Published for SISSA by Springer

Received: November 14, 2016
Accepted: January 1, 2017
Published: January 12, 2017

Next-to-soft corrections to high energy scattering in QCD and gravity

A. Luna,[a] S. Melville,[a] S.G. Naculich[b] and C.D. White[c]

[a] *SUPA, School of Physics and Astronomy, University of Glasgow, Glasgow G12 8QQ, Scotland, U.K.*
[b] *Department of Physics, Bowdoin College, Brunswick, ME 04011, U.S.A.*
[c] *Centre for Research in String Theory, School of Physics and Astronomy, Queen Mary University of London, 327 Mile End Road, London E1 4NS, U.K.*

E-mail: a.luna-godoy.1@research.gla.ac.uk, s.melville.1@research.gla.ac.uk, naculich@bowdoin.edu, Christopher.White@qmul.ac.uk

Abstract: We examine the Regge (high energy) limit of 4-point scattering in both QCD and gravity, using recently developed techniques to systematically compute all corrections up to next-to-leading power in the exchanged momentum i.e. beyond the eikonal approximation. We consider the situation of two scalar particles of arbitrary mass, thus generalising previous calculations in the literature. In QCD, our calculation describes power-suppressed corrections to the Reggeisation of the gluon. In gravity, we confirm a previous conjecture that next-to-soft corrections correspond to two independent deflection angles for the incoming particles. Our calculations in QCD and gravity are consistent with the well-known double copy relating amplitudes in the two theories.

Keywords: Models of Quantum Gravity, Perturbative QCD, Scattering Amplitudes

ArXiv ePrint: 1611.02172

doi:10.1007/JHEP01(2017)052

Contents

1 Introduction

Scattering amplitudes have many theoretical and phenomenological applications in (non-) abelian gauge theories and gravity, whilst also revealing how different theories are related. When studying amplitudes, it can be useful to consider particular kinematic limits of scattering processes, which allow all-order insights into the structure of perturbative quantum field theory. One such limit is the *Regge limit*, in which the centre of mass energy of the scattering far exceeds the momentum transfer. In nonabelian gauge theories, it is known that propagators for exchanged gauge bosons become dressed by a power-like growth in the centre of mass energy, a phenomenon known as *Reggeisation* (see e.g. [1]), leading to compact all-order forms for amplitudes. More recently, the Regge limit has been studied using Wilson lines [2–6], known factorisation properties of soft and collinear gluons [7–10], and effective field theory [11]. Reggeisation has also been examined in (super)-gravity (see e.g. [5] and references therein), where it is found to be kinematically subleading with respect to other contributions at high energy.

There are a number of motivations for studying the Regge limit in different theories. In QCD, the physics of Reggeisation (including non-linear corrections) has potential applications in parton physics (see e.g. [12–15]), multijet processes [16–28], and heavy ion physics [29]. In gravity, the Regge limit can be used to probe scattering at transplanckian energies [30–36], allowing one to address crucial conceptual issues of quantum gravity, such as the impact of non-renormalisability, the existence of a well-defined S-matrix, black hole physics [37–41], and connections to string theory [42–45]. As well as studying each type of field theory individually, there has been much recent interest in relating (non)-abelian gauge and gravity theories, motivated in part by the conjectured *double copy* underlying their respective scattering amplitudes [46–48]. The Regge limit (as well as more general soft limits) can be used to provide all-order insights into this correspondence [5, 49–52], as well as showing how qualitatively different physics in the two types of theory are related. To this end, it is useful to develop languages and techniques for gauge theories and gravity, that make their common traits particularly clear.

An elegant picture for describing the Regge limit of $2 \to 2$ scattering has been provided in refs. [2, 3]. When the momentum transfer is much less than the centre of mass energy, the incoming particles barely glance off each other, and thus follow approximately straight-line (classical) trajectories. They can thus be described by Wilson line operators, which take into account the gauge-covariant phase suffered by each particle as it exchanges soft (low-momentum) gauge bosons with the other. References [2, 3] considered 4-point scattering in QCD, and showed that known properties of the Regge limit (namely the one-loop Regge trajectory, and infrared singular part of the two-loop trajectory) can indeed be obtained from vacuum expectation values of Wilson line operators separated by a transverse distance $|\vec{z}|$, representing the impact parameter. In ref. [5] this setup was generalised to gravity, using appropriate gravitational Wilson line operators, introduced and studied in refs. [53–55] (see also ref. [56]). Existing results regarding the Regge limits of QCD and gravity were rederived in such a way as to make the relationship between them especially clear, and the same method also provided a proof of graviton Reggeisation in $2 \to n$ processes.

