Pre-potential in the $AdS_5 \times S^5$ type IIB superspace

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ABSTRACT: We found the pre-potential in the superspace with $AdS_5 \times S^5$ background. The pre-potential appears as part of the vielbeins, without derivatives. In both subspaces (AdS_5 and S^5) we used Poincaré coordinates. We picked one bulk coordinate in AdS_5 and one bulk coordinate in S^5 to define the space-cone gauge. Such space-cone gauge destroys the bulk Lorentz covariance. However, it still preserves boundary Lorentz covariance (and gives projective superspace) $SO(3,1) \otimes SO(4)$ and so symmetries of boundary CFT are manifest.

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[1]http://insti.physics.sunysb.edu/~siegel/plan.html.

Contents

1 Introduction

In the paper [1] we obtained the curvature tensor (previously discovered in [2, 3]) in a manifestly T-dual way. In the paper [4] we extended the techniques for the case of three dimensional $\mathcal{N} = 2$ T-dual extended superspace. There we correctly obtained the prepotential (as a part of vielbein), structure of linearised dilaton and field equations. The aim of this paper was to look at the full ten dimensional $\mathcal{N} = 2$ T-dually extended superspace in the flat and also in $AdS_5 \times S^5$ background, i.e. IIB string theory expanded around $AdS_5 \times S^5$ background. The AdS was earlier analysed in superspace in papers [5–7] and [8]. In this paper we discovered the projective (and also the chiral) pre-potential to sit in a certain combination of $H_{S\widetilde{S}}$ and $H_{D\widetilde{D}}$. This was first obtained in the flat case and later generalised for the $AdS_5 \times S^5$. We also performed the near horizon limit and derived the equation of motion for the pre-potential in that limit. This limit also picks out the projective prepotential instead of the chiral pre-potential, even though both pre-potentials are valid bulk solutions. The projective and harmonic superspaces were earlier analysed in [10] and [11].

2 Motivation

The massive development of the AdS/CFT correspondence (and its generalisations) started with foundational papers [14] and [15]. It is considered to be one of the best achievements in string theory. The correspondence promises the way of computing quantum effects in strongly coupled field theory using the classical (super) gravity theory. The AdS/CFT correspondence (and its generalisations) is currently used in many different areas, like study of confinement, condensed matter systems or investigation of non-equilibrium phenomenas in strongly coupled plasma.

In the AdS/CFT correspondence the strongly coupled quantum field theory (four dimensional $\mathcal{N} = 4$ super Yang-Mills theory) is related to the supergravity theory living on $AdS_5 \times S^5$ space. The strongly coupled conformal field theory is hard to work with. The correspondence provides feasible way to calculate effects in such theory by translating it to weakly coupled supergravity.

It is therefore of high interest to develop suitable framework to handle supergravity living on $AdS_5 \times S^5$ background. This framework could be later used as a base for perturbative calculations. In this paper our aim was to look at type IIB supergravity living on $AdS_5 \times S^5$ background. We are looking for solutions generated via field called pre-potential. The pre-potential is a basic field from which all physical fields (in the massless spectrum) arise. Thus we get the classical solution of supergravity living on the background $AdS_5 \times S^5$. This solution is interesting because of our later aim to use it in perturbative calculations. Our novel approach unites the way how this solution is found. The first interesting observation is universal usage of string based framework of T-dually extended superspaces, that started with foundational paper [2, 3] and later examined in [1] and [4]. Another interesting feature is the way how the $AdS_5 \times S^5$ pre-potential solution was obtained in this paper. The approach is actually an analogy of how we derived the flat ten dimensional pre-potential in this paper. We started with flat supergravity where it was

much easier to identify the pre-potential. Inspired by this solution we turned on the $AdS_5 \times S^5$ background, modified the equations and looked for the $AdS_5 \times S^5$ deformed solution. This gave us both technical and also geometrical advantages to look at the pre-potential as coming from the same field solution. The sanity checked worked here, if we flatten back the $AdS_5 \times S^5$ (i.e. the $R \to \infty$ limit) we would get back the flat space pre-potential solution. Moreover by looking at the near horizon limit ($R \to 0$) it was easy to naturally identify the equations of motion and find the correspondence between gravity fields and CFT fields.

We believe that the use of the string based T-dually extended superspaces in various contexts is fruitful and natural way to cast supergravity, as was observed in [1] and [2, 3]. In this particular work we showed how it can be effectively used to naturally find the pre-potential solution in the fully fledged type IIB supergravity on the $AdS_5 \times S^5$ background. That is a novel result that to our best knowledge has never been derived. In the future we would like to look how similar approach might shed some light on the supergravity solutions on different backgrounds, relevant for generalizations of the $AdS_5 \times S^5$. Moreover in this framework the calculations beyond linearized level are also feasible and are left for future work.

3 Type II superspace, notation and motivation

3.1 10 dimensional type II superspace

The T-duality is important duality we know to exist in (super)string theory. The low energy limit of superstring theory is version of ten dimensional supergravity. The T-duality then forms T-duality symmetry which is the symmetry of such low energy theory. That symmetry can be manifestly realized by doubled spacetime coordinates. This realization was first made in the paper [2, 3].

In the next introduction we are closely following paper [28]. To T-dualize ordinary space we start with space that can be built by the coset construction. In the procedure of T-dualizing the coset construction one starts with ordinary Lie algebra generated by G_I, where by I we mean some particular set of indices. We require this Lie algebra to have non-degenerate metric. As for the usual coset construction, one can exponentialize the Lie algebra and get the Lie group (more precisely the vicinity of the unit element), i.e. one constructs the Lie group element $g(Z^I)$, where Z^I are group coordinates. The covariant derivatives and symmetry generators (Lie derivatives) on that space are obtained by considering left and right actions of the group. Since left and right action on a group element commute so do the covariant derivatives and symmetry generators. More explicitly for ordinary (particle) construction we get:

Symmetry generators: $\widetilde{\nabla}_I = L_I{}^M \frac{1}{i} \partial_M$ || Covariant derivatives: $\overset{\circ}{\nabla}_I = R_I{}^M \frac{1}{i} \partial_M$ (3.1)

where $L_I{}^M$ and $R_M{}^I$ are matrices defining right and left invariant one forms: $(dg)g^{-1} = i\, dZ^M\, L_M{}^I\, G_I$ and $g^{-1}(dg) = i\, dZ^M\, R_M{}^I\, G_I$. The L matrix is used in left action generator and R matrix is used in right action generator in (3.1). It follows from construction that

the covariant derivatives $\overset{\circ}{\nabla}_I$ and symmetry generators $\widetilde{\nabla}_J$ satisfy the following Lie algebra:

$$[\widetilde{\nabla}_I, \widetilde{\nabla}_J\} = i\, f_{IJ}{}^K\, \widetilde{\nabla}_K \quad \| \quad [\overset{\circ}{\nabla}_I, \overset{\circ}{\nabla}_J\} = -i\, f_{IJ}{}^K\, \overset{\circ}{\nabla}_K \quad \| \quad [\overset{\circ}{\nabla}_I, \widetilde{\nabla}_J\} = 0 \qquad (3.2)$$

where we included graded commutators as an immediate superalgebra generalization. The graded bracket in (3.2) is anticommuting for two fermions and commuting for others. Having the covariant derivatives in hands we can curve given space arbitrarily via vielbeins $\nabla_A = E_A{}^I\, \overset{\circ}{\nabla}_I$.

The next step is to generalize the algebra (3.2) to the (super)string case. It is obtained by string extension of particle one parameter $Z^I(\tau)$ to stringy $Z^I(\tau, \sigma)$. By that we get string generalization of (3.2), the string affine Lie algebra:

$$[\widetilde{\rhd}_I, \widetilde{\rhd}_J\} = i f_{IJ}{}^K \widetilde{\rhd}_K + i\overset{\circ}{\eta}_{IJ} \partial_\sigma \delta(2-1) \qquad (3.3)$$
$$[\overset{\circ}{\rhd}_I, \overset{\circ}{\rhd}_J\} = -i f_{IJ}{}^K \overset{\circ}{\rhd}_K - i\overset{\circ}{\eta}_{IJ} \partial_\sigma \delta(2-1)$$
$$[\widetilde{\rhd}_I, \overset{\circ}{\rhd}_J\} = 0$$

where the metric $\overset{\circ}{\eta}_{IJ}$ is the (graded), constant, non-degenerate (super) Lie group metric and $\partial_\sigma \delta(2-1) \equiv \partial_{\sigma_2} \delta(2-1) \equiv \delta'(2-1)$. In analogy with (3.1) the left and right action generators could be solved explicitly in the string case:

$$\text{Symmetry generators: } \widetilde{\rhd}_I := L_I{}^M \left(\frac{1}{i} \partial_M + \partial_\sigma Z^N B_{NM} \right) - \partial_\sigma Z^M L_M{}^J \overset{\circ}{\eta}_{JI} \quad (3.4)$$

$$\text{Covariant derivatives: } \overset{\circ}{\rhd}_I := R_I{}^M \left(\frac{1}{i} \partial_M + \partial_\sigma Z^N B_{NM} \right) + \partial_\sigma Z^M R_M{}^J \overset{\circ}{\eta}_{JI}$$

where B_{MN} is the B-field. We can get the curved version of the string covariant derivatives (in paper [28] also called curved current) via vielbeins:

$$\rhd_A := E_A{}^I(Z(\tau, \sigma)) \overset{\circ}{\rhd}_I \equiv E_A{}^I \overset{\circ}{\rhd}_I \qquad (3.5)$$

Using (3.5) we can introduce geometric objects like torsions. Looking at stringy affine algebra in the curved background we get:

$$[\rhd_A, \rhd_B\} := -iT_{AB}{}^C \rhd_C \delta(2-1) - i\eta_{AB} \partial_\sigma \delta(2-1) \qquad (3.6)$$

where $T_{AB}{}^C$ is the string generalisation of torsion and η_{AB} is the curved group metric. In general are both functions of vielbeins. We will impose constraint on η_{AB} and require η_{AB} to be a constant metric $\overset{\circ}{\eta}_{AB}$. Such constraint does not impose any restrictions on physical content and makes calculations simpler.

In the next step we will describe particular realization of above construction. For the usual particle we can start with algebra of translations generated by $p_{\mathbf{m}}$, where $\mathbf{m} \in \{1, \ldots, \dim\}$. Next we can include the supersymmetry generator D_μ, where μ is fermionic index and range depends on dimensionality of space. By including D_μ we get the super-translations. Finally we can add the Lorentz generator $S_{\mathbf{mn}}$ and thus get a particle algebra of super-Poincaré transformations. All previous can be described by nice diagram in table 1, see also [28]. To build the T-dual analog of 1 for type II string we first need to double

translations	→ supertranslations	→ super-Poincaré
$p_{\mathbf{m}}$	$D_\mu,\, p_{\mathbf{m}}$	$S_{\mathbf{mn}},\, D_\mu,\, p_{\mathbf{m}}$

Table 1. Particle algebra generators.

translations	→ supertranslations	→ super-Poincaré
$P_{\mathbf{m}}$	$D_\mu,\, P_{\mathbf{m}},\, \Omega^\mu$	$S_{\mathbf{mn}},\, D_\mu,\, P_{\mathbf{m}},\, \Omega^\mu,\, \Sigma^{\mathbf{mn}}$
$P_{\widetilde{\mathbf{m}}}$	$D_{\widetilde\mu},\, P_{\widetilde{\mathbf{m}}},\, \Omega^{\widetilde\mu}$	$S_{\widetilde{\mathbf{mn}}},\, D_{\widetilde\mu},\, P_{\widetilde{\mathbf{m}}},\, \Omega^{\widetilde\mu},\, \Sigma^{\widetilde{\mathbf{mn}}}$

Table 2. String affine algebra generators.

Explicit:	$_M := (\,_{\mathbf{mn}},\,_\mu,\,_{\mathbf{m}},\,^\mu,\,^{\mathbf{mn}})$	carried by	$S_{\mathbf{mn}},\, D_\mu,\, P_{\mathbf{m}},\, \Omega^\mu,\, \Sigma^{\mathbf{mn}}$
	$_{\widetilde M} := (\,_{\widetilde{\mathbf{mn}}},\,_{\widetilde\mu},\,_{\widetilde{\mathbf{m}}},\,^{\widetilde\mu},\,^{\widetilde{\mathbf{mn}}})$	carried by	$S_{\widetilde{\mathbf{mn}}},\, D_{\widetilde\mu},\, P_{\widetilde{\mathbf{m}}},\, \Omega^{\widetilde\mu},\, \Sigma^{\widetilde{\mathbf{mn}}}$
Symbolic:	$_M := (\,S,\, D,\, P,\, \Omega,\, \Sigma\,)$		
	$_{\widetilde M} := (\,\widetilde S,\, \widetilde D,\, \widetilde P,\, \widetilde\Omega,\, \widetilde\Sigma\,)$		
Multiindices:	$\mathcal{M} := (\,S,\, \widetilde S,\, D,\, \widetilde D,\, P,\, \widetilde P,\, \Omega,\, \widetilde\Omega,\, \Sigma,\, \widetilde\Sigma\,) \;\equiv\; (\,_M,\,_{\widetilde M}\,)$		

Table 3. String affine algebra indices.

Covariant derivatives:	$\overset{\circ}{\rhd}_M := (\,S_{\mathbf{mn}},\, D_\mu,\, P_{\mathbf{m}},\, \Omega^\mu,\, \Sigma^{\mathbf{mn}}\,)$
	$\overset{\circ}{\rhd}_{\widetilde M} := (\,S_{\widetilde{\mathbf{mn}}},\, D_{\widetilde\mu},\, P_{\widetilde{\mathbf{m}}},\, \Omega^{\widetilde\mu},\, \Sigma^{\widetilde{\mathbf{mn}}}\,)$
Coordinates:	$Z^M := (\,u^{\mathbf{mn}},\, \theta^\mu,\, x^{\mathbf{m}},\, \varphi_\mu,\, v_{\mathbf{mn}}\,)$
	$Z^{\widetilde M} := (\,u^{\widetilde{\mathbf{mn}}},\, \theta^{\widetilde\mu},\, x^{\widetilde{\mathbf{m}}},\, \varphi_{\widetilde\mu},\, v_{\widetilde{\mathbf{mn}}}\,)$

Table 4. String covariant derivatives and coordinates.

the translation generators and also have in mind that those generators are forming an affine Lie algebra as seen in (3.3). We get the set of string translation generators $P_{\mathbf{m}}$, $P_{\widetilde{\mathbf{m}}}$, they generate left and right translations. Next we will include the supersymmetry generators that are for obvious reasons also doubled to D_μ, $D_{\widetilde\mu}$. Including those generates a small issue. Because we consider the non-degenerate group metric η_{IJ} and the affine Lie algebra of supertranslations has to satisfy super-Jacobi identities we need to include another independent fermionic generators (dual currents) Ω^μ, $\Omega^{\widetilde\mu}$. The necessity for the dual currents was discussed in great detail in [9]. Finally, if we include the Lorentz generators $S_{\mathbf{mn}}$, $S_{\widetilde{\mathbf{mn}}}$ we also discover yet another set of dual currents $\Sigma^{\mathbf{mn}}$, $\Sigma^{\widetilde{\mathbf{mn}}}$. We summarise the type II generators in next diagram: We can see that the whole set of generators is doubled for type II string affine algebra. That introduces space with high dimensionality. In the end the dimensional reduction, coset constraints and section condition are imposed on fields (living on such high dimensional space) to remove unphysical degrees of freedom.

To deal with indices and generators in table 2 we introduce notation for various forms of graded indices, see table 3. Using definition from table 3 we can define stringy super-Poincaré covariant derivatives (and string coordinates) in the sense of generic string affine (super) Lie algebra from (3.3). The table 4 covariant derivatives satisfy the following ex-

plicit string type II affine super-Poincaré algebra, for which the non-zero structure constants and the central charges are:

$$[\, S_{\mathbf{mn}}\,(1),\, S_{\mathbf{kl}}\,(2)\,] \quad = \quad -i\,\eta_{[\,\mathbf{m}\,[\,\mathbf{k}}\,S_{\mathbf{l}\,]\,\mathbf{n}\,]}\,\delta\,(\,2-1\,) \tag{3.7}$$

$$[\, S_{\mathbf{mn}}\,(1),\, D_\rho\,(2)\,] \quad = \quad -\,i\,\frac{1}{2}\,(\,\gamma_{\mathbf{mn}}\,)^\sigma{}_\rho\,D_\sigma\,\delta\,(\,2-1\,)$$

$$[\, S_{\mathbf{mn}}\,(1),\, P_{\mathbf{k}}\,(2)\,] \quad = \quad i\,\eta_{\mathbf{k}\,[\,\mathbf{m}}\,P_{\mathbf{n}\,]}\,\delta\,(\,2-1\,)$$

$$[\, S_{\mathbf{mn}}\,(1),\, \Omega^\rho\,(2)\,] \quad = \quad i\,\frac{1}{2}\,(\,\gamma_{\mathbf{mn}}\,)^\rho{}_\sigma\,\Omega^\sigma\,\delta\,(\,2-1\,)$$

$$[\, S_{\mathbf{mn}}\,(1),\, \Sigma^{\mathbf{kl}}\,(2)\,] \quad = \quad i\,\delta_{\mathbf{mn}}{}^{\mathbf{kl}}\,\delta'\,(\,2-1\,) - i\delta_{[\,\mathbf{m}}{}^{[\,\mathbf{k}}\,\eta_{\mathbf{n}\,]\,\mathbf{s}}\Sigma^{\mathbf{l}]\,\mathbf{s}}\,\delta\,(\,2-1\,)$$

$$\{\, D_\rho\,(1),\, D_\sigma(2)\,\} \quad = \quad 2\,(\,\gamma^{\mathbf{m}}\,)_{\rho\sigma}\,P_{\mathbf{m}}\,\delta\,(\,2-1\,)$$

$$[\, D_\rho\,(1),\, P_{\mathbf{m}}\,(2)\,] \quad = \quad 2\,(\,\gamma_{\mathbf{m}})_{\rho\sigma}\,\Omega^\sigma\,\delta\,(\,2-1\,)$$

$$\{\, D_\rho\,(1),\, \Omega^\sigma\,(2)\,\} \quad = \quad i\,\delta^\sigma_\rho\,\delta'\,(\,2-1\,) - i\,\frac{1}{4}\,(\,\gamma_{\mathbf{mn}}\,)^\sigma{}_\rho\,\Sigma^{\mathbf{mn}}\,\delta\,(\,2-1\,)$$

$$[\, P_{\mathbf{m}}\,(1),\, P_{\mathbf{n}}\,(2)\,] \quad = \quad i\,\eta_{\mathbf{mn}}\,\delta'\,(\,2-1\,) + i\,\eta_{\mathbf{m}\,\mathbf{h}}\,\eta_{\mathbf{n}\,\mathbf{s}}\Sigma^{\mathbf{hs}}\,\delta\,(\,2-1\,)$$

$$\text{left algebra} \to -\,\text{right algebra}$$

$$[\,\text{left, right}\,\} \quad = \quad 0.$$

As indicated above, the algebra for the right generators is the same up to the overall sign. We can assign the canonical dimensions to the generators: $\dim\,(\,S,\,D,\,P,\,\Omega,\,\Sigma\,) = (0,\,\frac{1}{2},\,1,\,\frac{3}{2},\,2)$. The S generators generate the $SO(\,9,1\,)\otimes SO(\,9,1\,)$ algebra, i.e. left (and right) local Lorentz transformations. The D generate left (and right) supersymmetry transformation and P left (and right) translations. The Ω and Σ are the left (and right) dual currents (corresponding to D and S), see also [2, 3]. We can see once again that the only non-vanishing terms (for left handed algebra, similarly for right handed algebra) in the metric and structure constants in (3.7) are (as can be guessed by dimensional analysis):

$$\eta_{PP},\, \eta_{S\Sigma},\, \eta_{D\Omega}; \quad f_{SPP},\, f_{SS\Sigma},\, f_{DDP},\, f_{SD\Omega} \tag{3.8}$$

where we have lowered the upper index on f with η to take advantage of its total (graded) antisymmetry. In that notation we explicitly have, for the left-handed algebra:

$$(\eta)_{\mathbf{mn}} = \eta_{\mathbf{mn}}, \qquad (\eta)_{\mathbf{mn}}{}^{\mathbf{pq}} = \delta_{\mathbf{mn}}{}^{\mathbf{pq}}, \qquad (\eta)_\sigma{}^\rho = \delta^\rho_\sigma \tag{3.9}$$

$$f_{\mathbf{mn}}{}^{\mathbf{p}\,\mathbf{q}} = -\delta_{\mathbf{mn}}{}^{\mathbf{pq}}, \qquad f_{\mathbf{mn}\,\mathbf{pq}}{}^{\mathbf{rs}} = \eta_{[\mathbf{m}[\mathbf{p}}\delta_{\mathbf{q}]\mathbf{n}]}{}^{\mathbf{rs}}, \qquad f_{\sigma\rho}{}^{\mathbf{m}} = 2\,(\,\gamma^{\mathbf{m}}\,)_{\sigma\rho}, \tag{3.10}$$

$$f_{\mathbf{mn}\,\sigma}{}^\rho = -\frac{1}{2}\,(\,\gamma_{\mathbf{mn}}\,)^\rho_\sigma$$

The type IIA and IIB theories are distinguished by the choice of ten dimensional fermionic coordinates. For IIA theory we pick fermionic coordinates with both ten dimensional chiralities $(\,Z^\mu,\, Z^{\widetilde\mu}\,) \equiv (\,\theta^\mu,\, \theta_\mu\,)$. Note that now the θ coordinate uses concrete ten dimensional fermionic index (in chiral representation), in contrast with generic fermionic index used in description of θ coordinate in table 4. Moreover those indices are ten dimensional chiral indices with respect to the common (diagonal) local Lorentz group (defined after the dimensional reduction). The IIB theory uses the indices of the same ten dimensional chirality $(\,Z^\mu,\, Z^{\widetilde\mu}\,) \equiv (\,\theta_1{}^\mu,\, \theta_2{}^\mu\,)$, therefore we denoted them by substript $_1$ and $_2$.

The above construction is very natural from the point of view of the superstring theory, where we know the T-duality is a symmetry of low energy theory. Moreover as we have seen in papers [1] and [4] this description has advantages to naturally introduce objects of interest. For example in the paper [1] we derived the T-dual version of Riemann curvature tensor using T-dually extended space (even though there without supersymmetry). That tensor has been previously discovered in [2, 3] but by indirect methods. In paper [4] we discovered that the pre-potential for $\mathcal{N} = 2$ three dimensional supergravity is naturally part of vielbeins living on the T-dually extended superspace (i.e. the three dimensional version of above construction). It was known how to find the pre-potential for $\mathcal{N} = 2$ three dimensional supergravity before, however some further differential constraints were needed (like the bisection condition) see [21]. In paper [4] all constraints are coming naturally from the torsion constraints (we will discuss them later in this paper as well). We will see that ten dimensional generalization of [1] and [4] is a fruitful way how to treat the pre-potential in ten dimensional flat and even $AdS_5 \times S^5$ space.

In next sections we will use the Wick rotation. We feel free to Wick rotate from Minkowski to Euclidean metric and back because in this paper we do not discuss the reality conditions, so the rotation is the matter of convenience.

Our first and most significant use of Wick rotation is in the description of space-cone gauge in the case of $AdS_5 \times S^5$. The procedure is described more precisely in [17, 18] and so we are giving just short overview. In order to introduce the space-cone basis in $AdS_5 \times S^5$ one can first Wick rotate the sphere S^5 to AdS_5. This is done by extension of one S^5 coordinate to complex numbers and then taking it to be purely imaginary. After that we have space $AdS_5 \times AdS_5$. At two corresponding Poincaré patches (at the $AdS_5 \times AdS_5$) we take two bulk coordinates (two space-like coordinates one from the original AdS_5 and another one from the Wick rotated sphere) and introduce the space-cone coordinates x^+ and x^- as their combinations. The near horizon limit is attained by $x^+ \to 0$. This limit turns the superspace into the projective superspace. In the text we will use those x^+ and x^- coordinates and related space-cone gauge (for vielbeins living on the extended space). The space-cone gauge destroys the explicit local Lorentz covariance but still preserves the boundary (near horizon limit) Lorentz covariance SO(3, 1) $\otimes$ SO(4) the symmetries of boundary CFT.

We will also use the Wick rotation whenever we find it easier to explain or define some notion. For example, we define ten dimensional Γ matrices as matrices for SO(10) Lorentz group. To get the gamma matrices for SO(9, 1) one can Wick rotate back.

3.2 Gamma matrices

The gamma matrices $(\gamma_{\mathbf{m}})_{\mu \nu}$ used in the algebra (3.7) are the $16 \otimes 16$ block gamma matrices from 10 dimensional $32 \otimes 32$ chiral representation:

$$\Gamma_{\mathbf{m}} = \begin{pmatrix} 0 & (\gamma_{\mathbf{m}})^{\mu \nu} \\ (\gamma_{\mathbf{m}})_{\mu \nu} & 0 \end{pmatrix} \quad \text{where} \quad \{ \Gamma_{\mathbf{m}}, \Gamma_{\mathbf{n}} \} = 2 \eta_{\mathbf{m n}} \delta. \tag{3.11}$$

Moreover the block gamma matrices satisfy:

$$(\gamma_{\mathbf{m}})_{\mu \nu} = (\gamma_{\mathbf{m}})_{\nu \mu} \quad || \quad (\gamma_{(\mathbf{m}})^{\mu \nu} (\gamma_{\mathbf{n}}))_{\nu \sigma} = 2 \eta_{\mathbf{m n}} \delta_{\sigma}^{\mu} \quad || \quad (\gamma_{\mathbf{m}})_{(\mu \nu} (\gamma^{\mathbf{m}})_{\sigma) \lambda} = 0 \tag{3.12}$$

The IIB fermion generators (in algebra (3.7)) are described by $16 \oplus 16$ chiral fermion generators (for left and right generators) with the same 10 dimensional chirality. For the future use we need to look closer at the structure of the matrices $(\gamma_{\mathbf{m}})_{\mu\nu}$ from equation (3.11). The gamma matrices from equation (3.11) could be constructed from $SO(9)$ gamma matrices or equivalently from $SO(8)$ gamma matrices and the chirality matrix. We can go one step down and construct the $SO(8)$ gamma matrices from $SO(6)$ gamma matrices. For the $SO(6)$ gamma matrices we use the Majorana representation of those matrices (they are purely imaginary). Thus we can get the Majorana-Weyl representation of the $SO(8)$ gamma matrices.

For the future reference we will define the following $16 \otimes 16$ matrix $\widetilde{\Gamma}_5$:

$$\widetilde{\Gamma}_5 := \gamma_{\mathbf{10}} \prod_{\mathbf{m}=1}^{4} \gamma_{\mathbf{m}} \tag{3.13}$$

where $\gamma_{\mathbf{m}}$ are block gamma matrices from (3.11).

3.3 Space-cone basis and indices

In the following sections we will use the space-cone basis we introduce it for the gamma matrices we constructed in previous subsection. We first notice that the block gamma matrices $\gamma_{\mathbf{m}}$ in equation (3.11) could have either upper indices $(\gamma_{\mathbf{m}})^{\mu\nu}$ or lower indices $(\gamma_{\mathbf{m}})_{\mu\nu}$. From the construction it follows that those matrices are equal up to the sign.

In the equation (3.11) let us further divide the (either upper or lower) 16 dimensional index μ to $8 \oplus 8$ pieces (they are the $SO(8)$ chiral indices), thus we introduce $\mu := (\mu, \mu')$. In another words we want to look how the block $\gamma_{\mathbf{m}}$ matrices look in the $SO(8)$ (Majorana-Weyl) basis. Furthermore we introduce the following space-cone combinations of the equation (3.11) block gamma matrices:

$$(\gamma_+)_{\mu\nu} = \frac{1}{2}\left(\gamma_{\mathbf{10}}+\gamma_{\mathbf{9}}\right)_{\mu\nu} = \begin{pmatrix} 0 & 0 \\ 0 & \delta_{\mu'\nu'} \end{pmatrix} \ \| \ (\gamma_-)_{\mu\nu} = \frac{1}{2}\left(\gamma_{\mathbf{10}}-\gamma_{\mathbf{9}}\right)_{\mu\nu} = \begin{pmatrix} \delta_{\mu\nu} & 0 \\ 0 & 0 \end{pmatrix} \tag{3.14}$$

$$(\gamma_+)^{\mu\nu} = \frac{1}{2}\left(\gamma_{\mathbf{10}}+\gamma_{\mathbf{9}}\right)^{\mu\nu} = \begin{pmatrix} \delta^{\mu\nu} & 0 \\ 0 & 0 \end{pmatrix} \ \| \ (\gamma_-)^{\mu\nu} = \frac{1}{2}\left(\gamma_{\mathbf{10}}-\gamma_{\mathbf{9}}\right)^{\mu\nu} = \begin{pmatrix} 0 & 0 \\ 0 & \delta^{\mu'\nu'} \end{pmatrix}$$

For the convenience we also write the remaining gamma matrices using the $SO(8)$ indices:

$$(\gamma_{\mathbf{i}})_{\mu\nu} = \begin{pmatrix} 0 & (\widetilde{\gamma}_{\mathbf{i}})_{\mu\nu'} \\ (\widetilde{\gamma}_{\mathbf{i}})_{\mu'\nu} & 0 \end{pmatrix} \ \| \ (\gamma_{\mathbf{i}})^{\mu\nu} = \begin{pmatrix} 0 & (\widetilde{\gamma}_{\mathbf{i}})^{\mu\nu'} \\ (\widetilde{\gamma}_{\mathbf{i}})^{\mu'\nu} & 0 \end{pmatrix} \tag{3.15}$$

where the $\widetilde{\gamma}_{\mathbf{i}}$ are the $SO(8)$ gamma matrices.

In the above introduced $8 \oplus 8$ basis the (3.13) looks like $\sigma_3 \otimes \mathbb{1}$ where $\mathbb{1}$ is the $8 \otimes 8$ unit matrix and σ_3 is the Pauli matrix.

