Strong coupling limit of a family of Chern-Simons-matter theories

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ABSTRACT: We investigate the strong coupling limit of a family of Chern-Simons-matter theories in the planar limit. The family consists of $\mathcal{N} = 3$ theories with the gauge group $U(N_1)_{k_1} \times U(N_2)_{k_2}$ coupled to n bi-fundamental hypermultiplets. All observables which can be determined from the planar resolvent turn out to have finite limits in the large 't Hooft coupling limit. Possible gravity duals are briefly discussed. We observe that Kac-Moody algebras govern the structure of the planar spectral curves of the theories.

KEYWORDS: Matrix Models, AdS-CFT Correspondence, Chern-Simons Theories

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1 Introduction

ABJM theory [1] provides a prototypical example of $\text{AdS}_4/\text{CFT}_3$ correspondence. It is an $\mathcal{N} = 6$ Chern-Simons-matter theory, and its gravity dual is M-theory on $\text{AdS}_4 \times S^7/\mathbb{Z}_k$. It is natural to expect that this correspondence could be generalized by replacing the internal manifold $S^7/\mathbb{Z}_k$ in the gravity side with a less symmetric seven-dimensional manifold B . One should choose a manifold B with some good properties in order to be able to perform the actual analysis. At the same time, one would like to choose B from a wide range of possible manifolds so that the insight into AdS/CFT correspondence [2] can be gained by examining various examples.

One such criterion for choosing B is to require that the corresponding CFT_3 possesses $\mathcal{N} = 3$ supersymmetry. Indeed, $\mathcal{N} = 3$ supersymmetry for boundary theory is powerful enough to control quantum corrections, and also it is flexible enough to allow the construction of theories with various gauge groups and matter representations [3]. A more detailed characterization of B is given as follows. It is known that M-theory on the background of the form $\mathbb{R}^{1,2} \times M$ preserves $\mathcal{N} = 3$ supersymmetry in three dimensions if and only if M is hyper-Kähler. Suppose that M has a conical singularity, and the base of the cone is B . One may put M2-branes to probe the singularity without breaking any supersymmetry,

as long as the orientation of the M2-branes is appropriately chosen [4]. The near-horizon geometry of the M2-branes is then $\text{AdS}_4 \times B$ which is expected to be the dual gravity background for the worldvolume theory of the M2-branes. Therefore, one should choose B to be a 3-Sasakian manifold (see e.g. [5]).

Suppose that the worldvolume theory of the M2-branes in the background discussed above is an $\mathcal{N} = 3$ Chern-Simons-matter theory. Then, the hyper-Kähler manifold M should be obtained as the moduli space of vacua of this theory. The moduli spaces were investigated in [6] for $\mathcal{N} = 3$ Chern-Simons-matter theories whose matter representation is specified by a circular quiver diagram. In [7], this analysis was extended to theories corresponding to more general quiver diagrams, and it was shown that the moduli space is hyper-Kähler. It turned out that the dimension of the moduli space of a given theory is determined by the number of loops in the quiver diagram, whose explicit formula depends on the Chern-Simons levels. According to the formula in [7], theories with A-type quiver diagrams have eight-dimensional moduli spaces, as was shown explicitly in [6], while those with DE-type quiver diagrams would have four-dimensional ones. Later, the analysis in [7] was generalized in [8], in order to treat non-toric hyper-Kähler manifolds as well, which showed that certain theories with D-type quivers also have eight-dimensional moduli spaces.

A check of the correspondence can be performed by calculating the free energy. It was shown in [9] that the free energy of ABJM theory behaves as

$$F \sim \frac{\sqrt{2}}{3} \sqrt{k} N^{\frac{3}{2}} \quad (1.1)$$

in the large N limit, which exactly reproduces the corresponding gravity result. This analysis can be extended to $\mathcal{N} = 3$ cases. From the gravity side, the leading large N behavior of the free energy is expected to be of the form [10]

$$F \sim N^{\frac{3}{2}} \sqrt{\frac{2\pi^6}{27 \text{vol}(B)}} \quad (1.2)$$

where $\text{vol}(B)$ is the volume of B . The formula for $\text{vol}(B)$ for a toric B was obtained in [12]. This volume formula was extended in [8] to non-toric ones. The large N behavior (1.2) of the free energy was reproduced from the corresponding $\mathcal{N} = 3$ Chern-Simons-matter theories exactly by employing the technique developed in [10]¹ based on the supersymmetric localization [13]. The technique was applied to various theories in [14–16]. Similar analyses can be performed for $\mathcal{N} = 2$ Chern-Simons-matter theories as well [17–24].

Interestingly, there is one example of $\mathcal{N} = 3$ duals which does not seem to have been investigated in detail, compared with the other examples. The M-theory background of this case consists of a 3-Sasakian manifold known as $N^{0,1,0}$ which is a coset $\text{SU}(3)/\text{U}(1)$. The cone over $N^{0,1,0}$ is $T^*\mathbb{P}^2$ which is a quite simple eight-dimensional hyper-Kähler manifold. It would be natural to expect that the dual field theory would be an $\mathcal{N} = 3$ Chern-Simons-matter theory.²

¹A similar technique was used in [11] in the 't Hooft limit.

²There are arguments on possible dual field theories [25–28].

With these developments in mind, in this paper, we investigate $\mathcal{N} = 3$ Chern-Simons-matter theories with the gauge group $U(N_1)_{k_1} \times U(N_2)_{k_2}$ coupled to $n \geq 3$ bi-fundamental hypermultiplets in the planar limit. Note that such a family of theories with $n = 2$ includes ABJM theory. Another family with $n = 1$ was recently discussed in [29]. In our previous paper [30], we analyzed the saddle point equations for the localized partition function, and obtained a closed formula for the resolvent in terms of the theta functions. From the resolvent, the free energy and the vevs of BPS Wilson loops, which are of the same kind investigated in [31–33] for ABJM theory, can be calculated as functions of the 't Hooft couplings. We investigate the behavior of these observables in the large 't Hooft coupling limit. Interestingly, we find that the behavior is quite different from the one observed in ABJM theory. It turns out that the free energy scales as N^2 , just like the ordinary gauge theories, and the vevs of BPS Wilson loops approach constant values in the limit. In addition, other observables which can be determined from the planar resolvent are found to exhibit similar behaviors. This difference from the case of ABJM theory is rather surprising since the difference at the level of Lagrangians is just to increase the number of the hypermultiplets. Curiously, a difference can be seen also at the level of spectral curves.

This paper is organized as follows. In section 2, we show that the above-mentioned results for the Chern-Simons-matter theories are derived from the resolvent obtained in [30]. Possible gravity duals for the theories are briefly reconsidered in section 3, taking into account the results obtained in the previous section. In section 4, we observe that differences among the $\mathcal{N} = 3$ Chern-Simons-matter theories investigated in [7] can be seen in their spectral curves which are governed by associated Kac-Moody algebras. Section 5 is devoted to discussion. Some details on our resolvent are reviewed in appendix A. Appendix B contains a proof for a fact on the spectral curve.