The aim of this paper is to extend the results of ref. [5] by systematically including all corrections that are suppressed by a single power of momentum transfer. Given the soft nature of the exchanged gauge bosons in the leading Regge limit (equivalently, the eikonal approximation for the incoming and outgoing particles), such corrections are referred to as *next-to-soft*, or *next-to-eikonal*. There are a number of motivations for doing this. Firstly, there has recently been a large amount of attention to amplitudes dressed by additional real emissions up to next-to-soft level (see e.g. [57–78]), as well as previous work from a more phenomenological point of view [79–86]. The present analysis provides an interesting testing ground for these methods and results. Secondly, corrections to the eikonal approximation in transplanckian scattering may have a role to play in furthering our knowledge of quantum gravity (e.g. regarding issues of black hole production [34, 36]). Thirdly, by calculating such corrections both in QCD and gravity, one may further probe the relationship between these two theories.

Next-to-soft corrections to the Regge limit in gravity have been previously considered in detail for massless particles [32–35], and also for the case of one particle asymptotically massive, and the other massless [87, 88]. Here we will consider a general situation in which both incoming particles have (possibly different) masses. The advantage of the massive situation relative to the completely massless case is that corrections to the eikonal approximation are enhanced, in that they are suppressed by fewer powers of the momentum transfer. The kinematic limits adopted in the previous literature will emerge as special cases.

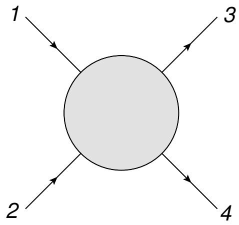

Figure 1. Particle labels used throughout for $2 \to 2$ scattering.

The structure of our paper is as follows. In the following section, we review the analysis of ref. [5] for obtaining the Regge limit from Wilson lines in position space. In section 3, we summarise the structure of next-to-soft corrections, before calculating these in both QCD and gravity. In section 4 we discuss and interpret our results, before concluding in section 5.

2 Eikonal analysis

Throughout, we consider $2 \to 2$ scattering with momenta defined as in figure 1, where we take $m_3 = m_1$, $m_4 = m_2$. One may then define the Mandelstam invariants

$$s = (p_1 + p_2)^2; \qquad t = (p_1 - p_3)^2; \qquad u = (p_1 - p_4)^2, \tag{2.1}$$

satisfying the momentum conservation constraint

$$s + t + u = 2(m_1^2 + m_2^2). \tag{2.2}$$

When nonzero masses are present, there is a choice regarding how to define the Regge limit. Following refs. [2, 3], we consider the ordering

$$s \gg m_i^2 \gg -t. \tag{2.3}$$

When the centre of mass energy dominates the momentum transfer, particles (1,2) and (3,4) become spacelike collinear to a first approximation. As discussed in the introduction and in detail in refs. [2, 3, 5], one may then represent the incoming and outgoing particles as two Wilson line operators separated by a transverse vector $\vec{z}$, where the latter constitutes the impact factor. This setup is depicted in figure 2, and results in the QCD amplitude

$$\mathcal{A} = \mathcal{A}_E \, \mathcal{A}_{\rm LO}, \tag{2.4}$$

where $\mathcal{A}_{\rm LO}$ is the leading order (Born) amplitude taken in the Regge limit, which becomes dressed by the eikonal amplitude (in position space)

$$\mathcal{A}_E = \langle 0 \,| \Phi(p_1, 0) \Phi(p_2, z) |\, 0 \rangle \,. \tag{2.5}$$

Here

$$\Phi(p, x) = \mathcal{P} \exp\left[-i g_s \mathbf{T}^a p^\mu \int ds A_\mu^a (sp + x) \right] \tag{2.6}$$

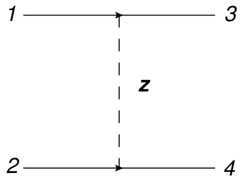

Figure 2. The Regge limit as two Wilson lines separated by a transverse distance $\vec{z}$.

