On the local frame in nonlinear higher-spin equations

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ABSTRACT: Properties of the resolution operator d_{loc}^* in higher-spin equations, that leads to local current interactions at the cubic order and minimally nonlocal higher-order corrections, are formulated in terms of the condition on the class of master fields of higher-spin theory that restricts both the dependence on the spinor Y , Z variables and on the contractions of indices between the constituent fields in bilinear terms. The Green function in the sector of zero-forms is found for the case of constituent fields carrying helicities of opposite signs. It is shown that the local resolution d_{loc}^* differs from the conventional De Rham resolution d_Z^* by a non-local shift.

KEYWORDS: Higher Spin Gravity, Higher Spin Symmetry

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Contents

1	Introduction	1
2	Free fields	3
3	Nonlinear higher-spin equations in AdS_4	5
4	Perturbative analysis and resolution operator	6
4.1	Perturbations	6
4.2	Homotopy trick	8
5	Perturbative analysis of locality	10
6	Cohomology shift and Green function	14
6.1	d_Z^* -induced nonlocal deformation in the zero-form sector	15
6.2	Green function	16
7	Discussion	20
8	Technical details of the Green function derivation	21

1 Introduction

Nonlinear field equations for massless fields of all spins in four dimensions were found in [1, 2]. The most symmetric vacuum solution to these equations describes AdS_4 . Due to the presence of AdS_4 radius as a dimensionful parameter, higher-spin (HS) interactions can contain infinite tails of higher-derivative terms. This can make the theory not local in the standard sense, raising the question which field variables lead to the local or minimally nonlocal setup in the perturbative analysis.

Geometric origin of the dimensionful parameter ρ has an important consequence that any HS gauge theory with unbroken HS symmetries does not allow a low-energy analysis because a dimensionless derivative $\rho \frac{\partial}{\partial x}$ where $\rho = \lambda^{-1}$ is the AdS radius, that appears in the expansion in powers of derivatives, cannot be treated as small. This is because the rescaled covariant derivatives $\mathcal{D} = \rho D$, which are non-commutative in the background AdS space-time of curvature ρ^{-2} , have commutator of order one, $[\mathcal{D}, \mathcal{D}] \sim 1$. As a result, all terms with higher derivatives may give comparable contributions.

Another important feature of HS theory is that it describes interactions of infinite towers of HS fields while HS symmetry transforms a spin- s field to fields of other spins. In particular, HS symmetries transform spin two to higher spins. As emphasized in [3], this has a consequence that such concepts of Riemannian geometry as space-time point and dimension are not necessarily applicable to interacting HS theories.

The importance of the proper definition of locality was originally emphasized in [4] where it was shown that by a seemingly local field redefinition it is possible to get rid of the currents from the nonlinear HS field equations in AdS_3 . The question what is a proper definition of a weakly local field redefinitions in HS nonlinear theories was addressed recently in several papers [5–10], in particular, in the context of the derivation of current interactions of massless fields from the nonlinear HS field equations of [2]. In [5] a proposal was put forward on the part of the problem associated with the exponential factors resulting from so-called inner Klein operators while the structure of the preexponential factors was only partially determined. In this paper we make a step towards completing the definition of locality in the setup of HS equations of [2].

Conclusions of the papers [6, 8, 10] and [9] were somewhat opposite. The authors of [6, 8, 10] argued that no preferred frame of field variables in HS theory exists, bringing current interactions to the proper form with the conclusion that using field redefinitions exhibiting the same asymptotic behaviour it is possible to obtain current interactions with arbitrary charges in front of different currents, including zero charges which means no interactions. In the absence of a further selection criterion such a conclusion would imply difficulties of the physical interpretation of the HS equations of [2].

The analysis of [6, 8, 10] was performed in the one-form gauge sector of equations of [2]. On the other hand, in [9] we considered the problem in the zero-form sector of the HS equations which is simpler in many respects being the same time fully informative, finding a simple field redefinition that brings the quadratic corrections to free field equations following from the nonlinear HS equations to the canonical local current form in agreement with the unfolded form of current interactions found earlier in [11]. In [12] these results were extended to the one-form sector. The results of [9] were then shown in [13] (for a related particular result see also [14]) to lead to correct AdS/CFT predictions at the boundary of AdS_4 thus resolving some of the puzzles of the analysis of HS holography conjectures of [15, 16] encountered in [17–19] (and references therein).

The resulting couplings for different HS currents were expressed unambiguously in [9, 12] via the coupling constant of the nonlinear HS equations. In this paper it will be shown that the choice of field variables of [9] is distinguished by the condition that the associated higher-order corrections in HS theory are minimally nonlocal. To see this it is necessary to take into account the dependence on the spinor variables Z^A in addition to that on Y^A considered in [6, 8, 10], that is insufficient to control higher-order locality. It will be shown that upon a proper redefinition of the resolution underlying the procedure of solving the nonlinear HS equations from the conventional one d_Z^* to d_{loc}^* the results of [9] come out directly with no need of further field redefinitions. This implies that the De Rham resolution d_Z^* is related to the local one d_{loc}^* by a nonlocal field redefinition.

In this paper we determine explicit form of d_{loc}^* in the sector of zero-forms and to the lowest order. Extension of these results to forms of higher degrees and to higher orders will be given elsewhere [20].

It should be stressed that, as explained in more detail below, the freedom in the definition of the resolution parametrizes the freedom in the choice of the homogeneous part of the solution to the differential equations in spinorial Z variables in the process of

solving nonlinear HS equations. As usual, this freedom has to be fixed by appropriate boundary conditions. In quantum mechanics proper boundary conditions for solutions to the Schrodinger equation are fixed from the normalizability condition. (Discarding the normalizability condition may lead to meaningless conclusions, making it impossible to find a physical spectrum in a problem in question.) In HS theory, the appropriate choice of the boundary conditions is determined by the minimal nonlocality condition (which in the cases considered so far implies locality of HS vertices). Since, as explained below, this demands the solution to belong to a proper functional class, it is natural to speculate that, eventually, this functional class can be determined by a certain normalizability condition. The results of this paper provide a starting point for exploring this problem setting.

The rest of the paper is organized as follows. In section 2 we recall unfolded formulation of free massless fields in four dimensions. The form of nonlinear HS equations is sketched in section 3. In section 4 details of the perturbative analysis of nonlinear equations with some emphasize on the homotopy technique are presented. The form of the local resolution d_{loc}^* leading to proper current interactions is found in section 5. The structure of the unfolded equations describing current interactions of massless fields of all spins in the zero-form sector is recalled in section 6.1. Green function for the zero-form sector of HS equations is found in section 6.2 for the case of opposite helicity signs of the constituent fields of the current. Conclusions and perspective are briefly discussed in section 7. Appendix contains details of the derivation of the Green function.

2 Free fields

In the frame-like formulation, the infinite set of $4d$ Fronsdal [21–24] massless fields of all spins $s = 0, 1, 2, \dots$ is described by a one-form $\omega(Y; K|x) = dx^n \omega_n(Y; K|x)$ [25, 26] and zero-form $C(Y; K|x)$ [27]

$$f(Y; K|x) = \frac{1}{2i} \sum_{n,m=0}^{\infty} \sum_{i,j=1}^2 \frac{1}{n!m!} y_{\alpha_1} \dots y_{\alpha_n} \bar{y}_{\dot{\beta}_1} \dots \bar{y}_{\dot{\beta}_m} k^i \bar{k}^j f_{ij}^{\alpha_1 \dots \alpha_n, \dot{\beta}_1 \dots \dot{\beta}_m}(x), \quad (2.1)$$

where x^n are $4d$ space-time coordinates, $Y^A = (y^\alpha, \bar{y}^{\dot{\alpha}})$ are auxiliary commuting spinor variables ($A = 1, \dots, 4$ is a Majorana spinorial index while $\alpha = 1, 2$ and $\dot{\alpha} = 1, 2$ are two-component ones) and the Klein operators $K = (k, \bar{k})$ satisfy

$$ky^\alpha = -y^\alpha k, \quad k\bar{y}^{\dot{\alpha}} = \bar{y}^{\dot{\alpha}} k, \quad \bar{k}y^\alpha = y^\alpha \bar{k}, \quad \bar{k}\bar{y}^{\dot{\alpha}} = -\bar{y}^{\dot{\alpha}} \bar{k}, \quad k^2 = \bar{k}^2 = 1, \quad k\bar{k} = \bar{k}k. \quad (2.2)$$

More precisely, to describe massless fields, the one-form $\omega(Y; K|x)$ should be even in $k, \bar{k}$ (i.e., $i = j$ in (2.1)) while the zero-form $C(Y; K|x)$ should be odd (i.e., $i + j = 1$).

The Central on-shell theorem states that unfolded system of field equations for free massless fields of all spins has the form [27]

$$\begin{aligned} R_1(Y; K|x) &= L(w, C) \\ &:= \frac{i}{4} \left(\eta \bar{H}^{\dot{\alpha}\dot{\beta}} \frac{\partial^2}{\partial \bar{y}^{\dot{\alpha}} \partial \bar{y}^{\dot{\beta}}} C(0, \bar{y}; K|x) k + \bar{\eta} H^{\alpha\beta} \frac{\partial^2}{\partial y^\alpha \partial y^\beta} C(y, 0; K|x) \bar{k} \right), \end{aligned} \quad (2.3)$$

$$\tilde{D}C(y, \bar{y}|x) = 0, \quad (2.4)$$

where

$$R_1(Y; K|x) := D^{ad}\omega_1(Y; K|x)$$

$$:= D^L\omega_1(Y; K|x) + \lambda h^{\alpha\dot{\beta}} \left(y_\alpha \frac{\partial}{\partial \bar{y}^{\dot{\beta}}} + \frac{\partial}{\partial y^\alpha} \bar{y}_{\dot{\beta}} \right) \omega_1(Y; K|x), \quad (2.5)$$

$$\tilde{D}C(Y; K|x) := D^L C(Y; K|x) - i\lambda h^{\alpha\dot{\beta}} \left(y_\alpha \bar{y}_{\dot{\beta}} - \frac{\partial^2}{\partial y^\alpha \partial \bar{y}^{\dot{\beta}}} \right) C(Y; K|x), \quad (2.6)$$

$$D^L f(Y; K|x) := df(Y; K|x) + \left(\omega_0^{\alpha\beta} y_\alpha \frac{\partial}{\partial y^\beta} + \bar{\omega}_0^{\dot{\alpha}\dot{\beta}} \bar{y}_{\dot{\alpha}} \frac{\partial}{\partial \bar{y}^{\dot{\beta}}} \right) f(Y; K|x), \quad d = dx^n \frac{\partial}{\partial x^n}. \quad (2.7)$$