2 Planar results on Chern-Simons-matter theories

In this section, we investigate theories in a family $\text{CSM}(n)$ of $\mathcal{N} = 3$ Chern-Simons-matter theories for a positive integer n by using the corresponding matrix models. Each theory in $\text{CSM}(n)$ has a gauge group of the form $U(N_1)_{k_1} \times U(N_2)_{k_2}$ and n hypermultiplets in the bi-fundamental representation of the gauge group. The action of the theory is completely specified by these data due to $\mathcal{N} = 3$ supersymmetry [3]. ABJM theory [1, 34] and GT theory [35] are members of $\text{CSM}(2)$. We are mainly interested in the theories in $\text{CSM}(3)$.

2.1 Planar resolvent

The partition function of a theory in $\text{CSM}(n)$ defined on S^3 can be given in terms of a finite-dimensional integral [13] as

$$Z = \int d^{N_1} u d^{N_2} w \exp \left[\frac{ik_1}{4\pi} \sum_{i=1}^{N_1} (u_i)^2 + \frac{ik_2}{4\pi} \sum_{a=1}^{N_2} (w_a)^2 \right] \frac{\prod_{i < j}^{N_1} \left(\sinh \frac{u_i - u_j}{2} \right)^2 \prod_{a < b}^{N_2} \left(\sinh \frac{w_a - w_b}{2} \right)^2}{\prod_{i=1}^{N_1} \prod_{a=1}^{N_2} \left(\cosh \frac{u_i - w_a}{2} \right)^n}. \quad (2.1)$$

This integral can be regarded as the partition function of a matrix model. As usual for matrix models, we may take the planar limit which makes the saddle point approximation for

the integral (2.1) exact. The planar limit is defined by introducing an auxiliary parameter k as

$$k \rightarrow \infty \quad \text{with} \quad t_\alpha := \frac{2\pi i N_\alpha}{k}, \quad \kappa_\alpha := \frac{k_\alpha}{k} \quad \text{fixed} \quad (\alpha = 1, 2). \quad (2.2)$$

In this limit, the integral in (2.1) is dominated by the saddle point which satisfies the following equations:

$$\frac{k_1}{2\pi i} u_i = \sum_{j \neq i}^{N_1} \coth \frac{u_i - u_j}{2} - \frac{n}{2} \sum_{a=1}^{N_2} \tanh \frac{u_i - w_a}{2}, \quad (2.3)$$

$$\frac{k_2}{2\pi i} w_a = \sum_{b \neq a}^{N_2} \coth \frac{w_a - w_b}{2} - \frac{n}{2} \sum_{i=1}^{N_1} \tanh \frac{w_a - u_i}{2}. \quad (2.4)$$

Let $\{\bar{u}_i, \bar{w}_a\}$ denote the solution of these equations. Because of the presence of i in the left-hand side, $\bar{u}_i$ and $\bar{w}_a$ are complex numbers in general. For later convenience, the eigenvalues $\bar{u}_i$ and $\bar{w}_a$ are labeled such that

$$\text{Re}(\bar{u}_i) \leq \text{Re}(\bar{u}_{i+1}), \quad \text{Re}(\bar{w}_a) \leq \text{Re}(\bar{w}_{a+1}) \quad (2.5)$$

holds. We assume that both $\bar{u}_i$ and $\bar{w}_a$ approach zero in the small 't Hooft coupling limit, since in this limit the left-hand sides dominate in the equations.

Some physical observables can be calculated in the planar limit. The free energy is obtained from the saddle point value of the integral (2.1). In addition, there are two BPS Wilson loops for $U(N_1)$ gauge fields and $U(N_2)$ gauge fields. Their vevs in the planar limit are given as

$$\langle W_1 \rangle = \frac{1}{N_1} \sum_{i=1}^{N_1} e^{\bar{u}_i}, \quad \langle W_2 \rangle = \frac{1}{N_2} \sum_{a=1}^{N_2} e^{\bar{w}_a}. \quad (2.6)$$

The information of the theory in the planar limit is encoded in a row-vector-valued resolvent [30]

$$v(z) := (v_1(z), v_2(z)). \quad (2.7)$$

The components $v_\alpha(z)$ are defined as

$$v_1(z) := \lim \frac{t_1}{N_1} \sum_{i=1}^{N_1} \frac{z + e^{\bar{u}_i}}{z - e^{\bar{u}_i}}, \quad v_2(z) := \lim \frac{t_2}{N_2} \sum_{a=1}^{N_2} \frac{z - e^{\bar{w}_a}}{z + e^{\bar{w}_a}}, \quad (2.8)$$

where $\lim$ indicates the planar limit (2.2). The function $v_1(z)$ has a square-root branch cut on an interval I_1 in $\mathbb{C}$, and holomorphic elsewhere, including the point at infinity. The branch points a_1 and b_1 on I_1 are given as

$$a_1 := e^{\bar{u}_1}, \quad b_1 := e^{\bar{u}_{N_1}}. \quad (2.9)$$

Due to the symmetry of the equations (2.3), (2.4), they satisfy

$$a_1 b_1 = 1. \quad (2.10)$$

The other function $v_2(z)$ has similar properties. The branch points a_2 and b_2 are given as

$$a_2 := -e^{\bar{w}_1}, \quad b_2 := -e^{\bar{w}_{N_2}} \quad (2.11)$$

which also satisfy

$$a_2 b_2 = 1. \quad (2.12)$$

Note that $|a_\alpha| \leq 1$ are satisfied.

The resolvent $v(z)$ is determined by specifying two complex parameters, say a_α . These two parameters are related to the values of the 't Hooft couplings t_α for given κ_α via

$$v(0) = -(t_1, t_2). \quad (2.13)$$

It was found in [30] that it is convenient to investigate a derivative $zv'(z)$ of the resolvent, instead of $v(z)$ itself, since the former can be obtained explicitly. In the following, we assume $n \neq 2$.

Let $c := (c_1, c_2)$ be the row vector satisfying

$$(2\kappa_1, 2\kappa_2) = (c_1, c_2) \begin{bmatrix} 2 & -n \\ -n & 2 \end{bmatrix}. \quad (2.14)$$

Note that the solution of this equation exists for $n \neq 2$. Let us introduce another row-vector-valued function $f(z)$ such that $zv'(z)$ can be written as

$$zv'(z) = c + \frac{1}{s(z)} f(z), \quad s(z) := \sqrt{(z-a_1)(z-b_1)(z-a_2)(z-b_2)}. \quad (2.15)$$

In order to determine the explicit form of $f(z)$, we introduce a new variable u defined as

$$u(z) := \frac{\varphi(z)}{2\varphi(b_1)}, \quad \varphi(z) := \int_{a_1}^z \frac{d\xi}{s(\xi)}, \quad (2.16)$$

where the integration contour for $\varphi(b_1)$ lies above the segment I_1 . The saddle point equations (2.3), (2.4) imply that, as a function of u , $f(z)$ can be written as

$$f(z(u)) = \left(\tilde{f}(u), \tilde{f}(-u) \right) S^{-1}, \quad S := \begin{bmatrix} 1 & 1 \\ -e^{\pi i \nu} & -e^{-\pi i \nu} \end{bmatrix} \quad (2.17)$$

where ν is related to n by $n = 2 \cos \pi \nu$. The scalar function $\tilde{f}(u)$ is required to satisfy

$$\tilde{f}(u+1) = \tilde{f}(u), \quad \tilde{f}(u+\tau) = e^{2\pi i \nu} \tilde{f}(u) \quad (2.18)$$

where $\tau := 2u(a_2)$. Therefore, $\tilde{f}(u)$ can be written in terms of the theta functions. The explicit form of $\tilde{f}(u)$ is given in appendix A.

