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Comments on real tachyon vacuum solution without square roots

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ABSTRACT: We analyze the consistency of a recently proposed real tachyon vacuum solution without square roots in open bosonic string field theory. We show that the equation of motion contracted with the solution itself is satisfied. Additionally, by expanding the solution in the basis of the curly $\mathcal{L}_0$ and the traditional L_0 eigenstates, we evaluate numerically the vacuum energy and obtain a result in agreement with Sen's conjecture.

KEYWORDS: String Field Theory, Tachyon Condensation

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1 Introduction

In open string field theory [1], we say that a string field Ψ is real if obeys the following reality condition

$$\Psi^\dagger = \Psi, \quad (1.1)$$

where the double dagger denotes a composition of Hermitian and BPZ conjugation introduced in Gaberdiel and Zwiebach’s seminal work [2].

Analytic tachyon vacuum solutions that satisfy the above reality condition (1.1) exist in the literature [3, 4], however they carry some technical complications. For instance, Schnabl’s original solution is real, but has some subtleties, the solution contains a singular, projector-like state known as the phantom term [5].

Solutions without the phantom term, known as simple solutions or Erler-Schnabl’s type solutions have been proposed [6–10], but they often fail to satisfy the reality condition. By performing a gauge transformation over a non-real simple solution, a real phantom-less solution has been constructed in reference [6]. However, as noted in reference [11], the cost of having this real solution is the introduction of somewhat awkward square roots.

It would be desirable to have a solution that is both real and simple, namely without square roots and phantom terms. This is precisely the issue that has been studied in a recent paper [11], where the author has presented an alternative prescription to obtain a real solution from a non-real one which does not make use of a similarity transformation. Basically, it has been shown that given a tachyon vacuum solution Υ together with its corresponding homotopy operator A [12–14], the string field defined by $\Phi = \text{Re}(\Upsilon) + \text{Im}(\Upsilon) A \text{Im}(\Upsilon)$ is a real solution for the tachyon vacuum.

Applying this prescription for the case of the non-real Erler-Schnabl’s tachyon vacuum solution [6]

$$\Phi_{\text{Er-Sch}} = c(1 + K)Bc \frac{1}{1 + K}, \quad (1.2)$$

the corresponding real solution [11] has been constructed

$$\Phi = \frac{1}{4} \left(\frac{1}{1+K} c + c \frac{1}{1+K} + c \frac{B}{1+K} c + \frac{1}{1+K} c \frac{1}{1+K} \right) + Q_B\text{-exact terms}, \quad (1.3)$$

where the Q_B -exact terms are given by

$$\frac{1}{2} \left[Q_B(Bc) \frac{1}{1+K} + \frac{1}{1+K} Q_B(Bc) \right] + \frac{1}{4} \frac{1}{1+K} Q_B(Bc) \frac{1}{1+K}. \quad (1.4)$$

For this real solution the corresponding energy has been computed and shown that the value is in agreement with the value predicted by Sen's conjecture [15, 16].

Nevertheless, for the evaluation of the energy, the equation of motion contracted with the solution itself was simply assumed to be satisfied. In this paper, we compute the cubic term of the action for the real solution (1.3) and discuss the validity of the previous assumption. Additionally, by expanding the solution in the basis of curly $\mathcal{L}_0$ eigenstates, we evaluate the energy numerically and obtain a result in agreement with Sen's conjecture. Since the numerical evaluation of the energy by means of the curly $\mathcal{L}_0$ level expansion of the solution is not a trivial task, in order to automate the computations of relevant correlation functions defined on the sliver frame, we have developed conservation laws.

This paper is organized as follows. In section 2, we evaluate the cubic term of the action for the real solution and test the validity of the equation of motion when contracted with the solution itself. In section 3, in order to automate the computations involved in the numerical evaluation of the energy associated with the solution, we developed conservation laws for operators defined on the sliver frame. In sections 4, and 5, we compute the energy by means of the curly $\mathcal{L}_0$ and the standard Virasoro L_0 level expansion of the solution and after using Padé approximants we show that the numerical results obtained for the energy are in agreement with Sen's conjecture. In section 6, a summary and further directions of exploration are given.

2 Computing the cubic term for the real solution

In reference [11], a new real solution for the tachyon vacuum has been proposed. This solution in the KBc subalgebra [17, 18] takes the form

$$\Phi = \frac{1}{4} \left(\frac{1}{1+K} c + c \frac{1}{1+K} + c \frac{B}{1+K} c + \frac{1}{1+K} c \frac{1}{1+K} \right) + Q_B\text{-exact terms}, \quad (2.1)$$

By evaluating the kinetic term of the action, it has been shown that the energy

$$E(\Phi) = \frac{1}{6} \text{tr}[\Phi Q_B \Phi] \quad (2.2)$$

associated with the solution (2.1) correctly reduces to a value which is in accordance with Sen's conjecture.

However, to derive the above equation (2.2) for the energy, it has been assumed that the equation of motion holds when contracted with the solution itself. We know from

experience with other solutions [9, 18–20] that this assumption is not a trivial one. In general, a priori there is no justification for assuming the validity of

$$\text{tr}[\Phi Q_B \Phi + \Phi \Phi \Phi] = 0 \quad (2.3)$$

without an explicit calculation. Therefore the cubic term of the action must be evaluated.

The computation of the kinetic term has been already done in reference [11] given as a result

$$\text{tr}[\Phi Q_B \Phi] = -\frac{3}{\pi^2}. \quad (2.4)$$

Thus, for equation (2.3) to be valid, we must show that

$$\text{tr}[\Phi \Phi \Phi] = \frac{3}{\pi^2}. \quad (2.5)$$

To explicitly compute this cubic term, we need to include the Q_B -exact terms of the real solution (2.1). Recall that these terms were not necessary in the evaluation of the kinetic term. The Q_B -exact terms in (2.1) are given by

$$\frac{1}{2} \left[Q_B(Bc) \frac{1}{1+K} + \frac{1}{1+K} Q_B(Bc) \right] + \frac{1}{4} \frac{1}{1+K} Q_B(Bc) \frac{1}{1+K}. \quad (2.6)$$

Inserting the solution (2.1) which includes the Q_B -exact terms (2.6) into the cubic interaction term $\text{tr}[\Phi \Phi \Phi]$, after a lengthy algebraic manipulations, we arrive to

$$\begin{aligned} \text{tr}[\Phi \Phi \Phi] = \text{tr} \bigg[& -\frac{1}{16} cKc \frac{1}{1+K} c \frac{1}{(1+K)^2} - \frac{3}{16} cKc \frac{1}{1+K} c \frac{1}{1+K} \\ & + \frac{1}{16} B \frac{1}{(1+K)^2} cKc \frac{1}{(1+K)^2} cKc + \frac{3}{16} B \frac{1}{(1+K)^2} cKc \frac{1}{1+K} cKc \\ & + \frac{1}{8} B \frac{1}{1+K} cKc \frac{1}{(1+K)^2} cKc + \frac{1}{8} B \frac{1}{1+K} cKc \frac{1}{1+K} cKc \\ & - \frac{1}{16} B \frac{1}{1+K} c \frac{1}{1+K} cKc \frac{1}{1+K} c + \frac{1}{16} B \frac{1}{1+K} c \frac{1}{1+K} c \frac{1}{1+K} cKc \bigg]. \quad (2.7) \end{aligned}$$

All the correlators appearing in the evaluation of the cubic term (2.7) can be computed by means of the following basic correlators

$$\begin{aligned} \text{tr}[ce^{-t_1 K} ce^{-t_2 K} ce^{-t_3 K}] \\ = \frac{(t_1 + t_2 + t_3)^3 \sin\left(\frac{\pi t_2}{t_1 + t_2 + t_3}\right) \sin\left(\frac{\pi t_3}{t_1 + t_2 + t_3}\right) \sin\left(\frac{\pi(t_2 + t_3)}{t_1 + t_2 + t_3}\right)}{\pi^3}, \quad (2.8) \end{aligned}$$

$$\begin{aligned} \text{tr}[Be^{-t_1 K} ce^{-t_2 K} ce^{-t_3 K} ce^{-t_4 K} c] \\ = \frac{s^2(t_2 + t_3 + t_4)}{\pi^3} \sin\left(\frac{\pi t_3}{s}\right) \sin\left(\frac{\pi t_4}{s}\right) \sin\left(\frac{\pi(t_3 + t_4)}{s}\right) \\ - \frac{s^2(t_3 + t_4)}{\pi^3} \sin\left(\frac{\pi t_4}{s}\right) \sin\left(\frac{\pi(t_2 + t_3)}{s}\right) \sin\left(\frac{\pi(t_2 + t_3 + t_4)}{s}\right) \\ + \frac{s^2 t_4}{\pi^3} \sin\left(\frac{\pi t_2}{s}\right) \sin\left(\frac{\pi(t_3 + t_4)}{s}\right) \sin\left(\frac{\pi(t_2 + t_3 + t_4)}{s}\right), \quad (2.9) \end{aligned}$$

where $s = t_1 + t_2 + t_3 + t_4$.