ω_1 describes first-order fluctuations of the HS gauge fields. Background AdS_4 space of radius $\lambda^{-1} = \rho$ is described by a flat $sp(4)$ connection $w = (\omega_{0\alpha\beta}, \bar{\omega}_{0\dot{\alpha}\dot{\beta}}, h_{\alpha\dot{\beta}})$ containing Lorentz connection $\omega_{0\alpha\beta}$, $\bar{\omega}_{0\dot{\alpha}\dot{\beta}}$ and vierbein $h_{\alpha\dot{\beta}}$ that obey the equations

$$R_{\alpha\beta} = 0, \quad \bar{R}_{\dot{\alpha}\dot{\beta}} = 0, \quad R_{\alpha\dot{\alpha}} = 0, \quad (2.8)$$

where, here and after discarding the wedge product symbols,

$$R_{\alpha\beta} := d\omega_{0\alpha\beta} + \omega_{0\alpha\gamma}\omega_{0\beta}^\gamma - \lambda^2 H_{\alpha\beta}, \quad \bar{R}_{\dot{\alpha}\dot{\beta}} := d\bar{\omega}_{0\dot{\alpha}\dot{\beta}} + \bar{\omega}_{0\dot{\alpha}\dot{\gamma}}\bar{\omega}_{0\dot{\beta}}^{\dot{\gamma}} - \lambda^2 \bar{H}_{\dot{\alpha}\dot{\beta}}, \quad (2.9)$$

$$R_{\alpha\dot{\beta}} := dh_{\alpha\dot{\beta}} + \omega_{0\alpha\gamma}h_{\dot{\beta}}^\gamma + \bar{\omega}_{0\dot{\beta}\dot{\delta}}h_{\alpha}^{\dot{\delta}}. \quad (2.10)$$

Two-component indices are raised and lowered by $\varepsilon_{\alpha\beta} = -\varepsilon_{\beta\alpha}$, $\varepsilon_{12} = 1$: $A^\alpha = \varepsilon^{\alpha\beta}A_\beta$, $A_\alpha = A^\beta\varepsilon_{\beta\alpha}$ and analogously for dotted indices. $H^{\alpha\beta} = H^{\beta\alpha}$ and $\bar{H}^{\dot{\alpha}\dot{\beta}} = \bar{H}^{\dot{\beta}\dot{\alpha}}$ are the basis two-forms

$$H^{\alpha\beta} := h^{\alpha\dot{\alpha}}h^{\beta\dot{\beta}}\dot{\alpha}\dot{\beta}, \quad \bar{H}^{\dot{\alpha}\dot{\beta}} := h^{\alpha\dot{\alpha}}h_{\alpha}^{\dot{\beta}}. \quad (2.11)$$

η and $\bar{\eta}$ are free complex conjugated parameters.

Due to the dependence on the Klein operators the one-form HS connection $\omega(Y; K|x)$ contains a doubled set of HS gauge fields. For spins $s \geq 1$, equation (2.3) expresses the Weyl 0-forms $C(Y; K|x)$ via gauge invariant combinations of derivatives of the HS gauge connections. More precisely, the primary-like Weyl 0-forms are just the holomorphic and antiholomorphic parts $C(y, 0; K|x)$ and $C(0, \bar{y}; K|x)$ which appear on the r.h.s. of eq. (2.3).

$C(y, 0; K|x)$ and $C(0, \bar{y}; K|x)$ describe gauge invariant combinations of derivatives of the gauge fields of spins $s \geq 1$ and the matter fields of spins $s = 0$ or $1/2$. For $s = 1, 2$, $C(y, 0; K|x)$ and $C(0, \bar{y}; K|x)$ parameterize Maxwell and Weyl tensors respectively. Those associated with higher powers of auxiliary variables y and $\bar{y}$ describe on-shell nontrivial combinations of derivatives of the generalized Weyl tensors as is obvious from equations (2.4), (2.6) relating second derivatives in $y, \bar{y}$ to the x derivatives of $C(Y; K|x)$ of lower degrees in Y . Higher derivatives in the nonlinear system result from the components of $C(Y; K|x)$ of higher degrees in Y . AdS geometry induces filtration with respect to space-time derivatives rather than gradation as would be the case for massless fields in Minkowski space free of a dimensional parameter.

System (2.3), (2.4) decomposes into subsystems of different spins, with a massless spin s described by the one-forms $\omega(y, \bar{y}; K|x)$ and zero-form $C(y, \bar{y}; K|x)$ obeying the homogeneity conditions

$$\omega(\mu y, \mu \bar{y}; K|x) = \mu^{2(s-1)} \omega(y, \bar{y}; K|x), \quad C(\mu y, \mu^{-1} \bar{y}; K|x) = \mu^{\pm 2s} C(y, \bar{y}; K|x), \quad (2.12)$$

where $+$ and $-$ signs correspond to selfdual and anti-selfdual parts of the generalized Weyl tensors $C(y, \bar{y}|x)$ with helicities $h = \pm s$.

3 Nonlinear higher-spin equations in AdS_4

The master fields of the construction of nonlinear equations of [2] consist of the zero-form $B(Z; Y; K|x)$, space-time one-form $W(Z; Y; K|x)$ and an additional spinor field $S_A(Z; Y; K|x)$. It is convenient to introduce anticommuting Z -differentials θ^A , $\theta^A \theta^B = -\theta^B \theta^A$, to interpret $S_A(Z; Y; K|x)$ as a one-form in Z direction,

$$S = \theta^A S_A(Z; Y; K|x). \quad (3.1)$$

HS equations determining dependence on the variables Z_A in terms of “initial data”

$$\omega(Y; K|x) = W(0; Y; K|x), \quad C(Y; K|x) = B(0; Y; K|x), \quad (3.2)$$

are formulated in terms of the associative star product $*$ acting on functions of two spinor variables

$$(f * g)(Z; Y) = \int \frac{d^4 U d^4 V}{(2\pi)^4} \exp[iU^A V^B C_{AB}] f(Z + U; Y + U) g(Z - V; Y + V), \quad (3.3)$$

where $C_{AB} = (\epsilon_{\alpha\beta}, \bar{\epsilon}_{\dot{\alpha}\dot{\beta}})$ is the $4d$ charge conjugation matrix and U^A, V^B are real integration variables. 1 is a unit element of the star-product algebra, i.e., $f * 1 = 1 * f = f$. Star product (3.3) provides a particular realization of the Weyl algebra

$$[Y_A, Y_B]_* = -[Z_A, Z_B]_* = 2iC_{AB}, \quad [Y_A, Z_B]_* = 0, \quad [a, b]_* := a * b - b * a. \quad (3.4)$$

The Klein operators satisfy

$$k * w^\alpha = -w^\alpha * k, \quad k * \bar{w}^{\dot{\alpha}} = \bar{w}^{\dot{\alpha}} * k, \quad \bar{k} * w^\alpha = w^\alpha * \bar{k}, \quad \bar{k} * \bar{w}^{\dot{\alpha}} = -\bar{w}^{\dot{\alpha}} * \bar{k}, \quad (3.5)$$

$$k * k = \bar{k} * \bar{k} = 1, \quad k * \bar{k} = \bar{k} * k \quad (3.6)$$

with $w^\alpha = (y^\alpha, z^\alpha, \theta^\alpha)$, $\bar{w}^{\dot{\alpha}} = (\bar{y}^{\dot{\alpha}}, \bar{z}^{\dot{\alpha}}, \bar{\theta}^{\dot{\alpha}})$. These relations extend the action of the star product to the Klein operators.

The nonlinear HS equations are [2]

$$dW + W * W = 0, \quad (3.7)$$

$$dB + W * B - B * W = 0, \quad (3.8)$$

$$dS + W * S + S * W = 0, \quad (3.9)$$

$$S * B = B * S, \quad (3.10)$$

$$S * S = i(\theta^A \theta_A + \theta^\alpha \theta_\alpha F_*(B) * k * \kappa + \bar{\theta}^{\dot{\alpha}} \bar{\theta}_{\dot{\alpha}} \bar{F}_*(B) * \bar{k} * \bar{\kappa}), \quad (3.11)$$

where $F_*(B)$ is some star-product function of the field B . The simplest choice of the linear function

$$F_*(B) = \eta B, \quad \bar{F}_*(B) = \bar{\eta} B, \quad (3.12)$$

where η is a complex parameter

$$\eta = |\eta| \exp i\varphi, \quad \varphi \in [0, \pi), \quad (3.13)$$

leads to a class of pairwise nonequivalent nonlinear HS theories. The cases of $\varphi = 0$ and $\varphi = \frac{\pi}{2}$ correspond to so called A and B HS models distinguished by the property that they respect parity [28].