4 The AdS background in the T-dually extended superspace

4.1 Short review of the theory in curved background

In the treatment of the theory (of T-dually extended superspaces) the curved background is introduced via vielbeins $E_{\mathcal{A}}{}^{\mathcal{M}}(Z^{\mathcal{N}})$, see papers [1–4]. Note that the index $\mathcal{M}$ was introduced in table 3:

$$\rhd_{\mathcal{A}} = E_{\mathcal{A}}{}^{\mathcal{M}}(Z^{\mathcal{N}}) \overset{\circ}{\rhd}_{\mathcal{M}} \tag{4.1}$$

where $\overset{\circ}{\rhd}_{\mathcal{M}}$ are generators of the flat algebra (3.7). The affine Lie algebra for the curved covariant derivatives $\rhd_{\mathcal{A}}$ can be written as:

$$[\rhd_{\mathcal{A}}, \rhd_{\mathcal{C}}\} \equiv -i\eta_{\mathcal{AC}}\,\delta'(2-1) - iT_{\mathcal{AC}}{}^{\mathcal{E}}\rhd_{\mathcal{E}}\,\delta(2-1) \tag{4.2}$$

where $T_{\mathcal{AC}}{}^{\mathcal{E}}$ is the superstring generalisation of the torsion, see [1]:

$$T_{\mathcal{AC}}{}^{\mathcal{E}} = E_{[\mathcal{A}}{}^{\mathcal{M}}(D_{\mathcal{M}}E_{\mathcal{C})}{}^{\mathcal{N}})E_{\mathcal{N}}^{-1\mathcal{E}} + \frac{1}{2}\eta^{\mathcal{ED}}E_{\mathcal{D}}{}^{\mathcal{M}}(D_{\mathcal{M}}E_{[\mathcal{A}|}{}^{\mathcal{N}})E_{\mathcal{N}}^{-1\mathcal{F}}\eta_{\mathcal{F}|\mathcal{C})} + E_{\mathcal{A}}{}^{\mathcal{M}}E_{\mathcal{C}}{}^{\mathcal{N}}E_{\mathcal{P}}^{-1\mathcal{E}}f_{\mathcal{MN}}{}^{\mathcal{P}} \tag{4.3}$$

where $[\mathcal{A}||\mathcal{C})$ indicates graded anti-symmetrization in only those indices. By $D_{\mathcal{M}}$ in (4.3) and in the whole next text we mean the group covariant derivatives of the (non-affine) part of algebra (3.7): $[D_{\mathcal{M}}, D_{\mathcal{N}}\} = i f_{\mathcal{MN}}{}^{\mathcal{E}}D_{\mathcal{E}}$.

Note that the super-Jacobi identities imply the total graded antisymmetry of the torsion, just as for the structure constants. We can set the coefficient of the Schwinger term (the central charge in algebra (4.2)) to be the flat metric $\overset{\circ}{\eta}$ (now we rename it to η, to simplify notation). After that the vielbein is forced to obey the orthogonality constraints:

$$E_{\mathcal{A}}{}^{\mathcal{M}}\eta_{\mathcal{MN}}E_{\mathcal{C}}{}^{\mathcal{N}} \equiv \eta_{\mathcal{AC}} \tag{4.4}$$

This choice does not affect the physics, and simplifies many of the expressions. For example, it implies the total graded antisymmetry of the torsion, when the upper index is implicitly lowered with η:

$$T_{\mathcal{ABC}} = \frac{1}{2}E_{[\mathcal{A}|}{}^{\mathcal{M}}(D_{\mathcal{M}}E_{|\mathcal{B}}{}^{\mathcal{N}})E_{\mathcal{C})\mathcal{N}} + E_{\mathcal{A}}{}^{\mathcal{M}}E_{\mathcal{B}}{}^{\mathcal{N}}E_{\mathcal{C}}{}^{\mathcal{P}}f_{\mathcal{MNP}} \tag{4.5}$$

where we have used $E_{\mathcal{M}}^{-1}{}^{\mathcal{A}} = \eta^{\mathcal{AB}}\eta_{\mathcal{MN}}E_{\mathcal{B}}{}^{\mathcal{N}}$. Also note that in the first term the graded anti-symmetrization can be written as a cyclic sum without the 1/2, since it is already graded antisymmetric in the last two indices. Because of orthogonality, the vielbein is like (the exponential of) a super 2-form, while the torsion is a super 3-form. Similarly the Bianchi identities are a super 4-form.

To solve the theory (in terms of pre-potential, to get physical fields and equation of motion) orthogonality condition (4.4) has to be solved explicitly (or at least at linearised level). Moreover there is a huge gauge group invariance that should be fixed:

$$\delta_{\Lambda}\rhd_{\mathcal{A}} = [-i\Lambda, \rhd_{\mathcal{A}}\} \quad \text{where} \quad \Lambda := \int d\sigma\,\lambda^{\mathcal{M}}(Z)\,D_{\mathcal{M}} \tag{4.6}$$

At the top of the orthogonality condition and the gauge invariance, we should include the torsion constraints. The torsion constraints are additional constraints on vielbein. They

are imposed by putting some of the torsions in (4.5) to zero. Of course, not all torsions in (4.5) are zero. The relevant torsion constraints have been carefully analysed in [28]. All possible constraints on torsions are coming from curved space version of the **A B C D** (first class) constraints: **A** Virasoro, (string world-sheet) diffeomorphism constraints, **B** and **C** and **D** are the first class fermionic κ symmetry constraints, for further details see [19, 20, 22–25] and [28]. The rule of thumb is that at least the torsions of negative (10 dimensional) engineering dimension should be zero.

We will not try to solve the full nonlinear version of the theory. We linearise the theory around some background. In previous papers [1, 4] we linearised around the flat background. In this paper however we linearise the theory around the $AdS_5 \times S^5$ solution of the classical supergravity (reformulated in the language of the doubled algebra).

After the linearisation we rewrite the orthogonality constraints (4.4) and torsions (4.5) using the vielbein expansion $E_{\mathcal{C}}{}^{\mathcal{D}} = \delta_{\mathcal{C}}{}^{\mathcal{D}} + E^{(1)}{}_{\mathcal{C}}{}^{\mathcal{D}} + \mathcal{O}(E^{(2)})$. Let us for simplicity rename the first fluctuation $E^{(1)}{}_{\mathcal{C}}{}^{\mathcal{D}} \equiv H_{\mathcal{C}}{}^{\mathcal{D}}$. Then the equation (4.4) is just statement that: $H_{(\mathcal{C}}{}^{\mathcal{D}} \eta_{D|E]} \equiv H_{(CE]} = 0$ and the structure of linearised torsion (4.5):

$$T_{\mathcal{ABC}} = f_{\mathcal{ABC}} + T^{(1)}{}_{\mathcal{ABC}} + \mathcal{O}(T^{(2)}) \tag{4.7}$$

$$\text{where} \qquad T^{(1)}{}_{\mathcal{ABC}} \equiv \frac{1}{2} D_{[\mathcal{A}} H_{\mathcal{BC})} + \frac{1}{2} H_{[\mathcal{A}}{}^{\mathcal{M}} f_{\mathcal{BC})\mathcal{M}}$$

4.2 $AdS_5 \times S^5$ background

In the expansion (4.7) we need to have the concrete structure constants $f_{\mathcal{ABC}}$ (i.e. vacuum values of torsions). We are interested in solving the theory (at least identifying the pre-potential) around this $AdS_5 \times S^5$ background. The relevant structure constants for the T-dually extended superspace in the context of the $AdS_5 \times S^5$ background were analysed in the last section of the paper [28]. We are repeating them here for the convenience. In the next section we will embed this $AdS_5 \times S^5$ version of T-dual algebra (see [28]) to a certain larger algebra that will be actually used in computations. The relevant non-vanishing $AdS_5 \times S^5$ torsions from [28] are:

$$\dim 0 : T_{SS\Sigma} = f_{SS\Sigma} \| T_{SD\Omega} = f_{SD\Omega} \| T_{SPP} = f_{SPP} \| T_{DDP} = f_{DDP}$$
$$\dim 1 : T_{D\widetilde{D}\Sigma} = R_{D\widetilde{D}\Sigma} \| T_{P\widetilde{D}\Omega} = R_{P\widetilde{D}\Omega}$$
$$\dim 2 : T_{\Omega\Omega P} = R_{\Omega\Omega P} \| T_{\Omega\Omega P} = R_{\Omega\Omega\widetilde{P}} \| T_{P\widetilde{P}\Sigma} = R_{P\widetilde{P}\Sigma}$$
$$\dim 3 : T_{\Omega\widetilde{\Omega}\Sigma} = R_{\Omega\widetilde{\Omega}\Sigma} \tag{4.8}$$

note that the left and right index notation was introduced in table 3.

The $f_{\mathcal{ABC}}$ in (4.8) are usual flat superspace structure constants for the (flat) T-dually extended superspace with the (common) local Lorentz group SO(4, 1) ⊗ SO(5). The nontrivial curvatures from table (4.8): $R_{D\widetilde{D}\Sigma}, R_{P\widetilde{D}\Omega} \ldots$ are defined using the dimension 1 torsion $T_{P\widetilde{D}\Omega}$. The $T_{P\widetilde{D}\Omega} \equiv T_{\mathbf{a}\widetilde{\alpha}}{}^{\beta} = \gamma_{\mathbf{a}\,\alpha\sigma} F^{\beta\widetilde{\sigma}}$, where the R-R field strength $F_{\Omega\widetilde{\Omega}} \equiv F^{\alpha\widetilde{\beta}} = \frac{1}{r_{\text{AdS}}} (\widetilde{\Gamma}_5)^{\alpha\beta}$. Note that the $\widetilde{\Gamma}_5$ was defined in (3.13) and the new parameter r_{AdS} is the AdS_5 space radius (note, $r_{\text{AdS}} = r_S$, i.e. the radius of S^5 is the same of AdS_5, so

that Ricci scalar $R = 0$). More specifically some of the table (4.8) curvatures:

$$\begin{aligned}
\dim 1 : \quad & R_{D \widetilde{D} \Sigma} \equiv R_{\alpha \widetilde{\beta}}{}^{\mathbf{a b}} = T_{\widetilde{\beta}}{}^{\sigma [\mathbf{a}} \gamma^{\mathbf{b}]}{}_{\alpha \sigma} & (4.9) \\
\dim 1 : \quad & R_{P \widetilde{D} \Omega} \equiv R_{\mathbf{a} \widetilde{\alpha}}{}^{\beta} = T_{\mathbf{a} \widetilde{\alpha}}{}^{\beta} = \gamma_{\mathbf{a} \ \alpha \sigma} F^{\beta \widetilde{\sigma}} \\
\dim 2 : \quad & R_{P \widetilde{P} \Sigma} \equiv R_{\mathbf{a} \widetilde{\mathbf{b}}}{}^{\mathbf{c d}} \propto \ T_{\widetilde{\mathbf{b}} \alpha}{}^{\widetilde{\beta}} R_{\widetilde{\beta} \gamma}{}^{\mathbf{c d}} \gamma_{\mathbf{a}}{}^{\alpha \gamma} \\
\dim 3 : \quad & R_{\Omega \widetilde{\Omega} \Sigma} \equiv R^{\alpha \widetilde{\beta} \ \mathbf{a b}} \propto \ (T^{\widetilde{\mathbf{d}} \ \widetilde{\beta}}{}_{\sigma} R_{\widetilde{\mathbf{d}} \mathbf{e}}{}^{\mathbf{a b}} + T_{\mathbf{e}}{}^{\widetilde{\beta} \nu} R_{\widetilde{\sigma} \nu}{}^{\mathbf{a b}}) \gamma^{\mathbf{e} \ \sigma \alpha}
\end{aligned}$$

where the dim 2 curvature is proportional with the constant $2^{-\frac{D}{2}+1}$ and the dim 3 curvature is proportional with a constant D (where D is 10 in our case).

All the other curvatures in (4.9) are obtained using the appropriate Bianchi identities (one can obtain all curvatures from $T_{P \widetilde{D} \Omega}$ using Bianchi identities). We note that the torsions (4.8) and curvatures (4.9) are consistent with torsions and curvatures given in the [16].

4.3 Extended $AdS_5 \times S^5$ T-dual algebra

To identify the pre-potential in the generalised vielbeins, i.e. solving the spectrum of the theory (on linearised level) we want to proceed as described in earlier papers [1, 4, 28]. There the vielbeins were introduced as in (4.1) and linearised as above the equation (4.7). This procedure means the expansion of generally curved superspace around some (in those papers just a flat) background. Moreover the gauge invariance was completely fixed (in referenced papers the covariant gauge was considered) and after that the pre-potential was identified as a part of vielbein (acting on by derivatives, the physical spectrum is produced).

Here we want to proceed in similar way. We want to introduce the vielbeins and linearise the theory around the $AdS_5 \times S^5$ background. We have tried to use solely the algebra described in the previous sub-section, i.e. to take the algebra (4.8) and introduce the vielbeins, gauge fix and linearise. Even though we still believe that the pre-potential is sitting in that theory in some combination of vielbeins, it was not easy to identify it. The reason was that to identify the pre-potential we need to find a scalar contraction of some linear combination of vielbeins that is anihilated by the D_v and $D_{\bar{v}}$ operators. The D_v and $D_{\bar{v}}$ are certain combinations of $D_{\alpha'}$ and $D_{\widetilde{\alpha}'}$ (see the index notation above the (3.14), i.e. they are a particular chiral part of the SO(8) chiral decomposition of the 16 supersymmetry translations D_α defined at the beginning, see (3.7) and section above (3.14)).

Because the metric is in $H_{P \widetilde{P}}$ vielbein, we expected the pre-potential to be (at least a part of it) in $\operatorname{Tr} H_{P \widetilde{P}}$. The problem with $\operatorname{Tr} H_{P \widetilde{P}}$ is that it already has dimension 0. To show that it is annihilated by D_v and $D_{\bar{v}}$ operators we would need to use torsion constraints of dimension $\frac{1}{2}$. Moreover we also know that the pre-potential has to be annihilated by the suitably defined $\mathcal{P}_+$ operator ($\mathcal{P}_+ \propto (P_+ + P_{\widetilde{+}})$ where $P_+ \equiv D_{P_+}$ and $P_{\widetilde{+}} \equiv D_{P_{\widetilde{+}}}$), in a light cone basis introduced in (3.14) and in the near horizon limit (defined later)). The high dimensionality of $\operatorname{Tr} H_{P \widetilde{P}}$ then requires to use at least dimension 1 torsion constraints to prove that $\mathcal{P}_+$ vanishes (in the near horizon limit). That seemed to be problematic to analyse in the theory based just on the algebra (4.8) and (4.9) because of the degauging procedure. The degauging appears since the theory coming from algebras (4.8) and (4.9) is really coming from the full SO(10) $\otimes$ SO(10) so there are some missing

Lorentz connections. By simple dimensional analysis it is evident that the missing Lorentz connections are first appearing at dimension 1 (for example the appearance of the full H_{PS} in the dim 1 torsion $T_{PPP} \propto \ldots + H_{[P|S}f_{PP]\Sigma} + \ldots$).

For that reason we extended the algebra (4.8) to include the original (Wick rotated) local Lorentz group $\mathrm{SO}(10) \otimes \mathrm{SO}(10)$. All the other structure constants and curvatures in (4.9) and (4.8) stay the same. Except now we have separate left local Lorentz $S_{\mathbf{ab}}$ generator together with the right local Lorentz generator $S_{\widetilde{\mathbf{ab}}}$, where $\mathbf{a} \in \{1, \ldots 10\}$. The common (Wick rotated) $\mathrm{SO}(5) \otimes \mathrm{SO}(5)$ Lorentz group of original $AdS_5 \times S^5$ algebra (4.8) is then the subgroup in the diagonal $\mathrm{SO}(10)$ subgroup of $\mathrm{SO}(10) \otimes \mathrm{SO}(10)$. The extension procedure can be viewed from the different picture. We could start with the full 10 dimensional string superspace as introduced in [2, 3] and [26, 27]. Then introduce the curved version of that space via vielbeins and then linearise around some background as described earlier. The extension of (4.8) and (4.9) to full $\mathrm{SO}(10) \otimes \mathrm{SO}(10)$ is then just a choice of some particular background that is consistent with the original $AdS_5 \times S^5$. This has an advantage that now we have a natural place where to put the troubling (part) of the pre-potential $\mathrm{Tr}\, H_{P\widetilde{P}}$. Because of the additional $S_{\mathbf{ab}}$ and $S_{\widetilde{\mathbf{ab}}}$ we can have $\mathrm{Tr}\, H_{+\mathbf{a}+\widetilde{\mathbf{b}}} \equiv \mathrm{Tr}\, H_{S\widetilde{S}}$ instead of the $\mathrm{Tr}\, H_{P\widetilde{P}}$. The $\mathrm{Tr}\, H_{P\widetilde{P}} \equiv H_{\mathbf{a}\widetilde{\mathbf{b}}}$ is related to $\mathrm{Tr}\, H_{S\widetilde{S}} \equiv \mathrm{Tr}\, H_{+\mathbf{a}+\widetilde{\mathbf{b}}}$ by an action of $D_{P_-} \equiv P_-$ and $D_{P_{\simeq}} \equiv P_{\simeq}$ (they are both invertible operators). The vielbein $H_{S\widetilde{S}}$ is of the dimension -2 and so there is no need to use the higher dimensional torsion constraints. Moreover in the full $\mathrm{SO}(10) \otimes \mathrm{SO}(10)$ theory we do not have to do the degauging procedure.

This extension comes with the cost. The mixed pieces of the AdS algebra (4.9) are breaking the explicit $\mathrm{SO}(10) \otimes \mathrm{SO}(10)$ invariance (they are not the $\mathrm{SO}(10) \otimes \mathrm{SO}(10)$ invariant tensors). We still have present the full $D_S \equiv S$ generators. Those derivatives could hit the (non-invariant) curvatures. The solution of this is to keep the explicit mixed curvature dependence (as generic mixed curvatures) till the S derivatives are not being explicitly evaluated. We will describe this procedure in detail later.

5 Gauge fixing

We want to fix the space-cone gauge (T-dual super space-cone gauge) for the first fluctuation $H_{\mathcal{AB}}$, i.e. like in the usual light-cone we have $D_{P_-} \equiv P_-$ operator invertible, now we have P_- and $P_{\simeq}$ invertible (where $D_{P_-} \equiv P_{\simeq}$).

First we look at the gauge variation (4.6) more closely and at the linearised level:

$$\delta_\Lambda H_{\mathcal{AB}} = D_{[\mathcal{A}}\lambda_{\mathcal{B})} + f_{\mathcal{AB}}{}^{\mathcal{C}}\lambda_{\mathcal{C}} \tag{5.1}$$

In the light-cone gauge we in general pick a vielbein with an P_- or $P_{\simeq}$ index, put that vielbein to zero. In order to maintain that gauge we need to fix the particular gauge parameter. For simplicity we call $P_- \equiv -$ and $P_{\simeq} \equiv \widetilde{-}$ then:

$$H_{-\mathcal{A}} = 0 \;\Rightarrow\; \delta_\Lambda H_{-\mathcal{A}} = 0 \;\Rightarrow\; P_-\lambda_{\mathcal{A}} - D_{\mathcal{A}}\lambda_- + f_{-\mathcal{A}}{}^{\mathcal{C}}\lambda_{\mathcal{C}} = 0 \tag{5.2}$$

$$\text{then } \lambda_{\mathcal{A}} = \frac{1}{P_-}\left(D_{\mathcal{A}}\lambda_- + f_{-\mathcal{A}}{}^{\mathcal{C}}\lambda_{\mathcal{C}}\right)$$

Note that there are more possibilities to fix the particular gauge parameters $\lambda_{\mathcal{A}}$. To fix $\lambda_{\mathcal{A}}$ we could also put $H_{\underline{-}\,\mathcal{A}} = 0$ and use the invertibility of $P_{\underline{-}}$. Of course we can not fix some gauge parameter twice. We have to decide which vielbeins we are going to fix in this "double" light-cone gauge.

We picked the approach where we used the mixed vielbeins to vanish by the double light-cone gauge fixing, i.e. we put $H_{\underline{-}\,\mathcal{A}} = 0$ for $\mathcal{A} \in \{S, D, P, \Omega, \Sigma\} \equiv$ left part of algebra. Together with $H_{-\,\widetilde{\mathcal{A}}} = 0$ for $\widetilde{\mathcal{A}} \in \{\widetilde{S}, \widetilde{D}, \widetilde{P}, \widetilde{\Omega}, \widetilde{\Sigma}\} \equiv$ right part of algebra. By that choice we fully fix the gauge parameters $\lambda_{\mathcal{A}}$ and $\lambda_{\widetilde{\mathcal{A}}}$ in terms of λ_-. That parameter can be fixed by the gauge invariance of the gauge invariance.

The motivation for the previously described mixed left right light-cone gauge fixing came from the flat space (just the extended $AdS_5 \times S^5$ space with $r_{\mathrm{AdS}} \to \infty$, i.e. the flat $\mathrm{SO}(10) \otimes \mathrm{SO}(10)$ T-dual superspace). After picking this type of the light-cone gauge the mixed torsion constraints of the type $T_{\underline{-}\,\mathcal{A}\mathcal{B}} = 0$ are as algebraic as possible:

$$T_{\underline{-}\,\mathcal{A}\mathcal{B}} = P_{\underline{-}}\, H_{\mathcal{A}\mathcal{B}} + D_{\mathcal{B}}\, H_{\underline{-}\,\mathcal{A}} + D_{\mathcal{A}}\, H_{\mathcal{B}\,\underline{-}} + H_{\underline{-}\,\mathcal{M}}\, \eta^{\mathcal{M}\mathcal{N}} f_{\mathcal{A}\mathcal{B}\mathcal{N}} + H_{[\mathcal{A}|\mathcal{M}}\eta^{\mathcal{M}\mathcal{N}} f_{\mathcal{B})\,\underline{-}\,\mathcal{N}} \quad (5.3)$$

$$T_{\underline{-}\,\mathcal{A}\mathcal{B}} = P_{\underline{-}}\, H_{\mathcal{A}\mathcal{B}} = 0 \ \Rightarrow\ H_{\mathcal{A}\mathcal{B}} = 0 \ \text{for}\, \mathcal{A},\, \mathcal{B} : T_{\underline{-}\,\mathcal{A}\mathcal{B}} = 0 \quad (5.4)$$

where we used our mixed light-cone gauge and $r_{\mathrm{AdS}} \to \infty$ of extended algebra in (5.4).

The same as in (5.3) and (5.4) holds if one fully swaps left and right indices. For finite r_{AdS} we can have the mixed structure constants nonzero (i.e. $f_{\mathcal{A}\mathcal{B}\,\underline{-}} \neq 0$) and so we would have a right hand side in (5.4). Note also that there could be the contribution from S derivatives hitting the mixed structure constants. Even though the right hand side in (5.4) is not generally vanishing for finite r_{AdS} we found that the mixed left-right light-cone gauge is still useful in the $AdS_5 \times S^5$ case.

6 Torsion constraints

6.1 $AdS_5 \times S^5$ curvatures and D_S derivatives

As we noted in the introduction section. Because we have enhanced our superspace, we have to take special care when the local Lorentz derivatives $D_S \equiv S$ are hitting the mixed curvatures (4.9). This problem arises because the curvatures in (4.9) are not full $\mathrm{SO}(10) \otimes \mathrm{SO}(10)$ invariant. The solution is to keep the non-invariant torsions (4.9) generic and explicitly act by the $D_S \equiv S$ derivatives on those torsions. Only after this explicit S action we can evaluate those torsions (or curvatures) and be fixed as in (4.9).

Let's take an example, from the equation (5.3) we can see that in the $AdS_5 \times S^5$ case in the mixed light-cone gauge the $H_{\mathcal{A}\mathcal{B}}$ is determined as:

$$H_{\mathcal{A}\mathcal{B}} = \frac{1}{P_{\underline{-}}}\, H_{[\mathcal{A}|\,\mathcal{M}}\, \eta^{\mathcal{M}\mathcal{N}} f_{\mathcal{B})\,\underline{-}\,\mathcal{N}} \quad (6.1)$$

In many instances in this paper we use similar relation as in (6.1) to fix some particular vielbein in terms of another vielbeins. If all structure constants f would be $\mathrm{SO}(10) \otimes \mathrm{SO}(10)$ invariant tensors then there is not a problem and we can treat the f structure constants as genuine constants also with respect to the S derivatives. In our case however

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the mixed f structure constants (that we call also the curvatures) in (4.9) explicitly break the SO(10) ⊗ SO(10) local Lorentz invariance down to the SO(5) ⊗ SO(5) local Lorentz (as it should be in the $AdS_5 \times S^5$ case). One possibility is to restrict our superspace local Lorentz invariance (the S derivatives) to SO(5) ⊗ SO(5). Then we would return back to the PSU (2, 2 | 4) that we wanted to avoid in the first place (in order to have $H_{+\mathbf{a}\,\widetilde{+\mathbf{b}}}$ instead of $H_{\mathbf{a}\widetilde{\mathbf{b}}}$). The alternative, that we picked, is to work with the full SO(10) ⊗ SO(10) local Lorentz group. But then the structure constants that are breaking that invariance are not invariant tensors and so the action of those S derivatives on the mixed structure constants has to be accounted for. So we should keep the mixed f structure constants and when needed explicitly act by the S derivatives on them. We will evaluate them as the very last step in our calculations. Let's look at the example in (6.1) and look at the action of $S_{\widetilde{+\mathbf{a}}}$, where $\mathbf{a} \in \{ 1 \ldots 8\}$:

$$S_{\widetilde{+\mathbf{a}}} \, H_{\mathcal{A}\mathcal{B}} = S_{\widetilde{+\mathbf{a}}} \left(\frac{1}{P_{\underline{-}}} \, H_{[\mathcal{A}|\,\mathcal{M}} \, \eta^{\mathcal{M}\mathcal{N}} \right) f_{\mathcal{B})\,\underline{\simeq}\,\mathcal{N}} + \frac{1}{P_{\underline{-}}} \, H_{[\mathcal{A}|\,\mathcal{M}} \, \eta^{\mathcal{M}\mathcal{N}} \left(S_{\widetilde{+\mathbf{a}}} \, f_{\mathcal{B})\,\underline{\simeq}\,\mathcal{N}} \right) \quad (6.2)$$

$$= S_{\widetilde{+\mathbf{a}}} \left(\frac{1}{P_{\underline{-}}} \, H_{[\mathcal{A}|\,\mathcal{M}} \, \eta^{\mathcal{M}\mathcal{N}} \right) f_{\mathcal{B})\,\underline{\simeq}\,\mathcal{N}} + \eta_{+-} \, \frac{1}{P_{\underline{-}}} \, H_{[\mathcal{A}|\,\mathcal{M}} \, \eta^{\mathcal{M}\mathcal{N}} \, f_{\mathcal{B})\,\widetilde{\mathbf{a}}\,\mathcal{N}}$$

We will evaluate the $f_{\mathcal{B}\,\widetilde{\mathbf{a}}\,\mathcal{N}}$ in the second equation just after all the (possibly future) S derivatives have already acted. We also should bear in mind that whenever we are acting by the S derivative on some vielbein (that is determined by another vielbeins) there might be the above described issue. The second term can (and it will) nontrivially contribute to our calculations.

6.2   Torsion constraints and H_{SS} vielbein

The torsion constraints are (mainly) given by the curved version of the **A B C D** (first class) constraints, see [19] and [28]. There are further constraints called $\widetilde{T}_{\mathcal{A}} = 0$ coming from requirement of partial integration in the presence of the dilaton measure, see [2, 3], and [4]. There is also a strong constraint: on every field in the double field theory one has to require $D^{\mathcal{A}} D_{\mathcal{A}} = 0$. There are also a dimensional reduction constraints, as we see later.

   Our aim is to analyse the necessary constraints consistent with the above constraints by which we can identify the pre-potential. Following the analysis given in [17, 18], we identify the pre-potential as a scalar super-field (given by some super-trace of possibly a combination of vielbeins), that is annihilated by certain combination of the $D_{\nu'}$ and $D_{\widetilde{\nu}'}$. The precise combination of $D_{\nu'}$ and $D_{\widetilde{\nu}'}$ is also going to be determined from the constraints.

   As usual, we start to eliminate the lowest dimensional vielbeins. The vielbeins of the lowest dimension are H_{SS}, $H_{S\widetilde{S}}$, $H_{\widetilde{S}\widetilde{S}}$. They are of the dimension -2 (we mean the ten dimensional dimension). Using equations (5.3) and (5.4) for indices $\mathcal{A} = S$ and $\mathcal{B} = S$ (also after change left ↔ right) we immediately get that $H_{SS} = 0 = H_{\widetilde{S}\widetilde{S}}$. Note, that even in the extended $AdS_5 \times S^5$ superspace the structure constant $f_{S\,\widetilde{\mathcal{A}}\mathcal{B}} = 0 \rightarrow f_{S\,\underline{\simeq}\,\mathcal{B}} = 0$.

   We mention an important observation that will help us simplify future calculations. As we saw in previous sub-section, we should keep the mixed structure constants generic and evaluate them at the end. Note however, that the mixed structure constants with the S indices are always zero (like the one we considered here: $f_{S\widetilde{\mathcal{A}}\mathcal{B}}$). The action of S derivatives

on them results in the mixed structure constants again with the S index and such are zero after the evaluation. So specifically, we can evaluate the mixed structure constants with the S indices to zero even before acting by S derivatives on them.

The mixed vielbein $H_{S\widetilde{S}}$ is not all zero and the claim is that the part of the pre-potential is in this particular vielbein. To see which parts are possibly nonzero we rewrite the $H_{S\widetilde{S}}$ in the double light-cone components:

$$H_{S\widetilde{S}} \equiv \left\{ H_{+\mathbf{a}\,\widetilde{+\mathbf{c}}},\, H_{+\mathbf{a}\,\widetilde{-\mathbf{b}}},\, H_{-\mathbf{a}\,\widetilde{-\mathbf{b}}},\, H_{+\mathbf{a}\,\widetilde{\mathbf{bc}}},\, H_{-\mathbf{a}\,\widetilde{\mathbf{bc}}},\, H_{\mathbf{ab}\,\widetilde{\mathbf{cd}}},\, H_{+-\,\widetilde{+-}},\, H_{+-\,\widetilde{+\mathbf{a}}} \quad (6.3)$$
$$H_{+-\,\widetilde{-\mathbf{a}}},\, H_{+-\,\widetilde{\mathbf{ab}}} \right\} \oplus \text{ swap}$$

where we might swap left index with the right index in (6.3). Also note that in all previous we have $\mathbf{a} \in \{1, \ldots, 8\}$. We remind that $P_+ \equiv + \propto 10 + 9$ and $P_- \equiv - \propto 10 - 9$.

