1\right)\right) \Gamma\left[\frac{1+\sqrt{1-\frac{\tilde{A}_i}{r_0^4}}}{4}\right]}{2\Gamma\left[\frac{3+\sqrt{1-\frac{\tilde{A}_i}{r_0^4}}}{4}\right]}, \quad (4.25)$$

$$q_2(\tilde{A}_i) = \frac{\sqrt{\pi} \cos\left(\frac{\pi}{4} \left(\sqrt{1-\frac{\tilde{A}_i}{r_0^4}} - 1\right)\right) \Gamma\left[\frac{3+\sqrt{1-\frac{\tilde{A}_i}{r_0^4}}}{4}\right]}{\Gamma\left[\frac{1+\sqrt{1-\frac{\tilde{A}_i}{r_0^4}}}{4}\right]}. \quad (4.26)$$

In order to find the conductivities we derive the relations which envisages the ingoing boundary conditions for the considered longitudinal currents

$$\delta A_z(u) = \tilde{E}_1 p_1(F) + \tilde{E}_2 q_1(F) + u(\tilde{E}_1 p_2(F) + \tilde{E}_2 q_2(F)), \quad (4.27)$$

$$\delta B_z(u) = \tilde{D}_1 p_1(B) + \tilde{D}_2 q_1(B) + u(\tilde{D}_1 p_2(B) + \tilde{D}_2 q_2(B)), \quad (4.28)$$

Using the definition of the conductivity [50], where the logarithmic divergence is removed with the adequate boundary counter-term in the gravity action [51], one attains

$$\sigma = \frac{2 A_\mu^{(0)}}{i \omega A_\mu^{(2)}} + \frac{i \omega}{2}, \quad (4.29)$$

where the gauge field fall off implies the following:

$$A_\mu = A_\mu^{(0)} + \frac{A_\mu^{(2)}}{r^2} + \dots, \quad (4.30)$$

(the similar condition holds for B_μ field). Consequently, by virtue of the equations (4.29) and (4.30), we obtain the expressions describing $\tilde{\sigma}_F$ and $\tilde{\sigma}_B$. They can be written as

$$\tilde{\sigma}_F = \frac{2r_0^2 (\tilde{E}_1 p_2 + \tilde{E}_2 q_2)}{i\omega (\tilde{E}_1 p_1 + \tilde{E}_2 q_1)} + \frac{i\omega}{2}, \quad (4.31)$$

$$\tilde{\sigma}_B = \frac{2r_0^2 (\tilde{D}_1 p_2 + \tilde{D}_2 q_2)}{i\omega (\tilde{D}_1 p_1 + \tilde{D}_2 q_1)} + \frac{i\omega}{2}. \quad (4.32)$$

Expansions of $\tilde{\sigma}_F$ and $\tilde{\sigma}_B$, at the leading order in ω , reveal the relations

$$\tilde{\sigma}_F = \left[\frac{i}{\omega} \frac{\tilde{A}}{4r_0^2} + \frac{\pi \tilde{A}}{16r_0^3 \cos\left(\frac{\pi}{2}\sqrt{1-\frac{\tilde{A}}{r_0^4}}\right)} \right] \frac{\Gamma\left[\frac{3-\sqrt{1-\frac{\tilde{A}}{r_0^4}}}{4}\right] \Gamma\left[\frac{3+\sqrt{1-\frac{\tilde{A}}{r_0^4}}}{4}\right]}{\Gamma\left[\frac{5-\sqrt{1-\frac{\tilde{A}}{r_0^4}}}{4}\right] \Gamma\left[\frac{5+\sqrt{1-\frac{\tilde{A}}{r_0^4}}}{4}\right]}, \quad (4.33)$$

$$\tilde{\sigma}_B = \left[\frac{i}{\omega} \frac{\tilde{B}}{4r_0^2} + \frac{\pi \tilde{B}}{16r_0^3 \cos\left(\frac{\pi}{2}\sqrt{1-\frac{\tilde{B}}{r_0^4}}\right)} \right] \frac{\Gamma\left[\frac{3-\sqrt{1-\frac{\tilde{B}}{r_0^4}}}{4}\right] \Gamma\left[\frac{3+\sqrt{1-\frac{\tilde{B}}{r_0^4}}}{4}\right]}{\Gamma\left[\frac{5-\sqrt{1-\frac{\tilde{B}}{r_0^4}}}{4}\right] \Gamma\left[\frac{5+\sqrt{1-\frac{\tilde{B}}{r_0^4}}}{4}\right]}, \quad (4.34)$$

