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Three-Higgs-doublet model under $A4$ symmetry implies alignment

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Abstract: A model with three scalar doublets can be conveniently accommodated within an $A4$ symmetric framework. The $A4$ symmetry permits only a restricted form for the scalar potential. We show that for the global minima of this potential alignment follows as a natural consequence. We also verify that in every case positivity and unitarity constraints are satisfactorily met.

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1 Introduction

The discovery at the LHC in 2012 of a spin-0 particle of mass around 125 GeV [1, 2] with properties closely matching that of the Standard Model (SM) Higgs boson is a major vindication of our understanding of the elementary particle properties. This colour neutral particle arises from a multiplet which transforms under the $SU(2)_L$ symmetry as a doublet. This is precisely what is required to generate masses for quarks and leptons and the gauge bosons while keeping $M_W/M_Z = \cos\theta_W$ in agreement with observations.

In spite of this success, the scalar sector of the Standard Model still retains several directions which merit investigation. A much discussed issue is that of naturalness, namely, there is no obvious reason for the protection of a light Higgs scalar mass. Different directions of addressing this impasse such as supersymmetry, compositeness, extra dimensions, clockwork, etc. have been under examination and experimental tests for these alternatives are being pursued with vigour.[1]

[1] There are other shortcomings of the Standard Model such as massless neutrinos and the lack of a dark matter candidate. Our focus in this work will be restricted to the scalar sector.

Besides naturalness there is also the question of minimality of the scalar field content. Is there only one scalar doublet as postulated in the Standard Model? Even though one scalar doublet serves most purposes rather well could there be in addition further scalar multiplets transforming under $\mathrm{SU}(2)_L$ either as doublets or as other representations? The simplest extension could be the addition of $\mathrm{SU}(2)_L$ singlet scalars.[2] Alternatively, seesaw models of neutrino mass of the Type-II variety rely on the introduction of an $\mathrm{SU}(2)_L$ triplet scalar multiplet [4]. But by far the most attention has been devoted to multi-doublet models among which justifiably the simplest two-Higgs-doublet model has been covered most extensively [5]. A specially important sub-class of these are the supersymmetric models which necessitate two $\mathrm{SU}(2)_L$ doublets with rather specific couplings. Models with n-Higgs doublets with $n > 2$ have also been under study [6–8].

In this work we consider a model in which there are three scalar doublets which transform as a triplet under the discrete symmetry $A4$. Models with three Higgs doublets have been of interest in their own right and have been examined from various angles [9–11]. The scalar spectrum of such models with possible discrete and continuous symmetries have been investigated in [12] while the potential minima and CP-violation options have been examined in [13, 14]. For a three-Higgs-doublet model with $S3$ symmetry novel scalar decays [15], the spectrum of the scalar sector and its consequences [16], and the high energy behaviour of the potential [17] have been explored. $A4$ as a flavour symmetry for lepton and quark masses was first considered in [18] who introduced three scalar doublets transforming as a triplet of the $A4$ group and wrote down the most general potential consistent with the symmetry. They showed that a choice of the symmetry breaking where the vacuum expectation value (vev) for the three doublets were equal led to a lepton mass model with attractive features. Closer to the spirit of this work, the authors in [19] also consider the same model with three scalar doublets transforming as an $A4$ triplet. They extract the scalar mass spectra for different vacuum expectation value patterns, to which our calculations agree, and examine their implications on gauge boson decays and on oblique corrections. Our primary focus in this work is different; it is to establish the “alignment” feature as discussed below. Another work with the same particle content but with soft symmetry breaking terms has been the subject of [20]. A model with several $A4$-triplet scalars and additional discrete symmetries has also been studied [21].

We consider a model with one $A4$-triplet consisting of $\mathrm{SU}(2)_L$-doublet scalars. We do not allow any soft $A4$ breaking terms. Demanding that the scalar potential respects $A4$ symmetry imposes restrictions on the allowed terms and relates them. We find that for all global minima of the potential these relations automatically imply vacuum alignment without any fine-tuning whatsoever. In every case in the mass eigenstate basis of the scalar fields, the so-called ‘Higgs basis’, the vacuum expectation value is restricted to only one of the three multiplets [22]. This multiplet has a massive neutral scalar, i.e., the SM Higgs boson analogue, and a massless neutral and a massless charged scalar, the Goldstone modes.[3] The other mass eigenstate scalars, all of non-zero mass, are superpositions of the

[2] There are a large number of recent papers on this subject. For an early view see, for example, ref. [3].

[3] Fermion masses, which are beyond the scope of this paper, also arise from their coupling to this multiplet.

remaining two scalar $\mathrm{SU}(2)_L$ doublets, with their exact composition varying case by case. We discuss the consequences on the model from requiring positivity of the potential and also demanding that tree-level s-wave unitarity be satisfied.

We stress here that this is at best only a toy model. Since the magnitude of the effective vev, v, is controlled by the gauge boson masses and it is the only mass parameter in the model, all scalars end up with either vanishing mass (the Goldstone states) or have mass $\mathcal{O}(v)$. Realistic models which incorporate quark and lepton masses usually have a richer scalar sector [18, 23–25].

In the next section we briefly review the $A4$ symmetry group. In the following section we write down the $A4$-symmetric scalar potential of the three-doublet model. The physics consequences of this model are presented in the two next sections where we discuss the scalar masses and alignment and the bounds arising from positivity and unitarity. We end with our conclusions and discussions.

2 The $A4$ group

The discrete group $A4$ comprises of twelve elements corresponding to the even permutations of four objects. Two basic permutations S and T which satisfy $S^2 = T^3 = (ST)^3 = \mathbb{I}$ and their nontrivial products generate $A4$. The inequivalent irreducible representations are four in number; one of 3 dimension and three of 1 dimension which we denote by $1, 1'$ and $1''$. The latter are singlets under S and transform under T as 1, ω, and ω^2 respectively, where ω is a complex cube root of unity. The one-dimensional representations satisfy

$$1' \times 1'' = 1\,. \tag{2.1}$$

For the remaining representation of dimension 3 one has

$$S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}\,. \tag{2.2}$$

As is seen from the above, in this basis the generator S is diagonal. We will use this basis. It is noteworthy that in the literature a basis in which the generator T is diagonal (with eigenvalues $1, \omega, \omega^2$) has also been used. The two bases are related by a unitary transformation by:

$$U_3 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{pmatrix}\,. \tag{2.3}$$

This matrix will reappear in our discussions later.

For the 3-dimensional representation the product rule is

$$3 \otimes 3 = 1 \oplus 1' \oplus 1'' \oplus 3 \oplus 3\,. \tag{2.4}$$

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The triplets $3_c, 3_d$ arising from the product of two triplets $3_a \equiv a_i$ and $3_b \equiv b_i$, where $i = 1, 2, 3$, can be represented as

$$c_i = (a_2 b_3, a_3 b_1, a_1 b_2) \;\text{ and }\; d_i = (a_3 b_2, a_1 b_3, a_2 b_1)\,. \tag{2.5}$$

In the same notation the other representations in the $3_a \otimes 3_b$ product are:

$$1 = a_1 b_1 + a_2 b_2 + a_3 b_3 \;\;,\;\; 1' = a_1 b_1 + \omega^2 a_2 b_2 + \omega a_3 b_3 \;\;,\;\; 1'' = a_1 b_1 + \omega a_2 b_2 + \omega^2 a_3 b_3\,. \tag{2.6}$$

More details of the $A4$ group can be found in [18, 23, 24].