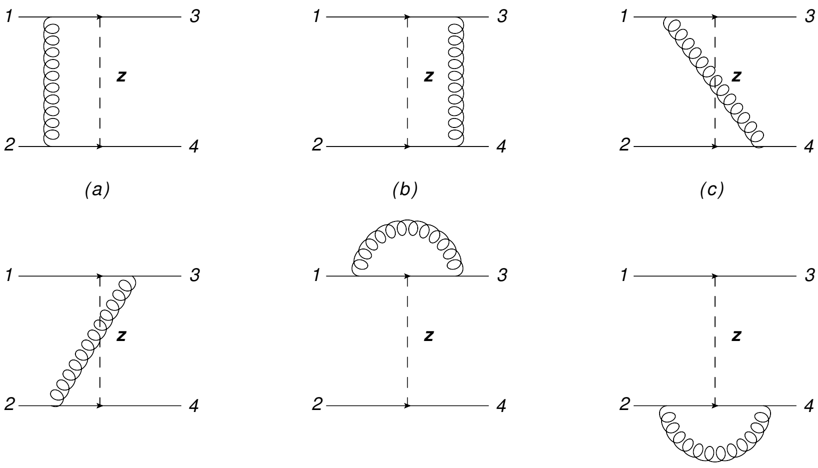

Figure 3. One-loop diagrams entering the calculation of eikonal amplitude $\mathcal{A}_E$.

is a Wilson line operator describing the emission of soft gluons from a straightline contour of momentum p^μ, and a constant offset x^μ. Equation (2.5) is then a vacuum expectation value of two Wilson lines, the second of which is displaced with respect to the first by the constant 4-vector z, which is taken to have non-zero components only in the transverse direction to the incoming particles. That is, one has[1]

$$z^2 = -\vec{z}^2. \tag{2.7}$$

Were the impact parameter to be zero, eq. (2.5) would correspond to the Regge limit of the *soft function* describing IR singularities in a scattering amplitude. As is well known, this soft function is exactly zero in dimensional regularisation, due to the cancellation of UV and IR singularities (see e.g. [89] for a review). The nonzero impact parameter acts as a UV regulator, so that any remaining singularities are manifestly of infrared origin.

One-loop diagrams[2] for the eikonal amplitude $\mathcal{A}_E$ are shown in figure 3. Diagrams (a)–(d) are regulated by the impact parameter, whereas diagrams (e)–(f) are rendered zero by the presence of an unregulated UV pole, which cancels the IR behaviour. One may impose a cutoff to regulate the UV region which, up to logs of the momentum scale choice, can be chosen to coincide with the same distance scale $|\vec{z}|$ that regulates the remaining

[1]We use the metric (+,−,−,−) throughout.

[2]As in reference [5], we do not include external self-energies, which lead to constant pieces irrelevant for the following discussion.

JHEP01(2017)052

graphs. Upon making this choice, the graphs of figure 3 evaluate (in $d = 4-2\epsilon$ dimensions, and taking the leading behaviour in s) to [5][3]

$$\begin{aligned}\mathcal{A}_E^{(1)} = \frac{g_s^2\,\Gamma(1-\epsilon)}{4\pi^{2-\epsilon}}\frac{(\mu^2\bar{z}^2)^\epsilon}{2\epsilon}\Bigg\{& i\pi\left[\mathbf{T}_1\cdot\mathbf{T}_2+\mathbf{T}_3\cdot\mathbf{T}_4\right]\\ &+\log\left(\frac{s}{m_1m_2}\right)\left[-\mathbf{T}_1\cdot\mathbf{T}_2-\mathbf{T}_3\cdot\mathbf{T}_4+\mathbf{T}_1\cdot\mathbf{T}_4+\mathbf{T}_2\cdot\mathbf{T}_3\right]\\ &+\mathbf{T}_1\cdot\mathbf{T}_3\log\left(-\frac{t}{m_1^2}\right)+\mathbf{T}_2\cdot\mathbf{T}_4\log\left(-\frac{t}{m_2^2}\right)\Bigg\},\end{aligned} \tag{2.8}$$

where $\mathbf{T}_i$ denotes a colour generator on line i, following the notation of refs. [90, 91], and satisfying the colour conservation condition