For instance, employing the correlator (2.8), let us explicitly compute the correlator

$$\begin{aligned} \text{tr} \left[cKc \frac{1}{1+K} c \frac{1}{(1+K)^2} \right] &= - \int_0^\infty dt_1 dt_2 t_2 e^{-t_1-t_2} \partial_s \text{tr} \left[ce^{-sK} ce^{-t_1K} ce^{-t_2K} \right] \Big|_{s=0} \\ &= - \int_0^\infty dt_1 dt_2 t_2 e^{-t_1-t_2} \frac{(t_1+t_2)^2 \sin\left(\frac{\pi t_1}{t_1+t_2}\right) \sin\left(\frac{\pi t_2}{t_1+t_2}\right)}{\pi^2}. \end{aligned} \quad (2.10)$$

To evaluate the above double integral, we perform the change of variables $t_1 \rightarrow uv$, $t_2 \rightarrow u - uv$, $\int_0^\infty dt_1 dt_2 \rightarrow \int_0^\infty du \int_0^1 dv u$, so that from equation (2.10), we obtain

$$\begin{aligned} \text{tr} \left[cKc \frac{1}{1+K} c \frac{1}{(1+K)^2} \right] &= \int_0^\infty du \int_0^1 dv \frac{e^{-u} u^4 (v-1) \sin^2(\pi v)}{\pi^2} \\ &= -\frac{6}{\pi^2}. \end{aligned} \quad (2.11)$$

To compute correlators containing the B string field, we proceed in the same manner. As an illustration, let us explicitly evaluate the correlator $\text{tr} \left[B \frac{1}{(1+K)^2} cKc \frac{1}{(1+K)^2} cKc \right]$. The integral representation of this correlator is given by

$$\int_0^\infty dt_1 dt_2 t_1 t_2 e^{-t_1-t_2} \partial_{s_1, s_2} \text{tr} \left[B e^{-t_1K} ce^{-s_1K} ce^{-t_2K} ce^{-s_2K} c \right] \Big|_{s_1=s_2=0}. \quad (2.12)$$

Using the correlator (2.9), from equation (2.12) we obtain

$$\int_0^\infty dt_1 dt_2 t_1 t_2 e^{-t_1-t_2} \frac{2 \sin\left(\frac{\pi t_2}{t_1+t_2}\right) \left((t_1+t_2) \sin\left(\frac{\pi t_2}{t_1+t_2}\right) - \pi t_2 \cos\left(\frac{\pi t_2}{t_1+t_2}\right) \right)}{\pi^2}. \quad (2.13)$$

Performing the change of variables $t_1 \rightarrow uv$, $t_2 \rightarrow u - uv$, $\int_0^\infty dt_1 dt_2 \rightarrow \int_0^\infty du \int_0^1 dv u$ into the above double integral (2.13), we get

$$\int_0^\infty du \int_0^1 dv \frac{2e^{-u} u^4 (1-v)v \sin(\pi v) (\sin(\pi v) - \pi(v-1) \cos(\pi v))}{\pi^2} = \frac{30}{\pi^4} + \frac{4}{\pi^2}. \quad (2.14)$$

Therefore, we have just shown that

$$\text{tr} \left[B \frac{1}{(1+K)^2} cKc \frac{1}{(1+K)^2} cKc \right] = \frac{30}{\pi^4} + \frac{4}{\pi^2}. \quad (2.15)$$

In this way, we can calculate all the relevant correlators appearing in the right hand side of equation (2.7). Let us list the results

$$\text{tr} \left[cKc \frac{1}{1+K} c \frac{1}{(1+K)^2} \right] = -\frac{6}{\pi^2}, \quad (2.16)$$

$$\text{tr} \left[cKc \frac{1}{1+K} c \frac{1}{1+K} \right] = -\frac{3}{\pi^2}, \quad (2.17)$$

$$\text{tr} \left[B \frac{1}{(1+K)^2} cKc \frac{1}{(1+K)^2} cKc \right] = \frac{30}{\pi^4} + \frac{4}{\pi^2}, \quad (2.18)$$

$$\text{tr} \left[B \frac{1}{(1+K)^2} c K c \frac{1}{1+K} c K c \right] = \frac{3}{\pi^2}, \quad (2.19)$$

$$\text{tr} \left[B \frac{1}{1+K} c K c \frac{1}{(1+K)^2} c K c \right] = \frac{6}{\pi^2}, \quad (2.20)$$

$$\text{tr} \left[B \frac{1}{1+K} c K c \frac{1}{1+K} c K c \right] = \frac{3}{\pi^2}, \quad (2.21)$$

$$\text{tr} \left[B \frac{1}{1+K} c \frac{1}{1+K} c K c \frac{1}{1+K} c \right] = \frac{15}{\pi^4} - \frac{4}{\pi^2}, \quad (2.22)$$

$$\text{tr} \left[B \frac{1}{1+K} c \frac{1}{1+K} c \frac{1}{1+K} c K c \right] = -\frac{15}{\pi^4} - \frac{2}{\pi^2}. \quad (2.23)$$

Employing these results (2.16)–(2.23) into equation (2.7) and adding up all terms, we obtain the value for the cubic term

$$\text{tr}[\Phi\Phi\Phi] = \frac{3}{\pi^2}. \quad (2.24)$$

Since we have explicitly shown that the equation of motion is satisfied when contracted with the solution itself, i.e. $\text{tr}[\Phi Q_B \Phi] + \text{tr}[\Phi\Phi\Phi] = 0$, it is guaranteed that the energy associated with the solution (2.1) is directly proportional to the kinetic term

$$E(\Phi) = -\mathcal{S}[\Phi] = \frac{1}{2} \text{tr}[\Phi Q_B \Phi] + \frac{1}{3} \text{tr}[\Phi\Phi\Phi] = \frac{1}{6} \text{tr}[\Phi Q_B \Phi]. \quad (2.25)$$

As a second test of consistency, we would like to analyze the solution from a numerical point of view, in particular, we will be interested in the numerical evaluation of the kinetic term by means of the curly $\mathcal{L}_0$ level expansion of the solution.

As we are going to show, when we insert the curly $\mathcal{L}_0$ level expansion of the solution into the kinetic term, we are required to evaluate two point vertices for string fields containing the operators $\hat{\mathcal{L}}$, $\hat{\mathcal{B}}$ and $\tilde{c}_p$. These two point vertices can be evaluated by means of the so-called conservation laws which will be studied in the next section.