Since what we have obtained is $zv'(z)$, the relation (2.13) cannot be used to determine t_α for a given a_α . An alternative way to recover t_α is to use the formula

$$t_\alpha = -\frac{1}{2} \int_{C_\alpha} \frac{dz}{2\pi i} \frac{\log z}{z} zv'_\alpha(z), \quad (2.19)$$

where C_1 (C_2) is a contour encircling the segment I_1 (I_2) counterclockwise.

In addition to t_α , the expansion of $zv'(z)$ provides the vevs $\langle W_\alpha \rangle$ of BPS Wilson loops as

$$zv'(z) = -2(t_1 \langle W_1 \rangle, -t_2 \langle W_2 \rangle) z + O(z^2). \quad (2.20)$$

Combining (2.19) and (2.20), the vevs $\langle W_\alpha \rangle$ can be given as functions of t_α .

2.2 Small 't Hooft coupling limit

Various quantities can be calculated perturbatively when the 't Hooft couplings t_α are small. For example, the vevs $\langle W_\alpha \rangle$ are given as [30]

$$\langle W_1 \rangle = 1 + \frac{t_1}{2\kappa_1} + \frac{1}{6} \left(\frac{t_1}{\kappa_1} \right)^2 \left(1 - \frac{1}{4N_1^2} \right) - \frac{n}{8} \frac{t_1 t_2}{\kappa_1^2} + O(t^3), \quad (2.21)$$

$$\langle W_2 \rangle = 1 + \frac{t_2}{2\kappa_2} + \frac{1}{6} \left(\frac{t_2}{\kappa_2} \right)^2 \left(1 - \frac{1}{4N_2^2} \right) - \frac{n}{8} \frac{t_1 t_2}{\kappa_2^2} + O(t^3), \quad (2.22)$$

which can be obtained by using the method developed in [13]. This result was reproduced in [30] from the resolvent $v(z)$ reviewed above by using the formulas (2.19), (2.20) for cases when $N_1 = N_2$ and $k_1 = \pm k_2$ are satisfied. This can be regarded as a non-trivial check of the validity of the resolvent $v(z)$ obtained in [30].

The method in [13], however, does not give us any information on the configuration of the eigenvalues $\{\bar{u}_i, \bar{w}_a\}$, and therefore, the positions a_α of the branch points of $v(z)$. To obtain such information, it is better to solve the saddle point equations (2.3), (2.4) perturbatively, as in [11].

In the following, we will restrict ourselves to the case $N_1 = N_2 =: N$ for simplicity. This implies that $t_1 = t_2 =: t$ holds. The analysis of this case should be the first step toward the understanding of the general case.

Recall that the saddle point equations in this case are

$$\frac{k_1}{2\pi i} u_i = \sum_{j \neq i}^N \coth \frac{u_i - u_j}{2} - \frac{n}{2} \sum_{a=1}^N \tanh \frac{u_i - w_a}{2}, \quad (2.23)$$

$$\frac{k_2}{2\pi i} w_a = \sum_{b \neq a}^N \coth \frac{w_a - w_b}{2} - \frac{n}{2} \sum_{i=1}^N \tanh \frac{w_a - u_i}{2}. \quad (2.24)$$

Assume that t is small, and introduce rescaled variables x_i and y_a defined such that

$$u_i = \sqrt{t} x_i, \quad w_a = \sqrt{t} y_a \quad (2.25)$$

holds. Then, the saddle point equations can be expanded in t . The leading order terms give the following simple equations:

$$\kappa_1 x_i = \frac{2}{N} \sum_{j \neq i}^N \frac{1}{x_i - x_j}, \quad \kappa_2 y_a = \frac{2}{N} \sum_{b \neq a}^N \frac{1}{y_a - y_b}. \quad (2.26)$$

Each of these equations are the same as the saddle point equations of the Gaussian matrix model. The planar solution of them is encoded in

$$\omega_1(z) = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N \frac{1}{z - x_i} = \frac{\kappa_1}{2} \left[z - \sqrt{z^2 - \frac{4}{\kappa_1}} \right], \quad (2.27)$$

$$\omega_2(z) = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{a=1}^N \frac{1}{z - y_a} = \frac{\kappa_2}{2} \left[z - \sqrt{z^2 - \frac{4}{\kappa_2}} \right]. \quad (2.28)$$

The perturbative corrections to this solution can be systematically calculated, as shown in [11] for ABJM theory.

The discontinuity of $\omega_1(z)$ and $\omega_2(z)$ encodes the distributions of $\{x_i\}$ and $\{y_a\}$ which then give the distributions of $\{u_i\}$ and $\{w_a\}$. The definitions (2.9), (2.11) of the parameters a_1 and a_2 imply that they are given as

$$a_1 \sim \exp\left(-2\sqrt{\frac{t}{\kappa_1}}\right), \quad a_2 \sim -\exp\left(-2\sqrt{\frac{t}{\kappa_2}}\right) \quad (2.29)$$

for small t .

One may notice that the relation between a_1 and a_2 becomes quite simple when $\kappa_1 = \kappa_2$ holds. In this case, the parameters are related simply as

$$a_1 = -a_2 =: a. \quad (2.30)$$

This relation is satisfied when the equalities $\bar{u}_i = \bar{w}_i$ hold. Since this equality is compatible with the full saddle point equations (2.23), (2.24) with $k_1 = k_2$, the relation (2.30) should hold beyond perturbation. Therefore, any observables which can be derived from the resolvent $v(z)$ are functions of a . Without loss of generality, we can choose $\kappa_1 = \kappa_2 = 1$.

Note that the explicit form of $zv'(z)$ shown in appendix A is written in terms of the parameters u_0, u_∞ , and τ . The definition of $u(z)$ implies

$$u_\infty = u_0 - \frac{1}{2}. \quad (2.31)$$

When the equality $a_1 = -a_2$ holds, u_0 is given by τ as

$$u_0 = \frac{1}{4}\tau. \quad (2.32)$$

Therefore, all the observables are also functions of τ . The relation between a and τ can be obtained from the inverse of $u(z)$ given as

$$z(u) = -\frac{\vartheta_1(u - u_0)\vartheta_1(u + u_0)}{\vartheta_1(u - u_\infty)\vartheta_1(u + u_\infty)}. \quad (2.33)$$

Since $z = a$ corresponds to $u = 0$ by definition, a is related to τ as

$$a = -\left(\frac{\vartheta_1\left(\frac{1}{4}\tau\right)}{\vartheta_1\left(\frac{1}{4}\tau - \frac{1}{2}\right)}\right)^2. \quad (2.34)$$

In the following, we choose τ as the parameter specifying the resolvent.

It is important to notice that not all values of τ are physically relevant. Since the 't Hooft couplings are defined as (2.2), their physical values are purely imaginary. Therefore, τ must be chosen such that the integrals (2.19) take purely imaginary values. The physical values of τ form a curve γ in the complex τ -plane.