Models of quarks and leptons based on $A4$ as a flavour symmetry group have been widely examined. Typical examples of such applications for the issue of neutrino masses can be found in [18, 23–25][4] and those for quark masses in [27–31].

3 The $A4$-symmetric scalar sector

We consider here a model where there is one scalar multiplet which transforms as an $A4$ triplet. The components of this muliplet are colour neutral and under the electroweak symmetry transform as three $\mathrm{SU}(2)_L$ doublets each with hypercharge $Y = 1$. We represent this collection of scalars as:

$$\Phi \equiv \begin{pmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \end{pmatrix} \equiv \begin{pmatrix} \phi_1^+ & \phi_1^0 \\ \phi_2^+ & \phi_2^0 \\ \phi_3^+ & \phi_3^0 \end{pmatrix} \;\;, \tag{3.1}$$

where the $\mathrm{SU}(2)_L$ symmetry acts horizontally while the $A4$ transformations do so vertically. We decompose the neutral fields into the scalar and pseudoscalar components: $\phi_i^0 = \phi_i + i\chi_i$.

Our objective is to explicitly show that alignment holds for the vev which have been identified as the global minima of the potential. This implies [22] that there exists a unitary transformation U such that if

$$U \begin{pmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \end{pmatrix} = \Psi \equiv \begin{pmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \end{pmatrix} \equiv \begin{pmatrix} \psi_1^+ & \psi_1^0 \\ \psi_2^+ & \psi_2^0 \\ \psi_3^+ & \psi_3^0 \end{pmatrix} \;\;, \tag{3.2}$$

then in the Ψ basis the vev is restricted to only one component, $\langle \psi_i^0 \rangle \neq 0$ and $\langle \psi_j^0 \rangle = 0$ for $j \neq i$. At the same time, the members of Ψ_i, namely, ψ_i^+ and $\psi_i^0 \equiv \eta_i^0 + i\xi_i^0$, are mass eigenstates with a massive neutral state and one massless neutral state along with a massless charged state. The other mass eigenstates are superpositions of the remaining states $\Psi_j \;\; (j \neq i)$. For most purposes Ψ_i mimics the Standard Model Higgs scalar doublet.

[4] For a review see for example ref. [26].

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3.1 The scalar potential

We will express the potential in terms of the components Φ_i, each of which is an $\mathrm{SU}(2)_L$ doublet scalar multiplet. $A4$-symmetry obviously implies a unique quadratic term, i.e., the same mass term for all components. No cubic terms are permitted by the electroweak symmetry. Turning now to the quartics it is useful to consider first the product of the triplet with itself and then the product of two such combinations. According to eq. (2.4) the product of two $A4$ triplets can give rise to two triplets (3_c and 3_d in eq. (2.5)) besides a 1, a $1'$, and a $1''$. Out of these, in the quartic term the two singlets together form a singlet as do the $1'$ with the $1''$ — see eq. (2.1). Two triplets can form a singlet but out of the four possibilities arising from 3_c and 3_d only two are independent. These are all the quartic terms allowed by the gauge and discrete symmetry. The potential in terms of components is then [18]:

$$\begin{aligned}V(\Phi_i) = m^2\left(\sum_{i=1}^{3}\Phi_i^\dagger\Phi_i\right) + \frac{\lambda_1}{2}\left(\sum_{i=1}^{3}\Phi_i^\dagger\Phi_i\right)^2 \\ +\frac{\lambda_2}{2}\left(\Phi_1^\dagger\Phi_1+\omega^2\Phi_2^\dagger\Phi_2+\omega\Phi_3^\dagger\Phi_3\right)\left(\Phi_1^\dagger\Phi_1+\omega\Phi_2^\dagger\Phi_2+\omega^2\Phi_3^\dagger\Phi_3\right) \\ +\frac{\lambda_3}{2}\left[\left(\Phi_1^\dagger\Phi_2\right)\left(\Phi_2^\dagger\Phi_1\right)+\left(\Phi_2^\dagger\Phi_3\right)\left(\Phi_3^\dagger\Phi_2\right)+\left(\Phi_3^\dagger\Phi_1\right)\left(\Phi_1^\dagger\Phi_3\right)\right] \\ +\lambda_4\left[\left(\Phi_1^\dagger\Phi_2\right)^2+\left(\Phi_2^\dagger\Phi_1\right)^2+\left(\Phi_2^\dagger\Phi_3\right)^2+\left(\Phi_3^\dagger\Phi_2\right)^2+\left(\Phi_3^\dagger\Phi_1\right)^2+\left(\Phi_1^\dagger\Phi_3\right)^2\right].\end{aligned} \tag{3.3}$$

We take all λ_i $(i = 1, \ldots 4)$ to be real. In general, only λ_4 can be complex. We comment, in passing, on the impact of this option.

We can rewrite the second term by using the property $1 + \omega + \omega^2 = 0$ to get:

$$\begin{aligned}V(\Phi_i) = m^2\left(\sum_{i=1}^{3}\Phi_i^\dagger\Phi_i\right) + \frac{\lambda_1+\lambda_2}{2}\left(\sum_{i=1}^{3}\Phi_i^\dagger\Phi_i\right)^2 \\ -\frac{3\lambda_2}{2}\left[\left(\Phi_1^\dagger\Phi_1\right)\left(\Phi_2^\dagger\Phi_2\right)+\left(\Phi_2^\dagger\Phi_2\right)\left(\Phi_3^\dagger\Phi_3\right)+\left(\Phi_3^\dagger\Phi_3\right)\left(\Phi_1^\dagger\Phi_1\right)\right] \\ +\frac{\lambda_3}{2}\left[\left(\Phi_1^\dagger\Phi_2\right)\left(\Phi_2^\dagger\Phi_1\right)+\left(\Phi_2^\dagger\Phi_3\right)\left(\Phi_3^\dagger\Phi_2\right)+\left(\Phi_3^\dagger\Phi_1\right)\left(\Phi_1^\dagger\Phi_3\right)\right] \\ +\lambda_4\left[\left(\Phi_1^\dagger\Phi_2\right)^2+\left(\Phi_2^\dagger\Phi_1\right)^2+\left(\Phi_2^\dagger\Phi_3\right)^2+\left(\Phi_3^\dagger\Phi_2\right)^2+\left(\Phi_3^\dagger\Phi_1\right)^2+\left(\Phi_1^\dagger\Phi_3\right)^2\right].\end{aligned} \tag{3.4}$$

We will be using this form in the subsequent calculations.