3 Conservation laws and the two point vertex in the sliver frame

The operators employed in the basis of curly $\mathcal{L}_0$ eigenstates are given in terms of the basic operators $\hat{\mathcal{L}}$, $\hat{\mathcal{B}}$ and $\tilde{c}_p$. These operators are related to the worldsheet energy momentum tensor $T(z)$, the $b(z)$ and $c(z)$ ghosts fields respectively. We are going to derive the conservation law for the $\hat{\mathcal{L}}$ operator

$$\hat{\mathcal{L}} = \oint \frac{dz}{2\pi i} (1+z^2) (\arctan z + \text{arccot} z) T(z). \quad (3.1)$$

Using the conformal map $\tilde{z} = \frac{2}{\pi} \arctan z$, we can write the expression of the $\hat{\mathcal{L}}$ operator in the sliver frame

$$\hat{\mathcal{L}} = \oint \frac{d\tilde{z}}{2\pi i} \varepsilon(\text{Re} \tilde{z}) \tilde{T}(\tilde{z}), \quad (3.2)$$

where $\varepsilon(x)$ is the step function equal to ± 1 for positive or negative values of its argument respectively.

For vertex operators ϕ_i defined on the sliver frame, the two functions f_1 and f_2 which appear in the definition of the two point vertex $\langle f_1 \circ \phi_1(0) f_2 \circ \phi_2(0) \rangle$ are given by

$$f_1(\tilde{z}_1) = \tan\left(\frac{\pi}{2}(1 + \tilde{z}_1)\right), \quad (3.3)$$

$$f_2(\tilde{z}_2) = \tan\left(\frac{\pi}{2}\tilde{z}_2\right). \quad (3.4)$$

We need conservation laws such that the operator $\hat{\mathcal{L}}$ acting on the two point vertex, which we denote as $\langle V_2 |$, can be expressed in terms of non-negative Virasoro modes defined on the sliver frame¹

$$\langle V_2 | \hat{\mathcal{L}}^{(2)} = \langle V_2 | \left(\sum_{n \geq 0} a_n \mathcal{L}_n^{(1)} + \sum_{n \geq 0} b_n \mathcal{L}_n^{(2)} \right), \quad (3.5)$$

where a_n and b_n are coefficients that will be determined below.

To derive a conservation law of the form (3.5), we need a vector field which behaves as $v^{(2)}(\tilde{z}_2) \sim \varepsilon(\text{Re}\tilde{z}_2) + O(\tilde{z}_2)$ around puncture 2, and has the following behavior in the other puncture, $v^{(1)}(\tilde{z}_1) \sim O(\tilde{z}_1)$. A vector field which does this job is given by

$$v(z) = (1 + z^2) \text{arccot} z. \quad (3.6)$$

The expression of the conservation law for Virasoro modes defined on the sliver frame is given by²

$$\langle V_2 | \sum_{j=1}^2 \oint_{\mathcal{C}_j} \frac{1}{2\pi i} v^{(j)}(\tilde{z}_j) \tilde{T}(\tilde{z}_j) d\tilde{z}_j = 0, \quad (3.7)$$

where $v^{(j)}(\tilde{z}_j) = (\partial_{\tilde{z}_j} f_j(\tilde{z}_j))^{-1} v(f_j(\tilde{z}_j))$, and $\mathcal{C}_j$ is a closed contour which encircles the j -puncture.

Using equations (3.3), (3.4) and (3.6) into the definition $v^{(j)}(\tilde{z}_j) = (\partial_{\tilde{z}_j} f_j(\tilde{z}_j))^{-1} v(f_j(\tilde{z}_j))$ of the vector fields $v^{(1)}(\tilde{z}_1)$ and $v^{(2)}(\tilde{z}_2)$, we find that

$$v^{(1)}(\tilde{z}_1) = -\tilde{z}_1 \quad (3.8)$$

$$v^{(2)}(\tilde{z}_2) = \varepsilon(\text{Re}\tilde{z}_2) - \tilde{z}_2. \quad (3.9)$$

Due to the presence of the step function we see that the vector field $v^{(2)}(\tilde{z}_2)$ is discontinuous around puncture 2, since we are interested in the conservation law of the operator defined in equation (3.2), this kind of discontinuity is what we want. Using (3.7) and noting that integration amounts to the replacement $v_n^{(i)} \tilde{z}_i^n \rightarrow v_n^{(i)} \mathcal{L}_{n-1}^{(i)}$, we can immediately write the conservation law

$$\langle V_2 | \left(-\mathcal{L}_0^{(1)} + \hat{\mathcal{L}}^{(2)} - \mathcal{L}_0^{(2)} \right) = 0. \quad (3.10)$$

¹We are going to use the following notation $\mathcal{O}^{(i)}$ to refer an operator $\mathcal{O}$ defined around the i -th puncture.

²This formula can be derived using the general prescription for conservation laws shown in references [21, 22].

We can write this conservation law (3.10) in the standard form as given in equation (3.5)

$$\langle V_2 | \hat{\mathcal{L}}^{(2)} = \langle V_2 | \left(\mathcal{L}_0^{(1)} + \mathcal{L}_0^{(2)} \right). \quad (3.11)$$

By the symmetry property of the two vertex, the same identity (3.11) holds after replacing (1) $\rightarrow$ (2)

$$\langle V_2 | \hat{\mathcal{L}}^{(1)} = \langle V_2 | \left(\mathcal{L}_0^{(2)} + \mathcal{L}_0^{(1)} \right). \quad (3.12)$$

Regarding the conservation law for the $\hat{\mathcal{B}}$ operator, since the b ghost is a conformal field of dimension two, the conservation laws for operators involving this field are identical to those for the Virasoro operators

$$\langle V_2 | \hat{\mathcal{B}}^{(2)} = \langle V_2 | \left(\mathcal{B}_0^{(1)} + \mathcal{B}_0^{(2)} \right), \quad (3.13)$$

$$\langle V_2 | \hat{\mathcal{B}}^{(1)} = \langle V_2 | \left(\mathcal{B}_0^{(2)} + \mathcal{B}_0^{(1)} \right). \quad (3.14)$$

Employing these conservation laws for the operators $\hat{\mathcal{L}}$ and $\hat{\mathcal{B}}$, together with the commutator and anti-commutator relations

$$[\mathcal{L}_0^{(i)}, \hat{\mathcal{L}}^{(j)}] = \delta^{ij} \hat{\mathcal{L}}^{(j)}, \quad [\mathcal{L}_0^{(i)}, \hat{\mathcal{B}}^{(j)}] = \delta^{ij} \hat{\mathcal{B}}^{(j)}, \quad [\mathcal{L}_0^{(i)}, \tilde{c}_p^{(j)}] = -\delta^{ij} p c_p^{(j)}, \quad (3.15)$$

$$[\mathcal{B}_0^{(i)}, \hat{\mathcal{L}}^{(j)}] = \delta^{ij} \hat{\mathcal{B}}^{(j)}, \quad \{\mathcal{B}_0^{(i)}, \hat{\mathcal{B}}^{(j)}\} = 0, \quad \{\mathcal{B}_0^{(i)}, \tilde{c}_p^{(j)}\} = \delta^{ij} \delta_{0,p}, \quad (3.16)$$

we can show that all two point correlation functions involving string fields constructed out of the operators $\hat{\mathcal{L}}$, $\hat{\mathcal{B}}$ and $\tilde{c}_p$ can be reduced to the evaluation of the following basic correlators

$$\begin{aligned} \langle V_2 | \tilde{c}_{p_1}^{(2)} \tilde{c}_{p_2}^{(2)} \tilde{c}_{p_3}^{(2)} \rangle &= \langle V_2 | \tilde{c}_{p_1}^{(1)} \tilde{c}_{p_2}^{(1)} \tilde{c}_{p_3}^{(1)} \rangle \\ &= \oint \frac{dx_1 dx_2 dx_3}{(2\pi i)^3} x_1^{p_1-2} x_2^{p_2-2} x_3^{p_3-2} \langle c(x_1) c(x_2) c(x_3) \rangle_{\mathcal{C}_2}, \end{aligned} \quad (3.17)$$

$$\begin{aligned} \langle V_2 | \tilde{c}_{p_1}^{(1)} \tilde{c}_{p_2}^{(2)} \tilde{c}_{p_3}^{(2)} \rangle &= \langle V_2 | \tilde{c}_{p_2}^{(1)} \tilde{c}_{p_3}^{(1)} \tilde{c}_{p_1}^{(2)} \rangle \\ &= \oint \frac{dx_1 dx_2 dx_3}{(2\pi i)^3} x_1^{p_1-2} x_2^{p_2-2} x_3^{p_3-2} \langle c(x_1 + 1) c(x_2) c(x_3) \rangle_{\mathcal{C}_2}, \end{aligned} \quad (3.18)$$

where the correlator $\langle c(x) c(y) c(z) \rangle_{\mathcal{C}_L}$ in general is given by

$$\langle c(x) c(y) c(z) \rangle_{\mathcal{C}_L} = \frac{L^3}{\pi^3} \sin \left(\frac{\pi(x-y)}{L} \right) \sin \left(\frac{\pi(x-z)}{L} \right) \sin \left(\frac{\pi(y-z)}{L} \right). \quad (3.19)$$