The left and right inner Klein operators

$$\kappa := \exp iz_\alpha y^\alpha, \quad \bar{\kappa} := \exp i\bar{z}_{\dot{\alpha}} \bar{y}^{\dot{\alpha}}, \quad (3.14)$$

which enter eq. (3.11), change a sign of undotted and dotted spinors, respectively,

$$(\kappa * f)(z, \bar{z}; y, \bar{y}) = \exp iz_\alpha y^\alpha f(y, \bar{z}; z, \bar{y}), \quad (\bar{\kappa} * f)(z, \bar{z}; y, \bar{y}) = \exp i\bar{z}_{\dot{\alpha}} \bar{y}^{\dot{\alpha}} f(z, \bar{y}; y, \bar{z}), \quad (3.15)$$

$$\kappa * f(z, \bar{z}; y, \bar{y}) = f(-z, \bar{z}; -y, \bar{y}) * \kappa, \quad \bar{\kappa} * f(z, \bar{z}; y, \bar{y}) = f(z, -\bar{z}; y, -\bar{y}) * \bar{\kappa}, \quad (3.16)$$

$$\kappa * \kappa = \bar{\kappa} * \bar{\kappa} = 1, \quad \kappa * \bar{\kappa} = \bar{\kappa} * \kappa. \quad (3.17)$$

4 Perturbative analysis and resolution operator

4.1 Perturbations

Perturbative analysis of eqs. (3.7)–(3.11) assumes their linearization around some vacuum solution. The simplest choice is

$$W_0(Z; Y; K|x) = w(Y|x), \quad S_0(Z; Y; K|x) = \theta^A Z_A, \quad B_0(Z; Y; K|x) = 0, \quad (4.1)$$

where $w(Y|x)$ is some solution to the flatness condition $dw + w * w = 0$. A flat connection $w(Y|x)$ bilinear in Y^A describes AdS_4

$$w(Y|x) = -\frac{i}{4} \omega^{AB}(x) Y_A Y_B = -\frac{i}{4} (\omega^{AB}(x) + h^{AB}(x)) Y_A Y_B, \quad (4.2)$$

$$\omega^{AB}(x) Y_A Y_B := \omega^{\alpha\beta}(x) y_\alpha y_\beta + \bar{\omega}^{\dot{\alpha}\dot{\beta}}(x) \bar{y}_{\dot{\alpha}} \bar{y}_{\dot{\beta}}, \quad h^{AB}(x) Y_A Y_B := 2h^{\alpha\dot{\beta}}(x) y_\alpha \bar{y}_{\dot{\beta}}. \quad (4.3)$$

Decomposing fields with respect to the Klein operator parity, $A^\pm(Z; Y; K|x) = \pm A^\pm(Z; Y; -K|x)$, HS gauge fields are W^+, S^+ and B^- while W^-, S^- and B^+ describe an infinite tower of topological fields with every AdS_4 irreducible field describing at most a finite number of degrees of freedom. (For more detail see [2, 29]). They can be treated as representing an infinite set of coupling constants in HS theory. In this paper all of these fields are truncated away.

The perturbative analysis goes as follows. Suppose that an order- n solution has been found

$$W = W_0 + W^{(n)}, \quad W^{(n)} = \sum_{k=1}^n W_k, \quad (4.4)$$

$$S = S_0 + S^{(n)}, \quad S^{(n)} = \sum_{k=1}^n S_k, \quad (4.5)$$

$$B^{(n)} = \sum_{k=1}^n B_k. \quad (4.6)$$

Then eq. (3.10) gives

$$[S_0, B_{n+1}]_* = - \sum_{k+l=n+1} [S_k, B_l]_* . \quad (4.7)$$

Using that S_0 has a trivial star-commutator with the Klein operators K , an elementary computation gives

$$[S_0, F(Y; Z; K|x)]_* = -2i d_Z F(Y; Z; K|x), \quad (4.8)$$

where

$$d_Z = \theta^A \frac{\partial}{\partial Z^A} \quad (4.9)$$

is the Z -space exterior derivative. As a result, eq. (4.7) is equivalent to

$$d_Z B_{n+1} = -\frac{i}{2} \sum_{k+l=n+1} [S_k, B_l]_* . \quad (4.10)$$

Since B is a zero-form, in the first nontrivial order this gives

$$B_1(Z; Y; K|x) = C(Y; K|x), \quad (4.11)$$

i.e., the zero-form $C(Y; K|x)$ of the free theory appears as Z -space De Rham cohomology.

Analogously, eq. (3.11) reads as

$$d_Z S_{n+1} = -\frac{1}{2} \left(i \sum_{k+l=n+1} S_k * S_l + \eta \theta^\alpha \theta_\alpha B_{n+1} * k * \kappa + \bar{\eta} \bar{\theta}^{\dot{\alpha}} \bar{\theta}_{\dot{\alpha}} B_{n+1} * \bar{k} * \bar{\kappa} \right). \quad (4.12)$$

In the lowest order this gives the equation

$$d_Z S_1 = -\frac{1}{2} \left(\eta \theta^\alpha \theta_\alpha C * k * \kappa + \bar{\eta} \bar{\theta}^{\dot{\alpha}} \bar{\theta}_{\dot{\alpha}} C * \bar{k} * \bar{\kappa} \right) \quad (4.13)$$

expressing S_1 in terms of C .

In the Z -space sector, perturbative analysis consists of solving repeatedly the equations of the form

$$d_Z f(Z; Y; K|x) = J(Z; Y; K|x). \quad (4.14)$$

Formal consistency of HS equations (3.11) and (3.10) guarantees that $J(Z; Y; K|x)$ is d_Z -closed

$$d_Z J(Z; Y; K|x) = 0, \quad (4.15)$$

implying formal consistency of eq. (4.14). However, it admits a solution only if J is d_Z -exact.

Analysis of the equations involving space-time one-forms is analogous. Firstly one resolves all equations that contain d_Z . The remaining equations in d_Z -cohomology produce dynamical equations on the dynamical fields ω and C which reproduce Central on-shell theorem (2.5), (2.6) along with all nonlinear corrections.

4.2 Homotopy trick

Let us now recall the standard homotopy trick. Let d be a differential (later on to be identified with d_Z) obeying

$$d^2 = 0 \quad (4.16)$$

as well as a homotopy operator ∂

$$\partial^2 = 0. \quad (4.17)$$

Then the operator

$$A := \{d, \partial\} \quad (4.18)$$

obeys

$$[d, A] = 0, \quad [\partial, A] = 0 \quad (4.19)$$

as a consequence of (4.16), (4.17). For diagonalizable A the standard Homotopy Lemma states that cohomology of d , denoted $H(d)$, is in the kernel of A

$$H(d) \subset \text{Ker} A. \quad (4.20)$$

In this case the projector $\hat{h}$ to $\text{Ker} A$

$$\hat{h}^2 = \hat{h} \quad (4.21)$$

can be defined to obey

$$[\hat{h}, d] = [\hat{h}, \partial] = 0. \quad (4.22)$$

Also we can introduce the operator A^* such that

$$A^* A = A A^* = Id - \hat{h}. \quad (4.23)$$

This allows us to define the *resolution operator*

$$d^* := A^* \partial = \partial A^* \quad (4.24)$$

that obeys

$$d^* d + d d^* = Id - \hat{h}, \quad (4.25)$$

which is equivalent to the resolution of identity $\{d, d^*\} + \hat{h} = Id$. This relation provides a general solution to the equation

$$d f = J \quad (4.26)$$

with d -closed J outside of $H(d)$, i.e., obeying $\hat{h} J = 0$,

$$J = d d^* J. \quad (4.27)$$

Hence

$$f = d^* J + d\epsilon + g, \quad (4.28)$$

where an exact part $d\epsilon$ and $g \in H(d)$ remain undetermined.

The simplest choice of the homotopy operator for the exterior differential $d = d_Z$ is

$$\partial = Z^A \frac{\partial}{\partial \theta^A}. \quad (4.29)$$

This gives

$$A = \theta^A \frac{\partial}{\partial \theta^A} + Z^A \frac{\partial}{\partial Z^A}, \quad (4.30)$$

$$A^* f(Z; Y; \theta) = \int_0^1 dt \frac{1}{t} f(tZ; Y; t\theta). \quad (4.31)$$

Eq. (4.23) is checked by using

$$t \frac{\partial}{\partial t} f(tx) = x \frac{\partial}{\partial x} f(tx). \quad (4.32)$$

$\text{Ker} A$ consists of Z , θ -independent functions and thus, by Poincaré lemma, relation (4.20) becomes an exact equality

$$H(d_Z) = \text{Ker} A. \quad (4.33)$$

Correspondingly,

$$\hat{h}J(Z; Y; \theta) = J(0; Y; 0), \quad (4.34)$$

while

$$d_Z^* J(Z; Y; \theta) = Z^A \frac{\partial}{\partial \theta^A} \int_0^1 dt \frac{1}{t} J(tZ; Y; t\theta). \quad (4.35)$$

In this construction, the freedom in the choice of the homotopy operator ∂ affects the homogeneous solution to equation (4.26), i.e., $d\epsilon + g$. Since dynamical fields C and ω are valued in the d_Z cohomology, this implies that going to a different ∂ may imply a field redefinition. Alternatively, one can directly redefine the resolution d^* by adding a closed form.

Though the conventional resolution operator d_Z^* looks simple and natural, it is known since [4] to lead to nonlocalities at the nonlinear order. In the next section we identify a resolution operator d_{loc}^* that leads directly to the local setup in the process of solving HS equations in the lowest nonlinear order. Its form is deduced from the results of [9]. In accordance with the general analysis the difference between the two approaches effectively results in a field redefinition associated with the difference between cohomological g -terms in the respective formulae (4.28).¹

Since d_{loc}^* gives a local result while d_Z^* leads to a nonlocal one, they should differ by nonlocal cohomological g -terms in (4.28). One can argue following the authors of [6, 8]

¹Retrospectively, it should be noted that the formulation originally found in [1] needed a field redefinition containing higher derivatives even at the free field level just because from the point of view of the later formulation of [2] it corresponded to the alternative choice of the homotopy operator $\partial^\pm = (Z \pm Y)^A \frac{\partial}{\partial \theta^A}$.

that one can equally well choose another resolution operator $d_{\text{loc}}' \neq d_{\text{loc}}^*$, that might lead to another local result. The weak point of this argument is however due to insufficient representativity of the lowest-order analysis of consistency insensitive to the specific coefficients in front of different vertices. Things do change drastically in the higher orders. As will be explained in the next section the choice of d_{loc}^* associated with the solution found in [9] is singled out by the condition that higher-order corrections remain minimally nonlocal (if nonlocal at all) which phenomenon is properly captured by the lowest-order corrections once the dependence on Z variables is taken into account. The analysis of [9] was based on the separation of variables taking into account that, in accordance with the fact that the left and right parts of d_Z form a bicomplex, in the left(right) sector the dependence on the right(left) spinors remains unchanged in the lowest order being governed by the original star product. At this condition the solution of [9] is the only one leading to local current interactions in the Y, x sector.

5 Perturbative analysis of locality

To see what distinguishes between the local frame of HS equations of [9] and other possible frames let us consider in more detail the perturbative analysis in the sector of zero-forms starting from De Rham resolution d_Z^* .