The asymptotic behavior of γ in the small 't Hooft coupling limit is determined as follows. When a approaches 1, the definition of τ implies that $\text{Im}(\tau)$ diverges to $+\infty$. In this limit, the relation (2.34) becomes

$$a \sim 1 - 4e^{\frac{1}{2}\pi i\tau}. \quad (2.35)$$

Figure 1. The plot of the curve γ for $n = 3$ on which t is purely imaginary. The curve γ approaches the line $\text{Re}(\tau) = \frac{1}{2}$ in the limit $t \rightarrow 0$. It terminates on the imaginary axis at which t diverges.

On the other hand, (2.29) implies

$$a \sim 1 - 2\sqrt{t} \quad (2.36)$$

for small t . Therefore, in the small 't Hooft coupling limit, the physical curve γ approaches the line

$$\text{Re}(\tau) = \frac{1}{2}. \quad (2.37)$$

2.3 Large 't Hooft coupling limit

Our interest is in the large 't Hooft coupling limit. The limit can be obtained by following the curve γ in the direction opposite to the small 't Hooft coupling limit discussed above. Recall that the curve γ is defined as $\text{Re}(t) = 0$. Since t is given in terms of the integral of a complicated function, it looks quite difficult to determine the curve γ analytically. Instead, we evaluate the integral (2.19) numerically, and find out where $\text{Re}(t) = 0$ holds in the τ -plane.

The plot of the curve γ for $n = 3$ is shown in figure 1. The curve γ terminates on the imaginary axis at which the 't Hooft coupling t diverges as

$$t \sim \frac{r}{\tau - \nu}, \quad r = 1.27i, \quad (2.38)$$

where $\nu = 0.306i$ for $n = 3$. This pole comes from the factor $(1 + z(u_\nu))^{-1}$ in the function $g(u)$ defined in appendix A. The singularity at $\tau = \nu$ was pointed out in [30].

The values of the vevs $\langle W_1 \rangle$ and $\langle W_2 \rangle$ are the same since $\bar{u}_i = \bar{w}_i$ holds. Their values in the large t limit are

$$\begin{aligned} \langle W_1 \rangle &= \langle W_2 \rangle = -\frac{1}{4\varphi(b)(1 + e^{\pi i \nu})t} [g'(u_0)G(u_0) + g(u_0)G'(u_0)] \\ &\rightarrow 5.77. \end{aligned} \quad (2.39)$$

Note that these are finite even in the large t limit, in contrast to ABJM theory and GT theory. The finiteness can be anticipated from the eigenvalue distribution. Recall that the large t limit corresponds to the limit $\tau \rightarrow \nu$. The relation (2.34) implies that $a = 1.19 \times 10^{-2}$ in the limit which is small but finite. Then, the value of $b = a^{-1}$ is also finite. The following inequality

$$|\langle W_1 \rangle| \leq \frac{1}{N_1} \sum_{i=1}^{N_1} e^{\text{Re}(\bar{u}_i)} \leq |b| \quad (2.40)$$

implies that $\langle W_1 \rangle$ must be finite.

In fact, it turns out that all the observables derived from the resolvent $v(z)$ are finite in the large t limit. To show this, consider a general situation in which $\rho(x; t)$ is a density function with a parameter t whose support is an interval I . The expectation value of $\mathcal{O}(x)$ is defined as

$$\langle \mathcal{O} \rangle_t := \int_I dx \rho(x; t) \mathcal{O}(x). \quad (2.41)$$

Define a resolvent

$$\omega(z; t) := t \int_I dx \rho(x; t) \frac{z+x}{z-x}. \quad (2.42)$$

The expectation value (2.41) can be written in terms of $\omega(z; t)$ as

$$\langle \mathcal{O} \rangle_t = \frac{1}{2t} \int_C \frac{dz}{2\pi i} \mathcal{O}(z) \frac{\omega(z; t)}{z}, \quad (2.43)$$

where C is a contour in the z -plane encircling I counterclockwise. Suppose that a function $\mathcal{F}(x)$ satisfies

$$\mathcal{F}'(x) = \frac{1}{x} \mathcal{O}(x). \quad (2.44)$$

Then, $\langle \mathcal{O} \rangle_t$ can be written as

$$\langle \mathcal{O} \rangle_t = -\frac{1}{2t} \int_C \frac{dz}{2\pi i} \frac{\mathcal{F}(z)}{z} \cdot z\omega'(z; t). \quad (2.45)$$

This integral has a finite value in the limit $t \rightarrow \infty$ if $t^{-1}z\omega'(z; t)$ has a finite limit at any points on the contour C .

In the case of the Chern-Simons-matter matrix models, the condition for the finiteness turns out to be

$$\lim_{t \rightarrow +\infty i} \left| t^{-1} \tilde{f}(u) \right| < \infty \quad \left(-\frac{1}{2} \leq u \leq +\frac{1}{2} \right). \quad (2.46)$$

This can be shown to be the case numerically. Since this implies that the eigenvalue distribution of this matrix model has a finite limit, all the quantities calculated by using this eigenvalue distribution must be finite. Recall that the planar free energy is equal to N^2 times a quantity calculated by the eigenvalue distribution. Therefore, it scales as N^2 , as for usual gauge theories.

If n is large, then $\text{Im}(\nu)$ becomes large. Then, since the pole of t is located at the point $\tau = \nu$, the q -expansion of the theta functions can be used for the analysis of the large t limit. The behavior of t around $\tau = \nu$ turns out to be

$$t \sim \frac{r}{\tau - \nu}, \quad r = \frac{2i}{\pi(e^{-\pi i \nu} - 1)}. \quad (2.47)$$

This pole structure is qualitatively the same as (2.38) observed in the numerical result for $n = 3$.

The vevs $\langle W_\alpha \rangle$ in the large t limit are

$$\langle W_1 \rangle = \langle W_2 \rangle \rightarrow \frac{1}{1 + e^{\pi i \nu}}. \quad (2.48)$$

This limiting value approaches 1 in the large n limit. This can be anticipated from the saddle point equations (2.23), (2.24). As $\text{Im}(\nu)$ becomes large, the attractive forces among the eigenvalues due to the second terms in the right-hand side of (2.23), (2.24) become strong. Then, the eigenvalue distribution shrinks to two points in the limit.

2.4 Small τ limit

We observed in the previous subsection that the behavior of the vevs $\langle W_\alpha \rangle$ for the theories in CSM($n \geq 3$) is quite different from that of ABJM theory and GT theory. From the matrix model point of view, the difference comes from the fact that, in the former theories, the extent of the eigenvalue distributions are kept finite in the large 't Hooft coupling limit, while in the latter theories, the eigenvalue distributions grow indefinitely. It is curious to know what happens if the eigenvalue distributions grow indefinitely in the former theories. It is not a strong coupling limit of the Chern-Simons-matter theories, but it could be realized by an analytic continuation of the parameters.

Since the branch points are constrained as (2.10), (2.12), an infinitely long distribution of the eigenvalues can be realized only when the parameter $a(= a_1 = -a_2)$ approaches zero. Small a limit corresponds to a small τ limit. Due to the relation (2.34), they are related as

$$-\log a \sim \frac{\pi i}{2\tau}. \quad (2.49)$$

It turns out that it is not straightforward to take the limit $\tau \rightarrow 0$. Since the factor $(1 + z(u_\nu))^{-1}$, which is the origin of the pole of t at $\tau = \nu$, has in fact infinitely many poles on the τ -plane for $n > 2$. The positions of the poles are

$$\tau = \frac{\nu}{2l+1} + \frac{m}{2l+1}, \quad (l, m \in \mathbb{Z}). \quad (2.50)$$

Therefore, the small τ limit taken along the imaginary axis, for example, is ill-defined.