To evaluate explicitly the above correlators (3.17) and (3.18), the following formulas will be very useful

$$S_{a,b} \equiv \oint \frac{dz}{2\pi i} z^a \sin(bz) = -\frac{b^{-a-1} \cos\left(\frac{\pi a}{2}\right)}{\Gamma(-a)}, \quad (3.20)$$

$$C_{a,b} \equiv \oint \frac{dz}{2\pi i} z^a \cos(bz) = -\frac{b^{-a-1} \sin\left(\frac{\pi a}{2}\right)}{\Gamma(-a)}. \quad (3.21)$$

For instance, let us compute correlator (3.17). Using (3.19) into equation (3.17), we have

$$\begin{aligned}
 \langle V_2 | \tilde{c}_{p_1}^{(2)} \tilde{c}_{p_2}^{(2)} \tilde{c}_{p_3}^{(2)} \rangle &= \langle V_2 | \tilde{c}_{p_1}^{(1)} \tilde{c}_{p_2}^{(1)} \tilde{c}_{p_3}^{(1)} \rangle \\
 &= \oint \frac{dx_1 dx_2 dx_3}{(2\pi i)^3} x_1^{p_1-2} x_2^{p_2-2} x_3^{p_3-2} \langle c(x_1) c(x_2) c(x_3) \rangle_{C_2} \\
 &= \frac{2}{\pi^3} \oint \frac{dx_1 dx_2 dx_3}{(2\pi i)^3} x_1^{p_1-2} x_2^{p_2-2} x_3^{p_3-2} \\
 &\quad \times \left[\sin(\pi x_1) \cos(\pi x_2) - \sin(\pi x_1) \cos(\pi x_3) \right. \\
 &\quad \left. + \sin(\pi x_2) \cos(\pi x_3) - \sin(\pi x_2) \cos(\pi x_1) \right. \\
 &\quad \left. + \sin(\pi x_3) \cos(\pi x_1) - \sin(\pi x_3) \cos(\pi x_2) \right]. \quad (3.22)
 \end{aligned}$$

It is clear that the above equation (3.22) can be written in terms of the functions (3.20) and (3.21), so that we arrive to an explicit expression for the correlator (3.17)

$$\begin{aligned}
 \langle V_2 | \tilde{c}_{p_1}^{(2)} \tilde{c}_{p_2}^{(2)} \tilde{c}_{p_3}^{(2)} \rangle &= \langle V_2 | \tilde{c}_{p_1}^{(1)} \tilde{c}_{p_2}^{(1)} \tilde{c}_{p_3}^{(1)} \rangle \\
 &= \frac{2}{\pi^3} \left[\delta_{1,p_3} S_{p_1-2,\pi} C_{p_2-2,\pi} - \delta_{1,p_2} C_{p_3-2,\pi} S_{p_1-2,\pi} \right. \\
 &\quad \left. - \delta_{1,p_3} C_{p_1-2,\pi} S_{p_2-2,\pi} + \delta_{1,p_1} C_{p_3-2,\pi} S_{p_2-2,\pi} \right. \\
 &\quad \left. + \delta_{1,p_2} C_{p_1-2,\pi} S_{p_3-2,\pi} - \delta_{1,p_1} C_{p_2-2,\pi} S_{p_3-2,\pi} \right]. \quad (3.23)
 \end{aligned}$$

In the same way, we can also derive the explicit expression for the correlator (3.18)

$$\begin{aligned}
 \langle V_2 | \tilde{c}_{p_1}^{(1)} \tilde{c}_{p_2}^{(2)} \tilde{c}_{p_3}^{(2)} \rangle &= \langle V_2 | \tilde{c}_{p_2}^{(1)} \tilde{c}_{p_3}^{(1)} \tilde{c}_{p_1}^{(2)} \rangle \\
 &= \frac{2}{\pi^3} \left[\delta_{1,p_2} S_{p_1-2,\pi} C_{p_3-2,\pi} - \delta_{1,p_3} C_{p_2-2,\pi} S_{p_1-2,\pi} \right. \\
 &\quad \left. - \delta_{1,p_2} C_{p_1-2,\pi} S_{p_3-2,\pi} + \delta_{1,p_3} C_{p_1-2,\pi} S_{p_2-2,\pi} \right. \\
 &\quad \left. + \delta_{1,p_1} C_{p_3-2,\pi} S_{p_2-2,\pi} - \delta_{1,p_1} C_{p_2-2,\pi} S_{p_3-2,\pi} \right]. \quad (3.24)
 \end{aligned}$$

To evaluate the kinetic term $\text{tr}[\Phi Q_B \Phi]$ for a string field Φ expanded in the basis of curly $\mathcal{L}_0$ eigenstates, it will be convenient to write the kinetic term in the language of a two point vertex

$$\text{tr}[\Phi Q_B \Phi] = \langle V_2 | \Phi^{(1)} Q_B \Phi^{(2)} \rangle. \quad (3.25)$$

Note that in addition to the conservation laws, we will be required to know the action of the BRST charge Q_B on the operators $\hat{\mathcal{L}}$, $\hat{\mathcal{B}}$ and $\tilde{c}_p$

$$[Q_B, \hat{\mathcal{L}}^{(j)}] = 0, \quad \{Q_B, \hat{\mathcal{B}}^{(j)}\} = \hat{\mathcal{L}}^{(j)}, \quad \{Q_B, \tilde{c}_p^{(j)}\} = \sum_{k=-\infty}^{\infty} (1-k) \tilde{c}_{p-k}^{(j)} \tilde{c}_k^{(j)}. \quad (3.26)$$

As an illustration of the use of conservation laws, we are going to compute a particular correlator involving the operators $\hat{\mathcal{B}}$ and $\hat{\mathcal{L}}$. We choose, as an example, the following string fields

$$\phi = \hat{\mathcal{B}} \hat{\mathcal{L}} \tilde{c}_0 \tilde{c}_1 |0\rangle, \quad \psi = \tilde{c}_1 |0\rangle. \quad (3.27)$$

Using these string fields, let us evaluate the correlator

$$\text{tr}[\phi Q_B \psi] = \langle V_2 | \phi^{(1)} Q_B \psi^{(2)} \rangle. \quad (3.28)$$

Inserting equation (3.27) into equation (3.28) and using (3.26), we obtain

$$\text{tr}[\phi Q_B \psi] = -\langle V_2 | \hat{\mathcal{B}}^{(1)} \hat{\mathcal{L}}^{(1)} \tilde{c}_0^{(1)} \tilde{c}_1^{(1)} \tilde{c}_0^{(2)} \tilde{c}_1^{(2)} \rangle. \quad (3.29)$$