$$\mathbf{T}_1+\mathbf{T}_2=\mathbf{T}_3+\mathbf{T}_4. \tag{2.9}$$

Here the $\log(-t/m_i^2)$ terms in eq. (2.8) originate from diagrams (e)–(f) in figure 3: had we chosen not to regulate the UV poles in these diagrams, the amplitude would contain logarithms of $s/(m_1m_2)$, rather than the expected combination $s/(-t)$ in the limit of eq. (2.3) (see e.g. ref [1]). That this combination indeed results upon keeping the diagrams involving only a single particle leg can be seen by defining the quadratic colour operators

$$\begin{aligned}\mathbf{T}_s^2&=(\mathbf{T}_1+\mathbf{T}_2)^2=(\mathbf{T}_3+\mathbf{T}_4)^2,\\ \mathbf{T}_t^2&=(\mathbf{T}_1-\mathbf{T}_3)^2=(\mathbf{T}_2-\mathbf{T}_4)^2,\\ \mathbf{T}_u^2&=(\mathbf{T}_1-\mathbf{T}_4)^2=(\mathbf{T}_2-\mathbf{T}_3)^2,\end{aligned} \tag{2.10}$$

which, from eq. (2.9), satisfy

$$\mathbf{T}_s^2+\mathbf{T}_t^2+\mathbf{T}_u^2=2C_1+2C_2, \qquad \mathbf{T}_1^2=\mathbf{T}_3^2=C_1, \qquad \mathbf{T}_2^2=\mathbf{T}_4^2=C_2. \tag{2.11}$$

Equation (2.8) then becomes

$$\begin{aligned}\mathcal{A}_E^{(1)} = \frac{g_s^2\,\Gamma(1-\epsilon)}{4\pi^{2-\epsilon}}\frac{(\mu^2\bar{z}^2)^\epsilon}{2\epsilon}\Bigg\{& i\pi\mathbf{T}_s^2+\mathbf{T}_t^2\log\left(\frac{s}{-t}\right)-i\pi(C_1+C_2)\\ &+C_1\log\left(\frac{-t}{m_1^2}\right)+C_2\log\left(\frac{-t}{m_2^2}\right)\Bigg\},\end{aligned} \tag{2.12}$$

thus one indeed sees that the colour non-diagonal terms involve a logarithm of $s/(-t)$. Given that vacuum expectation values of Wilson line operators exponentiate (see e.g. [89] for a review), one may immediately replace eq. (2.12) with

$$\begin{aligned}\mathcal{A}_E = \exp\Bigg\{&\frac{g_s^2\,\Gamma(1-\epsilon)}{4\pi^{2-\epsilon}}\frac{(\mu^2\bar{z}^2)^\epsilon}{2\epsilon}\Bigg[i\pi\mathbf{T}_s^2+\mathbf{T}_t^2\log\left(\frac{s}{-t}\right)-i\pi(C_1+C_2)\\ &+C_1\log\left(\frac{-t}{m_1^2}\right)+C_2\log\left(\frac{-t}{m_2^2}\right)\Bigg]\Bigg\}.\end{aligned} \tag{2.13}$$

[3]Reference [5] treats the case of $m_1 = m_2 \equiv m$ only. Here we modify the result slightly to encompass the unequal mass case.

JHEP01(2017)052

As discussed in refs. [5, 7, 8], the term in $\mathbf{T}_t^2$ acts as a *Reggeisation operator* on the Born amplitude in eq. (2.4), dressing the exchanged t-channel gluon by a power-like growth in $s/(-t)$, where the associated power involves the quadratic Casimir of the exchanged particle. The first term in the exponent in eq. (2.13) is a pure phase, and is associated with the formation of bound states in the s-channel [92, 93]. However, it dominates only if the quadratic Casimir associated with the t-channel exchange is zero (e.g. for photon exchange), given that the Reggeisation term is logarithmically enhanced in s. Note that eq. (2.13) has (logarithmic) singularities as either of the particle masses tends to zero. These are collinear singularities associated with the incoming and outgoing particles, and are usually absorbed into *impact factors* coupling the Reggeised gluon to the upper and lower particle lines (see e.g. [1]).