We want to use analog of equations (5.3) and (5.4) for the mixed $H_{S\widetilde{S}}$ and $r_{\text{AdS}} \neq \infty$:

$$T_{\widetilde{-}S\widetilde{S}} = 0 = P_{\widetilde{-}} H_{S\widetilde{S}} + D_{\widetilde{S}} H_{\widetilde{-}S} + D_S H_{\widetilde{S}\widetilde{-}} + H_{[\widetilde{-}|\mathcal{M}} \eta^{\mathcal{M}\mathcal{N}} f_{S\widetilde{S}]\mathcal{N}} \quad (6.4)$$

In equation (6.4) we have term $H_{\widetilde{-}S}$ zero by gauge choice. The vielbein $H_{\widetilde{S}\widetilde{-}}$ is proportional to $f_{S\widetilde{-}\mathcal{N}}$. We see that this term is zero after evaluating $f_{S\widetilde{-}\mathcal{N}} = 0$ by (4.9). Using the (5.3) and (5.4) for $\mathcal{A} = S$ and $\mathcal{B} = -$ and keeping $f_{-\widetilde{-}S}$ nonzero, we get $H_{S-} = \frac{1}{P_-} f_{-\widetilde{-}\mathcal{M}} \eta^{\mathcal{M}\mathcal{N}} H_{S\mathcal{N}}$ and similarly for $H_{\widetilde{S}\widetilde{-}}$ and so the third term in (6.4) is also fixed. The (6.4) is then:

$$T_{\widetilde{-}S\widetilde{S}} = 0 = P_{\widetilde{-}} H_{S\widetilde{S}} + D_S \left(\frac{1}{P_-} f_{-\widetilde{-}\mathcal{M}} \eta^{\mathcal{M}\mathcal{N}} H_{\widetilde{S}\mathcal{N}} \right) + H_{S\mathcal{M}} \eta^{\mathcal{M}\mathcal{N}} f_{\widetilde{S}\widetilde{-}\mathcal{N}} \quad (6.5)$$

Using equations (5.3), (5.4), (6.5) and the mixed light-cone gauge together with keeping the mixed structure constants and evaluating the explicit actions of the S and $\widetilde{S}$ derivatives we get the first important result for the structure of the $H_{S\widetilde{S}}$ vielbein (in the AdS case), see table 5 in appendices. From the table 5 we can see that the possibly nonzero $H_{S\widetilde{S}}$ in (6.5) are those for which $f_{\widetilde{S}\widetilde{-}\mathcal{N}}$ are nonzero (after evaluation of mixed structure constants). By simple left $\leftrightarrow$ right swap we get that for $H_{S\widetilde{S}}$ to be nonzero also $f_{S-\mathcal{N}}$ has to be nonzero. From the $[S, P]$ part of SO(10) $\otimes$ SO(10) extended algebra of (3.7) we can see the only possibility: $H_{+\mathbf{a}\,\widetilde{+\mathbf{c}}} \neq 0$. All the other components of $H_{S\widetilde{S}} = 0$ by (6.5) and table 5 after evaluation. This is a first hint that we are on the right track. The $H_{+\mathbf{a}\,\widetilde{+\mathbf{c}}} \neq 0$ is the only nonzero part (after evaluation of mixed structure constants) of $H_{S\widetilde{S}}$ piece, it has a scalar trace and we can easily relate it to $H_{\mathbf{a}\widetilde{\mathbf{b}}} \equiv H_{P\widetilde{P}}$, where we expect the part of the pre-potential to be (the symmetric part corresponds to the metric).

To see how $H_{P\widetilde{P}}$ is related to $H_{S\widetilde{S}}$ consider the third relation from table 5 and after evaluation of mixed structure constants:

$$\left(1 + \frac{1}{2(r_{\text{AdS}})^2} \frac{1}{P_- P_{\widetilde{-}}} \right) H_{+\mathbf{a}\widetilde{+\mathbf{b}}} = \frac{1}{P_-} H_{\mathbf{a}\,\widetilde{+\mathbf{b}}} \quad (6.6)$$

We want to reduce $H_{\mathbf{a}\,\widetilde{+\mathbf{b}}}$ further to get $H_{\mathbf{a}\widetilde{\mathbf{b}}}$. One can naïvely expect to just hit $H_{\mathbf{a}\widetilde{+\mathbf{b}}}$ with $P_{\widetilde{-}}$ to get rid of the $\widetilde{S}$ index (or alternatively hit by P_- the $H_{+\mathbf{a}\widetilde{\mathbf{b}}}$). It works but

one has to be more careful since in the $AdS_5 \times S^5$ space one has the mixed structure constant $f_{-\widetilde{\mathbf{b}}\mathcal{N}} \neq 0$. To see what is this structure constant (after the mixed structure constants evaluation) we remind that in (4.9) we saw that dimension 2 structure constant is given as $R_{P\widetilde{P}\Sigma} \equiv R_{\mathbf{a}\widetilde{\mathbf{b}}}{}^{\mathbf{cd}} \propto T_{\widetilde{\mathbf{b}}\alpha}{}^{\widetilde{\beta}} R_{\widetilde{\beta}\gamma}{}^{\mathbf{cd}} \gamma_{\mathbf{a}}{}^{\alpha\gamma}$. We have to be careful with the indices in the $R_{\mathbf{a}\widetilde{\mathbf{b}}}{}^{\mathbf{cd}}$. The Σ indices $\mathbf{cd}$ in (4.9) were indices for SO(5) $\otimes$ SO(4, 1) local Lorentz group (or its Wick rotated version SO(5) $\otimes$ SO(5)). But the index $\mathcal{N}$ in $f_{-\widetilde{\mathbf{b}}\mathcal{N}} \neq 0$ includes the indices for the full SO(10) $\otimes$ SO(10). We have already made the claim that the original local Lorentz group SO(5) $\otimes$ SO(5) is in the *diagonal* subgroup of the SO(10) $\otimes$ SO(10), i.e. we have the following (at the level of algebras) $so(10) \oplus so(10) \equiv \frac{1}{2}(so(10) + so(10)) \oplus \frac{1}{2}(so(10) - so(10)) := so(10)_{\mathbf{D}} \oplus so(10)_{\mathbf{Off}}$. (The meaning of previous is to do the operations on basis. The $so(10) - so(10)$ means for example to combine e.g. Lorentz generators like: $S - \widetilde{S} = S_{\mathbf{Off}}$ and similarly for another generators). Now the $so(5) \oplus so(5) \hookrightarrow so(10)_{\mathbf{D}}$. Let us write the last sequence of algebras more precisely. Using indices, the diagonal subgroup (subalgebra) is $so(10)_{\mathbf{D}} \equiv S^{\mathbf{D}}{}_{\mathbf{ab}} := \frac{1}{2}(S_{\mathbf{ab}} + S_{\widetilde{\mathbf{ab}}}) = (S^{\mathbf{D}}{}_{\mathbf{ij}}, S^{\mathbf{D}}{}_{\mathbf{kl}}, S^{\mathbf{D}}{}_{\mathbf{ik}})$, where $\mathbf{i} \in \{ 10, 1, 2, 3, 4 \}$ and $\mathbf{k} \in \{ 5, 6, 7, 8, 9 \}$ and the $\mathbf{a} = (\mathbf{i}, \mathbf{k}) \equiv \{ 1, \ldots, 10 \}$. The $S^{\mathbf{D}}{}_{\mathbf{ij}}$ and $S^{\mathbf{D}}{}_{\mathbf{kl}}$ are the generators of the SO(5) $\otimes$ SO(5). Moreover, the previous definitions give precise embedding of those operators.

Defining $S^{\mathbf{D}}{}_{\mathbf{ij}}$ and $S^{\mathbf{D}}{}_{\mathbf{kl}}$ we can see that the structure constant $f_{\mathbf{a}\widetilde{\mathbf{b}}\mathcal{N}}$ is either 0 or given by the appropriate $R_{P\widetilde{P}\Sigma_{\mathbf{D}}}$. We included a small subindex to the Σ coordinate just to remind us that the Σ coordinate is now for the SO(5) $\otimes$ SO(5) diagonal subgroup of SO(10)$_{\mathbf{D}}$ only.

Finally, taking definitions of $R_{\mathbf{a}\widetilde{\mathbf{b}}}{}^{\mathbf{cd}}$ and table 4.9 and our definitions we can see that $f_{\mathbf{a}\widetilde{\mathbf{b}}\mathcal{N}}$ is coming from the mixed commutator

$$[P_{\mathbf{a}}, P_{\widetilde{\mathbf{b}}}] \propto \begin{cases} (\frac{1}{r_{\mathrm{AdS}}})^2 S^{\mathbf{D}}{}_{\mathbf{ab}} & \text{if } \mathbf{a} \;\&\; \mathbf{b} \in \{ 10, 1, 2, 3, 4 \} \\ -(\frac{1}{r_{\mathrm{AdS}}})^2 S^{\mathbf{D}}{}_{\mathbf{ab}} & \text{if } \mathbf{a} \;\&\; \mathbf{b} \in \{ 5, 6, 7, 8, 9 \} \\ 0 & \text{otherwise} \end{cases}$$

The proportionality constant is $\mathbf{c_1} = -2$. With the previous definition and with $P_- \equiv - = \frac{1}{2}(10 - 9)$ and $\widetilde{\mathbf{b}} \in \{ 5, 6, 7, 8, 9 \}$ we will get the $[P_-, P_{\widetilde{\mathbf{b}}}] \propto S^{\mathbf{D}}{}_{\mathbf{9b}} = \frac{1}{2}(S_{\mathbf{9b}} + S_{\widetilde{\mathbf{9b}}})$ we can also use $P_{\mathbf{9}} = (P_+ - P_-)$ and so $[P_-, P_{\widetilde{\mathbf{b}}}] \propto (S_{+\mathbf{b}} + S_{\widetilde{+\mathbf{b}}} - S_{-\mathbf{b}} - S_{\widetilde{-\mathbf{b}}})$. Knowing the last relation we can proceed and hit the result of (6.6) by $P_{\simeq}$ and relate $H_{+\mathbf{a}\widetilde{+\mathbf{b}}}$ with $H_{\mathbf{a}\widetilde{\mathbf{b}}}$ for the evaluated version. For non-evaluated version we need to do the same for the non-evaluated version of (6.6), that is the third top equation in table 5. For vielbein $H_{\widetilde{+\mathbf{b}}\mathbf{a}}$ needed in this procedure we get:

$$0 = T_{P\widetilde{P}\widetilde{S}} \equiv T_{\mathbf{a}\simeq\widetilde{+\mathbf{b}}} = P_{[\mathbf{a}} H_{\simeq\widetilde{+\mathbf{b}}]} + H_{[\mathbf{a}|\mathcal{M}} \eta^{\mathcal{MN}} f_{\simeq\widetilde{+\mathbf{b}}]\mathcal{N}} \tag{6.7}$$
$$= -P_{\simeq} H_{\widetilde{+\mathbf{b}}\mathbf{a}} - P_{\mathbf{a}} H_{\simeq\widetilde{+\mathbf{b}}} - H_{\widetilde{+\mathbf{b}}\mathcal{M}} \eta^{\mathcal{MN}} f_{\mathbf{a}\simeq\mathcal{N}} - H_{\mathbf{a}\mathcal{M}} \eta^{\mathcal{MN}} f_{\simeq\widetilde{+\mathbf{b}}\mathcal{N}}$$

The term $H_{\simeq\mathbf{a}}$ vanishes because of mixed light-cone gauge, the term $H_{\simeq\widetilde{+\mathbf{b}}}$ is fixed by the torsion $T_{-\simeq\widetilde{+\mathbf{b}}} = 0$ and use of mixed light-cone gauge similarly as in equation (A.1). By that we get:

$$H_{\simeq\widetilde{+\mathbf{b}}} = -f_{-\simeq\mathcal{M}} \eta^{\mathcal{MN}} \frac{1}{P_-} H_{\widetilde{+\mathbf{b}}\mathcal{N}} \rightsquigarrow 0 \tag{6.8}$$

The last term in (6.7) is just $\eta_{-+}\,\eta_{\mathbf{a}\mathcal{N}}$, the analog term as in (6.6). The extra mixed term (after evaluation) in (6.7) is $H_{+\mathbf{a}\,\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\,f_{\widetilde{\mathbf{b}}-\mathcal{N}}\,\propto\,(\frac{1}{r_{\mathrm{AdS}}})^2\,H_{+\mathbf{a}\,\widetilde{+\mathbf{b}}}$. Plugging (6.8) into (6.7) and then the result (that is the fixed vielbein $H_{\widetilde{+\mathbf{b}}\,\mathbf{a}}$) into the third top equation in table 5 we obtain the non-evaluated relation between $H_{+\mathbf{a}\,\widetilde{+\mathbf{b}}}$:

$$P_-\,H_{+\mathbf{a}\,\widetilde{+\mathbf{b}}} = -\,f_{-\,\simeq\,\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\,S_{\widetilde{+\mathbf{b}}}\left(\frac{1}{P_{\simeq}}\,H_{+\mathbf{a}\mathcal{N}}\right) - f_{-\,\widetilde{\mathbf{b}}\,\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\,\frac{1}{P_{\simeq}}\,H_{+\mathbf{a}\mathcal{N}} \qquad (6.9)$$

$$-\,f_{-\,\simeq\,\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\,\frac{1}{P_{\simeq}}\,P_{\mathbf{a}}\,\frac{1}{P_-}\,H_{\widetilde{+\mathbf{b}}\mathcal{N}} + f_{\simeq\,\mathbf{a}\,\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\,\frac{1}{P_{\simeq}}\,H_{\widetilde{+\mathbf{b}}\mathcal{N}} + \frac{1}{P_{\simeq}}\,H_{\mathbf{a}\widetilde{\mathbf{b}}}$$

after evaluation of the mixed structure constants in (6.9) we get:

$$H_{\mathbf{a}\widetilde{\mathbf{b}}} = \left(P_-\,P_{\simeq} + \frac{1}{(r_{\mathrm{AdS}})^2}\right)H_{+\mathbf{a}\,\widetilde{+\mathbf{b}}} \qquad (6.10)$$

6.3 Torsion constraints and $H_{D\,S}$ vielbein

To identify what combination of vielbeins gives the pre-potential, we first repeat the properties we are looking for. We are looking for combination of vielbeins (of the low dimension), that has a scalar contraction and is annihilated by certain combination of $D_{\alpha'}$ and $D_{\widetilde{\alpha}'}$ (see indices defined above (3.14)). Moreover the combination has to be annihilated by the properly defined $\mathcal{P}_+$ operator in the $R \to 0$ limit (still to be defined).

To start, we have one nontrivial hint. We showed that the vielbein $H_{+\mathbf{a}\,\widetilde{+\mathbf{b}}}$ is nonzero and is related to the $H_{\mathbf{a}\widetilde{\mathbf{b}}}$. So we can examine what is the action of the $D_{\alpha'}$ on $H_{+\mathbf{a}\,\widetilde{+\mathbf{b}}}$, i.e. we look at the torsion constraint:

$$T_{D\,S\,\widetilde{S}}\equiv T_{\alpha'\,+\mathbf{a}\,\widetilde{+\mathbf{b}}}=0 = D_{[\alpha'}\,H_{+\mathbf{a}\,\widetilde{+\mathbf{b})}} + H_{[\alpha'\,|\,\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\,f_{+\mathbf{a}\,\widetilde{+\mathbf{b})}\mathcal{N}} \qquad (6.11)$$

$$= D_{\alpha'}\,H_{+\mathbf{a}\,\widetilde{+\mathbf{b}}} + S_{\widetilde{+\mathbf{b}}}\,H_{\alpha'\,+\mathbf{a}} + S_{+\mathbf{a}}\,H_{\widetilde{+\mathbf{b}}\,\alpha'} + H_{\widetilde{+\mathbf{b}}\,\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\,f_{\alpha'\,+\mathbf{a}\mathcal{N}}$$

$$= D_{\alpha'}\,H_{+\mathbf{a}\,\widetilde{+\mathbf{b}}} + S_{\widetilde{+\mathbf{b}}}\,H_{\alpha'\,+\mathbf{a}} + S_{+\mathbf{a}}\,H_{\widetilde{+\mathbf{b}}\,\alpha'} + \frac{1}{2}\,(\gamma_{+\mathbf{a}})_{\alpha'}{}^{\beta}\,H_{\widetilde{+\mathbf{b}}\,\beta} \qquad (6.12)$$

In the (6.12) we can see various terms with the S derivatives. If we could evaluate mixed structure constants before an action of S derivatives, those S terms in (6.12) would vanish (because the relevant vielbeins are proportional to vanishing mixed constants as we will see). Now they will nontrivially contribute. We note again that we still have the $f_{\widetilde{S}\,D\mathcal{N}} \equiv f_{\widetilde{+\mathbf{b}}\,\alpha'\mathcal{N}} = 0 = f_{S\,\widetilde{S}\mathcal{N}} \equiv f_{+\mathbf{a}\,\widetilde{+\mathbf{b}}\mathcal{N}}$. The vielbeins $H_{\alpha'\,+\mathbf{a}}$ and $H_{\alpha'\,\widetilde{+\mathbf{b}}}$ are fixed by torsion constraints $T_{\widetilde{P}\,D\,S} = T_{\simeq\,\alpha'\,+\mathbf{a}} = 0 = T_{P\,D\,\widetilde{S}} = T_{-\,\alpha'\,\widetilde{+\mathbf{b}}}$ and some few other torsion constraints, as will be shown. We note that as in (5.3) and (5.4) we almost always have the strategy to use invertibility of P_- and $P_{\simeq}$ together with our mixed left-right light cone gauge to eliminate/fix vielbeins. Sometimes its not enough and we need to explore some further torsion constraints. Let's look at the already mentioned set of torsion constraints:

$$T_{\widetilde{P}\,D\,S} = T_{\simeq\,\alpha'\,+\mathbf{a}} = 0 = P_{[\simeq}\,H_{\alpha'\,+\mathbf{a}]} + H_{[\simeq\,|\,\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\,f_{\alpha'\,+\mathbf{a}]\mathcal{N}} \qquad (6.13)$$

$$= P_{\simeq}\,H_{\alpha'\,+\mathbf{a}} + H_{+\mathbf{a}\,\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\,f_{\simeq\,\alpha'\mathcal{N}} \qquad (6.14)$$

$$\Rightarrow \quad H_{\alpha'\,+\mathbf{a}} = f_{\simeq\,\alpha'\,\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\,\frac{1}{P_{\simeq}}\,H_{+\mathbf{a}\mathcal{N}} \rightsquigarrow 0 \qquad (6.15)$$

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in (6.13) we used just the mixed light cone gauge and $f_{S \widetilde{P} \mathcal{N}} = 0$ (in flat case and also in AdS). To evaluate the last term in (6.14) that is present only in AdS case we have to take the AdS curvature $T_{\widetilde{P} D \widetilde{\Omega}} \equiv T_{\widetilde{\mathbf{a}} \alpha}{}^{\widetilde{\beta}} \equiv f_{\widetilde{\mathbf{a}} \alpha}{}^{\widetilde{\beta}} = \frac{1}{r_{\text{AdS}}} (\gamma_{\mathbf{a}})_{\alpha \nu} \widetilde{\Gamma}_5{}^{\nu \beta}$ as discussed above (4.9). For our specific indices we have $\frac{1}{r_{\text{AdS}}} (\gamma_-)_{\alpha' \nu'} \widetilde{\Gamma}_5{}^{\nu' \beta}$ but the $(\gamma_-)_{\alpha' \nu'} = 0$ as we can see in the construction of the light cone basis for the gamma matrices in (3.14). The vielbein $H_{\alpha' - \mathbf{a}}$ can be fixed in almost the same set of equations as $H_{\alpha' + \mathbf{b}}$. Fixing the vielbein $H_{\alpha' \widetilde{+\mathbf{b}}}$ (and similarly $H_{\alpha' \widetilde{-\mathbf{b}}}$) is also similar but a bit more profound. For that we first examine torsion analogous to (6.11) but for $H_{\alpha' \widetilde{+\mathbf{b}}}$ vielbein:

$$T_{P D \widetilde{S}} \equiv T_{- \alpha' \widetilde{+\mathbf{b}}} = 0 = P_{[-} H_{\alpha' \widetilde{+\mathbf{b}}]} + H_{[- | \mathcal{M}} \eta^{\mathcal{M} \mathcal{N}} f_{\alpha' \widetilde{+\mathbf{b}}] \mathcal{N}} \tag{6.16}$$

$$= P_- H_{\alpha' \widetilde{+\mathbf{b}}} + S_{\widetilde{+\mathbf{b}}} H_{- \alpha'} + H_{\widetilde{+\mathbf{b}} \mathcal{M}} \eta^{\mathcal{M} \mathcal{N}} f_{- \alpha' \mathcal{N}}$$

$$\Rightarrow \quad H_{\alpha' \widetilde{+\mathbf{b}}} = - \frac{1}{P_-} S_{\widetilde{+\mathbf{b}}} H_{- \alpha'} \tag{6.17}$$

The (6.16) structure constant $f_{- \alpha' \mathcal{N}} \propto (\gamma_-)_{\alpha' \nu'}$ but as before that particular piece of gamma matrix is zero (remember the non-mixed structure constants are not breaking the $\text{SO}(10) \otimes \text{SO}(10)$ so we can evaluate them without any concern). The other term in (6.16) is $H_{- \alpha'}$. That is fixed by the dim $\frac{1}{2}$ torsion constraint $T_{- \widetilde{-} \alpha'} = 0$:

$$T_{P \widetilde{P} D} = T_{- \widetilde{-} \alpha'} = 0 = P_{[-} H_{\widetilde{-} \alpha')} + H_{[- \mathcal{M}} \eta^{\mathcal{M} \mathcal{N}} f_{\widetilde{-} \alpha') \mathcal{N}} \tag{6.18}$$

$$= P_{\widetilde{-}} H_{\alpha' -} + H_{\alpha' \mathcal{M}} \eta^{\mathcal{M} \mathcal{N}} f_{- \widetilde{-} \mathcal{N}} \tag{6.19}$$

$$\Rightarrow \quad H_{\alpha' -} = - \frac{1}{P_{\widetilde{-}}} f_{- \widetilde{-} \mathcal{M}} \eta^{\mathcal{M} \mathcal{N}} H_{\alpha' \mathcal{N}} \rightsquigarrow 0 \tag{6.20}$$

We used in (6.18) the mixed light cone gauge, also the fact that $f_{- \alpha' \mathcal{M}} \propto (\gamma_-)_{\alpha' \beta'} = 0$. We note that the (6.18) evaluates to zero because the mixed structure constant $f_{- \widetilde{-} \mathcal{N}} = 0$. Moreover by the light-cone gauge the (6.18) term $H_{- \mathcal{M}} \eta^{\mathcal{M} \mathcal{N}} f_{\widetilde{-} \alpha' \mathcal{N}} = 0$ even in the non-evaluated regime. The reason is that the structure constant $f_{\widetilde{-} \alpha' \mathcal{N}}$ is zero after evaluation and the action of whatever S on this structure constant produces either zero or the right D index $\equiv \widetilde{D}$ (after the summation with the vielbein) i.e. the vielbein $H_{- \widetilde{D}}$ that is again zero by the light-cone gauge. Combining (6.17) and (6.20) we can get a fixed version of the $H_{\alpha' \widetilde{+\mathbf{b}}}$. By the similar equations as above we can fix $H_{\alpha' \widetilde{-\mathbf{b}}}$. That result and more detailed analysis is shown in the appendix, see table 6.

In the appendix, we also derived the equations (A.13) and (A.35). Those are the actions of the S and $\widetilde{S}$ derivatives that we need in the equation (6.12). Putting the results from (A.13) and (A.35) into (6.12) we get fixing of the $D_{\alpha'} H_{+\mathbf{a} \widetilde{+\mathbf{b}}}$, we note that this is an important result:

$$\begin{aligned}
D_{\alpha'} H_{+\mathbf{a} \widetilde{+\mathbf{b}}} = &- \frac{1}{g} \frac{1}{(r_{\text{AdS}}) P_{\widetilde{-}}} (\gamma_{\mathbf{b}})_{\alpha' \sigma} (\widetilde{\Gamma}_5)^{\sigma \beta} H_{\widetilde{\beta} + \mathbf{a}} \\
&+ \frac{1}{2 g} \left(1 - \frac{1}{f} \frac{1}{(r_{\text{AdS}})^2 P_{\widetilde{-}} P_-} \right) (\gamma_{+\mathbf{a}})_{\alpha'}{}^{\beta} H_{\beta \widetilde{+\mathbf{b}}} \\
&+ \frac{1}{f g} \frac{1}{2 (r_{\text{AdS}})^3 P_- (P_{\widetilde{-}})^2} (\gamma_{\mathbf{a}})_{\alpha' \nu} (\widetilde{\Gamma}_5)^{\nu \beta} H_{\widetilde{\beta} + \mathbf{b}}
\end{aligned} \tag{6.21}$$

where f and g are defined as follows:

$$f := \left(1 - \frac{1}{2\,(r_{\text{AdS}})^2\,P_{\simeq}\,P_-} \right) \tag{6.22}$$

$$g := \left(1 - \frac{1}{f}\,\frac{1}{2\,(r_{\text{AdS}})^2\,P_{\simeq}\,P_-} \right)$$

Changing left $\leftrightarrow$ right in (6.21) we get the equation for $D_{\widetilde{\alpha}'}\,H_{+\mathbf{a}\,\widetilde{+\mathbf{b}}}$. There is one simplification we can make in equations (6.21). Because only half of the block diagonal γ_+ matrix is nonzero and is proportional to the δ for the nonzero part. The $(\gamma_{+\mathbf{a}})_{\alpha'}{}^{\beta} = \delta_{\alpha'\,\nu'}\,(\gamma_{\mathbf{a}})^{\nu'\,\beta} \equiv (\gamma_{\mathbf{a}})_{\alpha'}{}^{\beta}$.

The observation from (6.21) and its left $\leftrightarrow$ right swap is that the action of the $D_{\alpha'}$ and $D_{\widetilde{\alpha}'}$ on $H_{+\mathbf{a}\,\widetilde{+\mathbf{b}}}$ is producing two new vielbeins $H_{\beta\,\widetilde{+\mathbf{a}}}$ and $H_{\widetilde{\beta}\,+\mathbf{a}}$. This hints that we need some another vielbein, such that the action of $D_{\alpha'}$ and $D_{\widetilde{\alpha}'}$ on it will effectively subtract the fields $H_{\beta\,\widetilde{+\mathbf{a}}}$ and $H_{\widetilde{\beta}\,+\mathbf{a}}$. We found such a vielbein, but before giving it we will look at the flat case superspace first to give a motivation. After that we will generalise it to the $AdS_5 \times S^5$ background.

7 Flat space solution

7.1 Flat space diagram

To see what could be possibly a missing vielbein that will subtract vielbeins in (6.21) (and its left $\leftrightarrow$ right change) we first solve the same problem in flat space background. That is the extended superspace with $r_{\text{AdS}} \to \infty$. Note that in flat superspace ($r_{\text{AdS}} \to \infty$) the relation (6.21) simplifies significantly, because there are no r_{AdS} dependent parts. The surviving part after $r_{\text{AdS}} \to \infty$ is just the second term on the right hand side of (6.21) with $g = 1$, i.e. $\frac{1}{2}\,(\gamma_{+\mathbf{a}})_{\alpha'}{}^{\beta}\,H_{\beta\,\widetilde{+\mathbf{b}}}$.

Let us therefore further examine an action of $D_{\alpha'}$ and $D_{\widetilde{\alpha}'}$ on $H_{\beta\,\widetilde{+\mathbf{a}}}$ and $H_{\widetilde{\beta}\,+\mathbf{a}}$ respectively:

$$T_{DD\widetilde{S}} \equiv T_{\alpha'\,\beta\,\widetilde{+\mathbf{a}}} = 0 = D_{[\alpha'}\,H_{\beta\,\widetilde{+\mathbf{a}})} + H_{[\alpha'\,\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\,f_{\beta\,\widetilde{+\mathbf{a}})\,\mathcal{N}} \tag{7.1}$$

$$= D_{\alpha'}\,H_{\beta\,\widetilde{+\mathbf{a}}} + S_{\widetilde{+\mathbf{a}}}\,H_{\alpha'\,\beta} - D_{\beta}\,H_{\widetilde{+\mathbf{a}}\,\alpha'} + H_{\alpha'\,\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\,f_{\beta\,\widetilde{+\mathbf{a}}\,\mathcal{N}}$$

$$+ H_{\widetilde{+\mathbf{a}}\,\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\,f_{\alpha'\,\beta\,\mathcal{N}} - H_{\beta\,\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\,f_{\widetilde{+\mathbf{a}}\,\alpha'\,\mathcal{N}}$$

The mixed terms in the f part of (7.1) are zero (note they are zero also in the AdS background). The structure constant $f_{\alpha'\,\beta\,\mathcal{N}} = 2\,(\gamma_{\mathbf{a}})_{\alpha'\,\beta}\,\delta^{\mathbf{a}}{}_{\mathcal{N}}$ (the same is in the AdS background). The vielbein $H_{\widetilde{+\mathbf{a}}\,\alpha'} \rightsquigarrow 0$ by the table 6. Then the equation (7.1) can be rewritten as:

$$0 = D_{\alpha'}\,H_{\beta\,\widetilde{+\mathbf{a}}} + S_{\widetilde{+\mathbf{a}}}\,H_{\alpha'\,\beta} + 2\,(\gamma^{\mathbf{c}})_{\alpha'\,\beta}\,H_{\widetilde{+\mathbf{a}}\,\mathbf{c}} \tag{7.2}$$

To evaluate the only S derivative term in (7.2) i.e. $S_{\widetilde{+\mathbf{a}}}\,H_{\alpha'\,\beta}$ we would need to work a bit, in the AdS superspace. The whole AdS analysis of the actions of $D_{\alpha'}$ and $D_{\widetilde{\alpha}'}$ on $H_{\beta\,\widetilde{+\mathbf{a}}}$ and $H_{\widetilde{\beta}\,+\mathbf{a}}$ is done in the appendix, see equations (A.39) till (A.63). In this section we would need only $r_{\text{AdS}} \to \infty$ limit of that analysis.

In the appendix we derived the equations (A.57) and (A.63). Those equations are telling us that in the AdS case (and so also in the flat case) the actions of $D_{\widetilde{\alpha}'}$ and $D_{\alpha'}$ result in a combination of $H_{+\mathbf{a}\,\widetilde{+\mathbf{b}}}$ and $H_{\alpha\,\widetilde{\beta}}$. This is actually a hint that we should add the trace of $H_{\alpha\,\widetilde{\beta}}$ to the trace of $H_{+\mathbf{a}\,\widetilde{+\mathbf{b}}}$ in order to subtract an action of a linear combination of $D_{\alpha'}$ and $D_{\widetilde{\alpha}'}$ (future D^v derivative) on trace $H_{+\mathbf{a}\,\widetilde{+\mathbf{b}}}$. In the rest of this paragraph and next section we will look at how the pre-potential is built up in a flat space limit, i.e. we consider equations (A.57) and (A.58) and (A.63) and (A.64) in the limit $r_{\mathrm{AdS}} \to \infty$. We find equations that are fixing pre-potential and vanishing D_w derivative.

Thus we repeat the flat space limits of the $D_{\widetilde{\alpha}'}$ and $D_{\alpha'}$ actions on $H_{\beta\,\widetilde{+\mathbf{a}}}$ and $H_{\widetilde{\beta}+\mathbf{a}}$ respectively, i.e. the equations (A.63) and (A.64) in $r_{\mathrm{AdS}} \to \infty$ limit:

$$0 = D_{\widetilde{\alpha}'} H_{\beta\,\widetilde{+\mathbf{a}}} + \frac{1}{2}\,(\gamma_{\mathbf{a}})_{\alpha'}{}^{\nu} H_{\beta\,\widetilde{\nu}} \tag{7.3}$$

$$0 = D_{\alpha'} H_{\widetilde{\beta}+\mathbf{a}} + \frac{1}{2}\,(\gamma_{\mathbf{a}})_{\alpha'}{}^{\nu} H_{\nu\,\widetilde{\beta}} \tag{7.4}$$

Similarly, we can also look at the equations (6.21) and (A.57) and (A.58) (in the flat space limit) and together with (7.3) and (7.4) we can observe the following interesting flat space diagram:

$$
\begin{array}{ccccc}
H_{+\mathbf{a}\,\widetilde{+\mathbf{b}}} & \xrightarrow{\;D_{\alpha'}\;} & H_{\beta\,\widetilde{+\mathbf{b}}} & \xrightarrow{\;D_{\alpha'}\;} & H_{\mathbf{c}\,\widetilde{+\mathbf{b}}} \\
\big\downarrow{\scriptstyle D_{\widetilde{\alpha}'}} & & \big\downarrow{\scriptstyle D_{\widetilde{\alpha}'}} & & \\
H_{+\mathbf{a}\,\widetilde{\beta}} & \xrightarrow{\;D_{\alpha'}\;} & H_{\beta\,\widetilde{\beta}} & & \\
\big\downarrow{\scriptstyle D_{\widetilde{\alpha}'}} & & & & \\
H_{+\mathbf{a}\,\widetilde{\mathbf{c}}} & & & &
\end{array}
\tag{7.5}
$$

The scheme (7.5) is nice and actually tells us what we should do next. Recall that the nodes $H_{+\mathbf{a}\,\widetilde{+\mathbf{b}}}$ and $H_{\mathbf{c}\,\widetilde{+\mathbf{b}}}$ and $H_{+\mathbf{a}\,\widetilde{\mathbf{c}}}$ could be identified by the use of invertible operators P_- and $P_{\widetilde{-}}$ see (6.6). Our original aim was find a field that has a scalar trace and could possibly subtract actions of $D_{\alpha'}$ and $D_{\widetilde{\alpha}'}$ on $H_{+\mathbf{a}\,\widetilde{+\mathbf{b}}}$. We will see that the missing field is exactly $H_{\beta\,\widetilde{\beta}}$ (in the flat space, the only nonzero part of $H_{D\,\widetilde{D}}$). The diagram (7.5) suggests what to do. We calculate the remaining arrows and fill the square.