The first step is to evaluate the first correction to S , i.e., S_1 . Using (3.15) and (3.16), eq. (4.31) gives

$$S_1 = S_{1\eta} + S_{1\bar{\eta}}, \quad (5.1)$$

where

$$S_{1\eta}(Z; Y; K|x) = -\eta z^\beta \theta_\beta \int_0^1 d\tau \tau \exp(i\tau z_\alpha y^\alpha) C(-\tau z, \bar{y}; K|x) * k, \quad (5.2)$$

$$S_{1\bar{\eta}}(Z; Y; K|x) = -\bar{\eta} \bar{z}^{\dot{\beta}} \bar{\theta}_{\dot{\beta}} \int_0^1 d\tau \tau \exp(i\tau \bar{z}_{\dot{\alpha}} \bar{y}^{\dot{\alpha}}) C(y, -\tau \bar{z}; K|x) * \bar{k}. \quad (5.3)$$

Resolution of (4.10) with the homotopy operator ∂_Z (4.29) gives for B_2 in the η -sector

$$\begin{aligned} B_{2\eta} &= -\frac{i}{2} d_Z^*([S_1, C]_*) \\ &= -\frac{i}{2} \eta \int d_+^4 \tau \delta\left(1 - \sum_{i=1}^4 \tau_i\right) \int d^4 U d^4 V \exp i(U_A V^A + (1 - \tau_3) z_\alpha y^\alpha) \\ &\quad \times \left[y^\beta u_\beta \delta(\tau_1) C(\tau_3 y - \tau_1 z + (\tau_3 + \tau_4) u, \bar{y} + \bar{u}; K) C(\tau_3 y + \tau_2 z + v, \bar{y} + \bar{v}; K) \right. \\ &\quad \left. - y^\beta v_\beta \delta(\tau_2) C(\tau_3 y - \tau_1 z + u, \bar{y} + \bar{u}; K) C(\tau_3 y + \tau_2 z + (\tau_3 + \tau_4) v, \bar{y} + \bar{v}) \right] * k * \kappa, \end{aligned} \quad (5.4)$$

where we use notation

$$d_+^4 \tau := d\tau_1 d\tau_2 d\tau_3 d\tau_4 \theta(\tau_1) \theta(\tau_2) \theta(\tau_3) \theta(\tau_4), \quad \theta(\tau) = 1(0) \quad \text{if } \tau \geq 0(\tau < 0) \quad (5.5)$$

with the convention that

$$\theta(\tau) \delta(\tau) = \delta(\tau). \quad (5.6)$$

Equivalently, this expression can be written in the differential form symmetric with respect the first and second factors of C

$$B_{2\eta} = \frac{1}{2}\eta \int d_+^4 \tau \delta\left(1 - \sum_{i=1}^4 \tau_i\right) y^\beta (\delta(\tau_1)\partial_{2\beta} + \delta(\tau_2)\partial_{1\beta}) \exp(X) C(Y_1; K) C(Y_2; K) \Big|_{Y_{1,2}=0} * k * \kappa, \quad (5.7)$$

where

$$X = i(1 - \tau_3)z_\alpha y^\alpha + \tau_3 y^\alpha (\partial_{1\alpha} + \partial_{2\alpha}) + z^\alpha (\tau_2 \partial_{2\alpha} - \tau_1 \partial_{1\alpha}) + i(\tau_3 + \tau_4)\partial_{1\alpha} \partial_2^\alpha + i\bar{\partial}_{1\dot{\alpha}} \bar{\partial}_2^{\dot{\alpha}}, \quad (5.8)$$

$$\partial_{i\alpha} := \frac{\partial}{\partial y_i^\alpha}, \quad \bar{\partial}_{i\dot{\alpha}} := \frac{\partial}{\partial \bar{y}_i^{\dot{\alpha}}}. \quad (5.9)$$

It is important that since $\tau_4 \geq 0$, and hence $\tau_3 + \tau_4 \geq \tau_3$, the expansion coefficients in powers of $\partial_{1\alpha} \partial_2^\alpha$ are larger than those of $y^\alpha \partial_{i\alpha}$, which as will be explained later, indicates nonlocality.

Let us now use Schouten identity

$$z_\alpha y^\alpha \partial_{1\beta} \partial_2^\beta - z^\alpha \partial_{1\alpha} y^\beta \partial_{2\beta} + z^\alpha \partial_{2\alpha} y^\beta \partial_{1\beta} = 0 \quad (5.10)$$

expressing the fact that antisymmetrization over any three two-component indices is zero to transform $B_{2\eta}$ to a different form. To this end, we observe that

$$i\partial_{1\alpha} \partial_2^\alpha \exp(X) = \frac{\partial}{\partial \tau_4} \exp(X), \quad (5.11)$$

$$z^\alpha \partial_{1\alpha} \exp(X) = -\frac{\partial}{\partial \tau_1} \exp(X), \quad (5.12)$$

$$z^\alpha \partial_{2\alpha} \exp(X) = \frac{\partial}{\partial \tau_2} \exp(X), \quad (5.13)$$

$$\left(z_\alpha y^\alpha + i y^\alpha (\partial_{1\alpha} + \partial_{2\alpha})\right) \exp(X) = -i \left(\frac{\partial}{\partial \tau_4} - \frac{\partial}{\partial \tau_3}\right) \exp(X). \quad (5.14)$$

Using these relations and integrating by parts we obtain

$$\begin{aligned} 0 &= - \int d_+^4 \tau \delta\left(1 - \sum_{i=1}^4 \tau_i\right) \left(z_\alpha y^\alpha \partial_{1\beta} \partial_2^\beta - z^\alpha \partial_{1\alpha} y^\beta \partial_{2\beta} + z^\alpha \partial_{2\alpha} y^\beta \partial_{1\beta}\right) \exp(X) \\ &= \int d_+^4 \tau \left((i y_\alpha z^\alpha \delta(\tau_4) + y^\beta (\partial_{2\beta} \delta(\tau_1) + \partial_{1\beta} \delta(\tau_2))) \delta\left(1 - \sum_{i=1}^4 \tau_i\right) \right. \\ &\quad \left. + (\delta(\tau_3) - \delta(\tau_4)) \delta'\left(1 - \sum_{i=1}^4 \tau_i\right) \right) \exp(X). \end{aligned} \quad (5.15)$$

Comparison of this with (5.7) allows us to represent $B_{2\eta}$ in the form

$$\begin{aligned} B_{2\eta} &= -\frac{1}{2}\eta \int d_+^4 \tau \left(i y_\alpha z^\alpha \delta(\tau_4) \delta\left(1 - \sum_{i=1}^4 \tau_i\right) + (\delta(\tau_3) - \delta(\tau_4)) \delta'\left(1 - \sum_{i=1}^4 \tau_i\right) \right) \\ &\quad \times \exp(X) C(Y_1; K) C(Y_2; K) \Big|_{Y_{1,2}=0} * k * \kappa. \end{aligned} \quad (5.16)$$

The terms with $\delta(\tau_3)$ and $\delta(\tau_4)$ have different meaning. The part with $\delta(\tau_3)$ is z -independent. Indeed, denoting it $\Delta C_{2\eta}$ we obtain using (3.15) and (3.16)

$$\begin{aligned} \Delta C_{2\eta} = & -\frac{1}{2}\eta \int d_+^4 \tau \delta(\tau_3) \delta' \left(1 - \sum_{i=1}^4 \tau_i \right) \\ & \times \exp(y^\alpha (\tau_1 \partial_{1\alpha} - \tau_2 \partial_{2\alpha}) + i\tau_4 \partial_{1\alpha} \partial_2^\alpha + i\bar{\partial}_{1\dot{\alpha}} \bar{\partial}_2^{\dot{\alpha}}) C(Y_1; K) C(Y_2; K) \Big|_{Y_{1,2}=0} * k. \end{aligned} \quad (5.17)$$

Since $\Delta C_{2\eta}$ is in the d_Z cohomology, according to (4.28) we can define a new resolution operator

$$d_{\text{loc}}^* B := d_Z^* B - \Delta C_{2\eta} \quad (5.18)$$

such that

$$B_{2\eta}^{\text{loc}} = -\frac{i}{2} d_{\text{loc}}^* ([S_1, C]_*) \quad (5.19)$$

has the form

$$\begin{aligned} B_{2\eta}^{\text{loc}} = & \frac{1}{2}\eta \int d_+^3 \tau \left(\delta' \left(1 - \sum_{i=1}^3 \tau_i \right) - i y_\alpha z^\alpha \delta \left(1 - \sum_{i=1}^3 \tau_i \right) \right) \\ & \times \exp(X^{\text{loc}}) C(Y_1; K) C(Y_2; K) \Big|_{Y_{1,2}=0} * k * \kappa, \\ X^{\text{loc}} = & i(1 - \tau_3) z_\alpha y^\alpha + \tau_3 y^\alpha (\partial_{1\alpha} + \partial_{2\alpha}) + z^\alpha (\tau_2 \partial_{2\alpha} - \tau_1 \partial_{1\alpha}) + i\tau_3 \partial_{1\alpha} \partial_2^\alpha + i\bar{\partial}_{1\dot{\alpha}} \bar{\partial}_2^{\dot{\alpha}}. \end{aligned} \quad (5.20)$$

Equivalently, $B_{2\eta}^{\text{loc}}$ can be represented in the integral form

$$\begin{aligned} B_{2\eta}^{\text{loc}} = & \frac{1}{2}\eta \int d_+^3 \tau \left(\delta' \left(1 - \sum_{i=1}^3 \tau_i \right) - i y_\alpha z^\alpha \delta \left(1 - \sum_{i=1}^3 \tau_i \right) \right) \\ & \times \int d^4 U d^4 V \exp i(U_A V^A + (1 - \tau_3) z_\alpha y^\alpha) \\ & \times C(\tau_3 y - \tau_1 z + \tau_3 u, \bar{y} + \bar{u}; K) C(\tau_3 y + \tau_2 z + v, \bar{y} + \bar{v}; K) * k * \kappa. \end{aligned} \quad (5.21)$$

Formula (5.18) just describes the nonlinear shift found in [9] to reduce the nonlocal bilinear corrections to the local form directly in the sector of x, y -variables.

We observe that $B_{2\eta}^{\text{loc}}$ (5.22) has the remarkable property that the coefficient in front of the term responsible for the index contraction between the first and second factors of C equals to those in front of the y variable in the arguments of C , which is analogous to the star product of Z -independent functions.