4 The four alternative global minima

For spontaneous symmetry breaking the neutral scalar fields in Φ develop vacuum expectation values. The following alternatives have been shown to be the only possible global

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minima of the potential [32, 33] and have commonly appeared in the literature:[5]

$$\langle\Phi\rangle_1 = \frac{v}{\sqrt{2}}\begin{pmatrix}0&1\\0&0\\0&0\end{pmatrix},\ \langle\Phi\rangle_2 = \frac{v}{2}\begin{pmatrix}0&1\\0&1\\0&0\end{pmatrix},\ \langle\Phi\rangle_3 = \frac{v}{\sqrt{6}}\begin{pmatrix}0&1\\0&1\\0&1\end{pmatrix},\ \langle\Phi\rangle_4 = \frac{v}{\sqrt{6}}\begin{pmatrix}0&1\\0&\omega\\0&\omega^2\end{pmatrix}. \tag{4.1}$$

Here $v = v_{SM} \sim 246\,\text{GeV}$. We examine each of these options in turn. We determine the condition under which any particular minimum arises from eq. (3.4) and then work out the mass matrices of the physical scalars that emerge. For this we use the following notation:

$$\mathcal{L}_{\text{mass}} = \frac{1}{2}(\chi_1\ \chi_2\ \chi_3)\ M^2_{\chi_i\chi_j}\begin{pmatrix}\chi_1\\\chi_2\\\chi_3\end{pmatrix} + \frac{1}{2}(\phi_1\ \phi_2\ \phi_3)\ M^2_{\phi_i\phi_j}\begin{pmatrix}\phi_1\\\phi_2\\\phi_3\end{pmatrix} + (\phi_1^-\ \phi_2^-\ \phi_3^-)\ M^2_{\phi_i^\mp\phi_j^\pm}\begin{pmatrix}\phi_1^+\\\phi_2^+\\\phi_3^+\end{pmatrix}. \tag{4.2}$$

In every case we verify that alignment is a consequence.

4.1 Case 1: $\langle\phi_i^0\rangle = \frac{v}{\sqrt{2}}(1,0,0)$

We begin with the case where $\langle\phi_1^0\rangle = \frac{v}{\sqrt{2}}$ and $\langle\phi_2^0\rangle = \langle\phi_3^0\rangle = 0$, i.e.,

$$\langle\Phi\rangle_1 = \frac{v}{\sqrt{2}}\begin{pmatrix}0&1\\0&0\\0&0\end{pmatrix}. \tag{4.3}$$

In this case alignment will be true if the components of Φ_1 be mass eigenstates, of which the charged and a neutral scalar become Goldstone modes of zero mass. We show that this is indeed the case.

From eq. (3.4) we find that the minimisation condition that must be satisfied for the vev in eq. (4.3) is:

$$m^2 + \frac{v^2}{2}\left[\lambda_1 + \lambda_2\right] = 0\,. \tag{4.4}$$

Using this condtion and the full scalar potential in eq. (3.4) we can find the mass matrices for the charged scalars ($\phi_i^\pm$) and the neutral scalars (ϕ_i) and pseudoscalars (χ_i). The ij-th off-diagonal entry of any mass matrix depends on the combination v_iv_j and since in this case $v_2 = v_3 = 0$ the mass matrices are all diagonal.

For the charged scalar sector the mass-squared matrix is:

$$M^2_{\phi_i^\mp\phi_j^\pm} = diag(0\,,\ r_+\,,\ r_+)\ \text{ where }\ r_+ = \frac{v^2}{4}\,(-3\lambda_2)\,. \tag{4.5}$$

[5]If λ_4 is complex then a more general form $\langle\Phi\rangle_2 = \frac{v}{2}\begin{pmatrix}0&1\\0&e^{i\alpha}\\0&0\end{pmatrix}$ is possible, where $\sin 2\alpha \propto Im(\lambda_4)$ as discussed in the appendix.

The Goldstone state $\phi_1^\pm$ becomes the longitudinal mode of the charged gauge boson. The mass-squareds of the two remaining degenerate states will be positive if $\lambda_2 < 0$. We show in the next section that such a choice is consistent with the positivity of the potential and in agreement with unitarity bounds. Since the couplings $|\lambda_i| \leq \mathcal{O}(16\pi)$ from perturbativity, the massive charged scalars can be in the 100 GeV to a TeV range.

The vev and the λ_i being real the neutral scalar (ϕ_i) and pseudoscalar (χ_i) sectors remain independent. We get for the neutral pseudoscalars:

$$M^2_{\chi_i\chi_j} = diag(0\,,\ p\,,\ p) \ \text{ where } \ p = \frac{v^2}{4}\,(-3\lambda_2 + \lambda_3 - 4\lambda_4)\ . \tag{4.6}$$

We can readily identify the zero mass Goldstone mode, χ_1, while $\chi_{2,3}$ are massive degenerate states. We show in the following section that positivity and unitarity constraints do allow positive mass-squareds for these scalars.

Finally, for the neutral real scalars we have:

$$M^2_{\phi_i\phi_j} = diag(q\,,\ r_0\,,\ r_0) \ \text{ where } \ q = v^2\ (\lambda_1 + \lambda_2)\,,\ r_0 = \frac{v^2}{4}\,(-3\lambda_2 + \lambda_3 + 4\lambda_4)\ . \tag{4.7}$$

Positivity of the scalar potential requires $(\lambda_1 + \lambda_2)$ to be positive. So, ϕ_1 has a positive mass-squared. Further, $r_0 = m^2_{\phi_2,\phi_3}$ is also positive. In other words, alignment is manifest and the unitary transformation in eq. (3.2) for this case is the unit matrix. The defining basis is also the Higgs basis.

If we had taken λ_4 to be complex, then the charged sector would be unaffected as would be ϕ_1 and χ_1. The other mass eigenstates would be orthogonal superpositions of ϕ_2 with χ_2 and ϕ_3 with χ_3, the mixing angle being proportional to $Im(\lambda_4)$.

4.2 Case 2: $\langle\phi_i^0\rangle = \frac{v}{2}(1,1,0)$

This is the global minimum for which $\langle\phi_1^0\rangle = \langle\phi_2^0\rangle = \frac{v}{2}$ and $\langle\phi_3^0\rangle = 0$, i.e.,

$$\langle\Phi\rangle_2 = \frac{v}{2}\begin{pmatrix} 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}\ . \tag{4.8}$$

One can get this minimum if the potential satisfies:

$$m^2 + \frac{v^2}{4}\left[\lambda_1 + \frac{1}{4}\lambda_2 + \frac{1}{4}\lambda_3 + \lambda_4\right] = 0\ . \tag{4.9}$$

As in the previous case we can obtain the mass matrices for the scalar fields starting from the potential in eq. (3.4). For example, using eq. (4.9) one obtains for the neutral pseudoscalars $(\chi_1\ \chi_2\ \chi_3)$:

$$M^2_{\chi_i\chi_j} = \left(\frac{v^2}{4}\right)\begin{pmatrix} -2\lambda_4 & 2\lambda_4 & 0 \\ 2\lambda_4 & -2\lambda_4 & 0 \\ 0 & 0 & -3\lambda_2/4 + \lambda_3/4 - 3\lambda_4 \end{pmatrix}\ . \tag{4.10}$$

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Similarly, for the real scalars $(\phi_1\ \phi_2\ \phi_3)$ one has:

$$M^2_{\phi_i\phi_j} = \left(\frac{v^2}{4}\right)\begin{pmatrix} \lambda_1+\lambda_2 & \lambda_1-\lambda_2/2+\lambda_3/2+2\lambda_4 & 0 \\ \lambda_1-\lambda_2/2+\lambda_3/2+2\lambda_4 & \lambda_1+\lambda_2 & 0 \\ 0 & 0 & -3\lambda_2/4+\lambda_3/4+\lambda_4 \end{pmatrix}. \tag{4.11}$$