To fill the remaining arrows we need to calculate the action of $D_{\alpha'}$ and $D_{\widetilde{\alpha}'}$ on $H_{\alpha\,\widetilde{\beta}}$ together with some another arrows that will be discussed later. We consider the dimension $\frac{1}{2}$ torsion constraint $T_{D\,D\,\widetilde{D}} \equiv T_{\alpha'\,\beta\,\widetilde{\sigma}} = 0$. We note again that for now on we are working in the flat space. Later we will generalise the procedure for the AdS space:

$$T_{D\,D\,\widetilde{D}} \equiv T_{\alpha'\,\beta\,\widetilde{\sigma}} = 0 = D_{(\alpha'} H_{\beta\,\widetilde{\sigma})} + H_{(\alpha'\,|\,\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}} f_{\beta\,\widetilde{\sigma})\mathcal{N}} \tag{7.6}$$

$$= D_{\alpha'} H_{\beta\,\widetilde{\sigma}} + 2\,(\gamma^{\mathbf{a}})_{\alpha'\,\beta}\,H_{\widetilde{\sigma}\,\mathbf{a}} \tag{7.7}$$

where we used that $H_{\alpha'\,\beta} = H_{\alpha'\,\widetilde{\sigma}} = 0$ in flat space (see (A.40) and (A.45) and do flat space limit). We also have a left $\leftrightarrow$ right swap of (7.7). The vielbein $H_{\widetilde{\sigma}\,\mathbf{c}}$ in (7.7) is related to $H_{+\mathbf{c}\,\widetilde{\sigma}}$. For that consider torsion constraint $T_{P\,S\,\widetilde{D}} \equiv T_{-+\mathbf{c}\,\widetilde{\sigma}} = 0$:

$$T_{P\,S\,\widetilde{D}} \equiv T_{-+\mathbf{c}\,\widetilde{\sigma}} = 0 = P_{[-} H_{+\mathbf{c}\,\widetilde{\sigma})} + H_{[-\,|\,\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}} f_{+\mathbf{c}\,\widetilde{\sigma})\mathcal{N}} \tag{7.8}$$

$$= P_- H_{+\mathbf{c}\,\widetilde{\sigma}} + \eta_{+-}\,H_{\widetilde{\sigma}\,\mathbf{c}} \tag{7.9}$$

where we used left-right light-cone gauge, together with $H_{-+\mathbf{c}} = 0$ that is shown in the appendix and holds even in AdS, see (A.20).

To fill the diagram (7.5) we need to calculate two more torsion constraints that are providing the actions of $D_{\alpha'}$ on $H_{+\mathbf{a}\widetilde{\mathbf{c}}}$ and on $H_{\sigma\widetilde{\mathbf{c}}}$. We first consider $T_{DS\widetilde{P}} \equiv T_{\alpha'+\mathbf{a}\widetilde{\mathbf{c}}} = 0$:

$$T_{DS\widetilde{P}} \equiv T_{\alpha'+\mathbf{a}\widetilde{\mathbf{c}}} = 0 = D_{[\alpha'}H_{+\mathbf{a}\widetilde{\mathbf{c}})} + H_{[\alpha'|\mathcal{M}}\eta^{\mathcal{M}\mathcal{N}}f_{+\mathbf{a}\widetilde{\mathbf{c}})\mathcal{N}} \tag{7.10}$$

$$= D_{\alpha'}H_{+\mathbf{a}\widetilde{\mathbf{c}}} + \frac{1}{2}(\gamma_{+\mathbf{a}})_{\alpha'}{}^{\sigma}H_{\widetilde{\mathbf{c}}\sigma} \tag{7.11}$$

$$= D_{\alpha'}H_{+\mathbf{a}\widetilde{\mathbf{c}}} + \frac{1}{2}(\gamma_{\mathbf{a}})_{\alpha'}{}^{\sigma}H_{\widetilde{\mathbf{c}}\sigma} \tag{7.12}$$

where we used that $H_{\alpha'+\mathbf{a}} = 0$ (holds even in the AdS, see table 6). We also used that $H_{\alpha'\widetilde{\mathbf{c}}} = 0$ (that is enough in a flat space to have $S_{+\mathbf{a}}H_{\alpha'\widetilde{\mathbf{c}}} = 0$). To see that $H_{\alpha'\widetilde{\mathbf{c}}} = 0$ we use the torsion $T_{\widetilde{P}\widetilde{S}D} \equiv T_{\widetilde{-}\widetilde{+\mathbf{c}}\alpha'} = 0$:

$$T_{\widetilde{P}\widetilde{S}D} \equiv T_{\widetilde{-}\widetilde{+\mathbf{c}}\alpha'} = 0 = P_{[\widetilde{-}}H_{\widetilde{+\mathbf{c}}\alpha')} + H_{[\widetilde{-}|\mathcal{M}}\eta^{\mathcal{M}\mathcal{N}}f_{\widetilde{+\mathbf{c}}\alpha')\mathcal{N}} \tag{7.13}$$

$$= P_{\widetilde{-}}H_{\widetilde{+\mathbf{c}}\alpha'} + \eta_{+-}H_{\alpha'\widetilde{\mathbf{c}}} \tag{7.14}$$

and previously we saw that $H_{\widetilde{+\mathbf{c}}\alpha'} = 0$ (even in the AdS case, see table 6). From (7.14) in the flat case follows that $H_{\alpha'\widetilde{\mathbf{c}}} = 0$. Examining the (7.14) in the AdS case one also finds that $H_{\alpha'\widetilde{\mathbf{c}}} = 0$ (after evaluation). The (7.12) however could have some additional term in the AdS case. The structure constant $f_{\alpha'\widetilde{\mathbf{c}}\mathcal{N}} \neq 0$ and so the term proportional to that structure constant in the AdS case is $\frac{1}{r_{\mathrm{AdS}}}(\gamma_{\mathbf{c}})_{\alpha'}{}_{\sigma}(\widetilde{\varGamma}_5)^{\sigma\nu}H_{+\mathbf{a}\widetilde{\nu}}$. That term is nonzero in the AdS case. Moreover, in the (7.10) one finds one more AdS term, coming from evaluated action $S_{+\mathbf{a}}H_{\alpha'\widetilde{\mathbf{c}}}$. Those terms are not of a big concern right now (doing the flat space first), we will see them later in the section where we generalise to AdS case.

Last torsion constraint to examine in order to fill the (7.5) is the one that determines the action of $D_{\alpha'}$ on $H_{\beta\widetilde{\mathbf{c}}}$. Consider therefore the dimension $\frac{1}{2}$ torsion $T_{DD\widetilde{P}} \equiv T_{\alpha'\beta\widetilde{\mathbf{c}}} = 0$:

$$T_{DD\widetilde{P}} \equiv T_{\alpha'\beta\widetilde{\mathbf{c}}} = 0 = D_{[\alpha'}H_{\beta\widetilde{\mathbf{c}})} + H_{[\alpha'|\mathcal{M}}\eta^{\mathcal{M}\mathcal{N}}f_{\beta\widetilde{\mathbf{c}})\mathcal{N}} \tag{7.15}$$

$$= D_{\alpha'}H_{\beta\widetilde{\mathbf{c}}} + 2(\gamma^{\mathbf{a}})_{\alpha'\beta}H_{\widetilde{\mathbf{c}}\mathbf{a}} \tag{7.16}$$

where we used the $H_{\alpha'\beta} = 0$ (holds also in AdS after the evaluation) and $H_{\widetilde{\mathbf{c}}\alpha'} = 0$ (also holds in AdS after the evaluation). In the AdS case in the equation (7.16) we have two additional terms. They come from $f_{\beta\widetilde{\mathbf{c}}\mathcal{N}} \neq 0$ and also $f_{\alpha'\widetilde{\mathbf{c}}\mathcal{N}} \neq 0$. Those terms will be further analysed in future sections, let just write their structure as $\frac{1}{r_{\mathrm{AdS}}}(\gamma_{\mathbf{c}})_{\beta\nu'}(\widetilde{\varGamma}_5)^{\nu'\sigma'}H_{\widetilde{\sigma}'\alpha'}$ and $\frac{1}{r_{\mathrm{AdS}}}(\gamma_{\mathbf{c}})_{\alpha'\nu}(\widetilde{\varGamma}_5)^{\nu\sigma}H_{\widetilde{\sigma}\beta}$. The vielbein $H_{\widetilde{\sigma}'\alpha'} = 0$ (in flat case and also in AdS after the evaluation) as can be calculated from torsion constraints $T_{-\widetilde{\sigma}'\alpha'} = 0$ and $T_{\widetilde{-}-\widetilde{\sigma}'} = 0$ and the use of the double light-cone gauge. The term $H_{\widetilde{\sigma}\beta}$ is nonzero ($H_{\widetilde{\sigma}\beta}$ vielbein is a part of a pre-potential).

We can add results of (flat space) equations (7.12) and (7.16) together with (7.7) and their left ↔ right swaps to the diagram (7.5) and find the following square diagram:

$$
\begin{array}{ccccc}
H_{+\mathbf{a}\,\widetilde{+\mathbf{b}}} & \xrightarrow{D_{\alpha'}} & H_{\beta\,\widetilde{+\mathbf{b}}} & \xrightarrow{D_{\alpha'}} & H_{\mathbf{c}\,\widetilde{+\mathbf{b}}} \\
\downarrow{\scriptstyle D_{\widetilde{\alpha}'}} & & \downarrow{\scriptstyle D_{\widetilde{\alpha}'}} & & \downarrow{\scriptstyle D_{\widetilde{\alpha}'}} \\
H_{+\mathbf{a}\,\widetilde{\beta}} & \xrightarrow{D_{\alpha'}} & H_{\beta\,\widetilde{\beta}} & \xrightarrow{D_{\alpha'}} & H_{\mathbf{c}\,\widetilde{\beta}} \\
\downarrow{\scriptstyle D_{\widetilde{\alpha}'}} & & \downarrow{\scriptstyle D_{\widetilde{\alpha}'}} & & \downarrow{\scriptstyle D_{\widetilde{\alpha}'}} \\
H_{+\mathbf{a}\,\widetilde{\mathbf{c}}} & \xrightarrow{D_{\alpha'}} & H_{\beta\,\widetilde{\mathbf{c}}} & \xrightarrow{D_{\alpha'}} & H_{\mathbf{c}\,\widetilde{\mathbf{c}}}
\end{array}
\tag{7.17}
$$

As we saw before the (7.17) nodes $\{\,H_{+\mathbf{a}\,\widetilde{+\mathbf{b}}},\,H_{\mathbf{c}\,\widetilde{+\mathbf{b}}},\,H_{+\mathbf{a}\,\widetilde{\mathbf{c}}},\,H_{\mathbf{c}\,\widetilde{\mathbf{c}}}\,\}$ should be identified (as one node). We proved that using various torsion constraints, mixed light-cone gauge and invertibility of P_- and $P_{\widetilde{\,}}$. The same way the nodes $\{\,H_{+\mathbf{a}\,\widetilde{\beta}},\,H_{\mathbf{c}\,\widetilde{\beta}}\,\}$ and independently nodes $\{\,H_{\beta\,\widetilde{+\mathbf{b}}},\,H_{\beta\,\widetilde{\mathbf{c}}}\,\}$ should be identified (as two independent nodes). The vielbein $H_{\beta\,\widetilde{\beta}}$ is then just a single node. After the described identifications the diagram (7.17) could be rewritten in the simpler and more informative form.

$$\tag{7.18}$$

Note, the dashed arrows stand for action of $D_{\widetilde{\alpha}'}$ and solid arrows stand for action of $D_{\alpha'}$. From the nice flat space diagram (7.18) it is obvious that in order to have a vanishing derivative we have to combine $D_{\alpha'}$ with $D_{\widetilde{\alpha}'}$ and that combination should act on the combination of traces of $H_{+\mathbf{a}\,\widetilde{+\mathbf{b}}}$ with $H_{\beta\,\widetilde{\beta}}$.

7.2   The $\mathbb{H}$ matrix

The diagram (7.18) could be rewritten in the matrix form. The observation is that each action of the derivatives in the (7.18) is given by some matrix. The derivatives are mixing fields just as in (7.18). Let us introduce the $2 \otimes 2$ block matrix $\mathbb{H}$:

$$
\mathbb{H} := \begin{pmatrix} H_{+\mathbf{a}\,\widetilde{+\mathbf{b}}} & H_{+\mathbf{a}\,\widetilde{\beta}} \\ H_{\beta\,\widetilde{+\mathbf{b}}} & H_{\beta\,\widetilde{\beta}} \end{pmatrix}
\tag{7.19}
$$

The action of $D_{\alpha'}$ is then given as the left action of some constant (up to P_- operator) block off diagonal matrix $\Gamma_{\alpha'}$:

$$
D_{\alpha'} \begin{pmatrix} H_{+\mathbf{a}\,\widetilde{+\mathbf{b}}} & H_{+\mathbf{a}\,\widetilde{\beta}} \\ H_{\beta\,\widetilde{+\mathbf{b}}} & H_{\beta\,\widetilde{\beta}} \end{pmatrix} \equiv \begin{pmatrix} 0 & \frac{1}{2}(\gamma_{\mathbf{a}})_{\alpha'}{}^{\sigma} \\ 2\,P_-\,(\gamma^{\mathbf{c}})_{\alpha'\beta} & 0 \end{pmatrix} \begin{pmatrix} H_{+\mathbf{c}\,\widetilde{+\mathbf{b}}} & H_{+\mathbf{c}\,\widetilde{\beta}} \\ H_{\sigma\,\widetilde{+\mathbf{b}}} & H_{\sigma\,\widetilde{\beta}} \end{pmatrix}
\tag{7.20}
$$

$$
D_{\alpha'}\,\mathbb{H} = \Gamma_{\alpha'}\,\mathbb{H}
\tag{7.21}
$$

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The action of $D_{\widetilde{\alpha}'}$ on $\mathbb{H}$ is given as a right action of similar matrix $\varGamma_{\widetilde{\alpha}'}$:

$$D_{\widetilde{\alpha}'} \begin{pmatrix} H_{+\mathbf{a}\,\widetilde{+\mathbf{b}}} & H_{+\mathbf{a}\,\widetilde{\beta}} \\ H_{\beta\,\widetilde{+\mathbf{b}}} & H_{\beta\,\widetilde{\beta}} \end{pmatrix} \equiv \begin{pmatrix} H_{+\mathbf{a}\,\widetilde{+\mathbf{c}}} & H_{+\mathbf{a}\,\widetilde{\sigma}} \\ H_{\beta\,\widetilde{+\mathbf{c}}} & H_{\beta\,\widetilde{\sigma}} \end{pmatrix} \begin{pmatrix} 0 & 2\,P_{\widetilde{-}}\,(\gamma^{\mathbf{c}})_{\alpha'\,\beta} \\ \frac{1}{2}\,(\gamma_{\mathbf{b}})_{\alpha'}{}^{\sigma} & 0 \end{pmatrix} \tag{7.22}$$

$$D_{\widetilde{\alpha}'}\,\mathbb{H} = \mathbb{H}\,\varGamma_{\widetilde{\alpha}'} \tag{7.23}$$

Now we will proceed to the main step. We arbitrarily linearly combine $D_{\alpha'}$ and $D_{\widetilde{\alpha}'}$, i.e. we multiply the $D_{\widetilde{\alpha}'}$ with some unknown nonsingular matrix $\mathcal{M}_{\alpha'}{}^{\beta'}$:

$$\mathcal{D}_v \equiv \mathcal{D}^v{}_{\alpha'} := (\,D_{\alpha'} - \mathcal{M}_{\alpha'}{}^{\beta'}\,D_{\widetilde{\beta}'}\,) \tag{7.24}$$

Moreover we impose that in the matrix version of $D_{\widetilde{\alpha}'}$ action the matrix $\mathcal{M}$ acts as follows:

$$\mathcal{M}_{\alpha'}{}^{\beta'}\,\varGamma_{\widetilde{\beta}'} := \mathbb{A}\,\varGamma_{\widetilde{\alpha}'}\,\mathbb{B} \tag{7.25}$$

for some nonsingular matrices $\mathbb{A}$ and $\mathbb{B}$. Combining (7.20) and (7.22) together with (7.24) and (7.25) we get:

$$\mathcal{D}^v{}_{\alpha'}\,\mathbb{H} = \varGamma_{\alpha'}\,\mathbb{H} - \mathbb{H}\,\mathbb{A}\,\varGamma_{\widetilde{\alpha}'}\,\mathbb{B} \qquad /\,\mathbb{B}^{-1} \tag{7.26}$$

$$\mathcal{D}^v{}_{\alpha'}\,\mathbb{H}\,\mathbb{B}^{-1} = \varGamma_{\alpha'}\,\mathbb{H}\,\mathbb{B}^{-1} - \mathbb{H}\,\mathbb{A}\,\varGamma_{\widetilde{\alpha}'} \qquad /\,\mathrm{Str} \tag{7.27}$$

$$\mathcal{D}^v{}_{\alpha'}\,\mathrm{Str}\,(\,\mathbb{H}\,\mathbb{B}^{-1}\,) = \mathrm{Str}\left(\,(\,\mathbb{B}^{-1}\,\varGamma_{\alpha'} - \mathbb{A}\,\varGamma_{\widetilde{\alpha}'}\,)\,\mathbb{H}\,\right) \tag{7.28}$$

by the Str we mean the super-trace. We put to zero the $\mathrm{Str}\left(\,(\,\mathbb{B}^{-1}\,\varGamma_{\alpha'} - \mathbb{A}\,\varGamma_{\widetilde{\alpha}'}\,)\,\mathbb{H}\,\right) = 0$ by finding the suitable matrices $\mathbb{B}$ and $\mathbb{A}$ and the matrix $\mathcal{M}$. By that we get the equation:

$$\mathcal{D}^v{}_{\alpha'}\,\mathrm{Str}\,(\,\mathbb{H}\,\mathbb{B}^{-1}\,) = 0 \tag{7.29}$$

thus the equation (7.29) defines the $\mathrm{Str}\,(\,\mathbb{H}\,\mathbb{B}^{-1}\,)$ as the scalar field on which particular combination of $D_{\alpha'}$ and $D_{\widetilde{\alpha}'}$ now called $\mathcal{D}^v{}_{\alpha'}$ vanishes. So, we found a pre-potential $V := \mathrm{Str}\,(\,\mathbb{H}\,\mathbb{B}^{-1}\,)$. We note that even though the equation (7.28) might seem easy to solve just by putting $\mathbb{B}^{-1} = \mathbb{A}$. It is not that simple since $\varGamma_{\alpha'} \neq \varGamma_{\widetilde{\alpha}'}$. Therefore some more involved solution has to be found.

7.3 Solution via the gamma matrix identity

We solve the equation (7.28) using the following identity:

$$\mathcal{A}_{\alpha'}{}^{\sigma'}\,\mathcal{B}_{\mathbf{a}}{}^{\mathbf{c}}\,\mathcal{C}_{\beta}{}^{\nu}\,(\gamma_{\mathbf{c}})_{\sigma'\,\nu} = (\gamma_{\mathbf{a}})_{\alpha'\,\beta} \tag{7.30}$$

has two SO(4) invariant solutions:

$$\mathbb{I}.: \quad \mathcal{A}_{\alpha'}{}^{\beta'} = \delta_{\alpha'}{}^{\beta'} \;\;||\;\; \mathcal{C}_{\alpha}{}^{\beta} = \delta_{\alpha}{}^{\beta} \;\;||\;\; \mathcal{B}_{\mathbf{a}}{}^{\mathbf{b}} = \delta_{\mathbf{a}}{}^{\mathbf{b}} \tag{7.31}$$

$$\mathbb{II}.: \quad \mathcal{A}_{\alpha'}{}^{\beta'} = (\widetilde{\varGamma}_5)_{\alpha'}{}^{\beta'} \;\;||\;\; \mathcal{C}_{\alpha}{}^{\beta} = (\widetilde{\varGamma}_5)_{\alpha}{}^{\beta} \;\;||\;\; \mathcal{B}_{\mathbf{a}}{}^{\mathbf{b}} = (\widetilde{\varGamma}_5)_{\mathbf{a}}{}^{\mathbf{b}} \tag{7.32}$$

The solution (7.31) is trivial, the solution (7.32) is based on property of the $\widetilde{\varGamma}_5$ matrix: $[\,\widetilde{\varGamma}_5,\,\gamma_{\mathbf{a}}\,] = 0$ for $\mathbf{a} \in \{\,10, 1, 2, 3, 4\,\}$ and $\{\,\widetilde{\varGamma}_5,\,\gamma_{\mathbf{a}}\,\} = 0$ for $\mathbf{a} \in \{\,5, 6, 7, 8, 9\,\}$. The

previous follows directly from the definition of $\widetilde{\Gamma}_5$, see (3.13). The new matrix $(\widetilde{\Gamma}_5)_{\mathbf{a}}{}^{\mathbf{b}}$ in (7.32) is defined by the (7.30) to fix the signs. Note that the indices $\mathbf{a}$ in (7.30) have a range: $\mathbf{a} \in \{1, \ldots, 8\}$.

Next, we look explicitly at the equation:

$$\mathrm{Str}\left(\left(\mathbb{B}^{-1}\, \Gamma_{\alpha'} \, - \, \mathbb{A}\, \Gamma_{\widetilde{\alpha}'} \right) \mathbb{H} \right) \equiv \mathrm{Str}\, \mathbb{X}_{\alpha'} \, = \, 0 \tag{7.33}$$

let's rename the members of the matrix $\mathbb{H}$:

$$\mathbb{H} \equiv \begin{pmatrix} H_{+\mathbf{a}\,\widetilde{+\mathbf{b}}} & H_{+\mathbf{a}\,\widetilde{\beta}} \\ H_{\beta\,\widetilde{+\mathbf{b}}} & H_{\beta\,\widetilde{\beta}} \end{pmatrix} \equiv \begin{pmatrix} H_{S\,\widetilde{S}} & H_{S\,\widetilde{D}} \\ H_{D\,\widetilde{S}} & H_{D\,\widetilde{D}} \end{pmatrix} \tag{7.34}$$

Let us define the matrices $\mathbb{A}$ and $\mathbb{B}^{-1}$ to be block diagonal matrices. This is a consistent choice with the fact that we want to have a pre-potential build out of $H_{S\,\widetilde{S}}$ and $H_{D\,\widetilde{D}}$. The pre-potential is in (7.29) given as $\mathrm{Str}\,(\,\mathbb{H}\,\mathbb{B}^{-1}\,)$. We do not want to mix in some off diagonal $\mathbb{H}$ fields by the action of $\mathbb{B}^{-1}$. Thus $\mathbb{A}$ and $\mathbb{B}^{-1}$ are:

$$\mathbb{A} \equiv \begin{pmatrix} A_{S\,\widetilde{S}} & 0 \\ 0 & A_{D\,\widetilde{D}} \end{pmatrix} \quad || \quad \mathbb{B}^{-1} \equiv \begin{pmatrix} B^{-1}{}_{S\,\widetilde{S}} & 0 \\ 0 & B^{-1}{}_{D\,\widetilde{D}} \end{pmatrix} \tag{7.35}$$

With definitions (7.35) we get the equation (7.33) into the following matrix equation:

$$\mathrm{Str}\left(\left(\begin{matrix} \frac{1}{2}\, B^{-1}{}_{S\,\widetilde{S}}\, \gamma\, H_{D\,\widetilde{S}} & \cdots \\ \cdots & 2\, P_{-}\, B^{-1}{}_{D\,\widetilde{D}}\, \gamma\, H_{S\,\widetilde{D}} \end{matrix} \right) - \left(\begin{matrix} 2\, P_{\sim}\, A_{S\,\widetilde{S}}\, \gamma\, H_{D\,\widetilde{S}} & \cdots \\ \cdots & \frac{1}{2}\, A_{D\,\widetilde{D}}\, \gamma\, H_{S\,\widetilde{D}} \end{matrix} \right) \right) = 0 \tag{7.36}$$

Then from (7.36) we get two equations (since fields $H_{D\,\widetilde{S}}$ and $H_{S\,\widetilde{D}}$ are independent):

$$\frac{1}{2}\, B^{-1}{}_{S\,\widetilde{S}} - 2\, P_{\sim}\, A_{S\,\widetilde{S}} = 0 \Rightarrow A_{S\,\widetilde{S}} = \frac{1}{4\, P_{\sim}}\, B^{-1}{}_{S\,\widetilde{S}} \tag{7.37}$$

$$P_{-}\, B^{-1}{}_{D\,\widetilde{D}} - A_{D\,\widetilde{D}} = 0 \Rightarrow A_{D\,\widetilde{D}} = 4\, P_{-}\, B^{-1}{}_{D\,\widetilde{D}} \tag{7.38}$$

Now we are prepared to examine the equation (7.25) using the $\mathbb{A}$ and $\mathbb{B}$ constructed above. Then the matrix equation (7.25) can be (schematically) written:

$$\mathcal{M} \begin{pmatrix} 0 & 2\, P_{\sim}\, \gamma \\ \frac{1}{2}\, \gamma & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{4 P_{\sim}}\, B^{-1}{}_{S\,\widetilde{S}} & 0 \\ 0 & 4\, P_{-}\, B^{-1}{}_{D\,\widetilde{D}} \end{pmatrix} \begin{pmatrix} 0 & 2\, P_{\sim}\, \gamma \\ \frac{1}{2}\, \gamma & 0 \end{pmatrix} \begin{pmatrix} B_{S\,\widetilde{S}} & 0 \\ 0 & B_{D\,\widetilde{D}} \end{pmatrix} \tag{7.39}$$

$$\mathcal{M} \begin{pmatrix} 0 & 2\, P_{\sim}\, \gamma \\ \frac{1}{2}\, \gamma & 0 \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{2}\, B^{-1}{}_{S\,\widetilde{S}}\, \gamma\, B_{D\,\widetilde{D}} \\ 2\, P_{-}\, B^{-1}{}_{D\,\widetilde{D}}\, \gamma\, B_{S\,\widetilde{S}} & 0 \end{pmatrix} \tag{7.40}$$

We now do the following re-scalings $\mathcal{M} \to \frac{1}{\lambda}\mathcal{M}$ and $B_{S\,\widetilde{S}} \to \Delta\, B_{S\,\widetilde{S}}$ and $B_{D\,\widetilde{D}} \to \rho\, B_{D\,\widetilde{D}}$. Rescaled $\mathcal{M}$ and $B_{S\,\widetilde{S}}$ and $B_{D\,\widetilde{D}}$ belong to one of the two solutions of identity (7.30). Then we get the version of (7.40):

$$\mathcal{M} \begin{pmatrix} 0 & 2\, P_{\sim}\, \gamma \\ \frac{1}{2}\, \gamma & 0 \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{2}\, \frac{\lambda \rho}{\Delta}\, B^{-1}{}_{S\,\widetilde{S}}\, \gamma\, B_{D\,\widetilde{D}} \\ 2\, \frac{\lambda \Delta}{\rho}\, P_{-}\, B^{-1}{}_{D\,\widetilde{D}}\, \gamma\, B_{S\,\widetilde{S}} & 0 \end{pmatrix} \tag{7.41}$$

Now we want the λ and ρ and Δ to satisfy:

$$\frac{\lambda \rho}{\Delta} = 4 P_{\simeq} \quad \text{and} \quad \frac{\lambda \Delta}{\rho} = \frac{1}{4 P_-} \quad \Rightarrow \quad \lambda = \pm \sqrt{\frac{P_{\simeq}}{P_-}} \quad \text{and} \quad \frac{\rho}{\Delta} = \pm 4 \sqrt{P_{\simeq} P_-} \tag{7.42}$$

The (7.41) is just a matrix equation:

$$\mathcal{M} \begin{pmatrix} 0 & P_{\simeq} \gamma \\ \gamma & 0 \end{pmatrix} = \begin{pmatrix} 0 & A^{-1}{}_{S \widetilde{S}} P_{\simeq} \gamma A_{D \widetilde{D}} \\ A^{-1}{}_{D \widetilde{D}} \gamma A_{S \widetilde{S}} & 0 \end{pmatrix} \tag{7.43}$$

that can be solved by (7.30). Even though we saw the appearance of the nasty square roots in the (7.42) and so in the definition of $\mathcal{D}^v{}_{\alpha'}$ and in the pre-potential via super-trace of $\mathbb{H} \mathbb{B}^{-1}$. We will see in the AdS case solution that there is a way how to get rid of it.

8 $AdS_5 \times S^5$ solution

8.1 $AdS_5 \times S^5$ diagram

In the previous sub-sections we saw how to find the pre-potential in the flat case. We are really interested in the AdS case. Along the way we analysed the flat case in the previous sub-sections we mentioned also changes one has to make in the AdS case. We repeat them here again since they are scattered over the previous flat case sub-sections and in the appendix. First change has already been worked out in the relation between $H_{+\mathbf{a}\,\widetilde{+\mathbf{b}}}$ and $H_{\mathbf{a}\,\widetilde{\mathbf{b}}}$ in (6.10). We also note that there are AdS contributions in equations (6.21) also in (A.57) and (A.58). The nontrivial contributions also appeared in equations (A.63) and (A.64).

We could visualise the relations (6.21) and (A.57) and (A.58) and (A.63) and (A.64) by the similar diagram as used in flat case, see (7.5). The structure is very similar just with more arrows between nodes. Since the AdS diagram is messier we will not provide it. The idea is however the same as in the flat case. In order to determine the vanishing $D^v{}_{\alpha'}$ derivative and the pre-potential we need to combine $D_{\alpha'}$ and $D_{\widetilde{\alpha}'}$ for D^v derivative and $H_{+\mathbf{a}\,\widetilde{+\mathbf{b}}}$ together with $H_{\alpha \widetilde{\beta}}$ for pre-potential.