The results of this paper prove that this is the only option consistent with locality. Indeed, we have shown that the dependence on y in the arguments of C in (5.22) contains the same dependence on the homotopy parameter as u , that determines the contractions of spinorial indices. This has to be compared with the original (non-local) source (5.5) where the coefficients in front of u (or v) determining contractions are larger than those in front of y . Clearly, since $\tau_4 \geq 0$ the solution (5.22) is strictly minimally nonlocal, having the same type of nonlocality as the original star product in y variables.

The following comments are now in order.

An important feature of $B_{2\eta}^{\text{loc}}$ (5.22) is that it contains the rightmost star-product factor $k * \kappa$. Because, by (3.15), (3.16), star product with κ exchanges y and z variables, eq. (5.22) can be equivalently rewritten in the form

$$\begin{aligned} B_{2\eta}^{\text{loc}} = & \frac{1}{2} \eta \int d_+^3 \tau \left(\delta' \left(1 - \sum_{i=1}^3 \tau_i \right) + i y_\alpha z^\alpha \delta \left(1 - \sum_{i=1}^3 \tau_i \right) \right) \\ & \times \int d^4 U d^4 V \exp i(U_A V^A + \tau_3 z_\alpha y^\alpha) \\ & \times C(\tau_1 y - \tau_3 z + \tau_3 u, \bar{y} + \bar{u}; K) C(-\tau_3 z - \tau_2 y + v, \bar{y} + \bar{v}; K) * k. \end{aligned} \quad (5.23)$$

In this form the coefficient in front of u that governs contractions is dominated by that in front of $z_\alpha y^\alpha$ in the exponential, as well as the coefficients in front of z in the arguments of C . We observe that the relevant terms disappear at $Z = 0$, not allowing to distinguish between proper and improper nonlinear corrections in attempt to analyze the issue of locality at $Z = 0$ as in [6, 8]. This feature is in agreement with the well-known fact that the overall coefficients in front of different currents are not determined by the lower-order consistency and hence one can seemingly freely go from one set of coefficients to another by a nonlocal field redefinition of the original variables which are those associated with the Y dependence at $Z = 0$. Such arguments led some of the authors of [6, 8] to claims that it is impossible to compute current vertices from the HS equations of [2]. (See e.g. [30] for such interpretation of conclusions of [8]). In fact, the meaning of the results of [8] is that the authors were using a bad luck *ad hoc* assumption that the problem can be analyzed with the help of the conventional resolution operator d_Z^* (4.35). To make a proper choice, higher-order effects have to be taken into account.

Indeed, our results imply that the setup of [9] not only leads to the local result in the first nontrivial approximation but, most significantly, it will lead to the minimally nonlocal setup in the higher orders. To see this it is important to have expressions for the bilinear corrections that account for the full dependence on both Y and Z . The computation of higher-order corrections will involve star products of the expressions like $B_{2\eta}^{\text{loc}}$ (5.22) or $B_{2\eta}$ (5.5) with $B_{2\eta}$ being more nonlocal than $B_{2\eta}^{\text{loc}}$. Also let us note that the factor of $k * \kappa$ is central and involutive. Hence it will cancel in particular the similar factor on the r.h.s. of (3.11) in the next order with the effect that no exchange of y and z variables will occur in (some of) the higher orders, i.e., the minimal order of nonlocality will be visible directly in the physical Y -space in the higher-order terms. From this perspective our approach is somewhat similar to the derivation of cubic HS vertices by Metsaev in [31] where the proper form of cubic HS vertices that precisely corresponds [12, 32] to that resulting from the application of the resolution d_{loc}^* to equations of [2] was deduced from the higher-order analysis.

One of the main results of this paper is the identification of the proper resolution operator d_{loc}^* (5.18) that directly leads to the formulation of the HS theory in the local (or minimally nonlocal in the higher orders) setup for the full system of nonlinear HS equations. So far, d_{loc}^* was found only in the lowest order of the 0-form sector. The goal

is to find its full fledged extension to all orders and all types of differential forms. This is the ongoing project [20].

Completion of the latter project will also imply the completion of the program of [5] of establishing a proper class of star-product functions associated with the minimally nonlocal setup in HS equations. Indeed, in [5] the proper dependence on Z and Y variables was established for the expressions like

$$f(Z; Y) = \int_0^1 d\tau \phi(\tau Z; (1 - \tau)Y; \tau) \exp i\tau Z_A Y^A \quad (5.24)$$

with $\phi(W; U; \tau)$ regular in W and U and integrable in τ . Being accompanied by the factor of τ and $1 - \tau$, the dependence on Z and Y on the r.h.s. of (5.24) trivializes at $\tau \rightarrow 0$ and $\tau \rightarrow 1$, respectively. In [5] the space of functions (5.24) called $\bar{V}_{0,0}$ was extended to the spaces $\bar{V}_{k,l}$ of such star-product elements (5.24) that $\phi(W; U; \tau)$ scales as τ^k at $\tau \rightarrow 0$ and $(1 - \tau)^l$ at $\tau \rightarrow 1$. More precisely, we allow (poly)logarithmic dependence on τ and $1 - \tau$ at $\tau \rightarrow 0$ and $\tau \rightarrow 1$, respectively, with the convention that it does not affect the indices k and l . In [5] $\bar{V}_{k,l}$ with both positive and negative k and/or l were considered.

The problem not considered in [5] was which restrictions on the inner structure of $\phi(W; U; \tau)$ have to be imposed to respect locality or minimal nonlocality. In particular, a question to be addressed is what is the proper dependence on τ_i in the expressions like

$$\int d^3_+ \tau \rho(\tau) \int d^2 u d^2 v \exp i(u_\alpha v^\alpha + t z_\alpha y^\alpha) C(\tau_3 y - \tau_1 z + \tau_5 u) C(\tau_4 y + \tau_2 z + v) * k * \kappa. \quad (5.25)$$

The answer combining the results of [5] with those of this paper is

$$\tau_3 \leq 1 - t, \quad \tau_4 \leq 1 - t, \quad \tau_1 \leq t, \quad \tau_2 \leq t, \quad \tau_5 \leq 1 - t, \quad (5.26)$$

where the new restriction is the last one dominating the dependence on τ_5 by that on the parameter t in front of the factor of $z_\alpha y^\alpha$ in the exponential.

Let us stress that not only solutions have to be of the form (5.25), (5.26) but also gauge transformation and field redefinitions (as explained in [5], the latter should even obey stronger conditions due to further restrictions on the measure $\rho(\tau)$). In any case, the Z -independent field redefinition (5.17) has $t = 1$ and hence does not belong to the proper class. Moreover, that $\tau_5 \leq 0$ for Z -independent functions obeying (5.26) implies that they should be distributions supported at $\tau_5 = 0$ hence being represented by a finite number of delta-function derivatives $\delta^n(\tau_5)$. In turn this means that the allowed class of Z -independent field redefinitions is genuinely local. Direct analysis of the next section also demonstrates that d_Z^* and d_{loc}^* are not related by a local field redefinition.

6 Cohomology shift and Green function

In this section we show that the cohomology shift (5.17) relating d_Z^* and d_{loc}^* is nonlocal. To this end we first recall in section 6.1 the structure of the nonlinear corrections to dynamical equations in the sector of Y variables (i.e., d_Z cohomology) resulting from d_Z^* , computing the zero-form Green function in section 6.2.

6.1 d_Z^* -induced nonlocal deformation in the zero-form sector

The lowest-order deformation of free equations (2.3), (2.4) has the form

$$d\omega + \omega * \omega + \mathcal{L}(w, \omega, C) + \Gamma_{\text{cur}}(w, J) = 0, \quad (6.1)$$

$$dC + \omega * C - C * \omega + \mathcal{H}_{\text{cur}}(w, J) = 0, \quad (6.2)$$

where $\mathcal{L}(w, \omega, C)$ is at most linear in both ω and C while the two-form $\Gamma_{\text{cur}}(w, J)$ and one-form $\mathcal{H}_{\text{cur}}(w, J)$ are some functionals of the background fields w and the current J

$$J(Y_1, Y_2; K|x) := C(Y_1; K|x)C(Y_2; K|x). \quad (6.3)$$

As a consequence of equations (2.4) on C , so defined current obeys the *current equation*

$$\left(D_L - i h^{\alpha\dot{\beta}} \left(y^1_{\alpha} \bar{y}^1_{\dot{\beta}} - y^2_{\alpha} \bar{y}^2_{\dot{\beta}} - \partial_{1\alpha} \bar{\partial}_{1\dot{\beta}} + \partial_{2\alpha} \bar{\partial}_{2\dot{\beta}} \right) \right) J(y^1, y^2; \bar{y}^1, \bar{y}^2; K|x) = 0 \quad (6.4)$$

at the convention that derivatives $\partial_{1\alpha}(\bar{\partial}_{1\dot{\alpha}})$ and $\partial_{2\alpha}(\bar{\partial}_{2\dot{\alpha}})$ over the first and second undotted(dotted) spinorial arguments of J are defined to anticommute with $k(\bar{k})$. The star product in (6.1), (6.2) results from the restriction of (3.3) to Z -independent functions.

Gauge invariant current interactions are associated with $\Gamma_{\text{cur}}(w, J)$ and $\mathcal{H}_{\text{cur}}(w, J)$ linear in J . Other ω -dependent terms bilinear in fluctuations, describe gauge non-invariant interactions which do not contribute if the spins s_1 and s_2 of two fields entering the bilinear terms on the right-hand sides of (6.1) and (6.2) and spin of the current s_J identified with the spin of the field contributing to the linear part of (6.1) and (6.2) (i.e., by its definition, the current of spin s_J contributes to the nonlinear corrections to the equations on the spin- s_J field) obey the condition

$$s_J \geq s_1 + s_2. \quad (6.5)$$

(For the derivation of (6.5) directly from eq. (6.2) see [33]).

The final result of [9] in the 0-form sector is

$$\begin{aligned} \mathcal{H}_{\text{cur}}(w, J) = & -\frac{i}{2} \int_0^1 d\tau \\ & \times \left(\eta \int \frac{d\bar{s} d\bar{t}}{(2\pi)^2} \exp i[\bar{s}_{\dot{\beta}} \bar{t}^{\dot{\beta}}] h(y, \tau \bar{s} + (1-\tau)\bar{t}) J(\tau y, -(1-\tau)y, \bar{y} + \bar{s}, \bar{y} + \bar{t}) \right. \\ & \left. + \bar{\eta} \int \frac{ds dt}{(2\pi)^2} \exp i[s_{\beta} t^{\beta}] h(\tau s - (1-\tau)t, \bar{y}) J(y + s, y + t, \tau \bar{y}, -(1-\tau)\bar{y}) \right), \end{aligned} \quad (6.6)$$

where

$$h(u, \bar{u}) = h^{\alpha\dot{\alpha}} u_{\alpha} \bar{u}_{\dot{\alpha}}. \quad (6.7)$$

This corresponds to $\mathcal{H}_{\text{cur}}(w, J)$ in (6.2) with the minimal number of derivatives which is finite for any spins s_1, s_2 and s_J .