The charged scalar mass matrix is found to be:

$$M^2_{\phi_i^\mp\phi_j^\pm} = \left(\frac{v^2}{4}\right)\begin{pmatrix} -\lambda_3/4-\lambda_4 & \lambda_3/4+\lambda_4 & 0 \\ \lambda_3/4+\lambda_4 & -\lambda_3/4-\lambda_4 & 0 \\ 0 & 0 & -3\lambda_2/4-\lambda_3/4-\lambda_4 \end{pmatrix}. \tag{4.12}$$

To go to the Higgs basis using eq. (3.2) one must use a unitary transformation by:

$$U_2 = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{4.13}$$

The mass matrices in eqs. (4.10)–(4.12) are all diagonalised by the same unitary transformation $U_2 M^2 U_2^\dagger = D^2$ where D is diagonal. We find:

$$D^2_{\chi_i\chi_j} = \left(\frac{v^2}{4}\right)\mathrm{diag}(0,-4\lambda_4,-3\lambda_2/4+\lambda_3/4-3\lambda_4), \tag{4.14}$$

$$D^2_{\phi_i\phi_j} = \left(\frac{v^2}{4}\right)\mathrm{diag}(2\lambda_1+\lambda_2/2+\lambda_3/2+2\lambda_4, 3\lambda_2/2-\lambda_3/2-2\lambda_4, -3\lambda_2/4+\lambda_3/4+\lambda_4), \tag{4.15}$$

and

$$D^2_{\phi_i^\mp\phi_j^\pm} = \left(\frac{v^2}{4}\right)\mathrm{diag}(0,\ -\lambda_3/2-2\lambda_4,\ -3\lambda_2/4-\lambda_3/4-\lambda_4). \tag{4.16}$$

It is important to note that we are defining the Higgs basis through:

$$\Psi = U_2\Phi\,, \tag{4.17}$$

where Φ is given in eq. (3.1) and Ψ is defined in eq. (3.2).

In this Higgs basis the scalars are mass eigenstates as expected, where ψ_1^+ and ξ_1 are massless Goldstones and η_1 is massive. Further, the vev is:

$$\langle\Psi\rangle_2 = \frac{v}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \tag{4.18}$$

which makes the alignment obvious.

Notice that the second and third eigenvalues of eq. (4.15) are proportional but with opposite sign. So, both cannot be made positive by any choice of the λ_i. This is an inadequacy which can be removed by choosing λ_4 to be complex, where alignment still continues to be valid. We demonstrate this in an appendix.

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4.3 Case 3: $\langle\phi_i^0\rangle = \frac{v}{\sqrt{6}}(1,1,1)$

Next we consider $\langle\phi_1^0\rangle = \langle\phi_2^0\rangle = \langle\phi_3^0\rangle = \frac{v}{\sqrt{6}}$, i.e.,

$$\langle\Phi\rangle_3 = \frac{v}{\sqrt{6}} \begin{pmatrix} 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} . \tag{4.19}$$

In this case the minimisation of the potential implies:

$$m^2 + \frac{v^2}{12}\left[3\lambda_1 + \lambda_3 + 4\lambda_4\right] = 0 . \tag{4.20}$$

Using the above one can calculate the mass matrices of the scalar fields. For example, for the neutral pseudoscalars $(\chi_1\ \chi_2\ \chi_3)$ one gets:

$$M^2_{\chi_i\chi_j} = 2\lambda_4\left(\frac{v^2}{6}\right)\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} . \tag{4.21}$$

The real scalar $(\phi_1\ \phi_2\ \phi_3)$ mass matrix in this case is:

$$M^2_{\phi_i\phi_j} = \left(\frac{v^2}{6}\right)\begin{pmatrix} y & z & z \\ z & y & z \\ z & z & y \end{pmatrix} , \tag{4.22}$$

where $y = (\lambda_1 + \lambda_2)$ and $z = (\lambda_1 - \lambda_2/2 + \lambda_3/2 + 2\lambda_4)$.

Finally, for the charged sector

$$M^2_{\phi_i^\mp\phi_j^\pm} = \left(\frac{v^2}{6}\right)\left(\lambda_4 + \frac{\lambda_3}{4}\right)\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} . \tag{4.23}$$

The above mass matrices are all diagonalised by a unitary transformation by the matrix U_3 defined in eq. (2.3). Thus, U_3 rotates the defining basis to the Higgs basis.[6]

The diagonal forms of the mass matrices are:

$$D^2_{\chi_i\chi_j} = \lambda_4\, v^2 \mathrm{diag}(0,\ -1,\ -1) , \tag{4.24}$$

$$D^2_{\phi_i^\mp\phi_j^\pm} = \left(\lambda_4 + \frac{\lambda_3}{4}\right)\left(\frac{v^2}{2}\right)\mathrm{diag}(0,\ -1,\ -1) , \tag{4.25}$$

[6] Notice that in all three scalar sectors there is double degeneracy and consequently the Higgs basis is non-unique. For example, in place of U_3 of eq. (2.3) one could just as well use the popular tribimaximal mixing matrix: $U_{TBM} = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ -\frac{\sqrt{2}}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$.

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and

$$D^2_{\phi_i\phi_j} = \left(\frac{v^2}{6}\right) \mathrm{diag}(y+2z,\ y-z,\ y-z)\ , \tag{4.26}$$

where $y+2z = 3\lambda_1 + \lambda_3 + 4\lambda_4$ and $y - z = 3\lambda_2/2 - \lambda_3/2 - 2\lambda_4$. Both these combinations can be positive while remaining consistent with positivity and unitarity. Similarly, λ_4 and $\lambda_4 + \lambda_3/4$, which appear in eqs. (4.24) and (4.25) respectively, can both be negative.

The fields in the Higgs basis are:

$$\Psi = U_3 \Phi\ , \tag{4.27}$$

where Φ is given in eq. (3.1). As before, we use $\psi_i^0 = \eta_i + i\xi_i$.

Further, in this basis in which the scalars are mass eigenstates, with ψ_1^+ and ξ_1 massless, the vev becomes

$$\langle\Psi\rangle_3 = \frac{v}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}\ . \tag{4.28}$$

Thus, alignment is again manifest.

4.4 Case 4: $\langle\phi_i^0\rangle = \frac{v}{\sqrt{6}}(1,\ \omega,\ \omega^2)$

The last alternative that we consider involves complex vacuum expectation values, namely, $\langle\phi_1^0\rangle = (v/\sqrt{6})$, $\langle\phi_2^0\rangle = (v/\sqrt{6})\omega$, and $\langle\phi_3^0\rangle = (v/\sqrt{6})\omega^2$, i.e.,

$$\langle\Phi\rangle_4 = \frac{v}{\sqrt{6}} \begin{pmatrix} 0 & 1 \\ 0 & \omega \\ 0 & \omega^2 \end{pmatrix}\ . \tag{4.29}$$

The vev is in this direction if

$$m^2 + \frac{v^2}{12}\left[3\lambda_1 + \lambda_3 - 2\lambda_4\right] = 0\ . \tag{4.30}$$