The only missing derivative in the set of AdS equations: (6.21) and (A.57) and (A.58) and (A.63) and (A.64), is an action of $D_{\alpha'}$ on $H_{\beta \widetilde{\sigma}}$. This action can be calculated from $T_{D D \widetilde{D}} \equiv T_{\alpha' \beta \widetilde{\sigma}} = 0$ torsion constraint. The AdS contribution in that constraint comes from $f_{\alpha' \widetilde{\sigma} \mathcal{N}}$ structure constant. We have already analysed this structure constant, see equations (A.46) and (A.47) till (A.50). We can thus directly write the constraint with an extra AdS term:

$$T_{D D \widetilde{D}} \equiv T_{\alpha' \beta \widetilde{\sigma}} = 0 = D_{[\alpha'} H_{\beta \widetilde{\sigma})} + H_{[\alpha' | \mathcal{M}} \eta^{\mathcal{M} \mathcal{N}} f_{\beta \widetilde{\sigma}) \mathcal{N}} \tag{8.1}$$

$$= D_{\alpha'} H_{\beta \widetilde{\sigma}} + 2 (\gamma^{\mathbf{a}})_{\alpha' \beta} H_{\widetilde{\sigma} \mathbf{a}} + \frac{1}{r_{\text{AdS}}} (\gamma^{[\mathbf{c}})_{\sigma \rho} (\widetilde{\Gamma}_5)^{\rho \nu} (\gamma^{\mathbf{d}]})_{\nu \alpha'} H_{\beta \mathbf{cd}}$$

where we again note that the Σ indices in the last expression of the (8.1) second line are from the SO(5) $\otimes$ SO(5) diagonal subgroup. The $H_{\beta \mathbf{cd}}$ vielbein has nonzero both $H_{\beta +\mathbf{b}}$ and also $H_{\beta \widetilde{+\mathbf{b}}}$. The second vielbein is the term already in the matrix $\mathbb{H}$ from the flat

section (ultimate goal is to rewrite the AdS case in the terms of matrix $\mathbb{H}$ and use the super-trace trick to get the pre-potential). The field $H_{\beta+\mathbf{b}}$ is related to the $H_{\widetilde{\rho}+\mathbf{c}}$ as we saw in table 7.

There is one last piece in the equation (8.1) that we did not relate to the fields in the $\mathbb{H}$ matrix. The field $H_{\widetilde{\sigma}\mathbf{a}}$. As we saw in the flat case, that field should be related to $H_{\widetilde{\sigma}+\mathbf{a}}$ via P_-. We have seen however (for example in (6.10)) that such relations are a bit changed in the AdS case. Consider the following torsion constraint (and use mixed light-cone and $H_{-+\mathbf{a}} \rightsquigarrow 0$):

$$T_{P\widetilde{D}S} \equiv T_{-\widetilde{\beta}+\mathbf{a}} = 0 = D_{[-} H_{\widetilde{\beta}+\mathbf{a})} + H_{[-|\mathcal{M}} \eta^{\mathcal{MN}} f_{\widetilde{\beta}+\mathbf{a})\mathcal{N}} \tag{8.2}$$

$$= P_- H_{\widetilde{\beta}+\mathbf{a}} + H_{\widetilde{\beta}\mathcal{M}} \eta^{\mathcal{MN}} f_{-+\mathbf{a}\mathcal{N}} + H_{+\mathbf{a}\mathcal{M}} \eta^{\mathcal{MN}} f_{-\widetilde{\beta}\mathcal{N}}$$

$$= P_- H_{\widetilde{\beta}+\mathbf{a}} + \eta_{-+} H_{\widetilde{\beta}\mathbf{a}} + \frac{1}{r_{\text{AdS}}} (\gamma_-)_{\beta\nu} (\widetilde{\Gamma}_5)^{\nu\sigma} H_{+\mathbf{a}\nu} \tag{8.3}$$

In the table 7 we derived the relation between $H_{+\mathbf{a}\nu}$ and $H_{+\mathbf{a}\widetilde{\nu}}$. That result together with (8.3) we get:

$$\left(P_- - \frac{1}{P_{\widetilde{-}}} \frac{1}{(r_{\text{AdS}})^2} \right) H_{+\mathbf{a}\widetilde{\alpha}} = H_{\mathbf{a}\widetilde{\alpha}} \tag{8.4}$$

With the equation (8.4) we succeeded to calculate the last missing derivative $D_{\alpha'} H_{\beta\widetilde{\sigma}}$ in terms of $\mathbb{H}$ vielbeins:

$$D_{\alpha'} H_{\beta\widetilde{\sigma}} - 2 \left(P_- - \frac{1}{P_{\widetilde{-}}} \frac{1}{(r_{\text{AdS}})^2} \right) (\gamma^{\mathbf{a}})_{\alpha'\beta} H_{+\mathbf{a}\widetilde{\sigma}} \tag{8.5}$$

$$- \frac{1}{2(r_{\text{AdS}})^2 P_{\widetilde{-}}} (\widetilde{\Gamma}_5)_\sigma{}^\nu (\gamma^{\mathbf{d}})_{\nu\alpha'} (\widetilde{\Gamma}_5)_\beta{}^\rho H_{\widetilde{\rho}+\mathbf{d}} - \frac{1}{2(r_{\text{AdS}})} (\widetilde{\Gamma}_5)_\sigma{}^\nu (\gamma^{\mathbf{d}})_{\nu\alpha'} H_{\beta\widetilde{+\mathbf{d}}} = 0$$

$$D_{\widetilde{\alpha}'} H_{\widetilde{\beta}\sigma} - 2 \left(P_{\widetilde{-}} - \frac{1}{P_-} \frac{1}{(r_{\text{AdS}})^2} \right) (\gamma^{\mathbf{a}})_{\alpha'\beta} H_{\widetilde{+\mathbf{a}}\sigma} \tag{8.6}$$

$$- \frac{1}{2(r_{\text{AdS}})^2 P_-} (\widetilde{\Gamma}_5)_\sigma{}^\nu (\gamma^{\mathbf{d}})_{\nu\alpha'} (\widetilde{\Gamma}_5)_\beta{}^\rho H_{\rho+\mathbf{d}} - \frac{1}{2(r_{\text{AdS}})} (\widetilde{\Gamma}_5)_\sigma{}^\nu (\gamma^{\mathbf{d}})_{\nu\alpha'} H_{\widetilde{\beta}+\mathbf{d}} = 0$$

Where the $(\widetilde{\Gamma}_5)_\sigma{}^\nu := (\gamma^+)_{\sigma\lambda} (\widetilde{\Gamma}_5)^{\lambda\nu}$. The AdS equations (6.21) and (A.57) and (A.58) and (A.63) and (A.64) and (8.5) and (8.6) could be summarised in the following diagram:

$$(H_{+\mathbf{a}\widetilde{+\mathbf{b}}}, H_{\alpha\widetilde{\beta}}) \xleftarrow[D_{\alpha'}]{\overset{D_{\widetilde{\alpha}'}}{\underset{D_{\widetilde{\alpha}'}}{\longleftarrow}}} (H_{\beta\widetilde{+\mathbf{b}}}, H_{+\mathbf{a}\widetilde{\beta}}) \tag{8.7}$$

From the above diagram is obvious that we again have to combine $H_{+\mathbf{a}\widetilde{+\mathbf{b}}}$ and $H_{\alpha\widetilde{\beta}}$ and derivatives $D_{\widetilde{\alpha}'}$ and $D_{\alpha'}$ to get a vanishing derivative on some scalar.

8.2 The $\mathbb{H}$ matrix in $AdS_5 \times S^5$

We want to repeat the section on the flat solution via the $\mathbb{H}$ matrix. The $\mathbb{H}$ matrix was defined in (7.34). We want to write the action $D_{\alpha'}$ and $D_{\widetilde{\alpha}'}$ on the $\mathbb{H}$. This was given in

components in equations: (6.21) and (A.57) and (A.58) and (A.63) and (A.64) and (8.5) and (8.6) and also graphically in (8.7). We expect that the resulting matrix equations have pieces given by the flat equations (7.20) and (7.22) plus purely AdS pieces (dependent as powers of $\frac{1}{r_{\text{AdS}}}$). We could write those equations in such explicit matrix form, but resulting equations are complicated and unnecessary for our purpose. We instead summarise the right hand side of $D_{\alpha'} \mathbb{H}$ and $D_{\widetilde{\alpha}'} \mathbb{H}$ using two new matrices $\mathbb{X}_{\alpha'}$ and $\mathbb{Y}_{\widetilde{\alpha}'}$ respectively. We propose matrix from of the AdS equations:

$$D_{\alpha'} \mathbb{H} = \mathbb{X}_{\alpha'} \tag{8.8}$$

$$D_{\widetilde{\alpha}'} \mathbb{H} = \mathbb{Y}_{\widetilde{\alpha}'} \tag{8.9}$$

The matrices $\mathbb{X}_{\alpha'}$ and $\mathbb{Y}_{\widetilde{\alpha}'}$ are fully fixed by (6.21) and (A.57) and (A.58) and (A.63) and (A.64) and (8.5) and (8.6). In the $r_{\text{AdS}} \to \infty$ the $\mathbb{X}_{\alpha'} \to \Gamma_{\alpha'} \mathbb{H}$ and $\mathbb{Y}_{\widetilde{\alpha}'} \to \mathbb{H} \Gamma_{\widetilde{\alpha}'}$, where the matrices $\Gamma_{\alpha'}$ and $\Gamma_{\widetilde{\alpha}'}$ are given in (7.20) and (7.22).

8.3 Chiral and projective solutions for $AdS_5 \times S^5$

In the next step we repeat the argument we gave in the flat case section but for the AdS equations (8.8) and (8.9). We define:

$$\mathcal{D}_v \equiv \mathcal{D}^v{}_{\alpha'} := (D_{\alpha'} + \mathcal{M}_{\alpha'}{}^{\beta'} D_{\widetilde{\beta}'}) \tag{8.10}$$

now we act by (8.10) on $\mathbb{H}$:

$$\mathcal{D}^v{}_{\alpha'} \mathbb{H} = (\mathbb{X}_{\alpha'} + \mathcal{M}_{\alpha'}{}^{\beta'} \mathbb{Y}_{\widetilde{\beta}'}) \tag{8.11}$$

we multiply by $\mathbb{B}$ and apply Str:

$$\mathcal{D}^v{}_{\alpha'} \operatorname{Str}(\mathbb{H}\mathbb{B}) = \operatorname{Str}((\mathbb{X}_{\alpha'} + \mathcal{M}_{\alpha'}{}^{\beta'} \mathbb{Y}_{\widetilde{\beta}'}) \mathbb{B}) \tag{8.12}$$

We will further analyse the structure of (8.12) in next discussion but before we note one change with respect to (7.24). In (7.24) we used $\mathbb{B}^{-1}$ here we are using (yet to be determined) matrix $\mathbb{B}$, the difference is purely conventional. As in the flat case, we want to put the right hand side of (8.12) to zero and by that obtain vanishing $\mathcal{D}^v{}_{\alpha'}$ on some scalar field $\operatorname{Str}(\mathbb{H}\mathbb{B})$, that will be called pre-potential. In the flat space it was crucial that we had the identity (7.30). It was used in the relation (7.25). Similarly in the AdS case the identity (7.30) will also be crucial.

In the solution of the vanishing (8.12) right hand side we still want to maintain the $SO(4) \otimes SO(4)$ invariance. Therefore the $\mathbb{B}$ matrix has a block-diagonal form:

$$\mathbb{B} := \begin{pmatrix} b_{+\mathbf{a}+\mathbf{b}} & 0 \\ 0 & b_{\alpha\beta} \end{pmatrix} \tag{8.13}$$

Let us also simplify the notation for the constants appearing in the equations (6.21) and (8.5) and similarly for their left-right conjugates. In (6.21) we redefine:

$$X_1 := -\frac{1}{g} \frac{1}{(r_{\text{AdS}}) P_{\sim}} \ || \ X_2 := \frac{1}{2\,g} \left(1 - \frac{1}{f} \frac{1}{(r_{\text{AdS}})^2 P_- P_{\sim}}\right) \tag{8.14}$$

$$X_3 := +\frac{1}{f\,g} \frac{1}{2\,(r_{\text{AdS}})^3 P_- (P_{\sim})^2} \ ||$$

where the f and g were defined in (6.22). In (8.5) we define:

$$Y_1 := -2\left(P_{\widetilde{-}} - \frac{1}{P_-}\frac{1}{(r_{\text{AdS}})^2}\right) \;||\; Y_2 := -\frac{1}{2\,(r_{\text{AdS}})^2\,P_-} \;||\; Y_3 := -\frac{1}{2\,(r_{\text{AdS}})} \quad (8.15)$$

With the definitions (8.13), (8.14) and (8.15) let us rewrite the right hand side of (8.12) explicitly:

$$0 = (X_1 - X_3)\,(\widetilde{\Gamma}_5)^{\beta\sigma}\,b_{+\mathbf{a}+\mathbf{b}}\,(\gamma^{\mathbf{b}})_{\sigma\alpha'} - \mathcal{M}_{\alpha'}{}^{\sigma'}\,X_2\,b_{+\mathbf{a}+\mathbf{b}}\,(\gamma^{\mathbf{b}})^{\beta}{}_{\sigma'} \qquad (8.16)$$
$$+\, Y_1\,b^{\beta\sigma}\,(\gamma_{\mathbf{a}})_{\sigma\alpha'} + Y_2\,b^{\nu\sigma}\,(\widetilde{\Gamma}_5)_{\sigma}{}^{\lambda}\,(\gamma_{\mathbf{a}})_{\lambda\alpha'}\,(\widetilde{\Gamma}_5)_{\nu}{}^{\beta} + \mathcal{M}_{\alpha'}{}^{\sigma'}\,Y_3\,b^{\beta\sigma}\,(\widetilde{\Gamma}_5)_{\sigma}{}^{\lambda}\,(\gamma_{\mathbf{a}})_{\lambda\sigma'}$$

$$0 = \mathcal{M}_{\alpha'}{}^{\nu'}\,(\tilde{X}_1 - \tilde{X}_3)\,(\widetilde{\Gamma}_5)^{\beta\sigma}\,b_{+\mathbf{a}+\mathbf{b}}\,(\gamma^{\mathbf{b}})_{\sigma\nu'} - X_2\,b_{+\mathbf{a}+\mathbf{b}}\,(\gamma^{\mathbf{b}})^{\beta}{}_{\alpha'} \qquad (8.17)$$
$$+\, \mathcal{M}_{\alpha'}{}^{\nu'}\,\tilde{Y}_1\,b^{\beta\sigma}\,(\gamma_{\mathbf{a}})_{\sigma\nu'} + \mathcal{M}_{\alpha'}{}^{\nu'}\,\tilde{Y}_2\,b^{\nu\sigma}\,(\widetilde{\Gamma}_5)_{\sigma}{}^{\lambda}\,(\gamma_{\mathbf{a}})_{\lambda\nu'}\,(\widetilde{\Gamma}_5)_{\nu}{}^{\beta} + Y_3\,b^{\beta\sigma}\,(\widetilde{\Gamma}_5)_{\sigma}{}^{\lambda}\,(\gamma_{\mathbf{a}})_{\lambda\alpha'}$$

where $\tilde{X}_1$, $\tilde{X}_3$ and $\tilde{Y}_1$, $\tilde{Y}_2$ are left-right conjugates of the constants defined in (8.14) and (8.15) and X_2 and Y_3 are the same after left-right swap.

The equations (8.16) and (8.17) are the AdS analogies of the flat space equations (7.43). To solve them we first multiply the equation (8.17) by matrix $\mathcal{M}_{\alpha'}{}^{\beta'}$. Thus we get the equation (8.17) into the form:

$$0 = \mathcal{M}^2{}_{\alpha'}{}^{\nu'}\,(\tilde{X}_1 - \tilde{X}_3)\,(\widetilde{\Gamma}_5)^{\beta\sigma}\,b_{+\mathbf{a}+\mathbf{b}}\,(\gamma^{\mathbf{b}})_{\sigma\nu'} - \mathcal{M}_{\alpha'}{}^{\nu'}\,X_2\,b_{+\mathbf{a}+\mathbf{b}}\,(\gamma^{\mathbf{b}})^{\beta}{}_{\nu'} \quad (8.18)$$
$$+\, \mathcal{M}^2{}_{\alpha'}{}^{\nu'}\,\tilde{Y}_1\,b^{\beta\sigma}\,(\gamma_{\mathbf{a}})_{\sigma\nu'} + \mathcal{M}^2{}_{\alpha'}{}^{\nu'}\,\tilde{Y}_2\,b^{\nu\sigma}\,(\widetilde{\Gamma}_5)_{\sigma}{}^{\lambda}\,(\gamma_{\mathbf{a}})_{\lambda\nu'}\,(\widetilde{\Gamma}_5)_{\nu}{}^{\beta}$$
$$+\, \mathcal{M}_{\alpha'}{}^{\nu'}\,Y_3\,b^{\beta\sigma}\,(\widetilde{\Gamma}_5)_{\sigma}{}^{\lambda}\,(\gamma_{\mathbf{a}})_{\lambda\nu'}$$

The equation (8.18) is almost identical to the (8.16) except of the left-right swapped constants and $\mathcal{M}^2$ matrix. By suitable choice of the $\mathcal{M}$ matrix we can turn (8.18) into (8.16) and thus reduce number of equations by half. By that we get the condition on the matrix $\mathcal{M}$:

$$\mathcal{M}^2{}_{\alpha'}{}^{\beta'} = q^2\,\delta_{\alpha'}{}^{\beta'} \qquad (8.19)$$

where the constant $q^2 = \frac{P_-}{P_{\widetilde{-}}}$. By that choice of the matrix $\mathcal{M}^2$ and constant q^2 we turn equation (8.18) into (8.16). Furthermore we should solve relation (8.19) for the matrix $\mathcal{M}$. As in the whole AdS section we ask for the $\text{SO}(4)\otimes\text{SO}(4)$ invariance. With that requirement we get two branches for the $\mathcal{M}$ matrix (actually we get four, as we will see, but the $\pm$ is not very important to us):

$$(\mathcal{M}^2)_{\alpha'}{}^{\beta'} = \frac{P_-}{P_{\widetilde{-}}}\,\delta_{\alpha'}{}^{\beta'} \;\Rightarrow\; \mathcal{M}_{\alpha'}{}^{\beta'} = \pm\sqrt{\frac{P_-}{P_{\widetilde{-}}}}\begin{cases}\delta_{\alpha'}{}^{\beta'}\\(\widetilde{\Gamma}_5)_{\alpha'}{}^{\beta'}\end{cases} \qquad (8.20)$$

We first notice few nice properties of (8.20). The solution is actually the same as in the flat case, see (7.42). We are in the AdS space but the matrix $\mathcal{M}$ that combines $D_{\alpha'}$ and $D_{\widetilde{\alpha}'}$ does not depend on the r_{AdS}. Unfortunately we got the same not very nice square root factor in (8.20). We would need to find some way to deal with it.

Having solved one half of equations (8.16) and (8.17). We solve the second half, that is just relation (8.16):

$$0 = (X_1 - X_3)\,(\widetilde{\Gamma}_5)^{\beta\sigma}\,b_{+\mathbf{a}+\mathbf{b}}\,(\gamma^{\mathbf{b}})_{\sigma\alpha'} - \mathcal{M}_{\alpha'}{}^{\sigma'}\,X_2\,b_{+\mathbf{a}+\mathbf{b}}\,(\gamma^{\mathbf{b}})^{\beta}{}_{\sigma'} \qquad (8.21)$$
$$+\, Y_1\,b^{\beta\sigma}\,(\gamma_{\mathbf{a}})_{\sigma\alpha'} + Y_2\,b^{\nu\sigma}\,(\widetilde{\Gamma}_5)_{\sigma}{}^{\lambda}\,(\gamma_{\mathbf{a}})_{\lambda\alpha'}\,(\widetilde{\Gamma}_5)_{\nu}{}^{\beta} + \mathcal{M}_{\alpha'}{}^{\sigma'}\,Y_3\,b^{\beta\sigma}\,(\widetilde{\Gamma}_5)_{\sigma}{}^{\lambda}\,(\gamma_{\mathbf{a}})_{\lambda\sigma'}$$

The claim is that given solution $\mathcal{M}$ the block matrices $b_{+\mathbf{a}+\mathbf{b}}$ and $b_{\alpha\beta}$ are fixed (up to the overall constant). We will again use the same identity (7.30) as in the flat case. We expect the solutions (we have two branches) will be certain r_{AdS} dependent deformation of the original flat space solutions. We also require to maintain the $\text{SO}(4) \otimes \text{SO}(4)$ invariance of the solution so the most general ansatz for the equation (8.21) is:

$$b_{+\mathbf{a}+\mathbf{b}} := A\,\delta_{\mathbf{a}\mathbf{b}} + B\,(\widetilde{\varGamma}_5)_{\mathbf{a}\mathbf{b}} \;\;||\;\; b_{\alpha\beta} := C\,\delta_{\alpha\beta} + D\,(\widetilde{\varGamma}_5)_{\alpha\beta} \tag{8.22}$$

Because of later importance we will first solve the $(\widetilde{\varGamma}_5)$ branch of the $\mathcal{M}$ solution (8.20). Later we will also provide solution for the δ branch of the (8.20). We plug $\mathcal{M}$ and (8.22) into (8.21) and solve for A, B, C and D using the identity (7.30), we remind that $q := \pm\sqrt{\frac{P_-}{P_-^-}}$. We get the following solutions:

$$\text{the } (\widetilde{\varGamma}_5)_{\alpha'}{}^{\beta'} \text{ branch:} \tag{8.23}$$
$$det := \left(\,(X_1 - X_3)^2 - (q\,X_2)^2\,\right) \;||$$
$$A = \frac{D}{det}\big((X_1 - X_3)(Y_1 + Y_2) + q^2 X_2 Y_3\big) \;||\; B = q\frac{D}{det}\big(X_2(Y_1 + Y_2) + (X_1 - X_3)Y_3\big)$$
$$C = 0 \;||$$
$$\text{the } \delta_{\alpha'}{}^{\beta'} \text{ branch:} \tag{8.24}$$
$$det := \left(\,(Y_1 + Y_2)^2 - (q\,Y_3)^2\,\right) \;||$$
$$|| \; B = 0$$
$$C = q\frac{A}{det}\big((Y_1 + Y_2)X_2 + (X_1 - X_3)Y_3\big) \;||\; D := -\frac{A}{det}\big((Y_1 + Y_2)(X_1 - X_3) + q^2 X_2 Y_3\big)$$

We can again see that as we do $r_{\text{AdS}} \to \infty$ limit in (8.23) we will get the flat solution (7.42), keeping the D (or A in δ branch) r_{AdS} independent in that limit.

8.4 Near horizon limit

In the previous section we found the structure of the linearised pre-potential (8.23) and (8.24) and also the construction of $\mathcal{D}^v{}_{\alpha'}$ that vanishes on the pre-potential (8.10) and (8.20). We will now introduce the complementary derivative $\mathcal{D}^w{}_{\alpha'}$ that is constructed after picking the $\mathcal{D}^v{}_{\alpha'}$ derivative (i.e. picking the matrix $\mathcal{M}$ in (8.20)) and changing the sign in front of the $\mathcal{M}$ (the second linearly independent combination). Thus we have:

$$\mathcal{D}^w{}_{\alpha'} := D_{\alpha'} - \mathcal{M}_{\alpha'}{}^{\beta'} D_{\widetilde{\beta}'} \tag{8.25}$$

The notation for the upper indices v and w in ((8.10) and (8.25)) comes from equivalent notation for d_v and d_w derivatives used in [17, 18], (and also for d_u and $d_{\bar{u}}$, whose analogies are to be defined later). In analogy with the paper [17, 18] we want to define the $\mathcal{P}_+$ operator that has $\mathcal{D}^v{}_{\alpha'}$ and $\mathcal{D}^w{}_{\alpha'}$ as eigenvectors with nonzero eigenvalues. We can solve for $\mathcal{P}_+$ in full generality, i.e. keeping the non-local square root factors in derivatives $\mathcal{D}^v{}_{\alpha'}$ and $\mathcal{D}^w{}_{\alpha'}$. This would introduce the non-local square root factors also into the definition of $\mathcal{P}_+$ and would cause further problems. What we will do instead is to restrict the coordinate dependence of the pre-potential V to be just the $\text{PSU}(2, 2\,|\,4)$. This is the

same algebra we wanted to use at the beginning of this project, but we were forced to extend it to the full SO(10) ⊗ SO(10) T-dually extended super-algebra. Now, we want to restrict just the coordinate dependence of the pre-potential. Doing so the $P_- = P_{\widetilde{-}}$ **on** pre-potential, not everywhere. That is enough to get rid of the non-local factors in $\mathcal{D}^v{}_{\alpha'}$ and $\mathcal{D}^w{}_{\alpha'}$ as they act on pre-potential. Then we can redefine (8.10) and (8.25) by saying that the new square root free $\mathcal{D}^v$ and $\mathcal{D}^w$ to be our new definitions. With this it is easy to see that the good definition of $\mathcal{P}_+$ is:

$$\mathcal{P}_+ := \frac{1}{2}\left(P_+ + P_{\widetilde{+}} \right) = P_+ \tag{8.26}$$

where the last equality holds on pre-potential.

Following the definitions in [17, 18] of the AdS boundary limit we propose that any operator $\mathcal{K}$ which is an eigenvector of $\mathcal{P}_+$ operator, i.e. $[\mathcal{P}_+, \mathcal{K}] = \mathbf{c}\,\mathcal{K}$, scales as $R^{\mathbf{c}}$ as we approach the boundary, i.e. $R \to 0$ limit, where R is a radial coordinate on the Poincaré patch. Another way how to state the limit is that by putting the $R \to 0$ we contract the isometry groups $SO(4,1)$ and $SO(4,1)$ to $ISO(3,1)$ and $ISO(3,1)$ (we Wick rotated the S^5 isometry group for the purpose of this limit). For more details on this limit (that can be stated also through the explicit coordinates on AdS_5 and S^5) see notes [17, 18].

Using the previous definitions of the AdS boundary limit we can analyse the different branches of the $\mathcal{D}^v$ solutions (8.20). Let's first pick the $\delta_{\alpha'}{}^{\beta'}$ branch (let's work with both $\pm$ sub-branches at once). Note that even on pre-potential the $D_{\alpha'} \neq D_{\widetilde{\alpha}'}$ as can be seen from the explicit construction of those derivatives in [28] in the section **5**. Then the commutator is:

$$[\mathcal{P}_+, D_{\alpha'} + D_{\widetilde{\alpha}'}] = \pm \frac{1}{r_{\text{AdS}}}\,(\gamma_+)_{\alpha'\,\beta'}\,(\widetilde{\Gamma}_5)^{\beta'\,\sigma'}\,(D_{\sigma'} \pm D_{\widetilde{\sigma}'}) + \dots \tag{8.27}$$

The $\dots$ part correspond to the current that vanishes in the supergravity limit (i.e. we do not see string parameter σ) and on pre-potential. We also used the commutators from (3.7) and the mixed AdS commutators from (4.9). We also used the explicit solution for the PSU$(2,2|4)$ (we are on pre-potential) derivatives in terms of τ and σ currents, see section **5** in [28]. More specifically we used that $D_\Omega \equiv D^{\alpha'} = \omega^{\alpha'} + \frac{1}{2}\frac{1}{r_{\text{AdS}}}\,(\widetilde{\Gamma}_5)^{\alpha'\,\beta'}\,D_{\widetilde{\beta}'}$, where the $\omega^{\alpha'}$ is the current proportional to σ derivative and it has to vanish in the supergravity limit. The equation (8.27) is very interesting. It tells us how the $\mathcal{D}^v$ scales for the $\pm\,\delta_{\alpha'}{}^{\beta'}$ branch of (8.20). We also notice that the scaling constant is r_{AdS} dependent and vanishes for $r_{\text{AdS}} \to \infty$. More importantly because of the $(\widetilde{\Gamma}_5)$ for fixed r_{AdS} and for fixed sub-branch of $\pm\delta_{\alpha'}{}^{\beta'}$ the scaling constant $\mathbf{c}$ is either $+(\frac{1}{r_{\text{AdS}}})$ for one half of SO(8) chiral index α' or $-(\frac{1}{r_{\text{AdS}}})$ for second half. And this is not good because by [17, 18] the $\mathcal{D}^v$ derivative should scale like $\frac{1}{R}$ and $\mathcal{D}^w$ should scale like R (put $r_{\text{AdS}} = 1$ for simplicity). In (8.27) we can see that just $\frac{1}{2}$ of derivatives scale properly. This boundary limit then distinguishes between two branches of (8.20). In the following we will see that the $(\widetilde{\Gamma}_5)$ branch has exactly right scaling properties so it corresponds to the right solution. Without this boundary limit we did not have a way how to pick a branch in (8.20). In the case of $(\widetilde{\Gamma}_5)$ branch we have one more $(\widetilde{\Gamma}_5)$ matrix in (8.27) thus we get:

$$[\mathcal{P}_+, D_{\alpha'} \pm (\widetilde{\Gamma}_5)_{\alpha'}{}^{\rho'} D_{\widetilde{\rho}'}] = \pm \frac{1}{r_{\text{AdS}}}\,(D_{\alpha'} \pm (\widetilde{\Gamma}_5)_{\alpha'}{}^{\rho'} D_{\widetilde{\rho}'}) + \dots \tag{8.28}$$

The equation (8.28) will give us the correct solution. From (8.28) we can see that for fixed r_{AdS} and for fixed $(\widetilde{\Gamma}_5)$ sub-branch we will have proper scaling for full SO(8) chiral index α'. Because we require $\mathcal{D}^v$ to scale like $\frac{1}{R}$ and $\mathcal{D}^w$ scale like R we have $\mathcal{D}^v{}_{\alpha'} = (D_{\alpha'} - (\widetilde{\Gamma}_5)_{\alpha'}{}^{\rho'} D_{\widetilde{\rho}'})$ and $\mathcal{D}^w{}_{\alpha'} = (D_{\alpha'} + (\widetilde{\Gamma}_5)_{\alpha'}{}^{\rho'} D_{\widetilde{\rho}'})$. The positive news is that the blowing-up derivative $\mathcal{D}^v$ is zero on the pre-potential by our construction, so there is no possible singularity arising as we approach the boundary.

Its easy to see how the derivatives D_α and $D_{\widetilde{\alpha}}$ scale. Because the $(\gamma_+)_{\alpha\beta} = 0$ the $[\mathcal{P}_+, D_\alpha] = [\mathcal{P}_+, D_{\widetilde{\alpha}}] = 0$. So they scale like 1. Those derivatives are building up the $\mathcal{D}^u$ and $\mathcal{D}^{\bar{u}}$, analogous derivatives to paper [17, 18] derivatives d_u and $d_{\bar{u}}$. The explicit forms of $\mathcal{D}^u$ and $\mathcal{D}^{\bar{u}}$ won't be needed in this paper so we do not provide them.