One way to avoid these singularities is to take the following limit:

$$\tau = \varepsilon e^{i\theta}, \quad \varepsilon \rightarrow 0 \quad (2.51)$$

with a fixed θ . It turns out that if θ is in the range

$$0 < \cot \theta < \frac{1}{2\text{Im}(\nu)}, \quad (2.52)$$

then all the poles can be avoided. At the same time, the expansion of the theta functions, after the transformation $\tau \rightarrow -\frac{1}{\tau}$, behaves well.

The integral formula (2.19) gives the small τ behavior of t . The leading order term turns out to be

$$t \sim \frac{C}{\tau}, \quad C := \frac{1}{\sinh\left(\frac{\pi}{2} \text{Im}(\nu)\right)} \sum_{n=1}^{\infty} \frac{(-i)^n}{n + \nu}. \quad (2.53)$$

Note that the sum in the coefficient C for $n = 3$ is finite:

$$\sum_{n=1}^{\infty} \frac{(-i)^n}{n + \nu} = -0.591 - 0.641 i. \quad (2.54)$$

Then, an upper bound of the vevs $\langle W_\alpha \rangle$ is obtained as

$$|\langle W_\alpha \rangle| \leq |b| \lesssim \left| \exp\left(\frac{\pi i}{2C} t\right) \right|. \quad (2.55)$$

As in the cases for ABJM theory and GT theory, this upper bound is actually saturated. The formula (2.20) implies

$$\langle W_\alpha \rangle \sim \exp\left(\frac{\pi i}{2C} t\right) \quad (2.56)$$

for large t . This behavior is different from the Wilson loops in ABJM theory and GT theory.

2.5 The case $k_1 = -k_2$

The weak coupling behavior (2.29) indicates that there is another case $k_1 = -k_2$ for which the analysis is rather simple. Assuming that t takes a physical value, that is $t \in i\mathbb{R}$, the parameters a_1 and a_2 are related as

$$a_1 = -a_2^*. \quad (2.57)$$

This is realized when the equalities $u_i = w_i^*$ are satisfied. This equality is compatible with the full saddle point equations (2.23), (2.24), and therefore, the relation (2.57) should hold beyond perturbation. Indeed, such eigenvalue distributions were discussed in [10, 11].

When the relation (2.57) holds, the parameters u_0, u_∞ , and τ in the resolvent $v(z)$ are related among them. In addition to the relation (2.31), one can show that τ is given as

$$\tau = 4i \text{Im}(u_0). \quad (2.58)$$

Therefore, all the observables derived from the resolvent $v(z)$ are functions of u_0 . Let u_0 be written as

$$u_0 = u_R + \frac{1}{4}\tau, \quad u_R \in \mathbb{R}, \quad (2.59)$$

the parameter a_1 is given as

$$a_1 \sim 1 - 4e^{\frac{1}{2}\pi i \tau} e^{2\pi i u_R} \quad (2.60)$$

for large $\text{Im}(\tau)$. The relation (2.29) for small t implies that the physical curve γ , now on the u_0 -plane, approaches the line

$$\text{Re}(u_0) = \frac{1}{8} \quad (2.61)$$

in the small t limit.

The large t limit can be analyzed in the same way as in the case $k_1 = k_2$. The plot of the curve γ for $n = 3$ turns out to be quite similar to figure 1. The curve γ terminates on the imaginary axis of the u_0 -plane. The intersection point is at

$$u_0 = \frac{1}{4}\nu, \quad (2.62)$$

where t diverges. The vevs $\langle W_\alpha \rangle$ are finite even in the large t limit.

3 On gravity dual

In this section, we briefly discuss possible gravity duals of theories in CSM($n \geq 3$).

The first step for the discussion is to examine the classical moduli space of vacua of the theory. The dimension D of the moduli space for a given Abelian Chern-Simons-matter theory is given by the matter representation and the Chern-Simons levels k_i [7]. Let l be the number of loops in the quiver diagram dictating the matter representation of the theory. The dimension D is given as

$$D = \begin{cases} 4(l+1), & (\sum_i k_i = 0) \\ 4l & (\sum_i k_i \neq 0) \end{cases}. \quad (3.1)$$

For example, ABJM theory corresponds to a quiver diagram with $l = 1$ and the two Chern-Simons levels are k and $-k$. Then, the above formula implies $D = 8$. This coincides with the dimension of the moduli space $\mathbb{C}^4/\mathbb{Z}_k$. For the theory in CSM($n \geq 3$), the corresponding quiver diagram has $l = n - 1$ loops. Therefore, the moduli space can be eight-dimensional only when $n = 3$ and $k_1 + k_2 \neq 0$ are satisfied. This is one reason that we focused our attention mainly on the theories with $n = 3$ and $k_1 = k_2$.

The gravity dual, however, seems to be rather different from the dual of ABJM theory, that is, M-theory on $\text{AdS}_4 \times S^7/\mathbb{Z}_k$. In the latter, the planar limit corresponds to a Type IIA limit since the limit involves $k \rightarrow \infty$ in which a circle direction in S^7 shrinks. On the other hand, in the former, the planar limit seems to be still eleven-dimensional.

To understand the origin of the difference, it would be instructive to recall how the moduli space of vacua is determined for Abelian Chern-Simons-matter theories. The relevant part of the action is

$$\frac{ik_1}{4\pi} \int A_1 \wedge dA_1 + \frac{ik_2}{4\pi} \int A_2 \wedge dA_2 - \int d^3x |\partial_\mu Y^I - i(A_1 - A_2)_\mu Y^I|^2 + \dots. \quad (3.2)$$

If the Chern-Simons levels satisfy $k_1 + k_2 = 0$, then by choosing a suitable linear combinations A and $\tilde{A}$ of the gauge fields A_1 and A_2 , the action can be written as

$$\frac{ik}{4\pi} \int A \wedge d\tilde{A} - \int d^3x |\partial_\mu Y^I - iA_\mu Y^I|^2 + \dots. \quad (3.3)$$

This action defines a theory of matters Y^I coupled to a $\mathbb{Z}_k$ gauge theory constructed in terms of A and $\tilde{A}$ [39]. This residual $\mathbb{Z}_k$ gauge symmetry makes the moduli space to be an orbifold. Since the superpotential vanishes for the case $k_1 = -k_2$, the moduli space is $\mathbb{C}^4/\mathbb{Z}_k$ for ABJM theory.

On the other hand, if $k_1 + k_2$ is nonzero, then there is no choice of A and $\tilde{A}$ for which the action becomes of the form (3.3). An alternative choice gives

$$\frac{ik_1k_2(k_1+k_2)}{4\pi} \int A \wedge dA + \frac{ik_1k_2}{4\pi} \int \tilde{A} \wedge d\tilde{A} - \int d^3x |\partial_\mu Y^I - i(k_1+k_2)AY^I|^2 + \dots \quad (3.4)$$

Here, the gauge field $\tilde{A}$ simply decouples. This implies that there is no orbifolding of the internal manifold. Therefore, the planar limit still seems to be eleven-dimensional.