Since the vev are complex there will be mixing terms involving neutral scalars and pseudoscalars. The (6×6) mass matrix in the $(\chi_1,\chi_2,\chi_3,\phi_1,\phi_2,\phi_3)$ basis is:

$$M^2_{\Phi_i^0\Phi_j^0} = \frac{v^2}{6} \begin{pmatrix} 2\lambda_4 & -\lambda_4 & -\lambda_4 & 0 & \sqrt{3}\lambda_4 & -\sqrt{3}\lambda_4 \\ -\lambda_4 & f_2 & f_1 & g_1 & g_2 & g_3 \\ -\lambda_4 & f_1 & f_2 & -g_1 & -g_3 & -g_2 \\ 0 & g_1 & -g_1 & (\lambda_1+\lambda_2) & h_1 & h_1 \\ \sqrt{3}\lambda_4 & g_2 & -g_3 & h_1 & h_3 & h_2 \\ -\sqrt{3}\lambda_4 & g_3 & -g_2 & h_1 & h_2 & h_3 \end{pmatrix}, \tag{4.31}$$

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where

$$
\begin{aligned}
f_1 &= -\frac{3}{4}\left[\lambda_1 - \lambda_2/2 + \lambda_3/2\right] - \lambda_4\,, & f_2 &= \frac{3}{4}(\lambda_1 + \lambda_2) + \frac{1}{2}\lambda_4\,,\\
g_1 &= \frac{\sqrt{3}}{4}\{2\lambda_1 - \lambda_2 + \lambda_3 - 4\lambda_4\}\,, & g_2 &= -\frac{\sqrt{3}}{4}\{\lambda_1 + \lambda_2 - 2\lambda_4\}\,,\\
g_3 &= -\frac{\sqrt{3}}{4}\{\lambda_1 - \lambda_2/2 + \lambda_3/2 - 4\lambda_4\}\,, & &\\
h_1 &= -\frac{1}{4}\{2\lambda_1 - \lambda_2 + \lambda_3 + 4\lambda_4\}\,, & h_2 &= \frac{1}{8}\{2\lambda_1 - \lambda_2 + \lambda_3 - 8\lambda_4\}\,,\\
h_3 &= \frac{1}{4}\{\lambda_1 + \lambda_2 + 6\lambda_4\}\,. & &
\end{aligned}
\tag{4.32}
$$

The matrix in eq. (4.31) has an eigenstate with zero eigenvalue. This state can be readily identified by changing the basis through a (6×6) unitary transformation by

$$
U_{6r} = \frac{1}{\sqrt{3}}\begin{pmatrix}
1 & 1 & 1 & 0 & 0 & 0\\
1 & -1/2 & -1/2 & 0 & -\sqrt{3}/2 & \sqrt{3}/2\\
1 & -1/2 & -1/2 & 0 & \sqrt{3}/2 & -\sqrt{3}/2\\
0 & 0 & 0 & 1 & 1 & 1\\
0 & \sqrt{3}/2 & -\sqrt{3}/2 & 1 & -1/2 & -1/2\\
0 & -\sqrt{3}/2 & \sqrt{3}/2 & 1 & -1/2 & -1/2
\end{pmatrix}.
\tag{4.33}
$$

The new basis thus obtained is:

$$
\begin{pmatrix}\xi_1\\ \xi_2\\ \xi_3\\ \eta_1\\ \eta_2\\ \eta_3\end{pmatrix}
= \frac{1}{\sqrt{3}}\begin{pmatrix}
\chi_1 + \chi_2 + \chi_3\\
\chi_1 - (\chi_2+\chi_3)/2 - \sqrt{3}(\phi_2-\phi_3)/2\\
\chi_1 - (\chi_2+\chi_3)/2 + \sqrt{3}(\phi_2-\phi_3)/2\\
\phi_1 + \phi_2 + \phi_3\\
\sqrt{3}(\chi_2-\chi_3)/2 + \phi_1 - (\phi_2+\phi_3)/2\\
-\sqrt{3}(\chi_2-\chi_3)/2 + \phi_1 - (\phi_2+\phi_3)/2
\end{pmatrix}.
\tag{4.34}
$$

It turns out that ξ_2 and η_2 are mass eigenstates with ξ_2 being the mass-zero mode. The rest of the mass matrix separates into two block diagonal forms, a (2×2) block for (ξ_1,ξ_3) and another for (η_1,η_3) which are:

$$
M^2_{\xi_1,\xi_3} = \frac{v^2}{6}\begin{pmatrix}\lambda_A & -\lambda_A\\ -\lambda_A & \lambda_A + 18\lambda_4\end{pmatrix}\,,\;
M^2_{\eta_1,\eta_3} = \frac{v^2}{6}\begin{pmatrix}\lambda_A & \lambda_A\\ \lambda_A & \lambda_A + 18\lambda_4\end{pmatrix}\,,
\tag{4.35}
$$

where $\lambda_A = \frac{9}{4}\lambda_2 - \frac{3}{4}\lambda_3 - 3\lambda_4$. $m^2_{\xi_2} = 0$ and $m^2_{\eta_2} = v^2\,(3\lambda_1/2 + \lambda_3/2 - \lambda_4)$. The eigenvalues and eigenvectors of the two matrices in eq. (4.35) are:

$$
\begin{aligned}
m_1^2 &= \frac{v^2}{6}\left[\lambda_A + 9\lambda_4 + \sqrt{\lambda_A^2 + 81\lambda_4^2}\right], \xi_1 = \chi_1\cos\alpha - \chi_3\sin\alpha, \eta_1 = \phi_1\cos\alpha + \phi_3\sin\alpha\\
m_3^2 &= \frac{v^2}{6}\left[\lambda_A + 9\lambda_4 - \sqrt{\lambda_A^2 + 81\lambda_4^2}\right], \xi_3 = \chi_1\sin\alpha + \chi_3\cos\alpha, \eta_3 = -\phi_1\sin\alpha + \phi_3\cos\alpha\,,
\end{aligned}
\tag{4.36}
$$

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where

$$\tan 2\alpha = \frac{\lambda_A}{9\lambda_4}\,. \tag{4.37}$$

At this stage we draw attention to the fact that the (6×6) unitary matrix, U_{6r} in eq. (4.33), acting on real fields (χ_i, ϕ_i) is nothing but a unitary transformation by U_3 of eq. (2.3) on the complex fields, $\phi_i^0 = \phi_i + i\chi_i$.

For the charged sector (after eliminating $\lambda_{1,2}$ using eq. (4.30)):

$$M^2_{\phi_i^{\mp}\phi_j^{\pm}} = \frac{v^2}{6}\begin{pmatrix} a & b & b^* \\ b^* & a & b \\ b & b^* & a \end{pmatrix}\,, \tag{4.38}$$

where $a = (2\lambda_4 - \lambda_3)/2$ and $b = (\omega^2\lambda_3 + 4\omega\lambda_4)/4$. This matrix is also diagonalised by going to the Ψ basis using eq. (3.2) with U_3 from eq. (2.3) and one has the eigenvalues $(a + 2Re(b)) = -\frac{v^2}{6}(3\lambda_3/4)$, $(a - Re(b) - \sqrt{3}Im(b)) = 0$, and $(a - Re(b) + \sqrt{3}Im(b)) = -\frac{v^2}{6}(3\lambda_3/4 - 3\lambda_4)$. The corresponding eigenstates are precisely:

$$\psi_1^{\pm} = (\phi_1^{\pm} + \phi_2^{\pm} + \phi_3^{\pm})/\sqrt{3}\,,\ \psi_2^{\pm} = (\phi_1^{\pm} + \omega\phi_2^{\pm} + \omega^2\phi_3^{\pm})/\sqrt{3}\,,\ \psi_3^{\pm} = (\phi_1^{\pm} + \omega^2\phi_2^{\pm} + \omega\phi_3^{\pm})/\sqrt{3}\,. \tag{4.39}$$

Thus $(\Psi_1,\ \Psi_2,\ \Psi_3)$ constitute the Higgs basis for this case with Ψ_2 mimicing the Standard Model scalar doublet in this case.