It is straightforward to generalise the above analysis to gravity [5]. By analogy with eq. (2.4), one defines a gravity amplitude

$$\mathcal{M} = \mathcal{M}_E \, \mathcal{M}_{\rm LO}. \tag{2.14}$$

Now

$$\mathcal{M}_E = \langle 0 \, | \Phi_g(p_1, 0) \Phi_g(p_2, z) | \, 0 \rangle \tag{2.15}$$

is a vacuum expectation value of two gravitational Wilson line operators, defined by [53–55]

$$\Phi_g(p, x) = \exp\left[\frac{i\kappa}{2} p^\mu \, p^\nu \int ds h_{\mu\nu}(sp + x)\right], \qquad\qquad \kappa^2 = 32\pi G_N, \tag{2.16}$$

where G_N is Newton's constant, and we have defined the graviton according to

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}. \tag{2.17}$$

Upon calculating the diagrams of figure 3 (including UV regularisation of graphs (e)–(f) as before), the gravitational eikonal function in the limit of eq. (2.3) is

$$\mathcal{M}_E = \exp\left\{-\left(\frac{\kappa}{2}\right)^2 \frac{\Gamma(1-\epsilon)}{4\pi^{2-\epsilon}} \frac{(\mu^2 \vec{z}^2)^\epsilon}{2\epsilon} \left[i\pi s + t \log\left(\frac{s}{-t}\right)\right]\right\} + \mathcal{O}(\epsilon^0). \tag{2.18}$$

This can also be obtained directly from eq. (2.13) by making the replacements

$$g_s \to \frac{\kappa}{2}, \qquad\qquad \mathbf{T}_s^2 \to s, \qquad\qquad \mathbf{T}_t^2 \to t, \qquad\qquad C_i \to m_i^2, \tag{2.19}$$

where terms $\propto m_i^2$ then vanish in the Regge limit.[4] As noted in ref. [5], these replacements are consistent with the double copy of refs. [46–48]. Note that in the gravity result one may take either mass smoothly to zero, consistent with the absence of collinear singularities in this theory [94–96]. Due to the replacements of quadratic colour Casimirs (in QCD) with Mandelstam invariants (gravity), the s-channel phase dominates over the Reggeisation term, which is power-suppressed. Indeed, the first term in the exponent of eq. (2.18) is the

[4] Reference [5] considered the limit $s \gg -t \gg m_i^2$ rather than that of eq. (2.3). In either case, one may neglect m_i^2 relative to s.

well-known gravitational eikonal phase, discussed in detail in refs. [30–36], so that in the limit of eq. (2.3) one may write

$$\mathcal{M}_E = e^{i\chi_{\rm E}}, \qquad\qquad \chi_{\rm E} = -\frac{sG_N}{\epsilon}(\mu^2 \vec{z}^2)^\epsilon + \mathcal{O}(\epsilon^0), \tag{2.20}$$

in agreement with e.g. ref. [87].[5]

One may connect eqs. (2.14) and (2.20) more directly with the literature as follows. The gravitational Born amplitude consists of a single t-channel graviton exchange, which in momentum space gives

$$\begin{aligned}\tilde{\mathcal{M}}_{\rm LO} &= -\frac{i\kappa^2\mu^{2\epsilon}}{2}\frac{(p_1\cdot p_2)(p_3\cdot p_4)+(p_1\cdot p_4)(p_2\cdot p_3)-(p_1\cdot p_3)(p_2\cdot p_4)+m_1^2 p_2\cdot p_4+m_2^2 p_1\cdot p_3-2m_1^2m_2^2}{(p_1-p_3)^2}\\ &= -8\pi i G_N \mu^{2\epsilon}\frac{s^2}{t}+\ldots,\end{aligned} \tag{2.21}$$

where the ellipsis denotes subleading terms as $s \gg -t, m_i^2$. In the Regge limit, the momentum transfer has components only in the transverse directions (see e.g. ref. [93]):

$$t \simeq -\vec{q}^2, \tag{2.22}$$

where $\vec{q}$ is the $(d-2)$-dimensional transverse momentum vector conjugate to the impact parameter $\vec{z}$. The Born amplitude in impact parameter space is then

$$\mathcal{M}_{\rm LO} = \int \frac{d^{d-2}\vec{q}}{(2\pi)^{d-2}}\tilde{\mathcal{M}}_{\rm LO}e^{i\vec{q}\cdot\vec{z}} = 2is\chi_{\rm E} \tag{2.23}$$

where

$$\chi_{\rm E} = -4\pi s G_N\mu^{2\epsilon}\int \frac{d^{d-2}\vec{k}}{(2\pi)^{d-2}}\frac{e^{i\vec{k}\cdot\vec{z}}}{-\vec{k}^2}. \tag{2.24}$$