8.5 Near horizon limit and field equations

Comparing result with [17, 18] we want to see that the field equations for the pre-potential in the near horizon limit (i.e. in the $R \to 0$) is just of the form $\mathcal{P}_+ V = 0 + \mathcal{O}(R)$. This will be our final confirmation that we discovered the right pre-potential. We first notice that the Lorentz generator scales like $\mathcal{O}(1)$, this can be seen from commutator $[S_{+\mathbf{a}}, \mathcal{P}_+] = [S_{\widetilde{+\mathbf{a}}}, \mathcal{P}_+] = 0$. To see what is $\mathcal{P}_+$ on pre-potential we could directly use some appropriate torsions (remember pre-potential is a linear combination of fields). We found it easier however to use a different approach. Let's look at the torsion constraint (6.11) but for the α index instead of α' (the α index is one of the SO(8) chiral indices):

$$T_{D\,S\,\widetilde{S}} \equiv T_{\alpha\,+\mathbf{a}\,\widetilde{+\mathbf{b}}} = 0 = D_{[\alpha} H_{+\mathbf{a}\,\widetilde{+\mathbf{b}})} + H_{[\alpha\,|\,\mathcal{M}} \eta^{\mathcal{M}\mathcal{N}} f_{+\mathbf{a}\,\widetilde{+\mathbf{b}})\mathcal{N}} \qquad (8.29)$$

$$= D_\alpha H_{+\mathbf{a}\,\widetilde{+\mathbf{b}}} + S_{\widetilde{+b}} H_{\alpha\,+\mathbf{a}} + S_{+\mathbf{a}} H_{\widetilde{+\mathbf{b}}\,\alpha} \qquad (8.30)$$

First notice that the structure of (8.30) is very different than the structure of (6.11). There is no f term in (8.30) and there is the full derivative term present. Even in the AdS case the f term is missing. This can be seen as follows. The $f_{+\mathbf{a}\,\widetilde{+\mathbf{b}}\mathcal{N}} = 0$ in AdS and also in flat case and also $f_{\alpha\,\widetilde{+\mathbf{b}}\mathcal{N}} = 0$. The only possibly nonzero f term is coming from $f_{\alpha\,+\mathbf{a}\mathcal{N}}$. The $H_{\widetilde{+\mathbf{b}}\,\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}} f_{\alpha\,+\mathbf{a}\mathcal{N}} \propto (\gamma_{+\mathbf{a}})_\alpha{}^{\nu'} H_{\nu'\,\widetilde{+\mathbf{b}}}$. The vielbein $H_{\nu'\,\widetilde{+\mathbf{b}}}$ is zero (also in the AdS) as was shown in the analysis under (6.11). Next, we can recognise the term $H_{\alpha\,\widetilde{+\mathbf{a}}}$ as a part of $\mathbb{H}$ matrix (7.19). The vielbein $H_{\alpha\,+\mathbf{a}}$ has also been analysed in table 7. It is related to $H_{+\mathbf{a}\,\widetilde{\alpha}}$, see table 7. We need to be more careful with that relation because in (8.30) we again discover the S derivative peculiarity, we saw earlier.

In general all fields in $\mathbb{H}$ (now better viewed as their irreducible pieces) could be obtained from the pre-potential V by an action of appropriate (irreducible) combination of $\mathcal{D}^w$ on the pre-potential. One could analyse in full detail what is the exact structure of those pieces and reproduce famous field content of $AdS_5 \times S^5$ supergravity first discovered in [12] and later used in [13]. This would lead us away from this paper real aim, so we postpone this analysis to next paper. The aim of this section is to show that on pre-potential V the operator $\mathcal{P}_+$ vanishes in the near horizon limit.

For this reason we notice following expansions:

$$H_{+\mathbf{a}\,\widetilde{+\mathbf{b}}} = \mathbf{c}_0\,V \,+\, \mathbf{c}_2\,(\mathcal{D}^w)^2\,V \,+\, \mathbf{c}_4\,(\mathcal{D}^w)^4\,V \,+\, \mathbf{c}_6\,(\mathcal{D}^w)^6\,V \,+\, \mathbf{c}_8\,(\mathcal{D}^w)^8\,V \tag{8.31}$$

$$H_{+\mathbf{a}\,\widetilde{\alpha}} = \mathbf{d}_1\,\mathcal{D}^w\,V \,+\, \mathbf{d}_3\,(\mathcal{D}^w)^3\,V \,+\, \mathbf{d}_5\,(\mathcal{D}^w)^5\,V \,+\, \mathbf{d}_7\,(\mathcal{D}^w)^7\,V \tag{8.32}$$

$$H_{\widetilde{+\mathbf{a}}\,\alpha} = \mathbf{e}_1\,\mathcal{D}^w\,V \,+\, \mathbf{e}_3\,(\mathcal{D}^w)^3\,V \,+\, \mathbf{e}_5\,(\mathcal{D}^w)^5\,V \,+\, \mathbf{e}_7\,(\mathcal{D}^w)^7\,V \tag{8.33}$$

where factors $\mathbf{c}_0$, $\mathbf{c}_2$ …, $\mathbf{d}_1$, $\mathbf{d}_3$ … and $\mathbf{e}_1$, $\mathbf{e}_3$ … are constant factors with appropriate index structure. Note that the $\mathbf{c}_0$ is non-zero. The important observation is that for each term in (8.31), (8.32) and (8.33) we know how it scales in the $R \to 0$ limit, because we know that $\mathcal{D}^w$ scales like R.

Next, we want to combine (8.30) with known scalings of all (8.30) objects to get an information how $D_\alpha V$ scales. On one hand it should scale like $\mathcal{O}(1)$ on the other hand the relation (8.30) relates it to different fields. What we obtain is a nontrivial relation that $D_\alpha V = \mathcal{O}(R)$ as we go to the boundary. It just means that $D_\alpha V = 0$ (and so also $D_{\widetilde{\alpha}} V = 0$) in the near horizon limit. Because of the anti-commutator $\{D_\alpha, D_\beta\} = 2\,(\gamma_-)_{\alpha\beta}\,P_+$. This is enough to see that $P_+ V \equiv \mathcal{P}_+ V = 0$ in the near horizon limit. There are two crucial steps. One is to relate the (8.30) term $S_{\widetilde{+\mathbf{b}}}\,H_{\alpha+\mathbf{a}}$ to $H_{+\mathbf{a}\,\widetilde{\alpha}}$. This is relatively straightforward using table 7 and explicit $S_{\widetilde{+\mathbf{b}}}$ derivative. Second step is to plug expansions (8.31), (8.32) and (8.33) and the scalings of particular pieces into (8.30). Doing that we get the following:

$$0 = D_\alpha\,H_{+\mathbf{a}\,\widetilde{+\mathbf{b}}} \,+\, S_{\widetilde{+\mathbf{b}}}\,H_{\alpha+\mathbf{a}} \,+\, S_{+\mathbf{a}}\,H_{\widetilde{+\mathbf{b}}\,\alpha} \tag{8.34}$$

$$= D_\alpha\,H_{+\mathbf{a}\,\widetilde{+\mathbf{b}}} \,+\, S_{\widetilde{+\mathbf{b}}}\left(-\,f_{\underset{\sim}{}\,\alpha\,\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\,\frac{1}{P_{\underset{\sim}{}}}\,H_{+\mathbf{a}\,\mathcal{N}} \right) \,+\, S_{+\mathbf{a}}\,H_{\widetilde{+\mathbf{b}}\,\alpha}$$

$$= D_\alpha\,H_{+\mathbf{a}\,\widetilde{+\mathbf{b}}} \,-\, \frac{\mathbf{c}}{r_{\mathrm{AdS}}}\,(\gamma_-)_{\alpha\nu}\,(\widetilde{\Gamma}_5)^{\nu\sigma}\,S_{\widetilde{+\mathbf{b}}}\,\frac{1}{P_{\underset{\sim}{}}}\,H_{+\mathbf{a}\,\widetilde{\sigma}} \,+\, S_{+\mathbf{a}}\,H_{\widetilde{+\mathbf{b}}\,\alpha} \tag{8.35}$$

$$= D_\alpha\left(\mathbf{c}_0\,V \,+\, \mathbf{c}_2\,(\mathcal{D}^w)^2\,V \,+\, \dots \right) \tag{8.36}$$

$$-\,\frac{1}{r_{\mathrm{AdS}}}\,(\gamma_-)_{\alpha\nu}\,(\widetilde{\Gamma}_5)^{\nu\sigma}\,S_{\widetilde{+\mathbf{b}}}\,\frac{1}{P_{\underset{\sim}{}}}\left(\mathbf{d}_1\,\mathcal{D}^w\,V \,+\, \mathbf{d}_3\,(\mathcal{D}^w)^3\,V \,+\, \dots \right)$$

$$+\,S_{+\mathbf{a}}\left(\mathbf{e}_1\,\mathcal{D}^w\,V \,+\, \mathbf{e}_3\,(\mathcal{D}^w)^3\,V \,+\, \dots \right)$$

The equation (8.35) contains all the right expressions to establish the near horizon limit. By the discussion below (8.28) the $\mathcal{D}^w{}_{\alpha'}$ derivative scales like $\mathcal{O}(R)$ (for the projective branch), we also have the scaling of $S_{+\mathbf{a}}$ and $S_{\widetilde{+\mathbf{b}}}$ that goes like a constant. Applying that knowledge we get the equation (8.35) in the near horizon limit:

$$0 = \mathbf{c}_0\,D_\alpha V \,+\, \mathcal{O}(R) \tag{8.37}$$

The $\mathbf{c}_0$ is nonzero constant (tensor) so it follows that $D_\alpha V = 0$ at the AdS boundary. From $\{D_\alpha, D_\beta\} = (\gamma_-)_{\alpha\beta}\,P_+$ we get the field equation for the pre-potential in the near horizon limit:

$$0 = P_+ V \,+\, \mathcal{O}(R) \tag{8.38}$$

$$\equiv \mathcal{P}_+ V \,+\, \mathcal{O}(R) \tag{8.39}$$

9 Conclusion

We outline results we have obtained: starting from the 10 dimensional IIB string theory. We embedded the $AdS_5 \times S^5$ background and expanded the theory around this background (we also considered a flat background, i.e. $AdS_5 \times S^5$ with $r_{\text{AdS}} \to \infty$). Our aim was to obtain (linearised) pre-potential with desired properties in the case of $AdS_5 \times S^5$ (also in the flat case). We succeeded and obtained pre-potential construction for flat and $AdS_5 \times S^5$ background. We derived only the linearised form, but the vielbein construction makes non-linearisation straightforward perturbation. The pre-potential (in flat and also in $AdS_5 \times S^5$, the projective and chiral) sits in the combination (without further derivatives) of vielbeins $H_{S\widetilde{S}}$ and $H_{D\widetilde{D}}$. By construction the D^v derivative vanishes in bulk on the pre-potential and the (projective) pre-potential satisfies the near horizon limit field equation $\mathcal{P}_+ V = 0 + \mathcal{O}(R)$ together with vanishing of $\mathcal{D}^u$ and $\mathcal{D}^{\bar{u}}$ on pre-potential in the near horizon limit. This near horizon limit picks out the projective pre-potential instead of chiral pre-potential (both were obtained as valid bulk solutions).

The vanishing of $\mathcal{P}_+$ at the boundary fixes the difference between the conformal weights ($\equiv \Delta$) and U(1) charges ($\equiv \Delta_Y$) of all boundary BPS operators, since $\mathcal{P}_+ \propto \Delta - \Delta_Y$. The $\mathcal{P}_- \propto \Delta + \Delta_Y$ and known expansion of $\mathbb{H}$ in powers of $\mathcal{D}^w$ from V, fixes the conformal weights and the U(1) charges for the boundary BPS operators, the relations important in the AdS/CFT correspondence, see [14, 15].

Acknowledgments

A $AdS_5 \times S^5$ structure of some vielbeins and their derivatives

A.1 The $H_{S\widetilde{S}}$

Using equations (5.3), (5.4), (6.5) and the mixed light-cone gauge together with keeping the mixed structure constants and evaluating the explicit actions of the S and $\widetilde{S}$ derivatives we derived the first important result for the structure of the $H_{S\widetilde{S}}$ vielbein (in the AdS case). Note that by the symbol $\rightsquigarrow$ in the in the whole text we denoted the evaluation of the mixed structure constants in the sense described in section 6.2. In the table 5 (and after the evaluation of mixed structure constants) we have heavily used the structure of the mixed structure constant $f_{\mathbf{a}\widetilde{\mathbf{b}}\mathcal{M}}$ that is analysed in the main text, see analysis before equation (6.7). Moreover we used one more torsion constrain to fix H_{PS} and $H_{\widetilde{P}\widetilde{S}}$ in the table 5. Let's take an example $H_{PS} = H_{\mathbf{a}\mathbf{b}\mathbf{c}}$. To fix that vielbein we consider $T_{\undertilde{-}\mathbf{a}\mathbf{b}\mathbf{c}} = 0$:

$$
\begin{aligned}
T_{\widetilde{P}PS} \equiv T_{\undertilde{-}\mathbf{a}\mathbf{b}\mathbf{c}} &= 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (\text{A.1})\\
&= P_{\undertilde{-}} H_{\mathbf{a}\mathbf{b}\mathbf{c}} + S_{\mathbf{b}\mathbf{c}} H_{\undertilde{-}\mathbf{a}} + P_{\mathbf{a}} H_{\mathbf{b}\mathbf{c}\undertilde{-}} + H_{\mathbf{b}\mathbf{c}\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}} f_{\undertilde{-}\mathbf{a}\mathcal{N}}\\
&= P_{\undertilde{-}} H_{\mathbf{a}\mathbf{b}\mathbf{c}} + f_{\undertilde{-}\mathbf{a}\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}} H_{\mathbf{b}\mathbf{c}\mathcal{N}}\\
\Rightarrow \quad H_{\mathbf{a}\mathbf{b}\mathbf{c}} &= - f_{\undertilde{-}\mathbf{a}\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}} \frac{1}{P_{\undertilde{-}}} H_{\mathbf{b}\mathbf{c}\mathcal{N}}
\end{aligned}
$$

$$H_{-\mathbf{a}\,\widetilde{-\mathbf{b}}} = -f_{-\,\widetilde{-}\,\mathcal{M}}\,\eta^{\mathcal{MN}}\,\frac{1}{P_{\widetilde{-}}}\,S_{-a}\left(\frac{1}{P_{\widetilde{-}}}\,H_{\widetilde{-b}\,\mathcal{N}}\right)\rightsquigarrow 0$$

$$H_{-\mathbf{a}\,\widetilde{+\mathbf{b}}} = -f_{-\,\widetilde{-}\,\mathcal{M}}\,\eta^{\mathcal{MN}}\,\frac{1}{P_{-}}\,S_{\widetilde{+\mathbf{b}}}\left(\frac{1}{P_{\widetilde{-}}}\,H_{-\mathbf{a}\,\mathcal{N}}\right) - f_{-\,\widetilde{\mathbf{b}}\,\mathcal{M}}\,\eta^{\mathcal{MN}}\,\frac{1}{P_{-}P_{\widetilde{-}}}\,H_{-\mathbf{a}\,\mathcal{N}}$$

$$\rightsquigarrow -\frac{1}{2\,(r_{\text{AdS}})^2}\,\frac{1}{P_{-}P_{\widetilde{-}}}\,H_{-\mathbf{a}\,\widetilde{+\mathbf{b}}} \Rightarrow H_{-\mathbf{a}\,\widetilde{+\mathbf{b}}} = 0$$

$$H_{+\mathbf{a}\,\widetilde{+\mathbf{b}}} = -f_{-\,\widetilde{-}\,\mathcal{M}}\,\eta^{\mathcal{MN}}\,\frac{1}{P_{-}}\,S_{\widetilde{+\mathbf{b}}}\left(\frac{1}{P_{\widetilde{-}}}\,H_{+\mathbf{a}\,\mathcal{N}}\right) - f_{-\,\widetilde{\mathbf{b}}\,\mathcal{M}}\,\eta^{\mathcal{MN}}\,\frac{1}{P_{-}P_{\widetilde{-}}}\,H_{+\mathbf{a}\,\mathcal{N}} + \frac{1}{P_{-}}\,H_{\widetilde{+\mathbf{b}}\,\mathbf{a}}$$

$$\rightsquigarrow -\frac{1}{2\,(r_{\text{AdS}})^2}\,\frac{1}{P_{-}P_{\widetilde{-}}}\,H_{+\mathbf{a}\,\widetilde{+\mathbf{b}}} + \frac{1}{P_{-}}\,H_{\widetilde{+\mathbf{b}}\,\mathbf{a}} \Rightarrow \left(1 + \frac{1}{2\,(r_{\text{AdS}})^2}\,\frac{1}{P_{-}P_{\widetilde{-}}}\right)H_{+\mathbf{a}\,\widetilde{+\mathbf{b}}} = \frac{1}{P_{-}}\,H_{\mathbf{a}\,\widetilde{+\mathbf{b}}}$$

$$H_{-\mathbf{a}\,\widetilde{\mathbf{bc}}} = -f_{-\,\widetilde{-}\,\mathcal{M}}\,\eta^{\mathcal{MN}}\,\frac{1}{P_{-}}\,S_{\widetilde{\mathbf{bc}}}\left(\frac{1}{P_{\widetilde{-}}}\,H_{-\mathbf{a}\,\mathcal{N}}\right)\rightsquigarrow 0$$

$$H_{+\mathbf{a}\,\widetilde{\mathbf{bc}}} = f_{\mathbf{a}\,\widetilde{-}\,\mathcal{M}}\,\eta^{\mathcal{MN}}\,\frac{1}{P_{\widetilde{-}}P_{-}}\,H_{\widetilde{\mathbf{bc}}\,\mathcal{N}} - f_{-\,\widetilde{-}\,\mathcal{M}}\,\eta^{\mathcal{MN}}\,\frac{1}{P_{\widetilde{-}}}\,S_{+\mathbf{a}}\left(\frac{1}{P_{-}}\,H_{\widetilde{\mathbf{bd}}\,\mathcal{N}}\right)$$

$$\rightsquigarrow -\frac{1}{2\,(r_{\text{AdS}})^2}\,\frac{1}{P_{\widetilde{-}}P_{-}}\,H_{\widetilde{\mathbf{bc}}\,+\mathbf{a}} \Rightarrow H_{+\mathbf{a}\,\widetilde{\mathbf{bc}}} = 0$$

$$H_{\mathbf{ab}\,\widetilde{\mathbf{cd}}} = -f_{-\,\widetilde{-}\,\mathcal{M}}\,\eta^{\mathcal{MN}}\,\frac{1}{P_{-}}\,S_{\widetilde{\mathbf{cd}}}\left(\frac{1}{P_{\widetilde{-}}}\,H_{\mathbf{ab}\,\mathcal{N}}\right)\rightsquigarrow 0$$

$$H_{+-\,\widetilde{+-}} = f_{-\,\widetilde{-}\,\mathcal{M}}\,\eta^{\mathcal{MN}}\,\frac{1}{P_{\widetilde{-}}P_{-}}\,H_{\widetilde{+-}\,\mathcal{N}} - f_{-\,\widetilde{-}\,\mathcal{M}}\,\eta^{\mathcal{MN}}\,\frac{1}{P_{\widetilde{-}}}\,S_{+-}\left(\frac{1}{P_{-}}\,H_{\widetilde{+-}\,\mathcal{N}}\right)\rightsquigarrow 0$$

$$H_{+-\,\widetilde{\mathbf{a}}} = f_{-\,\widetilde{-}\,\mathcal{M}}\,\eta^{\mathcal{MN}}\,\frac{1}{P_{\widetilde{-}}P_{-}}\,H_{\widetilde{\mathbf{a}}\,\mathcal{N}} - f_{-\,\widetilde{-}\,\mathcal{M}}\,\eta^{\mathcal{MN}}\,\frac{1}{P_{\widetilde{-}}}\,S_{+-}\left(\frac{1}{P_{-}}\,H_{\widetilde{\mathbf{a}}\,\mathcal{N}}\right)\rightsquigarrow 0$$

$$H_{+-\,\widetilde{+\mathbf{a}}} = f_{-\,\widetilde{\mathbf{a}}\,\mathcal{M}}\,\eta^{\mathcal{MN}}\,\frac{1}{P_{-}P_{\widetilde{-}}}\,H_{+-\,\mathcal{N}} - f_{-\,\widetilde{-}\,\mathcal{M}}\,\eta^{\mathcal{MN}}\,\frac{1}{P_{-}}\,S_{\widetilde{+\mathbf{a}}}\left(\frac{1}{P_{\widetilde{-}}}\,H_{+-\,\mathcal{N}}\right)$$

$$\rightsquigarrow \frac{1}{2\,(r_{\text{AdS}})^2}\,\frac{1}{P_{-}P_{\widetilde{-}}}\,H_{+-\,\widetilde{+\mathbf{a}}} \Rightarrow H_{+-\,\widetilde{+\mathbf{a}}} = 0$$

$$H_{+-\,\widetilde{\mathbf{ab}}} = f_{-\,\widetilde{-}\,\mathcal{M}}\,\eta^{\mathcal{MN}}\,\frac{1}{P_{\widetilde{-}}P_{-}}\,H_{\widetilde{\mathbf{ab}}\,\mathcal{N}} - f_{-\,\widetilde{-}\,\mathcal{M}}\,\eta^{\mathcal{MN}}\,\frac{1}{P_{\widetilde{-}}}\,S_{+-}\left(\frac{1}{P_{-}}\,H_{\widetilde{\mathbf{ab}}\,\mathcal{N}}\right)\rightsquigarrow 0$$

Table 5. $H_{S\,\widetilde{S}}$ vielbein.

$$H_{\alpha'\,-\mathbf{a}} = f_{\widetilde{-}\,\alpha'\,\mathcal{M}}\,\eta^{\mathcal{MN}}\,\frac{1}{P_{\widetilde{-}}}\,H_{-\mathbf{a}\,\mathcal{N}}\rightsquigarrow 0$$

$$H_{\alpha'\,+\mathbf{a}} = f_{\widetilde{-}\,\alpha'\,\mathcal{M}}\,\eta^{\mathcal{MN}}\,\frac{1}{P_{\widetilde{-}}}\,H_{+\mathbf{a}\,\mathcal{N}}\rightsquigarrow 0$$

$$H_{\alpha'\,\widetilde{-\mathbf{a}}} = -f_{-\,\widetilde{-}\,\mathcal{M}}\,\eta^{\mathcal{MN}}\,\frac{1}{P_{-}}\,S_{\widetilde{-\mathbf{a}}}\,\frac{1}{P_{\widetilde{-}}}\,H_{\alpha'\,\mathcal{N}}\rightsquigarrow 0$$

$$H_{\alpha'\,\widetilde{+\mathbf{a}}} = -S_{\widetilde{+\mathbf{a}}}\left(f_{-\,\widetilde{-}\,\mathcal{M}}\,\eta^{\mathcal{MN}}\,\frac{1}{P_{-}P_{\widetilde{-}}}\,H_{\alpha'\,\mathcal{N}}\right)\rightsquigarrow 0$$

Table 6. $H_{D\,\widetilde{S}}$ vielbein.

A.2 The $H_{S\,\widetilde{D}}$ and $H_{\widetilde{S}\,D}$

In the section 6.3 we analysed vielbein $H_{\alpha'\,+\mathbf{b}}$. By the similar set of equations as in the section 6.3 we can fix $H_{\alpha'\,\widetilde{-\mathbf{b}}}$. We summarise the structure of the fixed vielbeins from the section 6.3 discussion in the following table: Similarly we can calculate what is the table 6 with α' swapped with α. We will use the analogous analysis as in section 6.3 except sometimes instead of the equation (6.16) we use $T_{\widetilde{P}\,D\,\widetilde{S}}$ and also we fix the $H_{\widetilde{P}\,\widetilde{S}}$ using $T_{P\,\widetilde{P}\,\widetilde{S}}$ (or some left $-$ right swap of those). Let's look at two such examples and calculate what is $H_{\alpha\,-\mathbf{a}}$ and $H_{\widetilde{\alpha}\,-\mathbf{a}}$ respectively (we also use the mixed light-cone gauge):

$$T_{\widetilde{P}\,D\,S} \equiv T_{\widetilde{-}\,\alpha\,-\mathbf{a}} = 0 = P_{[\widetilde{-}}\,H_{\alpha\,-\mathbf{a})} + H_{[\widetilde{-}\,|\,\mathcal{M}}\,\eta^{\mathcal{MN}}\,f_{\alpha\,-\mathbf{a})\mathcal{N}} \tag{A.2}$$

$$= P_{\widetilde{-}}\,H_{\alpha\,-\mathbf{a}} + H_{-\mathbf{a}\,\mathcal{M}}\,\eta^{\mathcal{MN}}\,f_{\alpha\,\widetilde{-}\,\mathcal{N}} \tag{A.3}$$

$$\Rightarrow \quad H_{\alpha\,-\mathbf{a}} = -f_{\alpha\,\widetilde{-}\,\mathcal{M}}\,\eta^{\mathcal{MN}}\,\frac{1}{P_{\widetilde{-}}}\,H_{-\mathbf{a}\,\mathcal{N}}\rightsquigarrow (\gamma_{-})_{\alpha\,\nu}\,(\widetilde{\varGamma}_5)^{\nu\,\sigma}\,\frac{1}{(r_{\text{AdS}})\,P_{\widetilde{-}}}\,H_{-\mathbf{a}\,\widetilde{\sigma}}$$

$$H_{\alpha\,-\mathbf{a}} = -f_{\alpha\,\undertilde{-}\,\mathcal{M}}\,\eta^{\mathcal{MN}}\,\frac{1}{P_{\undertilde{-}}}\,H_{-\mathbf{a}\mathcal{N}}$$

$$\Rightarrow H_{\alpha\,-\mathbf{a}} \rightsquigarrow 0$$

$$H_{\alpha\,+\mathbf{a}} = f_{\undertilde{-}\,\alpha\,\mathcal{M}}\,\eta^{\mathcal{MN}}\,\frac{1}{P_{\undertilde{-}}}\,H_{+\mathbf{a}\mathcal{N}} \rightsquigarrow (\gamma_-)_{\alpha\,\nu}\,(\widetilde{\Gamma}_5)^{\nu\,\sigma}\,\frac{1}{(r_{\mathrm{AdS}})\,P_{\undertilde{-}}}\,H_{+\mathbf{a}\,\widetilde{\sigma}}$$

$$H_{\widetilde{\alpha}\,-\mathbf{a}} = f_{-\,\undertilde{-}\,\mathcal{M}}\,\eta^{\mathcal{MN}}\,\frac{1}{P_{\undertilde{-}}}\,D_{\widetilde{\alpha}}\,\frac{1}{P_{\undertilde{-}}}\,H_{-\mathbf{a}\mathcal{N}} + f_{-\,\widetilde{\alpha}\,\mathcal{M}}\,\eta^{\mathcal{MN}}\,\frac{1}{P_{\undertilde{-}}}\,H_{-\mathbf{a}\mathcal{N}}$$

$$\Rightarrow H_{\widetilde{\alpha}\,-\mathbf{a}} \rightsquigarrow 0$$

$$H_{\alpha\,\widetilde{+\mathbf{a}}} = -f_{-\,\undertilde{-}\,\mathcal{M}}\,\eta^{\mathcal{MN}}\,\frac{1}{P_{\undertilde{-}}}\,D_{\alpha}\,\frac{1}{P_{\undertilde{-}}}\,H_{\widetilde{+\mathbf{a}}\mathcal{N}} + f_{\undertilde{-}\,\alpha\,\mathcal{M}}\,\eta^{\mathcal{MN}}\,\frac{1}{P_{\undertilde{-}}}\,H_{\widetilde{+\mathbf{a}}\mathcal{N}} - \eta_{\undertilde{-}\,\widetilde{+}}\,\frac{1}{P_{\undertilde{-}}}\,H_{\alpha\,\widetilde{\mathbf{a}}}$$

$$\Rightarrow H_{\alpha\,\widetilde{+\mathbf{a}}} \rightsquigarrow (\gamma_-)_{\alpha\,\nu}\,(\widetilde{\Gamma}_5)^{\nu\,\sigma}\,\frac{1}{(r_{\mathrm{AdS}})\,P_{\undertilde{-}}}\,H_{\widetilde{+\mathbf{a}}\,\widetilde{\sigma}} + \frac{1}{P_{\undertilde{-}}}\,H_{\alpha\,\widetilde{\mathbf{a}}}$$

Table 7. $H_{D\,\widetilde{S}}$ vielbein.