In the limit when $B/T^2 \ll 1$ and $\tilde{B}_{\text{add}}/T^2 \ll 1$, which is equivalent to the condition that $\tilde{A} \ll 1$ and $\tilde{B} \ll 1$, one gains the formulae

$$\tilde{\sigma}_F = \sigma_F + \frac{i}{\omega} \frac{\tilde{A}}{2\pi^2 T^2} + \mathcal{O}\left(\frac{\tilde{A}^2}{T^4}\right), \quad \sigma_F = \frac{\pi T}{2} + \mathcal{O}\left(\frac{\tilde{A}^2}{T^4}\right), \quad (4.35)$$

$$\tilde{\sigma}_B = \sigma_B + \frac{i}{\omega} \frac{\tilde{B}}{2\pi^2 T^2} + \mathcal{O}\left(\frac{\tilde{B}^2}{T^4}\right), \quad \sigma_B = \frac{\pi T}{2} + \mathcal{O}\left(\frac{\tilde{B}^2}{T^4}\right). \quad (4.36)$$

Consequently, in the limit $\tilde{B}_{\text{add}} \rightarrow 0$, the conductivities imply

$$\tilde{\sigma}_F = \sigma_F + \frac{i}{\omega} \frac{(\tilde{\alpha} B)^2}{2\pi^2 T^2} + \mathcal{O}(\tilde{\alpha}^4 B^4), \quad \sigma_F = \pi T + \mathcal{O}(\tilde{\alpha}^4 B^4), \quad (4.37)$$

$$\tilde{\sigma}_B = \sigma_B + \frac{i}{\omega} \frac{(\tilde{\alpha} B)^2}{2\pi^2 T^2} + \mathcal{O}(\tilde{\alpha}^4 B^4), \quad \sigma_B = \pi T + \mathcal{O}(\tilde{\alpha}^4 B^4), \quad (4.38)$$

where we have defined

$$\tilde{\alpha}^2 = 64\alpha_1^2 + \frac{64}{9}(\alpha_3^2 + \alpha_3\alpha_4) + \frac{64}{3}\alpha_3\alpha_1, \quad (4.39)$$

$$\tilde{\tilde{\alpha}}^2 = \frac{64}{9}\alpha_4^2 + \frac{64}{9}(\alpha_3^2 + \alpha_3\alpha_1) + \frac{64}{3}\alpha_3\alpha_1. \quad (4.40)$$

Turning our attention to the limit when $\tilde{A} \gg 1$ and $\tilde{B} \gg 1$, one arrives at

$$\tilde{\sigma}_F = \sigma_F + \frac{i}{\omega} 2\sqrt{\tilde{A}} + \mathcal{O}\left(\frac{1}{2\sqrt{\tilde{A}}}\right), \quad \sigma_F = \frac{T}{2} e^{-\frac{\sqrt{\tilde{A}}}{2\pi T^2}} \left(\frac{\sqrt{\tilde{A}}}{T^2} + \mathcal{O}\left(\frac{1}{2\sqrt{\tilde{A}}}\right) \right), \quad (4.41)$$

$$\tilde{\sigma}_B = \sigma_B + \frac{i}{\omega} 2\sqrt{\tilde{B}} + \mathcal{O}\left(\frac{1}{2\sqrt{\tilde{B}}}\right), \quad \sigma_B = \frac{T}{2} e^{-\frac{\sqrt{\tilde{B}}}{2\pi T^2}} \left(\frac{\sqrt{\tilde{B}}}{T^2} + \mathcal{O}\left(\frac{1}{2\sqrt{\tilde{B}}}\right) \right). \quad (4.42)$$