If the gravity duals for theories in CSM(3) in the planar limit are really eleven-dimensional, then the dictionary between the AdS_4 gravity and CFT_3 should be quite different from the one for ABJM theory. Then, the behavior of observables in the large 't Hooft coupling limit might be quite different. For example, it is known that the dictionary for a CFT_3 whose gravity dual is a massive Type IIA theory is quite different from those for ABJM theory [40]. Due to this, the Wilson loop behaves as [41]

$$\langle W \rangle \sim \exp\left(ct^{\frac{1}{3}}\right) \quad (3.5)$$

for a constant c . The free energy scales as N^2 in the planar limit. The leading term of the free energy in the large 't Hooft coupling limit is multiplied by a power of t which reproduces the scaling in the M-theory limit [40]

$$F \sim c' N^{\frac{5}{3}}. \quad (3.6)$$

It seems that more detailed investigations are necessary to understand the issue of gravity duals for theories in CSM(3).

4 Spectral curves

In this section, we show that a difference among theories in $\text{CSM}(n)$ can be found in the structure of planar spectral curves obtained from the resolvent $zv'(z)$. This observation can be extended to more general Chern-Simons-matter theories. The structure turns out to be governed by an associated Kac-Moody algebra specified by the matter representation of the theory. The reader may consult [42] for results on Kac-Moody algebras mentioned below. The spectral curves of the theories in $\text{CSM}(2)$ were discussed in [43].

4.1 The theories in $\text{CSM}(n)$

It was shown in [30] that the resolvent $v(z)$ satisfies the following vector equations:

$$(2\kappa_1, 0) = x_1 v'(x_1^+) - x_1 v'(x_1^-) M_1, \quad (4.1)$$

$$(0, 2\kappa_1) = x_2 v'(x_2^+) - x_2 v'(x_2^-) M_2, \quad (4.2)$$

where $x_\alpha \in I_\alpha$, and x_α^+ (x_α^-) is a point slightly above (below) x_α . The matrices M_α are defined as

$$M_1 := \begin{bmatrix} -1 & 0 \\ n & 1 \end{bmatrix}, \quad M_2 := \begin{bmatrix} 1 & n \\ 0 & -1 \end{bmatrix}. \quad (4.3)$$

In many cases, the left-hand sides of (4.1), (4.2) can be eliminated. This can be done by introducing $\omega(z)$ such that

$$zv'(z) = c + \omega(z) \quad (4.4)$$

is satisfied, where the constant row vector c satisfies

$$(2\kappa_1, 2\kappa_2) = cA, \quad A := \begin{bmatrix} 2 & -n \\ -n & 2 \end{bmatrix}. \quad (4.5)$$

The solution exists as long as $n \neq 2$. Even in the case $n = 2$, there exists a solution of this equation if and only if

$$\kappa_1 + \kappa_2 = 0 \quad (4.6)$$

is satisfied. In the following, we assume that the equation (4.5) has a solution. Then, $\omega(z)$ satisfies

$$\omega(x_\alpha^+) = \omega(x_\alpha^-)M_\alpha. \quad (4.7)$$

It was shown in [30] that, for the case $n = 2$, the vector equations (4.7) imply a simple scalar equation

$$\omega_{\psi_2}(x_\alpha^+) = -\omega_{\psi_2}(x_\alpha^-), \quad (4.8)$$

where

$$\omega_\psi(z) := \omega(z) \cdot \psi, \quad \psi_2 := \begin{bmatrix} 1 \\ -1 \end{bmatrix}. \quad (4.9)$$

Therefore, $\omega_{\psi_2}(z)$ defines a two-sheeted covering of $\mathbb{C}$, that is, a torus.

For the case $n = 1$, one can check that the following equations are satisfied:

$$\begin{aligned} \omega_{\psi_1}(x_1^+) &= \omega_{M_1\psi_1}(x_1^-), & \omega_{\psi_1}(x_2^+) &= \omega_{\psi_1}(x_2^-), \\ \omega_{M_2M_1\psi_1}(x_1^+) &= \omega_{M_2M_1\psi_1}(x_1^-), & \omega_{M_1\psi_1}(x_2^+) &= \omega_{M_2M_1\psi_1}(x_2^-), \end{aligned} \quad (4.10)$$

where

$$\psi_1 := \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \quad (4.11)$$

The above equations imply that the three scalar functions $\omega_{\psi_1}(z), \omega_{M_1\psi_1}(z), \omega_{M_2M_1\psi_1}(z)$ define a three-sheeted covering of $\mathbb{C}$. After a compactification, the resulting curve is topologically a sphere. Note that these scalar functions are given in terms of the components of the resolvent $v(z)$ as

$$\omega_{\psi_1}(z) = zv'_1(z) - c_1, \quad (4.12)$$

$$\omega_{M_1\psi_1}(z) = zv'_2(z) - zv'_1(z) + c_1 - c_2, \quad (4.13)$$

$$\omega_{M_2M_1\psi_1}(z) = zv'_2(z) - c_2. \quad (4.14)$$

Similar spectral curves appeared in a different context [44, 45].

For the remaining cases $n \geq 3$, it can be shown that $\omega_\psi(z)$ do not define a finite-sheeted covering of $\mathbb{C}$ for any choice of ψ . A proof is given in appendix B. The best one can find is a finite covering with a twist, defined by the following equations:

$$\omega_{\psi_n}(x_1^+) = -\omega_{\psi'_n}(x_1^-), \quad \omega_{\psi_n}(x_2^+) = -e^{2\pi i\nu}\omega_{\psi'_n}(x_2^-), \quad (4.15)$$

where

$$\psi_n := \begin{bmatrix} 1 \\ -e^{\pi i \nu} \end{bmatrix}, \quad \psi'_n = \begin{bmatrix} 1 \\ -e^{-\pi i \nu} \end{bmatrix}. \quad (4.16)$$

These equations are reduced to (4.8) when $\nu = 0$, or in other words, $n = 2$.

In summary, we have found that the spectral curve defined in terms of $\omega(z)$ is a Riemann surface of genus $n - 1$ for the cases $n = 1, 2$, and otherwise, the curve is an infinite-covering of $\mathbb{C}$.

There is also a difference between the case $n = 1$ and $n = 2$. For the case $n = 1$, it can be shown that both $v'_1(z)$ and $v'_2(z)$ are algebraic functions, while for the case $n = 2$, only the combination $v'_1(z) - v'_2(z)$ is an algebraic function.

It is curious to notice that the matrix A in (4.5) is a generalized Cartan matrix. The corresponding Kac-Moody algebra is $\mathfrak{su}(3)$ for $n = 1$, affine $\mathfrak{su}(2)$ for $n = 2$ and an algebra of indefinite type for $n \geq 3$.

4.2 General quiver-type theories

The structure of the spectral curves for theories in $\text{CSM}(n)$ observed in the previous subsection can also be found in more general Chern-Simons-matter theories. Consider a Chern-Simons-matter theory with the gauge group $\prod_{a=1}^{n_g} \text{U}(N_a)_{k_a}$ coupled to bi-fundamental hypermultiplets discussed in [7, 38]. Let n_{ab} be the number of the hypermultiplets coupled to $\text{U}(N_a) \times \text{U}(N_b)$ factor. Note that n_{ab} is a symmetric matrix.