Note, that in this Higgs basis the vev becomes:

$$\langle\Psi\rangle_4 = \frac{v}{\sqrt{2}}\begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}\,. \tag{4.40}$$

The fact that in the Ψ-basis the vev takes the form in eq. (4.40) and that the components of Ψ_2, namely (ψ_2^+, ψ_2^0), are both mass eigenstates with massless charged and neutral modes and a massive neutral scalar exemplifies alignment in this case.

5 Positivity, Unitarity

The scalar potential in eq. (3.4) must be bounded from below. This gives rise to 'positivity' bounds on the couplings appearing in it. Further, tree-level unitarity of scalar-scalar scattering also gives rise to bounds on combinations of the same couplings. In the following we show that these constraints still do permit the λ_i to be chosen such that the mass-squareds of all scalars are either positive or vanishing.

5.1 Positivity limits

It is well-known that if the scalar potential of any model depends only on the squares of the fields then one has to consider the 'copositivity' constraints on the couplings. For a general model with three scalar doublets such constraints are available in the literature [34, 35]. We adopt these to the model with $A4$ symmetry.

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Because of the $A4$ symmetry, the scalar quartic couplings are few and related. If all vacuum expectation values are real one has to look for the copositivity of the matrix

$$M_{cop} = \begin{pmatrix} \lambda_P & \lambda_Q & \lambda_Q \\ \lambda_Q & \lambda_P & \lambda_Q \\ \lambda_Q & \lambda_Q & \lambda_P \end{pmatrix} , \tag{5.1}$$

where the combinations

$$\lambda_P = (\lambda_1 + \lambda_2)/2\,,\ \lambda_Q = (2\lambda_1 - \lambda_2 + \lambda_3 + 4\lambda_4)/4\,. \tag{5.2}$$

The conditions to be satisfied are:

$$\lambda_P \geq 0\,,\ \lambda_P + \lambda_Q \geq 0\,,\ \text{and}\ \sqrt{\lambda_P^3} + (3\lambda_Q)\ \sqrt{\lambda_P} + \sqrt{2(\lambda_P + \lambda_Q)^3}\ \geq 0\,. \tag{5.3}$$

To satisfy these conditions it is enough to demand $\lambda_P \geq 0$ and $\lambda_Q \geq -\frac{1}{2}\lambda_P$ which translate to:

$$\lambda_1 + \lambda_2 \geq 0\,,\ 3\lambda_1 + \lambda_3 + 4\lambda_4 \geq 0\,. \tag{5.4}$$

For complex vev, in general, one has to look at the *positivity* of the matrix. However, in the simpler situation in the Case 4 discussed earlier, where the phases of $\langle\Phi_1\rangle$, $\langle\Phi_2\rangle$ and $\langle\Phi_3\rangle$ are $(0, 2\pi/3, 4\pi/3)$, the copositivity criteria continue to apply. The matrix M_{cop} is the same as in eq. (5.1) except for

$$\lambda_Q \to \lambda_R = (2\lambda_1 - \lambda_2 + \lambda_3 - 2\lambda_4)/4\,. \tag{5.5}$$

In this case, one must satisfy

$$\lambda_1 + \lambda_2 \geq 0\,,\ 3\lambda_1 + \lambda_3 - 2\lambda_4 \geq 0\,. \tag{5.6}$$

5.2 s-wave unitarity

The potential in eq. (3.4) involves $\mathrm{SU}(2)_L$ doublet scalar fields ($I = 1/2$) with $Y = 1$ and their hermitian conjugates. The quartic terms in the potential can give rise to tree-level scalar-scalar scattering processes. At high energies one can classify the scattering states by their $\mathrm{SU}(2)_L$ and Y quantum numbers. The two-particle states can be in $\mathrm{SU}(2)_L$ singlet ($I = 0$) or $\mathrm{SU}(2)_L$ triplet ($I = 1$) channels and for both cases with $Y = 2$ (e.g., $\phi_i^+\phi_j^+$ initial/final states, $i, j = 1, 2, 3$) or 0 (e.g., $\phi_i^+\phi_j^{*0}$ initial/final states, $i, j = 1, 2, 3$). The scattering processes in every channel must respect limits arising from probabilitiy conservation, i.e., unitarity. This implies bounds on the amplitudes for each partial wave. Here we restrict ourselves to the bounds from s-wave scattering for the different channels [36, 37].

A discussion of the unitarity bounds for the two-scalar-doublet model along these lines can be found in [36]. It can be readily generalised to the three-scalar-doublet case under consideration here.

The results we obtained are displayed in table 1. Besides I and Y an initial or final state will carry two indices (i, j) when the two scalars are from Φ_i and Φ_j. We treat two cases separately. 'Diagonal' corresponds to states with $i = j$ while 'Off-diagonal' is for

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Quantum numbers		Type	Matrix size	Eigenvalues
$\mathrm{SU}(2)_L$	Y			
1	2	Diagonal	3×3	$\lvert(\lambda_1 - 2\lambda_4)\rvert\,,\ \lvert(\lambda_1 + 4\lambda_4)\rvert$
1	2	Off-diagonal	3×3	$\lvert(\lambda_3 - 3\lambda_2)/2\rvert$
0	2	Off-diagonal	3×3	$\lvert(\lambda_3 + 3\lambda_2)/2\rvert$
1	0	Diagonal	3×3	$\lvert(\lambda_1 - \lambda_3/2)\rvert\,,\ \lvert(\lambda_1 + \lambda_3)\rvert$
1	0	Off-diagonal	6×6	$\lvert(3\lambda_2 + 4\lambda_4)/2\rvert\,,\ \lvert(3\lambda_2 - 4\lambda_4)/2\rvert$
0	0	Diagonal	3×3	$\lvert(6\lambda_1 + 6\lambda_2 - \lambda_3)/2\rvert\,,\ \lvert(3\lambda_1 - 6\lambda_2 + \lambda_3)\rvert$
0	0	Off-diagonal	6×6	$\lvert(-3\lambda_2 + 2\lambda_3 - 12\lambda_4)/2\rvert\,,\ \lvert(-3\lambda_2 + 2\lambda_3 + 12\lambda_4)/2\rvert$

Table 1. The dimensionalities and eigenvalues of the tree-level scattering matrices for the different $\mathrm{SU}(2)_L$ and Y sectors. 'Diagonal' ('Off-diagonal') corresponds to $i = j$ ($i \neq j$) with $i, j = 1, 2, 3$. From unitarity the magnitude of each eigenvalue must be bounded by $1/8\pi$.

$i \neq j$. Since in all terms in the potential in eq. (3.4) any field, Φ_i, appears an even number of times, a 'Diagonal' initial state cannot scatter into an 'Off-diagonal' state and *vice versa*. So, the two sectors are completely decoupled for the $A4$-symmetric potential. Note that for the 'Diagonal' case there is no $\mathrm{SU}(2)_L$ singlet state with $Y = 2$ due to Bose statistics.