Using the conservation law (3.14) and the anti-commutator relations (3.16), from equation (3.29) we get

$$\text{tr}[\phi Q_B \psi] = -\langle V_2 | \hat{\mathcal{L}}^{(1)} \tilde{c}_0^{(1)} \tilde{c}_1^{(1)} \tilde{c}_1^{(2)} \rangle - \langle V_2 | \hat{\mathcal{L}}^{(1)} \tilde{c}_1^{(1)} \tilde{c}_0^{(2)} \tilde{c}_1^{(2)} \rangle - \langle V_2 | \hat{\mathcal{B}}^{(1)} \tilde{c}_0^{(1)} \tilde{c}_1^{(1)} \tilde{c}_0^{(2)} \tilde{c}_1^{(2)} \rangle. \quad (3.30)$$

Employing the conservation laws (3.12), (3.14) and the commutator and anti-commutator relations (3.15), (3.16), from equation (3.30) we arrive to

$$\text{tr}[\phi Q_B \psi] = \langle V_2 | \tilde{c}_0^{(1)} \tilde{c}_1^{(1)} \tilde{c}_1^{(2)} \rangle + \langle V_2 | \tilde{c}_1^{(1)} \tilde{c}_0^{(2)} \tilde{c}_1^{(2)} \rangle = 2 \langle V_2 | \tilde{c}_1^{(1)} \tilde{c}_0^{(2)} \tilde{c}_1^{(2)} \rangle = 2 \left(\frac{4}{\pi^2} \right) = \frac{8}{\pi^2}, \quad (3.31)$$

where we have used equation (3.24). These kind of computations can be automated in a computer. Next, we are going to apply the results shown in this section to evaluate the kinetic term by means of the curly $\mathcal{L}_0$ level expansion of the real solution (2.1).

4 Curly $\mathcal{L}_0$ level expansion analysis of the real solution

Since the kinetic term does not depend on the Q_B -exact terms, we are going to consider only the first term of Φ given in equation (2.1). Let us define this term as

$$\hat{\Phi} = \frac{1}{4} \left(\frac{1}{1+K} c + c \frac{1}{1+K} + c \frac{B}{1+K} c + \frac{1}{1+K} c \frac{1}{1+K} \right). \quad (4.1)$$

Using the integral representation of $1/(1+K)$

$$\frac{1}{1+K} = \int_0^\infty dt e^{-t(1+K)} = \int_0^\infty dt e^{-t\Omega^t}, \quad (4.2)$$

we can write (4.1) as

$$\hat{\Phi} = \frac{1}{4} \left[\int_0^\infty dt e^{-t} \left(\Omega^t c + c \Omega^t + c \Omega^t B c \right) + \int_0^\infty ds dt e^{-s-t} \Omega^s c \Omega^t \right]. \quad (4.3)$$

By writing the basic string fields K , B in terms of the operators $\hat{\mathcal{L}}$, $\hat{\mathcal{B}}$, and using the modes $\tilde{c}_p$ of the ghost field $c(z)$ defined in the $\tilde{z}$ -conformal frame $\tilde{z} = \frac{2}{\pi} \arctan z$, we can show that

$$\begin{aligned} \Omega^{t_1} c \Omega^{t_2} B c \Omega^{t_3} &= \sum_{n=0}^\infty \sum_{p=-\infty}^1 \frac{\beta^n}{2n!} (x^{1-p} + y^{1-p}) \hat{\mathcal{L}}^n \tilde{c}_p |0\rangle \\ &+ \sum_{n=0}^\infty \sum_{p=-\infty}^1 \sum_{q=-\infty}^1 \frac{\beta^n}{4n!} (x^{1-p} y^{1-q} - x^{1-q} y^{1-p}) \hat{\mathcal{B}} \hat{\mathcal{L}}^n \tilde{c}_p \tilde{c}_q |0\rangle, \end{aligned} \quad (4.4)$$

where

$$\beta = \frac{1}{2} - \frac{1}{2}(t_1 + t_2 + t_3), \quad x = \frac{1}{2}(t_3 - t_1 - t_2), \quad y = \frac{1}{2}(t_2 + t_3 - t_1). \quad (4.5)$$

Employing equation (4.4), it is possible to derive the curly $\mathcal{L}_0$ level expansion of the string field defined in equation (4.3). As a pedagogical illustration, let us explicitly compute the curly $\mathcal{L}_0$ level expansion of the last term appearing on the right hand side of equation (4.3)

$$\int_0^\infty ds dt e^{-s-t} \Omega^s c \Omega^t = \sum_{n=0}^\infty \sum_{p=-\infty}^1 \int_0^\infty ds dt e^{-s-t} \frac{\beta^n}{2n!} (x^{1-p} + y^{1-p}) \hat{\mathcal{L}}^n \tilde{c}_p |0\rangle, \quad (4.6)$$

where in this case

$$\beta = \frac{1}{2} - \frac{1}{2}(s + t), \quad x = y = \frac{1}{2}(t - s). \quad (4.7)$$

As we can see from equations (4.6) and (4.7), we are required to evaluate the following double integral

$$\int_0^\infty ds dt e^{-s-t} \frac{\beta^n}{n!} x^{1-p} = \int_0^\infty ds dt e^{-s-t} \frac{2^{-n+p-1} (-s-t+1)^n (t-s)^{1-p}}{n!}. \quad (4.8)$$

Performing the change of variables $s \rightarrow uv$, $t \rightarrow u - uv$, $\int_0^\infty ds dt \rightarrow \int_0^\infty du \int_0^1 dv u$ into the above integral (4.8), we obtain

$$\begin{aligned} \int_0^\infty ds dt e^{-s-t} \frac{\beta^n}{n!} x^{1-p} &= \int_0^\infty du \int_0^1 dv \frac{e^{-u} 2^{-n+p-1} (1-u)^n u^{2-p} (1-2v)^{1-p}}{n!} \\ &= \frac{((-1)^p - 1) 2^{-n+p-2}}{(p-2)n!} \int_0^\infty du e^{-u} (1-u)^n u^{2-p} \\ &= \frac{((-1)^p - 1) 2^{-n+p-2}}{(p-2)n!} \mathcal{F}(n, 2-p), \end{aligned} \quad (4.9)$$

where we have defined

$$\mathcal{F}(M, N) = \int_0^\infty du e^{-u} (1-u)^M u^N = \sum_{k=0}^M (-1)^{M-k} \binom{M}{k} (M+N-k)! \quad (4.10)$$

Proceeding in the same way, we can also calculate the curly $\mathcal{L}_0$ level expansion of the first terms appearing on the right hand side of equation (4.3). Adding up all the results, we show that the string field (4.1) has the following curly $\mathcal{L}_0$ level expansion

$$\hat{\Phi} = \sum_{n=0}^\infty \sum_{p=-\infty}^1 f_{n,p} \hat{\mathcal{L}}^n \tilde{c}_p |0\rangle + \sum_{n=0}^\infty \sum_{p=-\infty}^1 \sum_{q=-\infty}^1 f_{n,p,q} \hat{\mathcal{B}} \hat{\mathcal{L}}^n \tilde{c}_p \tilde{c}_q |0\rangle, \quad (4.11)$$

where the coefficients $f_{n,p}$ and $f_{n,p,q}$ are given by

$$f_{n,p} = \frac{(1 - (-1)^p) 2^{-n+p-4} \left(3\mathcal{F}(n, 1-p) + \frac{1}{2-p} \mathcal{F}(n, 2-p) \right)}{n!}, \quad (4.12)$$

$$f_{n,p,q} = \frac{((-1)^q - (-1)^p) 2^{-n+p+q-6} \mathcal{F}(n, 2-p-q)}{n!}. \quad (4.13)$$

To compute the kinetic term, we start by replacing the string field $\hat{\Phi}$ with $z^{\mathcal{L}_0}\hat{\Phi}$, so that states in the curly $\mathcal{L}_0$ level expansion will acquire different integer powers of z at different levels. As we are going to see, the parameter z is needed because we need to express the kinetic term as a formal power series expansion if we want to use Padé approximants. After doing our calculations, we will simply set $z = 1$.