Assume that the Chern-Simons levels k_a are chosen such that

$$2(\kappa_1, \dots, \kappa_{n_g}) = cA, \quad A_{bc} := 2\delta_{bc} - n_{bc} \quad (4.17)$$

has a solution. Again, the matrix A can be regarded as the generalized Cartan matrix of a Kac-Moody algebra $\mathfrak{g}_A$. By construction, A is always symmetric.

As in the previous subsection, the planar analysis of the Chern-Simons-matter theory is reduced to solving the following vector equation [38]:

$$\omega(x_a^+) = \omega(x_a^-)M_a, \quad (4.18)$$

where

$$\omega(z) := (\omega_1(z), \dots, \omega_{n_g}(z)) \quad (4.19)$$

is a row-vector-valued function, and the matrices M_a are defined as

$$(M_a)_{bc} := \delta_{bc} - \delta_{ac}A_{bc}, \quad (4.20)$$

where the repeated indices are not summed. It can be shown that $(M_a)^2 = I$ is satisfied for any a .

Let W be a group generated by these M_a . To construct a covering of $\mathbb{C}$, we choose a column vector $\psi_0 \in \mathbb{C}^{n_g}$ and consider the W -orbit:

$$W\psi_0 := \{w\psi_0 \mid w \in W\}. \quad (4.21)$$

Define scalar functions $\omega_\psi(z) := \omega(z) \cdot \psi$ for each $\psi \in W\psi_0$. Then, the equations (4.18) imply

$$\omega_\psi(x_a^+) = \omega_{M_a\psi}(x_a^-), \quad (4.22)$$

which define a covering of $\mathbb{C}$. If $W\psi_0$ is a finite set, then the covering gives a Riemann surface of a finite genus.

The following observation will be helpful for choosing an appropriate vector ψ_0 which defines a simple spectral curve. Consider the following vectors:

$$(\alpha_a)_b := A_{ab}. \quad (4.23)$$

The action of M_a on α_b is

$$M_a\alpha_b = \alpha_b - A_{ab}\alpha_a. \quad (4.24)$$

This indicates that M_a act on α_b as if α_b are fundamental roots of $\mathfrak{g}_A$, and M_a are the fundamental reflections of $\mathfrak{g}_A$. Then, W gives a representation of the Weyl group of $\mathfrak{g}_A$.

If the matrix A is non-degenerate, then α_a indeed define fundamental roots in a root system Φ of $\mathfrak{g}_A$, and M_a define a faithful representation of the Weyl group of $\mathfrak{g}_A$ on $\mathbb{C}^{n_g}$. Choosing ψ_0 as one of the roots $\alpha \in \Phi$, the W -orbit $W\psi_0$ is a subset of Φ .

Suppose that $\mathfrak{g}_A$ is of finite type, that is, $\mathfrak{g}_A$ is a finite-dimensional Lie algebra. Then, $W\psi_0$ is a finite set since Φ is a finite set, and the covering defined above gives a Riemann surface of a finite genus.

In fact, there is a simpler choice of ψ_0 . Recall that the Weyl group also acts on the set of weights in an irreducible representation. If ψ_0 is chosen to be the highest weight of the smallest representation, then the resulting Riemann surface is the simplest possible one. The covering for the case $n = 1$ given in the previous subsection is of this kind.

Next, suppose that $\mathfrak{g}_A$ is of affine type. In this case, the rank of A is $n_g - 1$. This implies that α_a are not linearly independent, and therefore, α_a cannot define fundamental roots in a root system Φ of $\mathfrak{g}_A$. It is known that Φ can be realized in $\mathbb{C}^{n_g+1}$.

The root system Φ of $\mathfrak{g}_A$ can be described explicitly as follows. Let $\mathfrak{g}_A^0$ be a finite-dimensional Lie algebra associated to $\mathfrak{g}_A$, and let Φ^0 be a root system of $\mathfrak{g}_A^0$. Then Φ is given as

$$\Phi = \{\alpha^0 + r\delta \mid \alpha^0 \in \Phi^0, r \in \mathbb{Z}\}, \quad (4.25)$$

where δ is defined as

$$\delta := \sum_{a=1}^{n_g} m_a \tilde{\alpha}_a \quad (4.26)$$

in terms of the fundamental roots $\tilde{\alpha}_a \in \Phi$ of $\mathfrak{g}_A$ and integers m_a satisfying

$$\sum_{b=1}^{n_g} A_{ab} m_b = 0, \quad (m_1, m_2, \dots, m_{n_g}) = 1. \quad (4.27)$$

It is known that the one-dimensional subspace $L \subset \mathbb{C}^{n_g+1}$ spanned by δ is invariant under the action of the Weyl group of $\mathfrak{g}_A$. Then, the Weyl group also acts on the quotient space $\mathbb{C}^{n_g+1}/L$. There exists an isomorphism of vector spaces

$$\pi : \mathbb{C}^{n_g+1}/L \rightarrow \mathbb{C}^{n_g}, \quad \pi(\tilde{\alpha}_a) = \alpha_a. \quad (4.28)$$

This is well-defined since

$$\pi(\delta)_b = \sum_{a=1}^{n_g} m_a (\alpha_a)_b = \sum_{a=1}^{n_g} m_a A_{ab} = 0 \quad (4.29)$$

is satisfied. By this isomorphism, the matrices M_a can be identified with the fundamental reflections of the Weyl group of $\mathfrak{g}_A$ acting on $\mathbb{C}^{n_g+1}/L$. In this quotient space, Φ becomes equivalent to Φ^0 which is a finite set. If ψ_0 is chosen to be one of α_a , then the W -orbit $W\psi_0$ is again finite. The covering for the case $n = 2$ given in the previous subsection is of this kind.

If $\mathfrak{g}_A$ is of indefinite type, there does not seem to exist a choice of ψ_0 which gives us a Riemann surface of a finite genus as a spectral curve. It would be interesting if it would be possible to find a “twisted” covering like the one given in the previous subsection for the cases $n \geq 3$. The existence of such coverings would open the possibility to determine the resolvent explicitly.

5 Discussion

We have discussed the large 't Hooft coupling limit of theories in $\text{CSM}(n \geq 3)$. We have found that all the observables which can be calculated from the planar resolvent have finite limit in the large 't Hooft coupling limit. As an example, the vevs of BPS Wilson loops are finite in the limit, quite different from the behavior observed in ABJM theory. The gravity dual of a theory in $\text{CSM}(3)$ was briefly discussed. In addition, we have found the structure of spectral curves of the theories which depends on n . The structure is governed by a Kac-Moody algebra associated to the matter representation of each Chern-Simons-matter theory.

It is curious that the behavior of observables of theories in $\text{CSM}(3)$ is quite different from the one observed in ABJM theory, although they seem to share many properties, like the dimension of the moduli space which would suggest the existence of a gravity dual. We briefly pointed out that the difference could be consistent with the observation that the dual gravity background would behave differently in the planar limit. To know more about the possible gravity duals of theories in $\text{CSM}(3)$, we need to know more detailed properties of the CFT_3 side. One issue to be clarified is the consistency of any claim with [35] since the Chern-Simons levels are chosen such that $k_1 + k_2 \neq 0$ is satisfied.