We denote the s-wave scattering amplitude by $S(I, Y)$. For each choice of I, I_3, and Y quantum numbers there are a fixed set of states determined by the available options for i and j. A scattering matrix can be obtained for an initial state from this set going over to a final state also from this set. s-wave unitarity requires every eigenvalue of the matrix to be bounded by $1/8\pi$. For example, for the $S(0, 2)$ 'Off-diagonal' case the states are $\Gamma_{12}(0, 2) \equiv (\phi_1^+\phi_2^0 - \phi_1^0\phi_2^+)/\sqrt{2}$, $\Gamma_{23}(0, 2) \equiv (\phi_2^+\phi_3^0 - \phi_2^0\phi_3^+)/\sqrt{2}$, $\Gamma_{31}(0, 2) \equiv (\phi_3^+\phi_1^0 - \phi_3^0\phi_1^+)/\sqrt{2}$. Scattering between these states[7] gives rise to a (3×3) matrix. The matrix is diagonal because no term in the potential in eq. (3.4) can cause an off-diagonal transition in this sector, e.g., $\Gamma_{12}(0, 2) \not\leftrightarrow \Gamma_{23}(0, 2)$, which involves an odd number of Φ_1 and Φ_3 fields. Also, the matrix is proportional to the identity due to the $A4$ symmetry. On the other hand, for the $S(0, 0)$ 'Off-diagonal' case for any i and j ($i \neq j$) the initial and final states can be any one of $(\phi_i^+\phi_j^- + \phi_i^0\phi_j^{*0})/\sqrt{2}$ or $(\phi_i^-\phi_j^+ + \phi_i^{*0}\phi_j^0)/\sqrt{2}$. This results in a (6×6) matrix which turns out to be of block diagonal form with three identical (2×2) blocks.

In table 1 we have listed the different channels, the corresponding scattering matrix dimensions, and their eigenvalues. We present below two typical examples of the matrices, corresponding to the first and fifth rows of table 1.

$$8\pi S(1, 2)_{\mathrm{diag}} = \begin{pmatrix} \lambda_1 & 2\lambda_4 & 2\lambda_4 \\ 2\lambda_4 & \lambda_1 & 2\lambda_4 \\ 2\lambda_4 & 2\lambda_4 & \lambda_1 \end{pmatrix},$$

[7] Notice that $\Gamma_{ij}(0, 2) = -\Gamma_{ji}(0, 2)$.

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$$8\pi S(1,0)_{\mathrm{off-diag}} = \begin{pmatrix} X & 0 & 0 \\ 0 & X & 0 \\ 0 & 0 & X \end{pmatrix}, X = \begin{pmatrix} -3\lambda_2/2 & 2\lambda_4 \\ 2\lambda_4 & -3\lambda_2/2 \end{pmatrix} . \tag{5.7}$$

For all cases at least one (or more) of the eigenvalues is degenerate due to the $A4$ symmetry requirement on the potential.

Note that the above discussion of tree-level unitarity has been in terms of the defining scalar fields $\Phi_{1,2,3}$. An alternate approach which also appears in the literature is to consider the unitarity bounds following from the scattering of physical mass eigenstate scalars. The results are equivalent.

5.3 Scalar mass-squareds

In section 4 we have obtained the mass eigenvalues of the charged and neutral scalars for the four alternate choices of the vevs. We find from the limits on the quartic couplings from positivity and from unitarity (table 1) that there is ample room to choose the λ_i such that all scalar mass-squareds are positive, i.e., all masses are real.

In particular, positivity of the mass-squared for all the physical fields requires:

$$\text{Case 1}: \lambda_1 + \lambda_2 > 0\,,\ \lambda_2 < 0\,,\ 3\lambda_2 - \lambda_3 + 4\lambda_4 < 0\,,\ 3\lambda_2 - \lambda_3 - 4\lambda_4 < 0\,.$$

Of the above conditions, the first is a requirement that must be met, see eq. (5.4), for the potential to be positive. Besides, one must make the choices $\lambda_2 < 0$ and $(3\lambda_2 - \lambda_3 + 4|\lambda_4|) < 0$.

$$\text{Case 2}: \text{ as noted in section 4.2, for real } \lambda_4 \text{ this case is inadmissible.}$$

In the appendix we show that this issue is removed if λ_4 is taken complex.

$$\text{Case 3}: \lambda_4 < 0\,,\ \lambda_3 + 4\lambda_4 < 0\,,\ 3\lambda_1 + \lambda_3 + 4\lambda_4 > 0\,,\ 3\lambda_2 - \lambda_3 - 4\lambda_4 > 0\,.$$

The third inequality in the above is again a consequence of the positivity of the potential, eq. (5.4). Along with this, the imposition of the first and second conditions imply the fourth.

$$\text{Case 4}: 3\lambda_1 + \lambda_3 - 2\lambda_4 > 0\,,\ \lambda_3 < 0\,,\ \lambda_3 - 4\lambda_4 < 0\,,\ \lambda_4\,(3\lambda_2 - \lambda_3 - 4\lambda_4) > 0\,.$$

The first condition is anyway satisfied for the positivity of the potential, see eq. (5.6). Further, this soultion is viable in the parameter region defined by the remaining conditions.

The constraints from unitarity in table 1 set bounds on the magnitude of some combinations of these couplings. Barring unnatural cancellations, this implies that none of the λ_i can be arbitrarily large in magnitude.

6 Conclusions

In this paper we have considered a three-Higgs doublet model with $A4$ symmetry. We have examined the vevs which have been identified as the *only* possible global minima of this potential. These choices have been used in realistic physics models. Here we have shown

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that in all these cases alignment is automatic due to the $A4$ symmetry. We demonstrate that the bounds on the quartic couplings from positivity and unitarity can be satisfied while keeping all scalar mass-squareds positive.

The attractive feature of $A4$ symmetry is that the terms that are allowed in the potential ensure distinctive textures for the scalar mass matrices. Alignment for the global minima vevs is a result of this. Another consequence is that the mixing matrices in the scalar sector are of a few standard forms.

Needless to say, since alignment is valid, with the minimal scalar content considered in this work, in the Higgs basis fermion masses will arise from the coupling to the analogue of the SM Higgs boson in this model. Usually, however, realistic models involve the inclusion of more scalar fields. Such a model based on $A4$ symmetry incorporating fermions and reproducing their observed mass and mixing patterns through Yukawa couplings is beyond the scope of this work.

In conclusion, this model exhibits several features which make it of interest for further exploration.

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A Case 2 generalisation

It was shown in section 4.2 that though alignment is valid for the vev $\langle\Phi_0\rangle = (v/2)(1,1,0)$ the physical scalar mass-squareds cannot all be made simultaneously positive when the coupling λ_4 is real. We show in this appendix that this issue is addressed when λ_4 is complex.

We take $\lambda_4 = |\lambda_4|e^{i\delta}$ and the vev of the neutral scalars as $(v/2)(1,\ e^{i\alpha},\ 0)$. In this case, the minimisation condition becomes:

$$m^2 + \frac{v^2}{4}\left[\lambda_1 + \frac{1}{4}\lambda_2 + \frac{1}{4}\lambda_3 - |\lambda_4|\right] = 0 \text{ and } \delta + 2\alpha = \pi \tag{A.1}$$

Note that the phase of λ_4 is related to that of the vev.