Next, examine:

$$T_{PS\widetilde{D}} \equiv T_{-\,-\mathbf{a}\,\widetilde{\alpha}} = 0 = P_{[-}\,H_{-\mathbf{a}\,\widetilde{\alpha})} + H_{[-\,|\,\mathcal{M}}\,\eta^{\mathcal{MN}}\,f_{-\mathbf{a}\,\widetilde{\alpha})\mathcal{N}} \tag{A.4}$$

$$= P_-\,H_{-\mathbf{a}\,\widetilde{\alpha}} + D_{\widetilde{\alpha}}\,H_{-\,-\mathbf{a}} + H_{-\mathbf{a}\,\mathcal{M}}\,\eta^{\mathcal{MN}}\,f_{\widetilde{\alpha}\,-\,\mathcal{N}} \tag{A.5}$$

the $H_{-\,-\mathbf{a}}$ is fixed by $T_{\widetilde{P}\,PS} \equiv T_{\undertilde{-}\,-\,-\mathbf{a}} = 0$:

$$T_{\widetilde{P}\,PS} \equiv T_{\undertilde{-}\,-\,-\mathbf{a}} = 0 = P_{[\undertilde{-}}\,H_{-\,-\mathbf{a})} + H_{[\undertilde{-}\,|\,\mathcal{M}}\,\eta^{\mathcal{MN}}\,f_{-\,-\mathbf{a})\mathcal{N}} \tag{A.6}$$

$$= P_{\undertilde{-}}\,H_{-\,-\mathbf{a}} + H_{-\mathbf{a}\,\mathcal{M}}\,\eta^{\mathcal{MN}}\,f_{\undertilde{-}\,-\,\mathcal{N}} \tag{A.7}$$

$$\Rightarrow \quad H_{-\,-\mathbf{a}} = f_{-\,\undertilde{-}\,\mathcal{M}}\,\eta^{\mathcal{MN}}\,\frac{1}{P_{\undertilde{-}}}\,H_{-\mathbf{a}\mathcal{N}} \rightsquigarrow 0 \tag{A.8}$$

plugging (A.8) into the (A.5) we get:

$$H_{-\mathbf{a}\,\widetilde{\alpha}} = -f_{-\,\undertilde{-}\,\mathcal{M}}\,\eta^{\mathcal{MN}}\,\frac{1}{P_-}\,D_{\widetilde{\alpha}}\,\frac{1}{P_{\undertilde{-}}}\,H_{-\mathbf{a}\mathcal{N}} - f_{-\,\widetilde{\alpha}\,\mathcal{M}}\,\eta^{\mathcal{MN}}\,\frac{1}{P_-}\,H_{-\mathbf{a}\mathcal{N}} \tag{A.9}$$

$$\Rightarrow H_{-\mathbf{a}\,\widetilde{\alpha}} \rightsquigarrow -(\gamma_-)_{\alpha\,\nu}\,(\widetilde{\Gamma}_5)^{\nu\,\sigma}\,\frac{1}{(r_{\mathrm{AdS}})\,P_-}\,H_{-\mathbf{a}\,\sigma} \tag{A.10}$$

We notice that combining the result (A.10) with (A.3) we get after the evaluation of the mixed structure constants that $H_{-\mathbf{a}\,\alpha} \rightsquigarrow 0$ and so also $H_{-\mathbf{a}\,\widetilde{\alpha}} \rightsquigarrow 0$. Similar analysis can be made for the rest of the vielbeins (we mean those from table 6, except α' switched with α). Thus we get the table 7. Let us repeat our goal. We wanted to determine the actions of $S_{\widetilde{+\mathbf{b}}}$ and $S_{+\mathbf{a}}$ on $H_{\alpha'\,+\mathbf{a}}$ and $H_{\widetilde{+\mathbf{b}}\,\alpha'}$ respectively. We wanted to do that because then the (6.12) gives the action of $D_{\alpha'}$ on $H_{+\mathbf{a}\,\widetilde{+\mathbf{b}}}$ (where at least the part of the pre-potential sits). The action of $S_{\widetilde{+\mathbf{b}}}$ on $H_{\alpha'\,+\mathbf{a}}$ is easily computed using our table 6. Taking the second top relation from the table 6 and by explicitly applying the $S_{\widetilde{+\mathbf{b}}}$ derivative we get:

$$S_{\widetilde{+\mathbf{b}}}\,H_{\alpha'\,+\mathbf{a}} = S_{\widetilde{+\mathbf{b}}}\left(f_{\undertilde{-}\,\alpha'\,\mathcal{M}}\,\eta^{\mathcal{MN}}\,\frac{1}{P_{\undertilde{-}}}\,H_{+\mathbf{a}\mathcal{N}}\right) \tag{A.11}$$

$$= \eta_{\undertilde{-}\,\widetilde{+}}\,f_{\widetilde{\mathbf{b}}\,\alpha'\,\mathcal{M}}\,\eta^{\mathcal{MN}}\,\frac{1}{P_{\undertilde{-}}}\,H_{+\mathbf{a}\mathcal{N}} + f_{\undertilde{-}\,\alpha'\,\mathcal{M}}\,\eta^{\mathcal{MN}}\,S_{\widetilde{+\mathbf{b}}}\left(\frac{1}{P_{\undertilde{-}}}\,H_{+\mathbf{a}\mathcal{N}}\right) \tag{A.12}$$

$$\Rightarrow S_{\widetilde{+\mathbf{b}}}\,H_{\alpha'\,+\mathbf{a}} \rightsquigarrow -(\gamma_{\mathbf{b}})_{\alpha'\,\nu}\,(\widetilde{\Gamma}_5)^{\nu\,\sigma}\,\frac{1}{(r_{\mathrm{AdS}})\,P_{\undertilde{-}}}\,H_{+\mathbf{a}\,\widetilde{\sigma}} \tag{A.13}$$

To evaluate $S_{+\mathbf{a}} H_{\widetilde{+\mathbf{b}}\,\alpha'}$ we need to work a bit more. One can directly use the last relation in the table 6. We found an easier way however. For that we need an alternative fixing of the vielbein $H_{\widetilde{+\mathbf{b}}\,\alpha'}$. This alternative fixing seems to be more suited for an explicit evaluation of the $S_{+\mathbf{a}}$ action (and $S_{-\mathbf{a}}$ action). An alternative way how to fix $H_{\alpha'\,\widetilde{+\mathbf{a}}}$ is to use $T_{\widetilde{P}\,D\,\widetilde{S}} \equiv T_{\underset{\sim}{-}\,\alpha'\,\widetilde{+\mathbf{a}}}$ instead of one that we used in (6.16) and (6.17). Similarly it will be useful to find an alternative fixing for $H_{\alpha'\,\underset{\sim}{-}\mathbf{a}}$. Again that could be done by considering torsion $T_{\widetilde{P}\,D\,\widetilde{S}} \equiv T_{\underset{\sim}{-}\,\alpha'\,\underset{\sim}{-}\mathbf{a}}$. Let's look at this alternative fixing more closely:

$$T_{\widetilde{P}\,D\,\widetilde{S}} \equiv T_{\underset{\sim}{-}\,\alpha'\,\underset{\sim}{-}\mathbf{a}} = 0 = P_{[\underset{\sim}{-}}\,H_{\alpha'\,\underset{\sim}{-}\mathbf{a})} + H_{[\underset{\sim}{-}\,|\,\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\,f_{\alpha'\,\underset{\sim}{-}\mathbf{a})\mathcal{N}} \tag{A.14}$$

$$= P_{\underset{\sim}{-}}\,H_{\alpha'\,\underset{\sim}{-}\mathbf{a}} + D_{\alpha'}\,H_{\underset{\sim}{-}\mathbf{a}\,\underset{\sim}{-}} + H_{\underset{\sim}{-}\mathbf{a}\,\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\,f_{\underset{\sim}{-}\,\alpha'\,\mathcal{N}} \tag{A.15}$$

The $H_{\underset{\sim}{-}\mathbf{a}\,\underset{\sim}{-}}$ type of vielbein has been fixed in (A.8). Plugging the fixing into (A.15) we get an alternative $H_{\alpha'\,\underset{\sim}{-}\mathbf{a}}$ fixing:

$$H_{\alpha'\,\underset{\sim}{-}\mathbf{a}} = -f_{-\,\underset{\sim}{-}\,\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\,\frac{1}{P_{\underset{\sim}{-}}}\,D_{\alpha'}\,\frac{1}{P_-}\,H_{\underset{\sim}{-}\mathbf{a}\,\mathcal{N}} - f_{\underset{\sim}{-}\,\alpha'\,\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\,\frac{1}{P_{\underset{\sim}{-}}}\,H_{\underset{\sim}{-}\mathbf{a}\,\mathcal{N}} \tag{A.16}$$

$$H_{\alpha'\,\underset{\sim}{-}\mathbf{a}} \rightsquigarrow 0 \tag{A.17}$$

again we can see the behaviour of the $H_{\alpha'\,\underset{\sim}{-}\mathbf{a}}$ in (A.17) as we evaluate the theory, as it should be comparing with its behaviour from the fixing in the table 6. The alternative fixing for the vielbein $H_{\widetilde{+\mathbf{b}}\,\alpha'}$ is calculated similarly:

$$T_{\widetilde{P}\,D\,\widetilde{S}} \equiv T_{\underset{\sim}{-}\,\alpha'\,\widetilde{+\mathbf{a}}} = 0 = P_{[\underset{\sim}{-}}\,H_{\alpha'\,\widetilde{+\mathbf{a}})} + H_{[\underset{\sim}{-}\,|\,\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\,f_{\alpha'\,\widetilde{+\mathbf{a}})\mathcal{N}} \tag{A.18}$$

$$= P_{\underset{\sim}{-}}\,H_{\alpha'\,\widetilde{+\mathbf{a}}} + D_{\alpha'}\,H_{\widetilde{+\mathbf{a}}\,\underset{\sim}{-}} + H_{\widetilde{+\mathbf{a}}\,\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\,f_{\underset{\sim}{-}\,\alpha'\,\mathcal{N}} + H_{\alpha'\,\widetilde{\mathbf{a}}} \tag{A.19}$$

The $H_{\widetilde{+\mathbf{a}}\,\underset{\sim}{-}}$ is fixed similarly to (A.8) resulting in:

$$H_{\widetilde{+\mathbf{a}}\,\underset{\sim}{-}} = f_{-\,\underset{\sim}{-}\,\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\,\frac{1}{P_-}\,H_{\widetilde{+\mathbf{a}}\,\mathcal{N}} \rightsquigarrow 0 \tag{A.20}$$

The $H_{\alpha'\,\widetilde{\mathbf{a}}}$ is fixed by the dim $\frac{1}{2}$ torsion constraint $T_{P\,D\,\widetilde{P}} \equiv T_{-\,\alpha'\,\widetilde{\mathbf{a}}} = 0$:

$$T_{P\,D\,\widetilde{P}} \equiv T_{-\,\alpha'\,\widetilde{\mathbf{a}}} = 0 = P_{[-}\,H_{\alpha'\,\widetilde{\mathbf{a}})} + H_{[-\,|\,\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\,f_{\alpha'\,\widetilde{\mathbf{a}})\mathcal{N}} \tag{A.21}$$

$$= P_-\,H_{\alpha'\,\widetilde{\mathbf{a}}} + P_{\widetilde{\mathbf{a}}}\,H_{-\,\alpha'} + H_{\alpha'\,\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\,f_{\widetilde{\mathbf{a}}\,-\,\mathcal{N}} \tag{A.22}$$

The last vielbein we need to fix is the $H_{-\,\alpha'}$, that is again fixed by the dim $\frac{1}{2}$ torsion constraint $T_{\widetilde{P}\,P\,D} \equiv T_{\underset{\sim}{-}\,-\,\alpha'} = 0$:

$$T_{\widetilde{P}\,P\,D} \equiv T_{\underset{\sim}{-}\,-\,\alpha'} = 0 = P_{[\underset{\sim}{-}}\,H_{-\,\alpha')} + H_{[\underset{\sim}{-}\,|\,\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\,f_{-\,\alpha')\mathcal{N}} \tag{A.23}$$

$$= P_{\underset{\sim}{-}}\,H_{-\,\alpha'} + H_{\alpha'\,\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\,f_{\underset{\sim}{-}\,-\,\mathcal{N}} \tag{A.24}$$

$$\Rightarrow \quad H_{-\,\alpha'} = f_{-\,\underset{\sim}{-}\,\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\,\frac{1}{P_{\underset{\sim}{-}}}\,H_{\alpha'\,\mathcal{N}} \rightsquigarrow 0 \tag{A.25}$$

Plugging (A.25) into (A.22) and that into (A.19) we finally get an alternative fixing for the $H_{\alpha'\,\widetilde{+a}}$:

$$H_{\alpha'\,\widetilde{+\mathbf{a}}} = -f_{-\,\underset{\sim}{-}\,\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\left(\frac{1}{P_{\underset{\sim}{-}}\,P_-}\,P_{\widetilde{a}}\,\frac{1}{P_{\underset{\sim}{-}}}\,H_{\alpha'\,\mathcal{N}} + \frac{1}{P_{\underset{\sim}{-}}}\,D_{\alpha'}\,\frac{1}{P_-}\,H_{\widetilde{+\mathbf{a}}\,\mathcal{N}}\right) - f_{\alpha'\,\underset{\sim}{-}\,\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\,\frac{1}{P_{\underset{\sim}{-}}}\,H_{\widetilde{+\mathbf{a}}\,\mathcal{N}}$$

$$- f_{-\,\widetilde{\mathbf{a}}\,\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\,\frac{1}{P_{\underset{\sim}{-}}\,P_-}\,H_{\alpha'\,\mathcal{N}} \tag{A.26}$$

$$S_{-\mathbf{b}}\, H_{\alpha'\,-\mathbf{a}} = -\,(\gamma_{-\mathbf{b}})_{\alpha'}{}^{\nu}\, f_{\widetilde{-}\,\nu\,\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\,\frac{1}{P_{\widetilde{-}}}\, H_{-\mathbf{a}\mathcal{N}} \;-\; f_{\widetilde{-}\,\alpha'\,\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\, S_{-\mathbf{b}}\,\frac{1}{P_{\widetilde{-}}}\, H_{-\mathbf{a}\mathcal{N}}$$

$$\Rightarrow S_{-\mathbf{b}}\, H_{\alpha'\,-\mathbf{a}} \rightsquigarrow 0$$

$$S_{\widetilde{-\mathbf{b}}}\, H_{\alpha'\,-\mathbf{a}} = -\, f_{\widetilde{-}\,\alpha'\,\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\, S_{\widetilde{-\mathbf{b}}}\,\frac{1}{P_{\widetilde{-}}}\, H_{-\mathbf{a}\mathcal{N}}$$

$$\Rightarrow S_{\widetilde{-\mathbf{b}}}\, H_{\alpha'\,-\mathbf{a}} \rightsquigarrow 0$$

$$S_{-\mathbf{b}}\, H_{\alpha'\,+\mathbf{a}} = -\,(\gamma_{-\mathbf{b}})_{\alpha'}{}^{\nu}\, f_{\widetilde{-}\,\nu\,\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\,\frac{1}{P_{\widetilde{-}}}\, H_{+\mathbf{a}\mathcal{N}} \;-\; f_{\widetilde{-}\,\alpha'\,\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\, S_{-\mathbf{b}}\,\frac{1}{P_{\widetilde{-}}}\, H_{+\mathbf{a}\mathcal{N}}$$

$$\Rightarrow S_{-\mathbf{b}}\, H_{\alpha'\,+\mathbf{a}} \rightsquigarrow 0$$

$$S_{\widetilde{-\mathbf{b}}}\, H_{\alpha'\,+\mathbf{a}} = -\, f_{\widetilde{-}\,\alpha'\,\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\, S_{\widetilde{-\mathbf{b}}}\,\frac{1}{P_{\widetilde{-}}}\, H_{+\mathbf{a}\mathcal{N}}$$

$$\Rightarrow S_{\widetilde{-\mathbf{b}}}\, H_{\alpha'\,+\mathbf{a}} \rightsquigarrow 0$$

$$S_{-\mathbf{b}}\, H_{\alpha'\,\widetilde{-\mathbf{a}}} = -\, f_{-\,\widetilde{-}\,\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\, S_{-\mathbf{b}}\,\frac{1}{P_{\widetilde{-}}}\, D_{\alpha'}\,\frac{1}{P_{-}}\, H_{\widetilde{-\mathbf{a}}\mathcal{N}} \;-\; (\gamma_{-\mathbf{b}})_{\alpha'}{}^{\nu}\, f_{\widetilde{-}\,\nu\,\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\,\frac{1}{P_{\widetilde{-}}}\, H_{\widetilde{-\mathbf{a}}\mathcal{N}}$$

$$-\, f_{\widetilde{-}\,\alpha'\,\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\, S_{-\mathbf{b}}\,\frac{1}{P_{\widetilde{-}}}\, H_{\widetilde{-\mathbf{a}}\mathcal{N}}$$

$$\Rightarrow S_{-\mathbf{b}}\, H_{\alpha'\,\widetilde{-\mathbf{a}}} \rightsquigarrow 0$$

$$S_{\widetilde{-\mathbf{b}}}\, H_{\alpha'\,\widetilde{-\mathbf{a}}} = -\, f_{-\,\widetilde{-}\,\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\, S_{\widetilde{-\mathbf{b}}}\,\frac{1}{P_{\widetilde{-}}}\, D_{\alpha'}\,\frac{1}{P_{-}}\, H_{\widetilde{-\mathbf{a}}\mathcal{N}} \;-\; f_{\widetilde{-}\,\alpha'\,\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\, S_{\widetilde{-\mathbf{b}}}\,\frac{1}{P_{\widetilde{-}}}\, H_{\widetilde{-\mathbf{a}}\mathcal{N}}$$

$$\Rightarrow S_{\widetilde{-\mathbf{b}}}\, H_{\alpha'\,\widetilde{-\mathbf{a}}} \rightsquigarrow 0$$

Table 8. The S action on $H_{D\,\widetilde{S}}$ vielbein.

Now we are ready to calculate an action of $S_{\widetilde{-\mathbf{a}}}$ and $S_{-\mathbf{a}}$ and $S_{\widetilde{+\mathbf{a}}}$ and $S_{+\mathbf{a}}$ on table 6 vielbeins (with the exception of $S_{+\mathbf{a}}\, H_{\widetilde{+\mathbf{b}}\,\alpha'}$ that we want to calculate in the end of this paragraph). We summarise those S (and $\widetilde{S}$) actions in the next tables: Now we calculate $S_{-\mathbf{b}}\, H_{\alpha'\,\widetilde{+\mathbf{a}}}$. The reasoning will be similar later for the final calculation of the $S_{+\mathbf{b}}\, H_{\alpha'\,\widetilde{+\mathbf{a}}}$ so we first do the former in order to see how it works. Calculation of the $S_{-\mathbf{b}}$ action on $H_{\alpha'\,\widetilde{+\mathbf{a}}}$ is straightforward. It's done using the relation (A.26) and applying $S_{-\mathbf{b}}$, thus we get:

$$S_{-\mathbf{b}}H_{\alpha'\widetilde{+\mathbf{a}}} = -f_{-\widetilde{-}\mathcal{M}}\eta^{\mathcal{M}\mathcal{N}}S_{-\mathbf{b}}\left(\frac{1}{P_{\widetilde{-}}P_{-}}P_{\widetilde{a}}\frac{1}{P_{\widetilde{-}}}H_{\alpha'\mathcal{N}} + \frac{1}{P_{\widetilde{-}}}D_{\alpha'}\frac{1}{P_{-}}H_{\widetilde{+\mathbf{a}}\mathcal{N}}\right) \tag{A.27}$$

$$-\frac{1}{2}\,(\gamma_{-\mathbf{b}})_{\alpha'}{}^{\nu}\, f_{\nu\,\widetilde{-}\,\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\,\frac{1}{P_{\widetilde{-}}}\, H_{\widetilde{+\mathbf{a}}\mathcal{N}} \;+\; f_{\alpha'\,\widetilde{-}\,\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\, S_{-\mathbf{b}}\,\frac{1}{P_{\widetilde{-}}}\, H_{\widetilde{+\mathbf{a}}\mathcal{N}}$$

$$-\, f_{-\,\widetilde{\mathbf{a}}\,\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\, S_{-\mathbf{b}}\,\frac{1}{P_{\widetilde{-}}\,P_{-}}\, H_{\alpha'\,\mathcal{N}}$$

Now, we want to evaluate equation (A.27). The terms proportional to $f_{-\,\widetilde{-}\,\mathcal{M}}$ and $f_{\alpha'\,\widetilde{-}\,\mathcal{M}}$ are vanishing by the AdS algebra. We write what's left over after evaluation:

$$S_{-\mathbf{b}}\, H_{\alpha'\,\widetilde{+\mathbf{a}}} = \frac{1}{2}\,(\gamma_{-\mathbf{b}})_{\alpha'}{}^{\nu}\,(\gamma_{-})_{\nu\,\sigma}\,(\widetilde{\Gamma}_5)^{\sigma\,\epsilon}\,\frac{1}{(r_{\text{AdS}})}\,\frac{1}{P_{\widetilde{-}}}\, H_{\widetilde{+\mathbf{a}}\,\widetilde{\epsilon}} \tag{A.28}$$

$$+\, \frac{1}{2\,(r_{\text{AdS}})^2}\,\frac{1}{P_{\widetilde{-}}\,P_{-}}\, S_{-\mathbf{b}}\,(\, H_{\alpha'\,+\mathbf{a}} \;+\; H_{\alpha'\,\widetilde{+\mathbf{a}}} \;\pm\; H_{\alpha'\,-\mathbf{a}} \;\pm\; H_{\alpha'\,\widetilde{-\mathbf{a}}}\,)$$

Note, the $\pm$ in last line in (A.28) is explained in the section above (6.7). According to the table 8 all actions of $S_{-\mathbf{b}}$ in the second line of (A.28) are evaluated to zero except of the $S_{-\mathbf{b}}\, H_{\alpha'\,+\mathbf{a}}$ that we want to determine. Then the (A.28) could be rewritten in a way that

$$S_{+\mathbf{b}}\, H_{\alpha'-\mathbf{a}} = -\tfrac{1}{2}(\gamma_{+\mathbf{b}})_{\alpha'}{}^{\nu} f_{\simeq\nu\,\mathcal{M}}\,\eta^{\mathcal{MN}}\tfrac{1}{P_{\simeq}} H_{-\mathbf{a}\mathcal{N}} + f_{\simeq\alpha'\mathcal{M}}\,\eta^{\mathcal{MN}} S_{+\mathbf{b}}\tfrac{1}{P_{\simeq}} H_{-\mathbf{a}\mathcal{N}}$$
$$\Rightarrow S_{+\mathbf{b}}\, H_{\alpha'-\mathbf{a}} \rightsquigarrow 0$$

$$S_{\widetilde{+\mathbf{b}}}\, H_{\alpha'-\mathbf{a}} = \eta_{\simeq\mp} f_{\widetilde{\mathbf{b}}\alpha'\mathcal{M}}\,\eta^{\mathcal{MN}}\tfrac{1}{P_{\simeq}} H_{-\mathbf{a}\mathcal{N}} + f_{\simeq\alpha'\mathcal{M}}\,\eta^{\mathcal{MN}} S_{\widetilde{+\mathbf{b}}}\tfrac{1}{P_{\simeq}} H_{-\mathbf{a}\mathcal{N}}$$
$$\Rightarrow S_{\widetilde{+\mathbf{b}}}\, H_{\alpha'-\mathbf{a}} \rightsquigarrow 0$$

$$S_{+\mathbf{b}}\, H_{\alpha'+\mathbf{a}} = -\tfrac{1}{2}(\gamma_{+\mathbf{b}})_{\alpha'}{}^{\nu} f_{\simeq\nu\,\mathcal{M}}\,\eta^{\mathcal{MN}}\tfrac{1}{P_{\simeq}} H_{+\mathbf{a}\mathcal{N}} + f_{\simeq\alpha'\mathcal{M}}\,\eta^{\mathcal{MN}} S_{+\mathbf{b}}\tfrac{1}{P_{\simeq}} H_{+\mathbf{a}\mathcal{N}}$$
$$\Rightarrow S_{+\mathbf{b}}\, H_{\alpha'+\mathbf{a}} \rightsquigarrow -(\gamma_{+\mathbf{b}})_{\alpha'}{}^{\nu}(\gamma_{-})_{\nu\sigma}(\widetilde{\Gamma}_5)^{\sigma\lambda}\tfrac{1}{(r_{\mathrm{AdS}})}\tfrac{1}{P_{\simeq}} H_{+\mathbf{a}\widetilde{\lambda}}$$

$$S_{\widetilde{+\mathbf{b}}}\, H_{\alpha'+\mathbf{a}} = \eta_{\simeq\mp} f_{\widetilde{\mathbf{b}}\alpha'\mathcal{M}}\,\eta^{\mathcal{MN}}\tfrac{1}{P_{\simeq}} H_{+\mathbf{a}\mathcal{N}} + f_{\simeq\alpha'\mathcal{M}}\,\eta^{\mathcal{MN}} S_{\widetilde{+\mathbf{b}}}\tfrac{1}{P_{\simeq}} H_{+\mathbf{a}\mathcal{N}}$$
$$\Rightarrow S_{\widetilde{+\mathbf{b}}}\, H_{\alpha'+\mathbf{a}} \rightsquigarrow (\gamma_{\mathbf{b}})_{\alpha'}{}^{\nu}(\widetilde{\Gamma}_5)^{\nu\sigma}\tfrac{1}{(r_{\mathrm{AdS}})}\tfrac{1}{P_{\simeq}} H_{+\mathbf{a}\widetilde{\sigma}}$$

Table 9. The S action on $H_{D\,\widetilde{S}}$ vielbein.

determines $S_{-\mathbf{b}}\, H_{\alpha'\,\widetilde{+\mathbf{a}}}$ (after evaluation):

$$\left(1 - \frac{1}{2\,(r_{\mathrm{AdS}})^2}\frac{1}{P_{\simeq}\,P_{-}}\right) S_{-\mathbf{b}}\, H_{\alpha'\,\widetilde{+\mathbf{a}}} = \frac{1}{2}(\gamma_{-\mathbf{b}})_{\alpha'}{}^{\nu}(\gamma_{-})_{\nu\sigma}(\widetilde{\Gamma}_5)^{\sigma\epsilon}\frac{1}{(r_{\mathrm{AdS}})}\frac{1}{P_{\simeq}} H_{\widetilde{+\mathbf{a}}\,\widetilde{\epsilon}} \quad (A.29)$$

We remind that the $H_{\widetilde{+\mathbf{a}}\,\widetilde{\epsilon}}$ vielbein is related to the $H_{\widetilde{+\mathbf{a}}\,\epsilon}$ vielbein by the second top line in the table 7. Similarly to the (A.27) and its evaluated version (A.29) we can calculate an action of $S_{\widetilde{-\mathbf{b}}}\, H_{\alpha'\,\widetilde{+\mathbf{a}}}$. The result is:

$$S_{\widetilde{-\mathbf{b}}}H_{\alpha'\widetilde{+\mathbf{a}}} = -f_{-\simeq\mathcal{M}}\eta^{\mathcal{MN}}S_{\widetilde{-\mathbf{b}}}\left(\frac{1}{P_{\simeq}P_{-}}P_{\widetilde{a}}\frac{1}{P_{\simeq}}H_{\alpha'\mathcal{N}} + \frac{1}{P_{\simeq}}D_{\alpha'}\frac{1}{P_{-}}H_{\widetilde{+\mathbf{a}}\mathcal{N}}\right) \quad (A.30)$$

$$-f_{\alpha'\simeq\mathcal{M}}\,\eta^{\mathcal{MN}} S_{\widetilde{-\mathbf{b}}}\frac{1}{P_{\simeq}} H_{\widetilde{+\mathbf{a}}\mathcal{N}} - \eta_{\widetilde{\mathbf{a}}\widetilde{\mathbf{b}}} f_{-\simeq\mathcal{M}}\,\eta^{\mathcal{MN}}\frac{1}{P_{\simeq}\,P_{-}} H_{\alpha'\mathcal{N}}$$

$$-f_{-\widetilde{\mathbf{a}}\mathcal{M}}\,\eta^{\mathcal{MN}} S_{\widetilde{-\mathbf{b}}}\frac{1}{P_{\simeq}\,P_{-}} H_{\alpha'\mathcal{N}} \quad (A.31)$$

and after evaluation, where we again use the results from table 8:

$$S_{\widetilde{-\mathbf{b}}}\, H_{\alpha'\,\widetilde{+\mathbf{a}}} \rightsquigarrow 0 \quad (A.32)$$

Finally the action of $S_{+\mathbf{b}}\, H_{\alpha'\,\widetilde{+\mathbf{a}}}$ is calculated as in (A.27). Now we know that by an analogy with the (A.27) and its evaluation we would need to know analogy of the table 8 except now for the $S_{+\mathbf{b}}$. Since calculations are very analogous to those that led to the table 8 we list just the resulting table(s): 9 and 10.

Now, we calculate the the missing piece in the equation (6.12), i.e. $S_{+\mathbf{b}}\, H_{\alpha'\,\widetilde{+\mathbf{a}}}$. In analogy with (A.27) we get:

$$S_{+\mathbf{b}}H_{\alpha'\widetilde{+\mathbf{a}}} = -f_{-\simeq\mathcal{M}}\eta^{\mathcal{MN}}S_{+\mathbf{b}}\left(\frac{1}{P_{\simeq}P_{-}}P_{\widetilde{a}}\frac{1}{P_{\simeq}}H_{\alpha'\mathcal{N}} + \frac{1}{P_{\simeq}}D_{\alpha'}\frac{1}{P_{-}}H_{\widetilde{+\mathbf{a}}\mathcal{N}}\right) \quad (A.33)$$

$$-\eta_{-+} f_{\mathbf{b}\simeq\mathcal{M}}\,\eta^{\mathcal{MN}}\left(\frac{1}{P_{\simeq}\,P_{-}}P_{\widetilde{a}}\frac{1}{P_{\simeq}}H_{\alpha'\mathcal{N}} + \frac{1}{P_{\simeq}}D_{\alpha'}\frac{1}{P_{-}}H_{\widetilde{+\mathbf{a}}\mathcal{N}}\right)$$

$$
\begin{array}{|c|}
\hline
S_{+\mathbf{b}}\, H_{\alpha'\,\widetilde{-}\mathbf{a}} \;=\; -\tfrac{1}{2}\,(\gamma_{+\mathbf{b}})_{\alpha'}{}^{\nu}\, f_{\widetilde{-}\,\nu\,\mathcal{M}}\,\eta^{\mathcal{MN}}\,\tfrac{1}{P_{\widetilde{-}}}\,H_{\widetilde{-}\mathbf{a}\mathcal{N}} \;+\; f_{\widetilde{-}\,\alpha'\,\mathcal{M}}\,\eta^{\mathcal{MN}}\, S_{+\mathbf{b}}\,\tfrac{1}{P_{\widetilde{-}}}\,H_{\widetilde{-}\mathbf{a}\mathcal{N}} \\[4pt]
+\,\eta_{-+}\, f_{\mathbf{b}\,\widetilde{-}\,\mathcal{M}}\,\eta^{\mathcal{MN}}\,\tfrac{1}{P_{\widetilde{-}}}\,D_{\alpha'}\,\tfrac{1}{P_{-}}\,H_{\widetilde{-}\mathbf{a}\mathcal{N}} \;+\; f_{-\,\widetilde{-}\,\mathcal{M}}\,\eta^{\mathcal{MN}}\, S_{+\mathbf{b}}\,\tfrac{1}{P_{\widetilde{-}}}\,D_{\alpha'}\,\tfrac{1}{P_{-}}\,H_{\widetilde{-}\mathbf{a}\mathcal{N}} \\[4pt]
\Rightarrow\; S_{+\mathbf{b}}\, H_{\alpha'\,\widetilde{-}\mathbf{a}} \;\rightsquigarrow\; 0 \\
\hline
S_{\widetilde{+}\mathbf{b}}\, H_{\alpha'\,\widetilde{-}\mathbf{a}} \;=\; \eta_{\widetilde{-}\,\widetilde{+}}\, f_{-\,\widetilde{\mathbf{b}}\,\mathcal{M}}\,\eta^{\mathcal{MN}}\,\tfrac{1}{P_{\widetilde{-}}}\,D_{\alpha'}\,\tfrac{1}{P_{-}}\,H_{\widetilde{-}\mathbf{a}\mathcal{N}} \;+\; f_{-\,\widetilde{-}\,\mathcal{M}}\,\eta^{\mathcal{MN}}\, S_{\widetilde{+}\mathbf{b}}\,\tfrac{1}{P_{\widetilde{-}}}\,D_{\alpha'}\,\tfrac{1}{P_{-}}\,H_{\widetilde{-}\mathbf{a}\mathcal{N}} \\[4pt]
+\,\eta_{\widetilde{-}\,\widetilde{+}}\, f_{\widetilde{\mathbf{b}}\,\alpha'\,\mathcal{M}}\,\eta^{\mathcal{MN}}\,\tfrac{1}{P_{\widetilde{-}}}\,H_{\widetilde{-}\mathbf{a}\mathcal{N}} \;+\; f_{\widetilde{-}\,\alpha'\,\mathcal{M}}\,\eta^{\mathcal{MN}}\, S_{\widetilde{+}\mathbf{b}}\,\tfrac{1}{P_{\widetilde{-}}}\,H_{\widetilde{-}\mathbf{a}\mathcal{N}} \\[4pt]
\Rightarrow\; S_{\widetilde{+}\mathbf{b}}\, H_{\alpha'\,\widetilde{-}\mathbf{a}} \;\rightsquigarrow\; 0 \\
\hline
\end{array}
$$

Table 10. The S action on $H_{D\,\widetilde{S}}$ vielbein.