The observation that Kac-Moody algebras may play important role in Chern-Simons-matter matrix models seems to be quite interesting. The relation of Kac-Moody algebras and Chern-Simons-matter theories would imply a classification of the latter in terms of the classification of the former. This also suggests that detailed knowledge of Kac-Moody algebras of indefinite type would give us a crew to investigate various Chern-Simons-matter theories. For example, if one could find a representation of a Kac-Moody algebra of indefinite type which may give the twisted covering like (4.15) given in subsection 4.1, then the planar resolvent of the corresponding Chern-Simons-matter theory could be obtained explicitly in terms of the theta functions on a higher-genus Riemann surface.

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A Details on the planar resolvent

In this appendix, we review some details on the planar resolvent obtained in [30].

The derivative of the planar resolvent can be written as

$$zv'(z) = c + \frac{1}{s(z)}f(z), \quad (\text{A.1})$$

where $s(z)$ is a square-root function defined in (2.15), and c is a constant vector satisfying (2.14). Explicitly,

$$c_1 = -\frac{4\kappa_1 + 2n\kappa_2}{n^2 - 4}, \quad c_2 = -\frac{2n\kappa_1 + 4\kappa_2}{n^2 - 4} \quad (\text{A.2})$$

for $n \neq 2$. The row-vector-valued function $f(z)$ is given in terms of a scalar-valued function $\tilde{f}(u)$ as in (2.17).

One can check that a function $G(u)$ defined as

$$G(u) := \frac{\vartheta_1(u - u_\nu)\vartheta_1(u - u_\nu + \frac{1}{2})}{\vartheta_1(u - u_\infty)\vartheta_1(u + u_\infty)} \quad (\text{A.3})$$

satisfies the conditions (2.18), provided that the parameters are defined as

$$u_0 := u(0), \quad u_\infty := u(\infty), \quad u_\nu := \frac{1}{2}\nu + \frac{1}{4}. \quad (\text{A.4})$$

In terms of $G(u)$, the function $\tilde{f}(u)$ is given as

$$\tilde{f}(u) = g(u)G(u) \quad (\text{A.5})$$

where $g(u)$ is an elliptic function. The function $g(u)$ is then determined by requiring that the derivative $zv'(z)$ has the right analytic structure. The explicit form of $g(u)$ is

$$g(u) = r_1(1 + z(u)) + r_3(g_3(u) + g_4(u)), \quad (\text{A.6})$$

where $z(u)$ is defined in (2.33), and

$$g_3(u) := \frac{\vartheta_1(u - u_0)\vartheta_1(u - u_\nu + \frac{1}{2})}{\vartheta_1(u - u_\infty)\vartheta_1(u - u_\nu)}, \quad g_4(u) := -\frac{\vartheta_1(u + u_0)\vartheta_1(u - u_\nu)}{\vartheta_1(u + u_\infty)\vartheta_1(u - u_\nu + \frac{1}{2})}. \quad (\text{A.7})$$

The coefficients r_1, r_3 are given as

$$r_1 = \frac{1}{g_3(-u_0) - g_4(u_0)} \left[\tilde{c}_1 \frac{g_3(-u_0)}{G(u_0)} - \tilde{c}_2 \frac{g_4(u_0)}{G(-u_0)} \right], \quad (\text{A.8})$$

$$r_3 = \frac{1}{g_3(-u_0) - g_4(u_0)} \left[-\frac{\tilde{c}_1}{G(u_0)} + \frac{\tilde{c}_2}{G(-u_0)} \right], \quad (\text{A.9})$$

where $\tilde{c} := (\tilde{c}_1, \tilde{c}_2)$ is defined as $\tilde{c} := cS$. Note that the equality

$$g_3(-u_0) - g_4(u_0) = g_3(-u_0)(1 + z(u_\nu)) \quad (\text{A.10})$$

holds.

B Infinite number of sheets for $n \geq 3$

Let W be a group generated by M_1 and M_2 . As explained in subsection 4.1, one can construct a spectral curve from the equations (4.7) in terms of the scalar functions

$$\omega_\psi(z), \quad \psi \in W\psi_0 \quad (\text{B.1})$$

for a choice of $\psi_0 \in \mathbb{C}^2$.

Suppose that $W\psi_0$ is finite for a suitable ψ_0 . Then, there exists an element $w \in W$ such that

$$w\psi_0 = \psi_0 \quad (\text{B.2})$$

holds. The element w has one of the following forms:

$$(M_1 M_2)^m, \quad (M_2 M_1)^m, \quad M_1 (M_2 M_1)^m, \quad M_2 (M_1 M_2)^m \quad (\text{B.3})$$

for a non-negative integer m . One may choose a basis of $\mathbb{C}^2$ such that M_1 and M_2 can be written as

$$M_1 = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 0 & -e^{-2\pi i \nu} \\ -e^{2\pi i \nu} & 0 \end{bmatrix}. \quad (\text{B.4})$$

For $n \geq 3$, the parameter ν is purely imaginary. We choose $\text{Im}(\nu) > 0$.

From (B.4), one obtains

$$(M_1 M_2)^m = \begin{bmatrix} e^{2\pi i m \nu} & \\ 0 & e^{-2\pi i m \nu} \end{bmatrix}. \quad (\text{B.5})$$

Apparently, if w is of this form, then the condition (B.2) cannot be satisfied for any ψ_0 . By the same reasoning, w is not of the form $(M_2 M_1)^m$.

Now, suppose $w = M_2 (M_1 M_2)^m$. The explicit form of this matrix is

$$\begin{bmatrix} 0 & -e^{2\pi i (m+1) \nu} \\ -e^{2\pi i (m+1) \nu} & 0 \end{bmatrix}. \quad (\text{B.6})$$

Then, ψ_0 is determined to be

$$\psi_0 = \begin{bmatrix} 1 \\ -e^{2\pi i (m+1) \nu} \end{bmatrix} \quad (\text{B.7})$$

up to an overall factor. This ψ_0 gives

$$(M_1 M_2)^l \psi_0 = \begin{bmatrix} e^{2\pi i l \nu} \\ -e^{2\pi i (m+1-l) \nu} \end{bmatrix}, \quad (\text{B.8})$$

$$M_2 (M_1 M_2)^l \psi_0 = \begin{bmatrix} e^{2\pi i (m-l) \nu} \\ -e^{2\pi i (l+1) \nu} \end{bmatrix} \quad (\text{B.9})$$

for $0 \leq l \leq m$. If $W\psi_0$ is finite, then $(M_2M_1)^l\psi_0$ for a positive integer l must have one of the above forms. However, this is impossible since its form is

$$(M_2M_1)^l\psi_0 = \begin{bmatrix} e^{-2\pi il\nu} \\ -e^{2\pi i(m+1+l)\nu} \end{bmatrix}. \quad (\text{B.10})$$

Therefore, w is not of the form $M_2(M_1M_2)^m$. The same argument excludes the possibility for w to be of the form $M_1(M_2M_1)^m$. It is concluded that an element $w \in W$ satisfying (B.2) does not exist, implying that $W\psi_0$ is infinite for any choice of ψ_0 .

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