Let us start with the evaluation of the kinetic term as a formal power series expansion in z . By inserting the expansion (4.11) of the string field $\hat{\Phi}$ into the kinetic term, and using the conservation laws studied in section 3 to evaluate the corresponding two point vertices, we obtain

$$\begin{aligned} \text{tr}[z^{\mathcal{L}_0}\hat{\Phi} Q_B(z^{\mathcal{L}_0}\hat{\Phi})] = & -\frac{4}{\pi^2 z^2} + \left(1 - \frac{2}{\pi^2}\right) - z + \left(\frac{3}{2} - \frac{3\pi^2}{8}\right)z^2 + \left(-\frac{7}{2} + \frac{19\pi^2}{8}\right)z^3 \\ & + \left(\frac{41}{4} - \frac{51\pi^2}{4} + \frac{\pi^4}{8}\right)z^4 + \left(-36 + \frac{279\pi^2}{4} - \frac{35\pi^4}{16}\right)z^5 \\ & + \left(\frac{293}{2} - \frac{1615\pi^2}{4} + \frac{825\pi^4}{32} - \frac{5\pi^6}{128}\right)z^6 + \dots \end{aligned} \quad (4.14)$$

Considering terms up to order z^6 , and setting $z = 1$, from equation (4.14) we get 3328% of the expected result (2.4). In principle, we can compute the curly $\mathcal{L}_0$ level expansion of the kinetic term up to any desired order, however as we increase the order, the involved tasks demand a lot of computing time. We have determined the series (4.14) up to order z^{18} , and setting $z = 1$, we obtain about $1.5036 \times 10^{15}\%$ of the expected result. As we can see, if we naively set $z = 1$ and sum the series, we are left with a non-convergent result.

Recall that in numerical curly $\mathcal{L}_0$ level truncation computations, a regularization technique based on Padé approximants provides desired results for gauge invariant quantities like the energy [6, 20, 23, 24]. Let us see if after applying Padé approximants, we can recover the expected result.

To start with Padé approximants, first let us define the normalized value of the kinetic term as follows

$$\hat{E}(z) \equiv \frac{\pi^2 z^2}{3} \text{tr}[z^{\mathcal{L}_0}\hat{\Phi} Q_B(z^{\mathcal{L}_0}\hat{\Phi})]. \quad (4.15)$$

Since the series for the kinetic term (4.14) is known up to order z^{18} , we can write the series for $\hat{E}(z)$ up to order z^{20} , and after considering a numerical value for π , we obtain

$$\begin{aligned} \hat{E}(z) = \sum E_k z^k = & -1.33333 + 2.6232z^2 - 3.28987z^3 - 7.24133z^4 + 65.601z^5 \\ & - 340.21z^6 + 1445.31z^7 - 4489.28z^8 - 1862.15z^9 + 218120.z^{10} \\ & - 2.84231 \times 10^6 z^{11} + 2.83085 \times 10^7 z^{12} - 2.4607 \times 10^8 z^{13} \\ & + 1.87127 \times 10^9 z^{14} - 1.1131 \times 10^{10} z^{15} + 1.91077 \times 10^{10} z^{16} \\ & + 9.10893 \times 10^{11} z^{17} - 2.20996 \times 10^{13} z^{18} + 3.69796 \times 10^{14} z^{19} \\ & - 5.29538 \times 10^{15} z^{20}. \end{aligned} \quad (4.16)$$

In general, to construct a Padé approximant of order $P_n^n(z)$ for the normalized value of the kinetic term (4.15), we need to truncate the series (4.16) up to order z^{2n} .

As an illustration, let us compute the normalized value of the kinetic term using a Padé approximant of order $P_2^2(z)$. First, we express $\hat{E}(z)$ as the rational function $P_2^2(z)$

$$\hat{E}(z) = P_2^2(z) = \frac{a_0 + a_1 z + a_2 z^2}{1 + b_1 z + b_2 z^2}. \quad (4.17)$$

Expanding the right hand side of (4.17) around $z = 0$ up to order z^4 and equating the coefficients of z^0, z^1, z^2, z^3, z^4 with the expansion (4.16), we get a system of algebraic equations for the unknown coefficients a_0, a_1, a_2, b_1 , and b_2 . Solving those equations we get

$$a_0 = -1.3333, \quad a_1 = -1.6721, \quad a_2 = -3.1546, \quad b_1 = 1.2541, \quad b_2 = 4.3333. \quad (4.18)$$

Replacing the value of these coefficients inside the definition of $P_2^2(z)$ (4.17), and evaluating this at $z = 1$, we get the following value

$$P_2^2(z = 1) = -0.935125008. \quad (4.19)$$

The results of our calculations are summarized in table 1. As we can see, the value of $\hat{E}(z)$ at $z = 1$ by means of Padé approximants confirms the expected analytical result $\hat{E}(1) = \frac{\pi^2}{3} \text{tr}[\hat{\Phi} Q_B \hat{\Phi}] \rightarrow -1$. Although the convergence to the expected answer gets irregular at $n = 4$, by considering higher level contributions, we will eventually reach to the right value.

Using an alternative resummation technique, we would like to confirm the expected answer for the normalized value of the kinetic term. We have used a second method which is based on a combination of Padé and Borel resummation. We replace the Borel transform of $\hat{E}(z)$, which is defined as $\hat{E}(z)_{\text{Borel}} = \sum E_k z^k / k!$, by its Padé approximant $P_n^n(z)_{\text{Borel}}$ and then evaluate the integral

$$\tilde{P}_n^n(z) = \int_0^\infty dt e^{-t} P_n^n(z t)_{\text{Borel}} \quad (4.20)$$

at $z = 1$. In the third column of table 1, we list the results obtained for $\hat{E}(1)$ by means of Padé-Borel approximation. Note that starting at the value of $n = 4$, Padé-Borel does a little better than Padé.

5 L_0 level expansion analysis of the real solution

To expand the string field (4.3) in the Virasoro basis of L_0 eigenstates, we are going to use the following formulas

$$\begin{aligned} & e^{-t_1 K} c e^{-t_2 K} B c e^{-t_3 K} \\ &= \frac{r \cos^2\left(\frac{\pi x}{r}\right) \left(\pi(r - 2y) - r \sin\left(\frac{2\pi y}{r}\right)\right)}{4\pi^2} \tilde{U}_r c \left(\frac{2 \tan\left(\frac{\pi x}{r}\right)}{r}\right) |0\rangle \\ &+ \frac{r \cos^2\left(\frac{\pi y}{r}\right) \left(\pi(r + 2x) + r \sin\left(\frac{2\pi x}{r}\right)\right)}{4\pi^2} \tilde{U}_r c \left(\frac{2 \tan\left(\frac{\pi y}{r}\right)}{r}\right) |0\rangle \end{aligned}$$

	P_n^n	$\tilde{P}_n^n$	P_0^{2n}
$n = 0$	-1.3333333333	-1.3333333333	-1.3333333333
$n = 2$	-0.9351250080	-0.6792579899	-9.2413341787
$n = 4$	-0.7462344772	-0.9160629680	-3327.8214730
$n = 6$	-0.9803952323	-0.9938587065	2.56791×10^7
$n = 8$	-0.9800827399	-1.0020031889	9.62763×10^9
$n = 10$	-0.9997340118	-1.0017620332	-4.94676×10^{15}

Table 1. The Padé and Padé-Borel approximations for the normalized value of the kinetic term $\hat{E}(z) = \frac{\pi^2 z^2}{3} \text{tr}[z^{\mathcal{L}_0} \hat{\Phi} Q_B(z^{\mathcal{L}_0} \hat{\Phi})]$ evaluated at $z = 1$. The second column shows the P_n^n Padé approximation. The third column shows the corresponding $\tilde{P}_n^n$ Padé-Borel approximation. In the last column, P_0^{2n} represents a trivial approximation, a naively summed series.