The deformation $\mathcal{H}(w, J)$ resulting from the nonlinear equations of [2] by virtue of the resolution operator d_Z^* is [6, 9]

$$\mathcal{H}(w, J) = \mathcal{H}_{\eta}(w, J) + \mathcal{H}_{\bar{\eta}}(w, J), \quad (6.8)$$

where

$$\begin{aligned} \mathcal{H}_\eta(w, J) = & -\frac{i}{2}\eta \int \frac{dSdT}{(2\pi)^4} \exp iS_AT^A \\ & \times \int_0^1 d\tau [h(s, \tau\bar{y} - (1-\tau)\bar{t})J(\tau s, -(1-\tau)y + t; \bar{y} + \bar{s}, \bar{y} + \bar{t}; K) \\ & - h(t, \tau\bar{y} - (1-\tau)\bar{s})J((1-\tau)y + s, \tau t, \bar{y} + \bar{s}; \bar{y} + \bar{t}; K)] * k, \end{aligned} \quad (6.9)$$

$$\begin{aligned} \mathcal{H}_{\bar{\eta}}(w, J) = & -\frac{i}{2}\bar{\eta} \int \frac{dSdT}{(2\pi)^4} \exp iS_AT^A \\ & \times \int_0^1 d\tau [h(\tau y - (1-\tau)t, \bar{s})J(y + s, y + t; \tau\bar{s}, -(1-\tau)\bar{y} + \bar{t}; K) \\ & - h(\tau y - (1-\tau)s, \bar{t})J(y + s, y + t; (1-\tau)\bar{y} + \bar{s}, \tau\bar{t}; K)] \bar{*} k. \end{aligned} \quad (6.10)$$

The integration over S and T in (6.9), (6.10) brings infinite tails of contracted indices, inducing by (2.4) and (2.6) an infinite expansion in higher space-time derivatives of the constituent fields. Hence, $\mathcal{H}$ (6.9), (6.10) differs from the conventional current interactions (6.6) which, being free of the integration over s_α and t_α , contains a finite number of derivatives for any s_1, s_2 and s_J .

To reproduce standard current interactions from those resulting from d_Z^* one has to find a field redefinition

$$C \rightarrow C'(Y; K|x) = C(Y; K|x) + \Phi(Y; K|x) \quad (6.11)$$

with Φ linear in J , bringing $\mathcal{H}$ (6.8) to the form (6.6), i.e.,

$$\mathcal{H}_\eta(w, J) = D_0(\Phi_\eta) + \mathcal{H}_{\eta \text{ cur}}(w, J). \quad (6.12)$$

The proper field redefinition found in [9] is $\Phi = \Delta C_{2\eta} + \Delta C_{2\bar{\eta}}$ with $\Delta C_{2\eta}$ (5.17) and $\Delta C_{2\bar{\eta}}$ being its complex conjugate. It should be stressed that formula (6.12) is valid for any spins s_1, s_2 and σ_J independently on whether the condition (6.5) holds or not.

6.2 Green function

To discuss locality properties of the field redefinition (6.11) it is useful to consider Green function that removes the current interactions. The goal is to find a solution to the equation

$$DG_C(J) = \mathcal{H}_{\eta \text{ cur}}(w, J) + \mathcal{H}_{\bar{\eta} \text{ cur}}(w, J). \quad (6.13)$$

Since $\mathcal{H}_{\eta \text{ cur}}(w, J)$ and $\mathcal{H}_{\bar{\eta} \text{ cur}}(w, J)$ describe an arbitrary local current with certain coefficients the resulting solution $G_C(J)$ describes Green function applied to J . Let us stress that this problem setting is only applicable at the condition (6.5), when the HS connections ω do not contribute. So, we will assume that (6.5) is true. Since the problem is linear, the terms proportional to η and $\bar{\eta}$ can be found separately. We consider the part $G_\eta(J)$ linear in η . The term $G_{\bar{\eta}}(J)$ with $\bar{\eta}$ can be obtained by complex conjugation.

Let us look for $G_\eta(J)$ in the most general Lorentz covariant form

$$G_\eta(J) = \phi(N_i, \bar{N}_i, M, \bar{M})J(Y; \bar{Y}; k, \bar{k}) \Big|_{Y_1=Y_2=0}, \quad (6.14)$$

where

$$N_i = y^\alpha \partial_{i\alpha} \quad \bar{N}_i = \bar{y}^{\dot{\alpha}} \bar{\partial}_{i\dot{\alpha}}, \quad (6.15)$$

$$M = \epsilon^{\alpha\beta} \partial_{1\alpha} \partial_{2\beta} \quad \bar{M} = \epsilon^{\dot{\alpha}\dot{\beta}} \bar{\partial}_{1\dot{\alpha}} \bar{\partial}_{2\dot{\beta}}, \quad (6.16)$$

$$\partial_{i\alpha} := \frac{\partial}{\partial y_i^\alpha} \quad \bar{\partial}_{i\dot{\alpha}} := \frac{\partial}{\partial \bar{y}_i^{\dot{\alpha}}}. \quad (6.17)$$

We use convention that both y^α and y_i^α along with the respective derivatives anticommute with the Klein operator k inside J , while $\bar{y}^{\dot{\alpha}}$ and $\bar{y}_i^{\dot{\alpha}}$ anticommute with $\bar{k}$.

Using (2.4) and (6.4) it is not difficult to obtain

$$\begin{aligned} DG_\eta(J) = i\lambda h^{\alpha\dot{\beta}} \Big\{ & y_\alpha \bar{y}_{\dot{\beta}} \left(-1 + \frac{\partial^2}{\partial N_1 \partial \bar{N}_1} + \frac{\partial^2}{\partial N_2 \partial \bar{N}_2} \right) - y_\alpha \bar{\partial}_{1\dot{\beta}} \frac{\partial^2}{\partial N_2 \partial \bar{M}} \\ & + y_\alpha \bar{\partial}_{2\dot{\beta}} \frac{\partial^2}{\partial N_1 \partial \bar{M}} - \partial_{1\alpha} \bar{y}_{\dot{\beta}} \frac{\partial^2}{\partial M \partial \bar{N}_2} + \partial_{2\alpha} \bar{y}_{\dot{\beta}} \frac{\partial^2}{\partial M \partial \bar{N}_1} + \partial_{i\alpha} \bar{\partial}_{j\dot{\beta}} \frac{\partial^2}{\partial N^i \partial \bar{N}^j} \\ & - \left(\partial_{1\alpha} \partial_{1\dot{\beta}} + \partial_{2\alpha} \partial_{2\dot{\beta}} \right) \left(1 - \frac{\partial^2}{\partial M \partial \bar{M}} \right) \Big\} \phi(N_i, \bar{N}_i, M, \bar{M}) J(Y_1; Y_2; K) \Big|_{Y_1=Y_2=0}. \end{aligned} \quad (6.18)$$

To reproduce $\mathcal{H}_{\eta cur}(w, J)$ (6.8) we have to demand that all terms on the r.h.s. of (6.18) should be zero except for those containing $y_\alpha \bar{\partial}_{i\dot{\beta}}$. However, demanding this, one should take into account that antisymmetrization over any three two-component indices gives zero. This yields the relations

$$M y_\alpha + N_2 \partial_{1\alpha} - N_1 \partial_{2\alpha} = 0, \quad \bar{M} \bar{y}_{\dot{\alpha}} + \bar{N}_2 \bar{\partial}_{1\dot{\alpha}} - \bar{N}_1 \bar{\partial}_{2\dot{\alpha}} = 0 \quad (6.19)$$

allowing to add the following expression to DG_η (6.18)

$$\begin{aligned} O(J) = i\lambda h^{\alpha\dot{\beta}} [& (M y_\alpha + N_2 \partial_{1\alpha} - N_1 \partial_{2\alpha}) (\bar{a} \bar{y}_{\dot{\beta}} + \bar{b}_1 \bar{\partial}_{1\dot{\beta}} + \bar{b}_2 \bar{\partial}_{2\dot{\beta}}) \\ & + (a y_\alpha + b_1 \partial_{1\alpha} + b_2 \partial_{2\alpha}) (\bar{M} \bar{y}_{\dot{\beta}} + \bar{N}_2 \bar{\partial}_{1\dot{\beta}} - \bar{N}_1 \bar{\partial}_{2\dot{\beta}})] J(Y_1; Y_2; K) \Big|_{Y_1=Y_2=0}, \end{aligned} \quad (6.20)$$

where $a, b_i, \bar{a}$ and $\bar{b}_i$ are arbitrary functions of $N_i, M, \bar{N}_i$ and $\bar{M}$. As a result,

$$\begin{aligned} DG_\eta(J) + O &= i\lambda h^{\alpha\dot{\beta}} \Big\{ y_\alpha \bar{y}_{\dot{\beta}} \left(-\phi + \frac{\partial^2 \phi}{\partial N_1 \partial \bar{N}_1} + \frac{\partial^2 \phi}{\partial N_2 \partial \bar{N}_2} + M \bar{a} + a \bar{M} \right) \\ & + y_\alpha \bar{\partial}_{1\dot{\beta}} \left(-\frac{\partial^2 \phi}{\partial N_2 \partial \bar{M}} + M \bar{b}_1 + \bar{N}_2 a \right) + y_\alpha \bar{\partial}_{2\dot{\beta}} \left(\frac{\partial^2 \phi}{\partial N_1 \partial \bar{M}} + M \bar{b}_2 - \bar{N}_1 a \right) \\ & + \partial_{1\alpha} \bar{y}_{\dot{\beta}} \left(-\frac{\partial^2 \phi}{\partial M \partial \bar{N}_2} + \bar{M} b_1 + N_2 \bar{a} \right) + \partial_{2\alpha} \bar{y}_{\dot{\beta}} \left(\frac{\partial^2 \phi}{\partial M \partial \bar{N}_1} + \bar{M} b_2 - N_2 \bar{a} \right) \\ & + \partial_{1\alpha} \bar{\partial}_{1\dot{\beta}} \left(\frac{\partial^2 \phi}{\partial N_1 \partial \bar{N}_1} + \frac{\partial^2 \phi}{\partial M \partial \bar{M}} - \phi + N_2 \bar{b}_1 + \bar{N}_2 b_1 \right) \\ & + \partial_{2\alpha} \bar{\partial}_{2\dot{\beta}} \left(\frac{\partial^2 \phi}{\partial N_2 \partial \bar{N}_2} + \frac{\partial^2 \phi}{\partial M \partial \bar{M}} - \phi - N_1 \bar{b}_2 - \bar{N}_1 b_2 \right) \\ & + \partial_{1\alpha} \bar{\partial}_{2\dot{\beta}} \left(\frac{\partial^2 \phi}{\partial N_1 \partial \bar{N}_2} + N_2 \bar{b}_2 - \bar{N}_1 b_1 \right) \\ & + \partial_{2\alpha} \bar{\partial}_{1\dot{\beta}} \left(\frac{\partial^2 \phi}{\partial N_2 \partial \bar{N}_1} + \bar{N}_2 b_2 - N_1 \bar{b}_1 \right) \Big\} J(Y_1; Y_2; K) \Big|_{Y_1=Y_2=0}. \end{aligned} \quad (6.21)$$