It will be interesting to analyze the effect of the additional $\mathbb{Z}_2$ topological charge on the transport properties of the holographic system under consideration. First we note that

the holographic approach allowed us the direct calculation of σ_F , the component of conductivity undetermined in the hydrodynamic attitude. In the literature it is sometimes called quantum critical component of conductivity albeit to call it the Boltzmann conductivity is perhaps more appropriate in the present context, as it can be obtained from the Boltzmann kinetic equation. One has to remember that the general formula (4.33) is valid to lowest order in frequency. The conductivity σ_F depends on the magnetic field B and also on the additional magnetic field B_{add} . In accordance with the paper [44], we assume here $B_{\text{add}} = 0$ arguing, after the cited paper, that $\mathbb{Z}_2$ topological charge, at least in some systems, is related to the spin projections and one does not expect spin analog of the B magnetic field.

In the upper left panel of the figure 2, we show the dependence of $\omega \text{Im}\sigma(\omega)$ on the magnetic field, and more exactly, on the parameter $\frac{CB}{\pi^2 T^2}$, for the two investigated cases. One curve ($C_3 = 0$) corresponds to the system with the chiral anomaly described by $C_1 = C \neq 0$ with $C_2 = C_3 = C_4 = 0$. The presence of $\mathbb{Z}_2$ topological charge in the figure represented by the curve with $C_3 = 1$ ($= C_2 = C_4$), has an effect of narrowing the dependence of $\omega \text{Im}\sigma(\omega)$ on the magnetic field. This result harmonizes with the conclusion obtained in the previous section by applying the hydrodynamic approach to the holographic system.

The upper right panel of the figure 2 illustrates similar dependence of the $\tilde{\sigma}_F$ on the scaled magnetic field for the system with only chiral symmetry ($C_3 = 0$) and for the type II Dirac semimetal with both chiral anomaly and $\mathbb{Z}_2$ topological charge. The real part of the DC-conductivity (i.e., $\omega = 0$) $\tilde{\sigma}_F$, shown in the main part of the figure, comprises the quantum critical (or Boltzmann) part and the frequency dependent part which remains nonzero if the system is characterized by the finite charge relaxation time $\tau_e = 0.05$ introduced phenomenologically by the substitution $\omega \rightarrow \omega + i/\tau_e$. The inset to the figure shows the dependence on $\frac{CB}{\pi^2 T^2}$ of the Boltzmann contribution to the total conductivity. Interestingly it features strong non-monotonic dependence on the B-field, which is only limited to relatively low fields. Summing it all up, we have obtained a non-monotonous overall dependence of the magneto-conductivity on the magnetic field. Again the presence of the $\mathbb{Z}_2$ topological charge reveals a narrowing of the magneto-conductivity line.

The lower panel of the figure 2 depicts the magnetoresistivity of the same system. One observes that for the very small magnetic field the DC-magnetoresistivity $[\tilde{\rho}_F(B) - \tilde{\rho}_F(0)]/\tilde{\rho}_F(0)$ (with $\tilde{\rho}_F = 1/\tilde{\sigma}_F$) is positive and only for higher fields it becomes negative. The low field negative magneto-conductivity and positive magnetoresistivity, followed by the negative magnetoresistivity at high fields, has been observed experimentally [34] in ZrTe_5 . The holographic attitude provides qualitative the explanation of the experimental findings competitive to the weak anti-localization proposal.

Besides narrowing of the $\sigma_F(B)$ curve, the auxiliary topological charge, changes the relative contribution of quantum critical and relaxation limited parts. This is visible as a shift of the minimum of the two magneto-conductivity curves. The existence of $\mathbb{Z}_2$ topological charge shifts that point towards lower magnetic fields. However, the detailed behavior also depends on the assumed value of the charge relaxation time, τ_c .