$$\begin{aligned}
 & + \sum_{k=1}^{\infty} \frac{(-1)^{k+1} 2^{2k-1} \left(\frac{1}{r}\right)^{2k-3} \cos^2\left(\frac{\pi x}{r}\right) \cos^2\left(\frac{\pi y}{r}\right)}{(4k^2 - 1) \pi^2} \\
 & \quad \times \tilde{U}_r b_{-2k} c\left(\frac{2}{r} \tan\left(\frac{\pi x}{r}\right)\right) c\left(\frac{2}{r} \tan\left(\frac{\pi y}{r}\right)\right) |0\rangle, \quad (5.1)
 \end{aligned}$$

$$r = t_1 + t_2 + t_3 + 1, \quad x = \frac{1}{2}(t_3 - t_1 - t_2), \quad y = \frac{1}{2}(t_2 + t_3 - t_1), \quad (5.2)$$

where the operator $\tilde{U}_r$ is defined as

$$\tilde{U}_r \equiv \dots e^{u_{10,r} L_{-10}} e^{u_{8,r} L_{-8}} e^{u_{6,r} L_{-6}} e^{u_{4,r} L_{-4}} e^{u_{2,r} L_{-2}}. \quad (5.3)$$

To find the coefficients $u_{n,r}$ appearing in the exponentials, we use

$$\begin{aligned}
 \frac{r}{2} \tan\left(\frac{2}{r} \arctan z\right) &= \lim_{N \rightarrow \infty} [f_{2,u_{2,r}} \circ f_{4,u_{4,r}} \circ f_{6,u_{6,r}} \circ f_{8,u_{8,r}} \circ f_{10,u_{10,r}} \circ \dots \circ f_{N,u_{N,r}}(z)] \\
 &= \lim_{N \rightarrow \infty} [f_{2,u_{2,r}}(f_{4,u_{4,r}}(f_{6,u_{6,r}}(f_{8,u_{8,r}}(f_{10,u_{10,r}}(\dots(f_{N,u_{N,r}}(z))\dots)))))], \quad (5.4)
 \end{aligned}$$

where the function $f_{n,u_{n,r}}(z)$ is given by

$$f_{n,u_{n,r}}(z) = \frac{z}{(1 - u_{n,r} n z^n)^{1/n}}. \quad (5.5)$$

Employing the set of equations (5.1)–(5.3) for the string field (4.3), we obtain

$$\begin{aligned}
 \hat{\Phi} &= \int_0^\infty dt \frac{e^{-t} r \sin^2\left(\frac{\pi}{2r}\right) (2\pi r - r \sin\left(\frac{\pi}{r}\right) + \pi)}{16\pi^2} \tilde{U}_r \left(c\left(-\frac{2 \tan\left(\frac{\pi t}{2r}\right)}{r}\right) + c\left(\frac{2 \tan\left(\frac{\pi t}{2r}\right)}{r}\right) \right) \\
 &+ \int_0^\infty dt \sum_{k=1}^{\infty} \frac{e^{-t} (-1)^{k+1} 2^{2k-3} \left(\frac{1}{r}\right)^{2k-3} \sin^4\left(\frac{\pi}{2r}\right)}{\pi^2 (4k^2 - 1)} \tilde{U}_r b_{-2k} c\left(-\frac{2 \tan\left(\frac{\pi t}{2r}\right)}{r}\right) c\left(\frac{2 \tan\left(\frac{\pi t}{2r}\right)}{r}\right) \\
 &+ \int_0^\infty ds \int_0^\infty dt \frac{e^{-s-t} (1+s+t)^2 \cos^2\left(\frac{\pi(t-s)}{2(1+s+t)}\right)}{8\pi} \tilde{U}_{1+s+t} c\left(\frac{2 \tan\left(\frac{\pi(t-s)}{2(1+s+t)}\right)}{1+s+t}\right), \quad (5.6)
 \end{aligned}$$

where $r = 1 + t$.

By writing the c ghost in terms of its modes $c(z) = \sum_m c_m/z^{m-1}$ and employing equations (5.3) and (5.6), the string field $\hat{\Phi}$ can be readily expanded and the individual coefficients can be numerically integrated. For instance, let us write the expansion of $\hat{\Phi}$ up to level four states

$$\begin{aligned} \hat{\Phi} = & 0.45457753c_1|0\rangle + 0.17214438c_{-1}|0\rangle - 0.03070678L_{-2}c_{-1}|0\rangle - 0.01400692b_{-2}c_0c_1|0\rangle \\ & - 0.00605891L_{-4}c_1|0\rangle + 0.02033379L_{-2}L_{-2}c_1|0\rangle + 0.16194599c_{-3}|0\rangle \\ & - 0.00976204b_{-2}c_{-2}c_1|0\rangle - 0.01053192L_{-2}c_{-1}|0\rangle + 0.00976204b_{-2}c_{-1}c_0|0\rangle \\ & + 0.00465417b_{-4}c_0c_1|0\rangle - 0.00308797L_{-2}b_{-2}c_0c_1|0\rangle + \dots \end{aligned} \quad (5.7)$$

As in the case of the curly $\mathcal{L}_0$ level expansion analysis, to evaluate the normalized value of the vacuum energy, first we perform the replacement $\hat{\Phi} \rightarrow z^{L_0}\hat{\Phi}$ and then using the resulting string field $z^{L_0}\hat{\Phi}$, we define, the analogue of equation (4.15)

$$\tilde{E}(z) = \frac{\pi^2 z^2}{3} \text{tr}[z^{L_0}\hat{\Phi} Q_B(z^{L_0}\hat{\Phi})]. \quad (5.8)$$

The normalized value of the vacuum energy is obtained just by setting $z = 1$. Since the kinetic term is diagonal in L_0 eigenstates, the coefficients of the energy (5.8) at order z^{2L} are exactly the contributions from fields at level L . We have expanded the string field $\hat{\Phi}$ given in equation (5.6) up to level twelve states, and hence the series of $\tilde{E}(z)$ can be determined up to the order z^{24}

$$\begin{aligned} \tilde{E}(z) = & -0.6798207 - 0.1505669z^4 - 0.2304622z^8 + 0.238568z^{12} - 0.3465834z^{16} \\ & + 0.4456892z^{20} - 0.58817204z^{24}. \end{aligned} \quad (5.9)$$

If we naively evaluate the truncated vacuum energy (5.9), i.e., setting $z = 1$ in the series before using Padé or Padé-Borel approximations, we obtain a non-convergent result. Note that the series (5.9) is less divergent than the series (4.16) that has been obtained in the case of the curly $\mathcal{L}_0$ level expansion analysis of the energy.

Let us re-sum the divergent series (5.9). To obtain the Padé or Padé-Borel approximation of order P_n^n for the energy, we will need to know the series expansion of $\tilde{E}(z)$ up to the order z^{2n} . The results of these numerical calculations are summarized in table 2.

6 Summary and discussion

We have analyzed the validity of the recently proposed real tachyon vacuum solution [11], in open bosonic string field theory. We have found that the solution solves in a non trivial way the equation of motion when contracted with itself. Let us point out that a similar test of consistency was performed by Okawa [18], Fuchs, Kroyter [19] and Arroyo [20] for the case of the original Schnabl's solution [3].

As a second test of consistency, we have analyzed the solution from a numerical point of view. Using either the curly $\mathcal{L}_0$, or the Virasoro L_0 level expansion of the solution, we have found that the expression representing the energy is given in terms of a divergent series,

	P_n^n	$\tilde{P}_n^n$	P_0^{2n}
$n = 0$	-0.6798207586	-0.6798207586	-0.6798207586
$n = 4$	-0.3960692519	-0.8200429863	-1.0608500359
$n = 8$	-0.9687853277	-0.9606366393	-1.1688654612
$n = 12$	-0.9782343686	-0.9537697224	-1.3113482982

Table 2. The Padé and Padé-Borel approximations for the normalized value of the vacuum energy $\tilde{E}(z) = \frac{\pi^2 z^2}{3} \text{tr}[z^{L_0} \hat{\Phi} Q_B(z^{L_0} \hat{\Phi})]$ evaluated at $z = 1$. The second column shows the P_n^n Padé approximation. The third column shows the corresponding $\tilde{P}_n^n$ Padé-Borel approximation. In the last column, P_0^{2n} represents a trivial approximation, a naively summed series.

which nevertheless can be re-summed, either by means of Padé technique or a combination of Padé-Borel resummation to bring the expected result in agreement with Sen’s conjecture.