Now we can solve the equation

$$DG_C(J) + O(J) = \mathcal{H}_{\eta \text{ cur}}(w, J) \quad (6.22)$$

ignoring relations (6.19). However, this equation admits a solution only at the condition (6.5) since otherwise it is simply inconsistent as long as the contribution of the HS connections ω is not taken into account.

To project the currents to appropriate helicities h_J , h_1 and h_2 , where

$$2h_J = y^\alpha \frac{\partial}{\partial y^\alpha} - \bar{y}^{\dot{\alpha}} \frac{\partial}{\partial \bar{y}^{\dot{\alpha}}}, \quad 2h_i = y_i^\alpha \frac{\partial}{\partial y_i^\alpha} - \bar{y}_i^{\dot{\alpha}} \frac{\partial}{\partial \bar{y}_i^{\dot{\alpha}}} \quad (6.23)$$

the function ϕ should be chosen appropriately. Firstly, we observe that the operators

$$x_i = N_i \bar{N}_i, \quad z := M \bar{M} \quad (6.24)$$

do not affect the helicities since they commute with the operators (6.23).

Ansatz that is consistent with condition (6.5), allowing to solve (6.22), is

$$G_{C s_1, s_2, s_J} = N_1^{n_1} N_2^{n_2} \bar{M}^t \varphi(x_i, z), \quad (6.25)$$

where, assuming for definiteness that $s_1 \geq s_2$ (the opposite case can be considered analogously)

$$n_1 = s_j + s_1 + s_2, \quad n_2 = s_j - s_1 - s_2, \quad t = s_j - s_1 + s_2. \quad (6.26)$$

This implies, in particular,

$$2s_1 \geq t \geq 2s_2 \quad (6.27)$$

and

$$h_1 = s_1, \quad h_2 = -s_2. \quad (6.28)$$

The complex conjugated Ansatz solves for $\mathcal{H}_{\bar{\eta} \text{ cur}}(w, J)$. In this paper we only consider the case of opposite helicities $h_1 h_2 \leq 0$ which is sufficient for our purposes. It would be interesting to extend the obtained results to helicities of equal signs as well. In particular, this is useful for the analysis of holography along the lines of [13].

It is also convenient to use the following Ansatz

$$a(N, M) = N_1^{n_1} N_2^{n_2} \bar{M}^{t-1} \alpha(x, z) \quad (6.29)$$

$$\bar{a}(N, M) = N_1^{n_1} N_2^{n_2} \bar{M}^{t+1} \bar{\alpha}(x, z) \quad (6.30)$$

$$b_1(N, M) = N_1^{n_1} N_2^{n_2+1} \bar{M}^t \beta_1(x, z) \quad (6.31)$$

$$b_2(N, M) = N_1^{n_1+1} N_2^{n_2} \bar{M}^t \beta_2(x, z) \quad (6.32)$$

$$\bar{b}_1(N, M) = N_1^{n_1} N_2^{n_2+1} \bar{M}^t \bar{\beta}_1(x, z) \quad (6.33)$$

$$\bar{b}_2(N, M) = N_1^{n_1-1} N_2^{n_2} \bar{M}^t \bar{\beta}_2(x, z) \quad (6.34)$$

Plugging this into (6.22) gives the following system of differential equations in the variables x_i and z

$$-\varphi + (n_1 + 1 + x_1 \partial_{x_1}) \partial_{x_1} \varphi + (n_2 + 1 + x_2 \partial_{x_2}) \partial_{x_2} \varphi + \alpha - z \bar{\alpha} = 0, \quad (6.35)$$

$$(n_2 + x_2 \partial_{x_2})(t + z \partial_z) \varphi - z \bar{\beta}_1 - x_2 \alpha = \psi_1(x), \quad (6.36)$$

$$(n_1 + x_1 \partial_{x_1})(t + z \partial_z) \varphi - z \bar{\beta}_2 - x_1 \alpha = \psi_2(x), \quad (6.37)$$

$$\partial_{x_2} \partial_z \varphi + \beta_1 + \bar{\alpha} = 0, \quad (6.38)$$

$$\partial_{x_1} \partial_z \varphi - \beta_2 + \bar{\alpha} = 0, \quad (6.39)$$

$$(n_1 + 1 + x_1 \partial_{x_1}) \partial_{x_1} \varphi - \varphi + (t + 1 + z \partial_z) \partial_z + \bar{\beta}_1 - x_2 \beta_1 = 0, \quad (6.40)$$

$$(n_2 + 1 + x_2 \partial_{x_2}) \partial_{x_2} \varphi - \varphi + (t + 1 + z \partial_z) \partial_z + \bar{\beta}_2 + x_1 \beta_2 = 0, \quad (6.41)$$

$$(n_1 + x_1 \partial_{x_1}) \partial_{x_2} - \bar{\beta}_2 + x_1 \beta_1 = 0, \quad (6.42)$$

$$(n_2 + x_2 \partial_{x_2}) \partial_{x_1} - \bar{\beta}_1 - x_2 \beta_2 = 0, \quad (6.43)$$

where

$$\partial_{x_i} := \frac{\partial}{\partial x_i}, \quad \partial_z := \frac{\partial}{\partial z}. \quad (6.44)$$

Here $\psi_1(x)$ are functions of x_i which should reproduce $\mathcal{H}_{\eta \text{ cur}}(w, J)$ (6.8). $\psi_1(x)$ are demanded to be independent of z since, containing no integration over s and t $\mathcal{H}_{\eta \text{ cur}}(w, J)$ (6.8) contains no contractions of undotted indices. In appendix we shall see that it is enough to demand this to derive the proper form of $\psi_1(x)$ that corresponds to $\mathcal{H}_{\eta \text{ cur}}(w, J)$ by solving the system (6.35)–(6.43). The final result is

$$\begin{aligned} G_\eta(J) = & \eta \sum_{r_1, 2, \bar{r}_1, 2, p, \bar{p}=0}^{\infty} \theta(r_1 - \bar{r}_1) \theta(r_2 - \bar{r}_2) \theta(\bar{p} - p) \theta(r_1 - \bar{r}_1 - \bar{p} + p) \theta(\bar{p} - p - r_2 + \bar{r}_2) \\ & \times \frac{i^{\bar{p}-p+1}}{\bar{r}_1! \bar{r}_2! \bar{p}! (r_1 + r_2 + p + 1)!} N_1^{r_1} N_2^{r_2} \bar{N}_1^{\bar{r}_1} \bar{N}_2^{\bar{r}_2} M^p \bar{M}^{\bar{p}} J(Y_1, Y_2; k, \bar{k}) \Big|_{Y_1=Y_2=0}. \end{aligned} \quad (6.45)$$

On the other hand, using the generalized beta-function formula [12]

$$\int d\tau^m \delta^{(k)} \left(1 - \sum_{i=1}^m \tau_i \right) \prod_{i=1}^m \theta(\tau_i) \tau_i^{n_i} = \frac{\prod_{i=1}^m n_i!}{\left(\sum_{i=1}^m n_i + m - 1 - k \right)!}, \quad \forall n_i, k \geq 0 \quad (6.46)$$

the field redefinition (5.18) can be rewritten in the form

$$\begin{aligned} \Phi_\eta(J) = & \eta \sum_{r_1, 2, \bar{r}_1, 2, p, \bar{p}=0}^{\infty} i^{\bar{p}-p+1} \frac{1}{\bar{r}_1! \bar{r}_2! \bar{p}! (r_1 + r_2 + p + 1)!} \\ & \times N_1^{r_1} N_2^{r_2} \bar{N}_1^{\bar{r}_1} \bar{N}_2^{\bar{r}_2} M^p \bar{M}^{\bar{p}} J(Y_1, Y_2; k, \bar{k}) \Big|_{Y_1=Y_2=0}. \end{aligned} \quad (6.47)$$

We observe that the Green function (6.45) differs from the field redefinition (5.18) only by the factors of θ restricting the field redefinition to the region (6.5), (6.26), (6.27).

The fact that in the allowed region of helicities the Green function (6.45) and field redefinition (6.47) coincide is not accidental. It is a consequence of the property that, as is easy to see, the contribution (6.9) resulting from application of the resolution d_Z^* to the nonlinear equations is zero in this sector. Hence, in this sector, the field redefinition (6.47) must have the form of the Green function applied to the resulting local current. This means that the field redefinition (5.18) is essentially nonlocal, i.e., in agreement with the conclusions of section 5 the resolution operators d_Z^* and d_{loc}^* are locally nonequivalent.

7 Discussion

We have identified the proper resolution operator in the space of spinorial Z^A -variables, that leads to local first nonlinear correction to HS equations. It is shown to correspond to certain class of functions of the type identified in [5] extended to the terms accounting contractions between indices of the field product factors. As shown in [13], the current interactions of [5] resulting from this resolution operator properly reproduce the anticipated HS holographic results.