$$
+\tfrac{1}{2}\,(\gamma_{+\mathbf{b}})_{\alpha'}{}^{\nu}\, f_{\nu\,\widetilde{-}\,\mathcal{M}}\,\eta^{\mathcal{MN}}\,\frac{1}{P_{\widetilde{-}}}\,H_{\widetilde{+}\mathbf{a}\mathcal{N}} \;-\; f_{\alpha'\,\widetilde{-}\,\mathcal{M}}\,\eta^{\mathcal{MN}}\, S_{+\mathbf{b}}\,\frac{1}{P_{\widetilde{-}}}\,H_{\widetilde{+}\mathbf{a}\mathcal{N}}
$$

$$
-\, f_{-\,\widetilde{\mathbf{a}}\,\mathcal{M}}\,\eta^{\mathcal{MN}}\, S_{+\mathbf{b}}\,\frac{1}{P_{\widetilde{-}}\,P_{-}}\,H_{\alpha'\mathcal{N}} \;+\; \eta_{-+}\, f_{\widetilde{\mathbf{a}}\,\mathbf{b}\,\mathcal{M}}\,\eta^{\mathcal{MN}}\,\frac{1}{P_{\widetilde{-}}\,P_{-}}\,H_{\alpha'\,\mathcal{M}}
$$

and the (partially) evaluate version of (A.33):

$$
S_{+\mathbf{b}}\, H_{\alpha'\,\widetilde{+}\mathbf{a}} \;\rightsquigarrow\; -\frac{1}{2\,(r_{\mathrm{AdS}})^2}\,\frac{1}{P_{\widetilde{-}}}\, D_{\alpha'}\,\frac{1}{P_{-}}\,H_{\widetilde{+}\mathbf{a}+\mathbf{b}} \;-\; (\gamma_{+\mathbf{b}})_{\alpha'}{}^{\nu}\,(\gamma_{-})_{\nu\,\sigma}\,(\widetilde{\varGamma}_5)^{\sigma\,\lambda}\,\frac{1}{(r_{\mathrm{AdS}})}\,\frac{1}{P_{\widetilde{-}}}\,H_{\widetilde{+}\mathbf{a}\,\widetilde{\lambda}}
$$
$$
+\,\frac{1}{2\,(r_{\mathrm{AdS}})^2}\, S_{+\mathbf{b}}\,\frac{1}{P_{\widetilde{-}}\,P_{-}}\,\Big(H_{\alpha'\,+\mathbf{a}} \,+\, H_{\alpha'\,\widetilde{+}\mathbf{a}} \,\pm\, H_{\alpha'\,-\mathbf{a}} \,\pm\, H_{\alpha'\,\widetilde{-}\mathbf{a}} \Big) \qquad (\text{A.34})
$$

We can see why we just partially evaluated the equation (A.33). The reason is that last term leads to an action of $S_{+\mathbf{b}}$. Fortunately for us we already computed all those actions in tables 9 and 10 except $S_{+\mathbf{b}}\, H_{\alpha'\,\widetilde{+}\mathbf{a}}$ that we want to calculate. Therefore the (A.34) leads to the evaluated version of the $S_{+\mathbf{b}}\, H_{\alpha'\,\widetilde{+}\mathbf{a}}$:

$$
\left(1 - \frac{1}{2(r_{\mathrm{AdS}})^2}\,\frac{1}{P_{\widetilde{-}}\,P_{-}}\right) S_{+\mathbf{b}}\, H_{\alpha'\,\widetilde{+}\mathbf{a}} \;\rightsquigarrow\; -\frac{1}{2\,(r_{\mathrm{AdS}})^2}\,\frac{1}{P_{\widetilde{-}}}\, D_{\alpha'}\,\frac{1}{P_{-}}\,H_{\widetilde{+}\mathbf{a}+\mathbf{b}} \qquad (\text{A.35})
$$
$$
-\,\frac{1}{2}\,(\gamma_{+\mathbf{b}})_{\alpha'}{}^{\nu}\,(\gamma_{-})_{\nu\,\sigma}\,(\widetilde{\varGamma}_5)^{\sigma\,\lambda}\,\frac{1}{(r_{\mathrm{AdS}})}\,\frac{1}{P_{\widetilde{-}}}\,H_{\widetilde{+}\mathbf{a}\,\widetilde{\lambda}}
$$
$$
-(\gamma_{+\mathbf{b}})_{\alpha'}{}^{\nu}(\gamma_{-})_{\nu\sigma}(\widetilde{\varGamma}_5)^{\sigma\lambda}\,\frac{1}{2(r_{\mathrm{AdS}})^3}\,\frac{1}{P_{\widetilde{-}}\,P_{-}\,P_{\widetilde{-}}}\,H_{+\mathbf{a}\,\widetilde{\lambda}}
$$

where we used results of tables 9 and 10. For completeness we provide (just the evaluated version) the last remaining part of tables 9 and 10, i.e. $S_{\widetilde{+}\mathbf{b}}\, H_{\alpha'\,\widetilde{+}\mathbf{a}}$:

$$
\left(1 - \frac{1}{2\,(r_{\mathrm{AdS}})^2}\,\frac{1}{P_{\widetilde{-}}\,P_{-}}\right) S_{\widetilde{+}\mathbf{b}}\, H_{\alpha'\,\widetilde{+}\mathbf{a}} \;\rightsquigarrow\; -\frac{1}{2\,(r_{\mathrm{AdS}})^2}\,\frac{1}{P_{\widetilde{-}}}\, D_{\alpha'}\,\frac{1}{P_{-}}\,H_{\widetilde{+}\mathbf{a}+\mathbf{b}} \qquad (\text{A.36})
$$
$$
-\,\frac{1}{2}\,(\gamma_{\mathbf{b}})_{\alpha'\,\nu}\,(\widetilde{\varGamma}_5)^{\nu\,\lambda}\,\frac{1}{(r_{\mathrm{AdS}})}\,\frac{1}{P_{\widetilde{-}}}\,H_{\widetilde{+}\mathbf{a}\,\widetilde{\lambda}}
$$
$$
-\,(\gamma_{\mathbf{b}})_{\alpha'\,\nu}\,(\widetilde{\varGamma}_5)^{\nu\,\lambda}\,\frac{1}{2\,(r_{\mathrm{AdS}})^3}\,\frac{1}{P_{\widetilde{-}}\,P_{-}\,P_{\widetilde{-}}}\,H_{+\mathbf{a}\,\widetilde{\lambda}}
$$

We repeat the first important relation we derived by the above analysis from (6.11) where we add results from (A.13) and (A.35):

$$D_{\alpha'}H_{+\mathbf{a}\widetilde{+\mathbf{b}}} = -\frac{1}{g}\frac{1}{(r_{\text{AdS}})P_{\widetilde{-}}}(\gamma_{\mathbf{b}})_{\alpha'\sigma}(\widetilde{\Gamma}_5)^{\sigma\beta}H_{\widetilde{\beta}+\mathbf{a}} + \frac{1}{2g}\left(1 - \frac{1}{f}\frac{1}{(r_{\text{AdS}})^2 P_{\widetilde{-}}P_-}\right)(\gamma_{+\mathbf{a}})_{\alpha'}{}^{\beta}H_{\beta\widetilde{+\mathbf{b}}}$$
$$+ \frac{1}{f\,g}\frac{1}{2\,(r_{\text{AdS}})^3\,P_-\,(P_{\widetilde{-}})^2}\,(\gamma_{\mathbf{a}})_{\alpha'\nu}\,(\widetilde{\Gamma}_5)^{\nu\,\beta}\,H_{\widetilde{\beta}+\mathbf{b}} \tag{A.37}$$

where f and g are defined as follows:

$$f := \left(1 - \frac{1}{2\,(r_{\text{AdS}})^2\,P_{\widetilde{-}}\,P_-}\right) \tag{A.38}$$
$$g := \left(1 - \frac{1}{f}\frac{1}{2\,(r_{\text{AdS}})^2\,P_{\widetilde{-}}\,P_-}\right)$$

A.3 The $H_{D\widetilde{D}}$

To obtain the AdS equation (7.2), we need to fix $H_{\alpha'\beta}$. This term is fixed by the zero dimensional torsion $T_{\widetilde{P}DD} \equiv T_{\widetilde{-}\,\alpha'\,\beta} = 0$:

$$T_{\widetilde{-}\,\alpha'\beta} = 0 = P_{[\widetilde{-}}H_{\alpha'\beta)} + H_{[\widetilde{-}|\mathcal{M}}\eta^{\mathcal{MN}}f_{\alpha'\beta)\mathcal{N}} \tag{A.39}$$
$$= P_{\widetilde{-}}H_{\alpha'\beta} + H_{(\beta|\mathcal{M}}\eta^{\mathcal{MN}}f_{\alpha')\widetilde{-}\mathcal{N}}$$
$$\Rightarrow \quad H_{\alpha'\beta} = -f_{\widetilde{-}(\alpha'|\mathcal{M}}\eta^{\mathcal{MN}}\frac{1}{P_{\widetilde{-}}}H_{\beta)\mathcal{N}} \rightsquigarrow 0 \tag{A.40}$$

In (A.39) we again used the mixed light-cone gauge. In the flat case the mixed f terms are zero so is $H_{\alpha'\beta}$. In the AdS case (after evaluation), term proportional to $f_{\alpha'\widetilde{-}\mathcal{N}}$ is zero because of $(\gamma_-)_{\alpha'\beta'} = 0$. But the term proportional to $f_{\beta\widetilde{-}\mathcal{N}}$ is nonzero. Luckily for us the $f_{\beta\widetilde{-}\mathcal{N}} \propto \frac{1}{r_{\text{AdS}}}(\gamma_-)_{\beta\sigma}(\widetilde{\Gamma}_5)^{\sigma\nu}\eta_{\widetilde{\nu}\mathcal{N}}$. That structure constant just eats up the β index and returns $\widetilde{\nu}$ index with some fixed constant dependence. The torsion constraint $T_{P\widetilde{D}D} \equiv T_{-\widetilde{\sigma}\alpha'} = 0$ relates $H_{\widetilde{\sigma}\alpha'}$ back to $H_{\sigma\alpha'}$ (after the evaluation). From that and assuming some wider invertibility (P_- and $P_{\widetilde{-}}$ are bigger than some constant lower bound in AdS) we get also in the AdS space $H_{\alpha'\beta} \rightsquigarrow 0$ (after the evaluation).

We apply $S_{\widetilde{+\mathbf{a}}}$ on the result of non-evaluated (A.40), thus get:

$$S_{\widetilde{+\mathbf{a}}}H_{\alpha'\beta} = -\eta_{\widetilde{-}\widetilde{+}}f_{\widetilde{\mathbf{a}}(\alpha'|\mathcal{M}}\eta^{\mathcal{MN}}\frac{1}{P_{\widetilde{-}}}H_{\beta)\mathcal{N}} - f_{\widetilde{-}(\alpha'|\mathcal{M}}\eta^{\mathcal{MN}}S_{\widetilde{+\mathbf{a}}}\frac{1}{P_{\widetilde{-}}}H_{\beta)\mathcal{N}} \tag{A.41}$$

$$S_{\widetilde{+\mathbf{a}}}H_{\alpha'\beta} \rightsquigarrow -(\gamma_-)_{\beta\nu}(\widetilde{\Gamma}_5)^{\nu\sigma}\frac{1}{r_{\text{AdS}}}S_{\widetilde{+\mathbf{a}}}\frac{1}{P_{\widetilde{-}}}H_{\alpha'\widetilde{\sigma}} + (\gamma_{\mathbf{a}})_{\alpha'\nu}(\widetilde{\Gamma}_5)^{\nu\sigma}\frac{1}{r_{\text{AdS}}}\frac{1}{P_{\widetilde{-}}}H_{\beta\widetilde{\sigma}} \tag{A.42}$$

The last term in (A.42) does not bother us too much (it will be a part of the pre-potential), the first term in (A.42) is actually something we need to evaluate. For that we need to fix $H_{\alpha'\widetilde{\sigma}}$. That could be done by the torsion constraint $T_{P\widetilde{D}D} \equiv T_{-\alpha'\widetilde{\sigma}} = 0$:

$$T_{-\alpha'\widetilde{\sigma}} = 0 = P_{[-}H_{\alpha'\widetilde{\sigma})} + H_{[-|\mathcal{M}}\eta^{\mathcal{MN}}f_{\alpha'\widetilde{\sigma})\mathcal{N}} \tag{A.43}$$
$$= P_-H_{\alpha'\widetilde{\sigma}} - D_{\widetilde{\sigma}}H_{-\alpha'} + H_{-\mathcal{M}}\eta^{\mathcal{MN}}f_{\alpha'\widetilde{\sigma}\mathcal{N}} + H_{\alpha'\mathcal{M}}\eta^{\mathcal{MN}}f_{\widetilde{\sigma}-\mathcal{N}} \tag{A.44}$$

moreover the $H_{-\,\alpha'}$ has been fixed in (A.25), plugging that into (A.44) we get fixing of $H_{\alpha'\,\widetilde\sigma}$:

$$H_{\alpha'\,\widetilde\sigma} = f_{-\,\widetilde{-}\,\mathcal{M}}\,\eta^{\mathcal{MN}}\,\frac{1}{P_-}\,D_{\widetilde\sigma}\,\frac{1}{P_{\widetilde{-}}}\,H_{\alpha'\,\mathcal{N}} \;-\; f_{\alpha'\,\widetilde\sigma\,\mathcal{M}}\,\eta^{\mathcal{MN}}\,\frac{1}{P_-}\,H_{-\,\mathcal{N}} \;+\; f_{\widetilde\sigma\,-\,\mathcal{M}}\,\eta^{\mathcal{MN}}\,\frac{1}{P_-}\,H_{\alpha'\,\mathcal{N}} \tag{A.45}$$

We are ready to calculate $S_{\widetilde{+\mathbf{a}}}\,H_{\alpha'\,\widetilde\sigma}$ i.e. the term needed in (A.42):

$$S_{\widetilde{+\mathbf{a}}}H_{\alpha'\,\widetilde\sigma} = \eta_{\widetilde{-}\,\widetilde{+}}\,f_{-\,\widetilde{\mathbf{a}}\,\mathcal{M}}\,\eta^{\mathcal{MN}}\,\frac{1}{P_-}D_{\widetilde\sigma}\,\frac{1}{P_{\widetilde{-}}}H_{\alpha'\,\mathcal{N}} + f_{-\,\widetilde{-}\,\mathcal{M}}\,\eta^{\mathcal{MN}}\,S_{\widetilde{+\mathbf{a}}}\,\frac{1}{P_-}D_{\widetilde\sigma}\,\frac{1}{P_{\widetilde{-}}}H_{\alpha'\,\mathcal{N}} \tag{A.46}$$

$$+\frac{1}{2}\,(\gamma_{+\mathbf{a}})_\sigma{}^{\nu'}\,f_{\alpha'\,\widetilde{\nu'}\,\mathcal{M}}\,\eta^{\mathcal{MN}}\,\frac{1}{P_-}\,H_{-\,\mathcal{N}} \;-\; f_{\alpha'\,\widetilde\sigma\,\mathcal{M}}\,\eta^{\mathcal{MN}}\,S_{\widetilde{+\mathbf{a}}}\,\frac{1}{P_-}\,H_{-\,\mathcal{N}}$$

$$-\frac{1}{2}\,(\gamma_{+\mathbf{a}})_\sigma{}^{\nu'}\,f_{\widetilde{\nu'}\,-\,\mathcal{M}}\,\eta^{\mathcal{MN}}\,\frac{1}{P_-}\,H_{\alpha'\,\mathcal{N}} \;+\; f_{\widetilde\sigma\,-\,\mathcal{M}}\,\eta^{\mathcal{MN}}\,S_{\widetilde{+\mathbf{a}}}\,\frac{1}{P_-}\,H_{\alpha'\,\mathcal{N}}$$

Now, we can evaluate (A.46), for clearness we include terms that we already know are evaluated to zero or are zero by the mixed light-cone gauge:

$$S_{\widetilde{+\mathbf{a}}}\,H_{\alpha'\,\widetilde\sigma} \rightsquigarrow \frac{1}{2\,(r_{\text{AdS}})^2}\,\frac{1}{P_-}\,D_{\widetilde\sigma}\,\frac{1}{P_{\widetilde{-}}}\,(\,H_{\widetilde{\alpha'}\,+\mathbf{a}} + H_{\widetilde{\alpha'}\,\widetilde{+\mathbf{a}}} \pm H_{\widetilde{\alpha'}\,-\mathbf{a}} \pm H_{\widetilde{\alpha'}\,\widetilde{-\mathbf{a}}}\,) \tag{A.47}$$

$$(\pm)\,\frac{1}{2}\,\frac{1}{r_{\text{AdS}}}\,(\gamma_{+\mathbf{a}})_\sigma{}^{\nu'}\,(\gamma^{\mathbf{cd}})_{\nu'}{}^{\rho'}\,(\widetilde{\Gamma}_5)_{\rho'\,\alpha'}\,\frac{1}{2\,P_-}\,(\,H_{-\,\mathbf{cd}} + H_{-\,\widetilde{\mathbf{cd}}}\,) \tag{A.48}$$

$$-(\pm)\,\frac{1}{r_{\text{AdS}}}\,(\gamma^{\mathbf{cd}})_\sigma{}^{\rho'}\,(\widetilde{\Gamma}_5)_{\rho'\,\alpha'}\,S_{\widetilde{+\mathbf{a}}}\,\frac{1}{2\,P_-}\,(\,H_{-\,\mathbf{cd}} + H_{-\,\widetilde{\mathbf{cd}}}\,) \tag{A.49}$$

$$-\frac{1}{r_{\text{AdS}}}\,(\gamma_-)_{\sigma\,\nu}\,(\widetilde{\Gamma}_5)^{\nu\,\rho}\,S_{\widetilde{+\mathbf{a}}}\,\frac{1}{P_-}\,H_{\alpha'\,\rho} \tag{A.50}$$

Note that in the lines (A.48) and (A.49) we have the $\pm$ symbol. It comes from the mixed structure constant $f_{D\,\widetilde{D}\,\Sigma^{\mathbf{d}}} \equiv f_{\underline{\alpha}\,\underline{\widetilde\beta}}{}^{\mathbf{cd}}$, where underline indices are now (and just now) the SO(10) chiral indices (for the left and right algebra), and $\Sigma^{\mathbf{d}}$ is the Σ index for the SO(5) $\otimes$ SO(5) diagonal subgroup of the original SO(10) $\otimes$ SO(10) group. The $(\pm)$ symbol determines to which SO(5) of the diagonal subgroup given $\Sigma^{\mathbf{d}}$ belongs. This mixed structure constant could be written without the $\pm$ symbols as $f_{\underline{\alpha}\,\underline{\widetilde\beta}}{}^{\mathbf{cd}} = \frac{1}{r_{\text{AdS}}}\,(\gamma^{[\mathbf{c}})_{\sigma\,\rho}\,(\widetilde{\Gamma}_5)^{\rho\,\nu}\,(\gamma^{\mathbf{d}]})_{\nu\,\alpha'}$. The $\pm$ then comes from the fact that by the construction $\widetilde{\Gamma}_5$ commutes with $\mathbf{a} \in \{10, 1 \ldots 4\}$ and anti-commutes with $\mathbf{a} \in \{5, \ldots 9\}$. We used the prior definition in this section for some convenience. In the final expressions we will always use the definition without the $\pm$ symbol.

Let's evaluate expressions (A.47), (A.48), (A.49) and (A.50). The line (A.47) is evaluated to 0 by the table 6. We note very important property in the lines (A.48) and (A.49). The summation over the $\mathbf{cd}$ indices is really just a summation over the SO(5) $\otimes$ SO(5) diagonal subgroup of the full SO(10) $\otimes$ SO(10). The line (A.48) is evaluated to 0 by the mixed light-cone gauge (second term) and by the following fixing of the $H_{-\,\mathbf{cd}}$ (coming from torsion constraint $T_{\widetilde{P}\,P\,S} \equiv T_{\widetilde{-}\,-\,\mathbf{ab}} = 0$):

$$H_{-\,\mathbf{ab}} = f_{-\,\widetilde{-}\,\mathcal{M}}\,\eta^{\mathcal{MN}}\,\frac{1}{P_{\widetilde{-}}}\,H_{\mathbf{ab}\,\mathcal{N}} \;\Rightarrow\; H_{-\,\mathbf{ab}} \rightsquigarrow 0 \tag{A.51}$$

The line (A.50) has an action $S_{\widetilde{+a}} H_{\alpha'\,\rho}$ that is exactly what we want to determine. The line (A.49) is fixed as follows. The vielbein $H_{-\,\widetilde{\mathbf{cd}}} = 0$ by the mixed light-cone gauge. The action $S_{\widetilde{+a}} H_{-\,\mathbf{cd}}$ is however nontrivial. We should take fixing (A.51) and apply $S_{\widetilde{+a}}$:

$$S_{\widetilde{+a}} H_{-\,\mathbf{cd}} = -f_{-\,\widetilde{\mathbf{a}}\,\mathcal{M}}\,\eta^{\mathcal{MN}}\,\frac{1}{P_{\widetilde{-}}}\,H_{\mathbf{cd}\,\mathcal{N}} + f_{-\,\widetilde{-}\,\mathcal{M}}\,\eta^{\mathcal{MN}}\,S_{+\mathbf{a}}\,\frac{1}{P_{\widetilde{-}}}\,H_{\mathbf{cd}\,\mathcal{N}} \tag{A.52}$$

$$S_{\widetilde{+a}} H_{-\,\mathbf{cd}} \rightsquigarrow \frac{1}{2\,(r_{\mathrm{AdS}})^2}\,\frac{1}{P_{\widetilde{-}}}\,(\,H_{\mathbf{cd}\,+\mathbf{a}} + H_{\mathbf{cd}\,\widetilde{+\mathbf{a}}} \pm H_{\mathbf{cd}\,-\mathbf{a}} \pm H_{\mathbf{cd}\,\widetilde{-\mathbf{a}}}\,) \tag{A.53}$$

By the table 5 the only nonzero term in (A.53) is $H_{+\mathbf{d}\,\widetilde{+\mathbf{a}}}$ thus we get:

$$S_{\widetilde{+a}} H_{-\,+\mathbf{d}} \rightsquigarrow \frac{1}{2\,(r_{\mathrm{AdS}})^2}\,\frac{1}{P_{\widetilde{-}}}\,H_{+\mathbf{d}\,\widetilde{+\mathbf{a}}} \tag{A.54}$$

Then finally the equation (A.47) till (A.50) is evaluated to:

$$S_{\widetilde{+a}} H_{\alpha'\,\widetilde{\sigma}} \rightsquigarrow \pm \frac{(-1)}{4\,(r_{\mathrm{AdS}})^3\,P_-\,P_{\widetilde{-}}}\,(\gamma^{+\mathbf{d}})_\sigma{}^{\rho'}\,(\widetilde{\Gamma}_5)_{\rho'\,\alpha'}\,H_{+\mathbf{d}\,\widetilde{+\mathbf{a}}} \tag{A.55}$$
$$-\frac{1}{(r_{\mathrm{AdS}})\,P_-}\,(\gamma_-)_{\sigma\,\nu}\,(\widetilde{\Gamma}_5)^{\nu\,\rho}\,S_{\widetilde{+a}} H_{\alpha'\,\rho}$$

Combining (A.55) and (A.42) we will get the following relation for the evaluated action of $S_{\widetilde{+a}} H_{\alpha'\,\beta}$:

$$\left(1 - \frac{1}{P_-\,P_{\widetilde{-}}\,(r_{\mathrm{AdS}})^2}\right) S_{\widetilde{+a}} H_{\alpha'\,\beta} = -\frac{1}{4(r_{\mathrm{AdS}})^4\,P_-\,P_{\widetilde{-}}{}^2}(\widetilde{\Gamma}_5)_{\beta\nu}(\gamma^{\mathbf{d}})^{\nu\rho'}(\widetilde{\Gamma}_5)_{\rho'\,\alpha'}(\widetilde{\Gamma}_5)_{\mathbf{d}}{}^{\mathbf{g}}H_{+\mathbf{g}\,\widetilde{+\mathbf{a}}}$$
$$+\frac{1}{(r_{\mathrm{AdS}})\,P_{\widetilde{-}}}\,(\gamma_{\mathbf{a}})_{\alpha'\,\nu}\,(\widetilde{\Gamma}_5)^{\nu\,\sigma}\,H_{\beta\,\widetilde{\sigma}} \tag{A.56}$$

where we simplified (A.55) by using the explicit property of γ_- and γ_+ being the unit or zero matrix (depending on specific indices), see (3.14). We used this simplification in another equations as well (for example in equation (6.21)). In the equation (A.56) we also used new matrix $(\widetilde{\Gamma}_5)_{\mathbf{d}}{}^{\mathbf{g}}$, that was be introduced in (7.32). We also used the identity (7.30) to simplify (A.56). Plugging the evaluated expression (A.56) into (7.2) we will get the action of $D_{\alpha'} H_{\beta\,\widetilde{+\mathbf{a}}}$:

$$0 = D_{\alpha'} H_{\beta\,\widetilde{+\mathbf{a}}} - \frac{1}{h\,4\,(r_{\mathrm{AdS}})^4\,P_-\,P_{\widetilde{-}}{}^2}\,(\gamma^{\mathbf{c}})_{\alpha'\,\beta}\,H_{+\mathbf{c}\,\widetilde{+\mathbf{a}}} \tag{A.57}$$
$$+2\,(\gamma^{\mathbf{c}})_{\alpha'\,\beta}\,H_{\widetilde{+\mathbf{a}}\,\mathbf{c}} + \frac{1}{h\,(r_{\mathrm{AdS}})\,P_{\widetilde{-}}}\,(\gamma_{\mathbf{a}})_{\alpha'\,\nu}\,(\widetilde{\Gamma}_5)^{\nu\,\sigma}\,H_{\beta\,\widetilde{\sigma}}$$
$$\text{left} \leftrightarrow \text{right} \tag{A.58}$$

where h is defined as:
$$h := \left(1 - \frac{1}{P_-\,P_{\widetilde{-}}\,(r_{\mathrm{AdS}})^2}\right) \tag{A.59}$$

The equations (A.57) and (A.58) are very interesting since after applying the $D_{\alpha'}$ (or $D_{\widetilde{\alpha}'}$) derivatives we are getting terms like $H_{\widetilde{+\mathbf{a}}\,\mathbf{c}}$ that is basically our original $H_{\widetilde{+\mathbf{a}}\,+\mathbf{c}}$, see (6.6).

Moreover we got also term $H_{\beta\widetilde{\sigma}}$ that is a new term and was important in sections where we constructed the pre-potential.

Another important derivatives are $D_{\widetilde{\alpha}'}$ on $H_{\beta\widetilde{+\mathbf{a}}}$ and $D_{\alpha'}$ on $H_{\widetilde{\beta}+\mathbf{a}}$. We will look at those closer:

$$T_{\widetilde{D}\,D\,\widetilde{S}} \equiv T_{\widetilde{\alpha}'\,\beta\,\widetilde{+\mathbf{a}}} = 0 = D_{[\widetilde{\alpha}'}\,H_{\beta\,\widetilde{+\mathbf{a}})} + H_{[\widetilde{\alpha}'\,|\,\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\,f_{\beta\,\widetilde{+\mathbf{a}})\,\mathcal{N}} \tag{A.60}$$

$$= D_{\widetilde{\alpha}'}\,H_{\beta\,\widetilde{+\mathbf{a}}} + S_{\widetilde{+\mathbf{a}}}\,H_{\widetilde{\alpha}'\,\beta} - D_{\beta}\,H_{\widetilde{+\mathbf{a}}\,\widetilde{\alpha}'} + H_{\widetilde{\alpha}'\,\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\,f_{\beta\,\widetilde{+\mathbf{a}}\,\mathcal{N}}$$
$$+ H_{\widetilde{+\mathbf{a}}\,\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\,f_{\widetilde{\alpha}'\,\beta\,\mathcal{N}} - H_{\beta\,\mathcal{M}}\,\eta^{\mathcal{M}\mathcal{N}}\,f_{\widetilde{+\mathbf{a}}\,\widetilde{\alpha}'\,\mathcal{N}} \tag{A.61}$$

The mixed f terms are zero in the flat superspace. In the AdS case the $f_{\widetilde{\alpha}'\,\beta\,\mathcal{N}} \neq 0$ and there is also AdS contribution coming from $S_{\widetilde{+\mathbf{a}}}\,H_{\widetilde{\alpha}'\,\beta}$. This contribution can be calculated by analogy with the equations (A.47), (A.48), (A.49), (A.50) and (A.56). Thus getting evaluated action $S_{\widetilde{+\mathbf{a}}}\,H_{\widetilde{\alpha}'\,\beta}$:

$$S_{\widetilde{+\mathbf{a}}}\,H_{\widetilde{\alpha}'\,\sigma} = \pm\frac{(-1)}{h\,4\,(r_{\mathrm{AdS}})^3\,P_{\simeq}\,P_{-}}\,(\gamma^{+\mathbf{d}})_{\alpha'}{}^{\nu}\,(\widetilde{\Gamma}_5)_{\nu\,\sigma}\,H_{\widetilde{+\mathbf{d}}+\mathbf{a}} \tag{A.62}$$
$$+ \frac{1}{h\,(r_{\mathrm{AdS}})^2\,P_{\simeq}\,P_{-}}\,(\gamma_{-})_{\sigma\,\nu}\,(\widetilde{\Gamma}_5)^{\nu\,\rho}\,H_{\widetilde{\rho}\beta}\,(\widetilde{\Gamma}_5)^{\beta\,\epsilon}\,(\gamma_{\mathbf{a}})_{\epsilon\,\alpha'}$$

where h was defined in (A.59). The (A.61) mixed structure constant $f_{\beta\,\widetilde{+\mathbf{a}}\,\mathcal{N}} = 0$ and the $f_{\widetilde{\alpha}'\,\beta\,\mathcal{N}}$ has been discussed before (see equations (A.46) and (A.47) till (A.50)). Moreover, the vielbein $H_{\widetilde{+\mathbf{a}}\,\widetilde{\alpha}'}$ is evaluated to 0, see table 6. Evaluating everything in (A.62) we get:

$$0 = D_{\widetilde{\alpha}'}\,H_{\beta\,\widetilde{+\mathbf{a}}} - \frac{1}{h\,4\,(r_{\mathrm{AdS}})^3\,P_{\simeq}\,P_{-}}\,(\gamma^{\mathbf{d}})_{\alpha'\,\epsilon}\,(\widetilde{\Gamma}_5)^{\epsilon\,\sigma}\,\gamma^{+}{}_{\sigma\,\beta}\,H_{\widetilde{+\mathbf{d}}+\mathbf{a}} \tag{A.63}$$
$$+ \frac{1}{r_{\mathrm{AdS}}}\,(\gamma^{\mathbf{d}})_{\alpha'\,\epsilon}\,(\widetilde{\Gamma}_5)^{\epsilon\,\sigma}\,(\gamma^{+})_{\sigma\,\beta}\,H_{+\mathbf{d}\,\widetilde{+\mathbf{a}}}$$
$$+ \frac{1}{h\,(r_{\mathrm{AdS}})^2\,P_{\simeq}\,P_{-}}\,(\gamma_{-})_{\beta\,\nu}\,(\widetilde{\Gamma}_5)^{\nu\,\rho}\,H_{\widetilde{\rho}\lambda}\,(\widetilde{\Gamma}_5)^{\lambda\,\epsilon}\,(\gamma_{\mathbf{a}})_{\epsilon\,\alpha'} - \frac{1}{2}\,(\gamma_{+\mathbf{a}})_{\alpha'}{}^{\nu}\,H_{\beta\,\widetilde{\nu}}$$
$$\text{left} \leftrightarrow \text{right} \tag{A.64}$$

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