It would be interesting to analyze other real solutions. For instance, the tachyon vacuum solution corresponding to the regularized identity based solution [8]. The real version of this solution, obtained by means of a similarity transformation, contains square roots and consequently the analytical and numerical computations of the energy become cumbersome [9, 23]. Employing the prescription studied in reference [11], it should be possible to find an alternative real version for this regularized identity based solution.

Finally, regarding to the modified cubic superstring field theory [25] and Berkovits non-polynomial open superstring field theory [26], since these theories are based on Witten’s associative star product, their mathematical setup shares the same algebraic structure of the open bosonic string field theory, and thus the prescription developed in reference [11] and the results shown in this paper should be extended to construct and study new real solutions in the superstring context like the ones discussed in references [24, 27–32].

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References

- [1] E. Witten, *Noncommutative geometry and string field theory*, *Nucl. Phys. B* **268** (1986) 253 [[INSPIRE](#)].
- [2] M.R. Gaberdiel and B. Zwiebach, *Tensor constructions of open string theories. 1: foundations*, *Nucl. Phys. B* **505** (1997) 569 [[hep-th/9705038](#)] [[INSPIRE](#)].
- [3] M. Schnabl, *Analytic solution for tachyon condensation in open string field theory*, *Adv. Theor. Math. Phys.* **10** (2006) 433 [[hep-th/0511286](#)] [[INSPIRE](#)].

- [4] M. Schnabl, *Algebraic solutions in open string field theory — A lightning review*, *Acta Polytechn.* **50** (2010) 102 [[arXiv:1004.4858](#)] [[INSPIRE](#)].
- [5] T. Erler and C. Maccaferri, *The phantom term in open string field theory*, *JHEP* **06** (2012) 084 [[arXiv:1201.5122](#)] [[INSPIRE](#)].
- [6] T. Erler and M. Schnabl, *A simple analytic solution for tachyon condensation*, *JHEP* **10** (2009) 066 [[arXiv:0906.0979](#)] [[INSPIRE](#)].
- [7] E.A. Arroyo, *Generating Erler-Schnabl-type solution for tachyon vacuum in cubic superstring field theory*, *J. Phys. A* **43** (2010) 445403 [[arXiv:1004.3030](#)] [[INSPIRE](#)].
- [8] S. Zeze, *Regularization of identity based solution in string field theory*, *JHEP* **10** (2010) 070 [[arXiv:1008.1104](#)] [[INSPIRE](#)].
- [9] E.A. Arroyo, *Comments on regularization of identity based solutions in string field theory*, *JHEP* **11** (2010) 135 [[arXiv:1009.0198](#)] [[INSPIRE](#)].
- [10] T. Erler and C. Maccaferri, *Connecting solutions in open string field theory with singular gauge transformations*, *JHEP* **04** (2012) 107 [[arXiv:1201.5119](#)] [[INSPIRE](#)].
- [11] M. Jokel, *Real tachyon vacuum solution without square roots*, [arXiv:1704.02391](#) [[INSPIRE](#)].
- [12] I. Ellwood, B. Feng, Y.-H. He and N. Moeller, *The identity string field and the tachyon vacuum*, *JHEP* **07** (2001) 016 [[hep-th/0105024](#)] [[INSPIRE](#)].
- [13] I. Ellwood and M. Schnabl, *Proof of vanishing cohomology at the tachyon vacuum*, *JHEP* **02** (2007) 096 [[hep-th/0606142](#)] [[INSPIRE](#)].
- [14] S. Inatomi, I. Kishimoto and T. Takahashi, *Homotopy operators and one-loop vacuum energy at the tachyon vacuum*, *Prog. Theor. Phys.* **126** (2011) 1077 [[arXiv:1106.5314](#)] [[INSPIRE](#)].
- [15] A. Sen, *Descent relations among bosonic D-branes*, *Int. J. Mod. Phys. A* **14** (1999) 4061 [[hep-th/9902105](#)] [[INSPIRE](#)].
- [16] A. Sen, *Universality of the tachyon potential*, *JHEP* **12** (1999) 027 [[hep-th/9911116](#)] [[INSPIRE](#)].
- [17] T. Erler, *Split string formalism and the closed string vacuum*, *JHEP* **05** (2007) 083 [[hep-th/0611200](#)] [[INSPIRE](#)].
- [18] Y. Okawa, *Comments on Schnabl's analytic solution for tachyon condensation in Witten's open string field theory*, *JHEP* **04** (2006) 055 [[hep-th/0603159](#)] [[INSPIRE](#)].
- [19] E. Fuchs and M. Kroyter, *On the validity of the solution of string field theory*, *JHEP* **05** (2006) 006 [[hep-th/0603195](#)] [[INSPIRE](#)].
- [20] E.A. Arroyo, *Cubic interaction term for Schnabl's solution using Padé approximants*, *J. Phys. A* **42** (2009) 375402 [[arXiv:0905.2014](#)] [[INSPIRE](#)].
- [21] L. Rastelli and B. Zwiebach, *Tachyon potentials, star products and universality*, *JHEP* **09** (2001) 038 [[hep-th/0006240](#)] [[INSPIRE](#)].
- [22] E.A. Arroyo, *Conservation laws and tachyon potentials in the sliver frame*, *JHEP* **06** (2011) 033 [[arXiv:1103.4830](#)] [[INSPIRE](#)].
- [23] E. Aldo Arroyo, *Level truncation analysis of regularized identity based solutions*, *JHEP* **11** (2011) 079 [[arXiv:1109.5354](#)] [[INSPIRE](#)].
- [24] E. Aldo Arroyo, *Level truncation analysis of a simple tachyon vacuum solution in cubic superstring field theory*, *JHEP* **12** (2014) 069 [[arXiv:1409.1890](#)] [[INSPIRE](#)].

- [25] I. Ya. Arefeva, P.B. Medvedev and A.P. Zubarev, *New representation for string field solves the consistency problem for open superstring field theory*, *Nucl. Phys. B* **341** (1990) 464 [[INSPIRE](#)].
- [26] N. Berkovits, *SuperPoincaré invariant superstring field theory*, *Nucl. Phys. B* **450** (1995) 90 [*Erratum ibid.* **B 459** (1996) 439] [[hep-th/9503099](#)] [[INSPIRE](#)].
- [27] E. Aldo Arroyo, *Comments on multibrane solutions in cubic superstring field theory*, *PTEP* **2014** (2014) 063B03 [[arXiv:1306.1865](#)] [[INSPIRE](#)].
- [28] E. Aldo Arroyo, *Multibrane solutions in cubic superstring field theory*, *JHEP* **06** (2012) 157 [[arXiv:1204.0213](#)] [[INSPIRE](#)].
- [29] E.A. Arroyo, *A singular one-parameter family of solutions in cubic superstring field theory*, *JHEP* **05** (2016) 013 [[arXiv:1602.00059](#)] [[INSPIRE](#)].
- [30] T. Erler, *Tachyon vacuum in cubic superstring field theory*, *JHEP* **01** (2008) 013 [[arXiv:0707.4591](#)] [[INSPIRE](#)].
- [31] T. Erler, *Exotic universal solutions in cubic superstring field theory*, *JHEP* **04** (2011) 107 [[arXiv:1009.1865](#)] [[INSPIRE](#)].
- [32] T. Erler, *Analytic solution for tachyon condensation in Berkovits' open superstring field theory*, *JHEP* **11** (2013) 007 [[arXiv:1308.4400](#)] [[INSPIRE](#)].