It should be stressed that the distinguished role of the resolution d_Z^* becomes manifest only if the dependence on Z^A affecting the higher-order nonlinear corrections to HS equations is taken into account. It is however less obvious at $Z^A = 0$ in which case relevant terms in the deformation trivialize.

Naive interpretation of our results might be that the field redefinition (6.47) relating the simplest resolution operator d_Z^* in the HS theory to d_{loc}^* is not allowed, being nonlocal. The proper interpretation however is just opposite: to reach a minimally nonlocal setup in HS theory (which is fully local to the order in question), one has to use the resolution d_{loc}^* with no reference to d_Z^* at all. It is d_{loc}^* that fulfils locality compatible boundary conditions in the process of solving HS equations with respect to Z^A -variables. Hence, the proper interpretation is that d_Z^* is related to the local resolution d_{loc}^* by a nonlocal field redefinition making the d_Z^* setup improper from the locality perspective.

This raises the question of the proper definition of the resolution operator d_{loc}^* at the higher orders and its extension to the sector of one-forms. These issues will be considered in [20]. The analysis of higher orders is also interesting in the context of conclusions of the recent paper [30] claiming that the level of nonlocality in HS gauge theories is somewhat extreme. This remains to be analyzed carefully from several perspectives, however.

One is that the space-time derivatives in AdS do not commute, having the commutator of order one in dimensionless units $[\lambda^{-1}D, \lambda^{-1}D] \sim 1$. This raises the question in which ordering prescription the properties of the functions of the covariant D'Alambertian $f(\square)$ are analyzed. As is well known, going from one ordering to another may significantly affect analytic properties of the function in question. For instance, being exponentials in the HS star product (3.3), inner Klein operators have a form of distributions in the Moyal-Weyl star product [34] which property in fact highlights the distinguished role of the HS star product (3.3) in the HS gauge theory.

Another is that in presence of an infinite tower of HS states even local field redefinitions at the level of quadratic corrections may induce nonlocal contributions at higher orders.

This phenomenon has to be properly taken into account in the locality analysis of the cubic corrections to the HS field equations.

Also it should be stressed that the relation between the form of local corrections in terms of spinor Y^A -variables and that in terms of space-time derivatives via (2.6) acquires nonlinear corrections. As a result, analysis of the problem in terms of spinors may have much simpler form, simultaneously providing a distinguished ordering prescription mentioned above.

Finally, as a byproduct of our consideration we have found explicit expression for the zero-form Green function in the case of the constituent fields of opposite helicity signs. It would be interesting to extend these results to the helicities of the same sign as well as to the sector of one-forms.

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8 Technical details of the Green function derivation

The system (6.35)–(6.43) contains nine equations on seven arbitrary functions α , β_i , $\bar{\alpha}$, $\bar{\beta}_i$ and φ of x_i and z . In fact, only five independent combinations of α , β_i , $\bar{\alpha}$, $\bar{\beta}_i$ enter the system since O (6.20) is invariant under the following transformations

$$a \rightarrow a + Mu, \quad b_1 \rightarrow b_1 + N_2 u, \quad b_2 \rightarrow b_2 - N_1 u \quad (8.1)$$

$$\bar{a} \rightarrow \bar{a} - \bar{M}u, \quad \bar{b}_1 \rightarrow \bar{b}_1 - \bar{N}_2 u, \quad \bar{b}_2 \rightarrow \bar{b}_2 + \bar{N}_1 u \quad (8.2)$$

with arbitrary u .

Solving equations (6.35), (6.38), (6.39), (6.40) and (6.41) one obtains

$$\alpha - z\bar{\alpha} = \varphi - (n_1 + 1 + x_1 \partial_{x_1}) \partial_{x_1} \varphi - (n_2 + 1 + x_2 \partial_{x_2}) \partial_{x_2} \varphi = 0, \quad (8.3)$$

$$\beta_1 = -\bar{\alpha} - \partial_{x_2} \partial_z \varphi, \quad \beta_2 = \bar{\alpha} + \partial_{x_1} \partial_z \varphi, \quad (8.4)$$

$$\bar{\beta}_1 = \varphi - (n_1 + 1 + x_1 \partial_{x_1}) \partial_{x_1} \varphi - (t + 1 + z \partial_z) \partial_z \varphi - x_2 \bar{\alpha} - x_2 \partial_{x_2} \partial_z \varphi, \quad (8.5)$$

$$\bar{\beta}_2 = \varphi - (n_2 + 1 + x_2 \partial_{x_2}) \partial_{x_2} \varphi - (t + 1 + z \partial_z) \partial_z \varphi - x_1 \bar{\alpha} - x_1 \partial_{x_1} \partial_z \varphi. \quad (8.6)$$

Plugging this into the remaining equations gives four equations on φ while, as anticipated, the dependence on $\bar{\alpha}$ drops out. Namely, eqs. (6.36), (6.37) yield

$$(n_2 + x_2 \partial_{x_2} + z \partial_z)(t + x_2 \partial_{x_2} + z \partial_z) \varphi - (z + x_2)(\varphi - (n_1 + x_1 \partial_{x_1} + 1) \partial_{x_1} \varphi) = \psi_1(x), \quad (8.7)$$

$$(n_1 + x_1 \partial_{x_1} + z \partial_z)(t + x_1 \partial_{x_1} + z \partial_z) \varphi - (z + x_1)(\varphi - (n_2 + x_2 \partial_{x_2} + 1) \partial_{x_2} \varphi) = \psi_2(x), \quad (8.8)$$

while eqs. (6.42) and (6.43) yield

$$\varphi - (n_1 + n_2 + x_1 \partial_{x_1} + x_2 \partial_{x_2} + 1) \partial_{x_2} \varphi - (t + z \partial_z + 1) \partial_z \varphi + x_1 (\partial_{x_2} - \partial_{x_1}) \partial_z \varphi = 0, \quad (8.9)$$

$$\varphi - (n_1 + n_2 + x_1 \partial_{x_1} + x_2 \partial_{x_2} + 1) \partial_{x_1} \varphi - (t + z \partial_z + 1) \partial_z \varphi + x_2 (\partial_{x_1} - \partial_{x_2}) \partial_z \varphi = 0. \quad (8.10)$$

Now we are in a position to solve these equations for φ which is not hard because the system (8.7)–(8.10) is largely overdetermined.

The difference between (8.9) and (8.10) gives

$$(n_1 + n_2 + x_1 \partial_{x_1} + x_2 \partial_{x_2} - (x_1 + x_2) + 1) (\partial_{x_1} - \partial_{x_2}) \varphi(x_1, x_2, z) = 0. \quad (8.11)$$

For $\varphi(x_1, x_2, z)$ expandable in power series of x_i and z this implies that

$$\varphi(x_1, x_2, z) = \tilde{\varphi}(x, z), \quad x := x_1 + x_2. \quad (8.12)$$

Plugging this back into (8.9) yields

$$\tilde{\varphi} - (n_1 + n_2 + x \partial_x + 1) \partial_x \tilde{\varphi} - (t + z \partial_z + 1) \partial_z \tilde{\varphi} = 0. \quad (8.13)$$

This equation gives

$$\tilde{\varphi} = \frac{1}{(n_1 + n_2 + x \partial_x)! (t + z \partial_z)!} e^x \chi(w), \quad w = x - z, \quad (8.14)$$

where $\chi(w)$ is an arbitrary function of a single variable. Plugging this into (8.7), (8.8) one finds that the necessary condition, that the left-hand sides of these equations are z -independent, is

$$\partial_w [(n_1 + n_2 + w \partial_w + w + 1) \chi(w)] = 0. \quad (8.15)$$

This equation is solved by

$$\chi(w) = \chi_0 \frac{(w \partial_w)!}{(n_1 + n_2 + w \partial_w + 1)!} e^{-w}, \quad (8.16)$$

where χ_0 is a constant. This yields

$$(n_1 + n_2 + w \partial_w + w + 1) \chi(w) = \frac{1}{(n_1 + n_2)!} \chi_0. \quad (8.17)$$

Finally, plugging (8.16) into equations (8.7) and (8.8) after some transformations one finds that they are indeed solved provided that

$$\psi_1 = \chi_0 \frac{(n_2 + x_2 \partial_{x_2})}{(t-1)! (n_1 + n_2 + x \partial_x + 1)! (n_1 + n_2)!} e^x, \quad (8.18)$$

$$\psi_2 = \chi_0 \frac{(n_2 + x_1 \partial_{x_1})}{(t-1)! (n_1 + n_2 + x \partial_x + 1)! (n_1 + n_2)!} e^x. \quad (8.19)$$

These reproduce (6.8) provided that

$$\chi_0 = i^{t+1} \eta (n_1 + n_2)! . \quad (8.20)$$

This determines the Green's function in the form (6.14), (6.25) with

$$\begin{aligned} \varphi(x_1, x_2, z) &= i^{t+1} \eta \frac{(n_1 + n_2)!}{(n_1 + n_2 + x \partial_x)!} \exp x \\ &\times \frac{(x \partial_x + z \partial_z)!}{(n_1 + n_2 + x \partial_x + z \partial_z + 1)! (t + z \partial_z)!} \exp(z - x) . \end{aligned} \quad (8.21)$$

Using that

$$\int_0^1 d\tau \tau^n (1 - \tau)^m = \frac{n! m!}{(n + m + 1)!} \quad (8.22)$$

this expression can be further evaluated as

$$\begin{aligned} \varphi(x_1, x_2, z) &= i^{t+1} \eta \int_0^1 d\tau \tau^{n_1 + n_2} \exp x \frac{1}{(n_1 + n_2 + x \partial_x)! (t + z \partial_z)!} \exp(1 - \tau)(z - x) \\ &= i^{t+1} \eta \int_0^1 d\tau \tau^{n_1 + n_2} \frac{1}{(n_1 + n_2 + x \partial_x)! (t + z \partial_z)!} \exp(\tau x + (1 - \tau)z) . \end{aligned} \quad (8.23)$$

Note that if conditions (6.26) are not true, equations (6.35)–(6.43) admit no polynomial solutions at all because some of involved factorials will diverge, i.e., the obtained formulae hold only in the chosen area of helicities. Plugging this expression into (6.25) we obtain (6.45). This does not mean however that the Green function cannot be constructed in the case of helicities h_1 and h_2 of the same sign in which case being formally nonpolynomial function of its arguments, the proper solution is anticipated to be regular in the allowed region of spins (6.5).

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