If λ_4 is complex then the charged scalar sector $(\phi_1^\pm,\ \phi_2^\pm, \phi_3^\pm)$ mass-squared matrix is:

$$M^2_{\phi_i^\mp\phi_j^\pm} = \left(\frac{v^2}{4}\right)\begin{pmatrix} -\lambda_3/4 + |\lambda_4| & e^{-i\alpha}(\lambda_3/4 - |\lambda_4|) & 0 \\ e^{i\alpha}(\lambda_3/4 - |\lambda_4|) & -\lambda_3/4 + |\lambda_4| & 0 \\ 0 & 0 & -3\lambda_2/4 - \lambda_3/4 + |\lambda_4| \end{pmatrix}. \tag{A.2}$$

This matrix is diagonalised by a unitary transformation of the fields:

$$\begin{pmatrix}\Psi_1\\ \Psi_2\\ \Psi_3\end{pmatrix} = U_{2c}\begin{pmatrix}\Phi_1\\ \Phi_2\\ \Phi_3\end{pmatrix} \quad \text{with } U_{2c} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & e^{-i\alpha} & 0\\ e^{i\alpha} & -1 & 0 \\ 0 & 0 & 1\end{pmatrix}. \tag{A.3}$$

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As before, we write $\Psi_i^0 = \eta_i + i\xi_i$. Notice that in this basis the vev is of the form $\langle\Psi^0\rangle = (v/\sqrt{2})(1,0,0)$.

The charged states $(\psi_1^\pm, \psi_2^\pm, \psi_3^\pm)$ have masses 0, $(v/2)\sqrt{-\lambda_3/2+2|\lambda_4|}$, $(v/2)\sqrt{-3\lambda_2/4-\lambda_3/4+|\lambda_4|}$ respectively. That $\psi_1^\pm$ is massless is indicative that the Ψ_i constitute the Higgs basis. To establish alignment we need to check that ξ_1 is massless and η_1 has a positive mass-squared.

Because the vev and λ_4 are now complex there is scalar-pseudoscalar mixing in the neutral sector. The neutral sector mass-squared matrix splits up into a (2×2) block for (χ_3, ϕ_3) which remains decoupled from the remaining (4×4) block. For the other neutral fields, i.e., $(\chi_1, \chi_2, \phi_1, \phi_2)$ states:

$$M^2_{\chi_1,\chi_2,\phi_1,\phi_2} = \frac{v^2}{4} 2|\lambda_4| \left[I + \begin{pmatrix} 0 & -\cos\alpha & 0 & \sin\alpha \\ -\cos\alpha & K\sin^2\alpha & J\sin\alpha & K\sin\alpha\cos\alpha \\ 0 & J\sin\alpha & K & J\cos\alpha \\ \sin\alpha & K\sin\alpha\cos\alpha & J\cos\alpha & K\cos^2\alpha \end{pmatrix} \right]. \tag{A.4}$$

Here $K = (\lambda_1+\lambda_2-2|\lambda_4|)/2|\lambda_4|$ and $J = (\lambda_1-\lambda_2/2+\lambda_3/2-2|\lambda_4|)/2|\lambda_4|$.

This matrix is diagonalised by a (4×4) unitary matrix which is the upper (2×2) block of U_{2c} in eq. (A.3) expressed in this $(\chi_{1,2},\ \phi_{1,2})$ basis, i.e.,:

$$U_{4r} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & \cos\alpha & 0 & -\sin\alpha \\ \cos\alpha & -1 & \sin\alpha & 0 \\ 0 & \sin\alpha & 1 & \cos\alpha \\ -\sin\alpha & 0 & \cos\alpha & -1 \end{pmatrix}. \tag{A.5}$$

We find

$$\begin{aligned} U_{4r}^\dagger\ [M^2_{\chi_1,\chi_2,\phi_1,\phi_2}]\ U_{4r} &= \frac{v^2}{4} 2|\lambda_4| \\ &\times \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2+(-1+K-J)\sin^2\alpha & 0 & (-1+K-J)\sin\alpha\cos\alpha \\ 0 & 0 & (1+K+J) & 0 \\ 0 & (-1+K-J)\sin\alpha\cos\alpha & 0 & 2+(-1+K-J)\cos^2\alpha \end{pmatrix}. \end{aligned} \tag{A.6}$$

Thus, as required for alignment, the masses of ξ_1 and η_1 are:

$$m_{\xi_1} = 0\,,\ m_{\eta_1} = \frac{v}{2}\sqrt{2\lambda_1+\lambda_2/2+\lambda_3/2-2|\lambda_4|} \tag{A.7}$$

The latter plays the role of the SM Higgs boson.

The remaining mass eigenstates (ξ_2' and η_2'), which can also be read off from eq. (A.6), are superpositions of ξ_2 and η_2 defined through a rotation by the angle α with masses

$$m_{\xi_2',\ \eta_2'} = v\sqrt{|\lambda_4|}\,,\ \frac{v}{2}\sqrt{3\lambda_2/2-\lambda_3/2+2|\lambda_4|}\,. \tag{A.8}$$

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The other neutral scalars, namely $(\chi_3, \phi_3) \equiv (\xi_3, \eta_3)$, are decoupled from the rest and have the mass matrix:

$$M^2_{\chi_3,\,\phi_3} = \frac{v^2}{4}\left[\left(-\frac{3}{4}\lambda_2 + \frac{1}{4}\lambda_3 + |\lambda_4|\right) I + 2|\lambda_4|\cos(3\alpha)\begin{pmatrix} \cos\alpha & -\sin\alpha \\ -\sin\alpha & -\cos\alpha \end{pmatrix}\right] . \tag{A.9}$$

The eigenvalues of this matrix are:

$$m_{\xi_3',\,\eta_3'} = \frac{v}{2}\sqrt{-3\lambda_2/4 + \lambda_3/4 + |\lambda_4|\{1 \mp 2\cos(3\alpha)\}}\ , \tag{A.10}$$

where ξ_3' and η_3' are obtained from ξ_3 and η_3 by a rotation through an angle $\alpha/2$.

A bound on the phase α is readily obtained from eqs. (A.8) and (A.10). One has:

$$m^2_{\xi_3',\,\eta_3'} = \frac{1}{2} m^2_{\xi_2'}\left[\frac{m^2_{\xi_2'} - m^2_{\eta_2'}}{m^2_{\xi_2'}} \mp \cos(3\alpha)\right] . \tag{A.11}$$

One can immediately conclude that:

$$|\cos(3\alpha)| \leq \frac{m^2_{\xi_2'} - m^2_{\eta_2'}}{m^2_{\xi_2'}} . \tag{A.12}$$

Thus one must have

$$m^2_{\xi_2'} > m^2_{\eta_2'} \ \Rightarrow 2|\lambda_4| > \frac{3}{2}\lambda_2 - \frac{1}{2}\lambda_3 . \tag{A.13}$$

In addition, one has the sum-rule:

$$m^2_{\eta_2'} + m^2_{\xi_3'} + m^2_{\eta_3'} = m^2_{\xi_2'} . \tag{A.14}$$

It is worth bearing in mind that the choices $\alpha = 0$ and $\alpha = \pi$, which correspond to the real limit, are inadmissible. In both cases $m_{\eta_2'}$ has to vanish, which is ruled out on physics grounds.

It is not difficult to ensure the reality of all the scalar masses at the same time by a suitable choice of the λ_i while satisfying the requirements of positivity of the potential.

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