Figure 2. The dependence of the $\omega \text{Im}\sigma(\omega)$ (upper left panel) and the real part of the DC conductivity $\bar{\sigma}_F(\omega = 0)$ (upper right panel) on $\frac{CB}{\pi^2 T^2}$, with $C = C_1$ being a chiral anomaly parameter. The conductivity has been calculated from the relation (4.33) for a holographic system with action (3.1). We used the relation $C_i = 8\alpha_i/3$ between anomaly parameters C_i in the hydrodynamic approach and corresponding parameters α_i adopted in (3.1). One of the two curves in each panel is obtained for all $C_i \neq 0$ (denoted $C_3 = 1$), while the other for $C_1 = C = 1$ and $C_3 = 0$ (denoted $C_3 = 0$). The existence of $\mathbb{Z}_2$ topological charge (corresponding to $C_3 = 1 = C_2 = C_4 = C_1$ and shown by the upper curve in the upper panel) leads to the narrowing of the magneto-conductivity. To obtain the magnetoconductivity shown in the upper right panel we have assumed that $T = 1$, $\tau_e = 0.05$ and $\omega = 0$. The inset shows the Boltzmann part σ_F^B . The lower panel of the figure shows the magnetoresistance of the same system.

5 Summary and conclusions

We have studied the behavior of the longitudinal conductivities in the Weyl semimetals with $\mathbb{Z}_2$ topological charge and chiral anomaly in the hydrodynamics limit, with the adequate magnetic fields which are connected with the ordinary Maxwell and the second U(1)-gauge fields. The auxiliary gauge field was introduced in order to envisage the $\mathbb{Z}_2$ anomalous charge.

Using the kinetic coefficients [45] for the system in question, being the functions of temperature, ρ , ρ_d , μ , μ_d , we found the DC-conductivities in the presence of magnetic

fields. In our considerations we take into account the dissipation terms, i.e., momentum, energy and charges dissipation terms. For the general case we obtain the complicated functions which depend on both magnetic fields, as well as, chiral anomaly and $\mathbb{Z}_2$ topological charge coefficients. In order to simplify these relations we have considered limiting cases. Namely, the cases of $\rho = \mu = 0$, $\rho_d = \mu_d = 0$ and $\rho = \mu = \rho_d = \mu_d = 0$ were elaborated. Moreover when one assumes that the one of the considered magnetic fields is equal to zero, we get the DC-conductivities which are functions of charge densities. Namely, if $B = 0$, the conductivity depends on ρ_d , and on the contrary, when $\tilde{B}_{\text{add}} = 0$, it is the function of ρ .

In the case when $\rho = \mu = \rho_d = \mu_d = 0$ and vanishing of both magnetic fields, one obtains the result that even at zero densities we have finite DC-conductivities. The received relations were only dissipated by the charge dissipation terms responsible for both U(1)-gauge fields. In the limit when the additional gauge field is equal to zero, one gets the results presented in [20].

The system was also examined by the holographic attitude, by means of five-dimensional Einstein-Chern-Simons gravity. The gauge Chern-Simons terms in gravitational action were associated with the possible interactions, playing the crucial role from the point of view of chiral anomaly and $\mathbb{Z}_2$ topological charge. As a background, in the probe limit attitude, we take the line element of five-dimensional AdS-Schwarzschild black brane. Starting from the small perturbations of the aforementioned black brane we have found the longitudinal conductivities for the considered model. We directly found that using Kubo formula, in the probe limit, the results match the hydrodynamical approach.

In the holographic description we did not include the dissipation terms. In addition, in the case when the additional field is equal to zero, we obtained the previously announced results [20].

The future experimental observations may put some restrictions on the constants C_i connected with the magneto-conductivities. Namely, the measurements similar to those conducted in [37] may shed some light on the problem in question. Our results show that the presence of $\mathbb{Z}_2$ topological charge causes the narrowing of the magneto-conductivity curve. On the other hand, the temperature dependence and the narrowing of the magneto-conductivity curve qualitatively agree with the experimental data [35, 52–54].

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