Carrying out the integral with $d = 4-2\epsilon$ shows that eq. (2.24) is in agreement with eq. (2.20). One may then expand $\mathcal{M}_E = e^{i\chi_E}$ and use eq. (2.24) to write[6]

$$\begin{aligned}\mathcal{M} = \mathcal{M}_E\,\mathcal{M}_{\rm LO} &= \left[\sum_{m=0}^{\infty}\frac{(-4\pi i s G_N\mu^{2\epsilon})^m}{m!}\prod_{i=1}^{m}\int\frac{d^2\vec{k}_i}{(2\pi)^{d-2}}\frac{e^{i\vec{k}_i\cdot\vec{z}}}{-\vec{k}_i^2}\right]\mathcal{M}_{\rm LO}\\ &= 2s\sum_{n=1}^{\infty}\frac{(-4\pi i s G_N\mu^{2\epsilon})^n}{n!}\prod_{i=1}^{n}\int\frac{d^2\vec{k}_i}{(2\pi)^{d-2}}\frac{e^{i\vec{k}_i\cdot\vec{z}}}{-\vec{k}_i^2}\\ &= 2s(e^{i\chi_{\rm E}}-1),\end{aligned} \tag{2.25}$$

in agreement with ref. [93].

[5]A similar result is provided in ref. [93], but using a fictitious mass for the graviton as an infrared regulator.

[6]Care must be taken with combinatorial factors here: in the second line of eq. (2.25), n represents the number of gluons being exchanged, including the Born gluon. An additional factor of n^{-1} is then needed in each term due to the fact that the symmetric product of integrals introduces an overcounting, by the number of ways one can choose which gluon is the Born one.

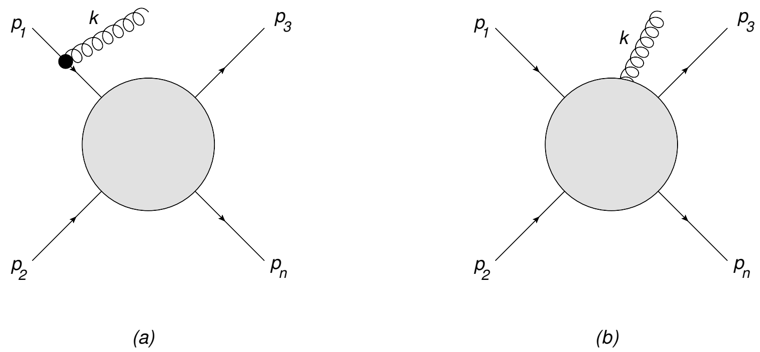

Figure 4. (a) External emisson of a (next-to) soft gluon; (b) Internal emission of a soft gluon.

3 Beyond the eikonal approximation

Having reviewed the eikonal calculation of QCD and gravity scattering in the Regge limit, we now turn to corrections beyond the leading soft approximation. To the best of our knowledge, this has not been previously studied in QCD. In gravity, refs. [32–35] considered corrections to the eikonal approximation when both incoming particles are strictly massless. A dimensional argument can then be used to show that such corrections are doubly subleading in the impact factor $|\vec{z}|$. First, one notes that $G_N E$ is the only classical length scale that one can form, where $E \sim \sqrt{s}$ is the energy of one of the incoming particles in the centre-of-mass frame. Then, analyticity of the amplitude requires only integer powers of s, so that the first subleading corrections

$$\sim \frac{G_N^2 s}{|\vec{z}|^2}, \tag{3.1}$$

with subsequent corrections also involving only even powers of the impact parameter. The corrections considered by the above references thus begin at two-loop order, and are beyond the scope of this paper.

Reference [87] considered the case of one strictly massless particle, and the other infinitely massive. In this case one evades the above dimensional argument due to the presence of an extra mass scale, such that the first subleading corrections to the eikonal are $\mathcal{O}(|\vec{z}|^{-1})$. Here, we will consider the general situation of two scalar particles with potentially different nonzero masses, such that the results of [87] emerge as a special case.[7]

To classify next-to-soft corrections, we will use the framework of refs. [54, 82] (see also ref. [97] for similar work in the eikonal approximation). The starting point is to consider an amplitude with n external hard particles (i.e. here the four-point amplitude of figure 1), to which an additional gluon or graviton emission is added. There are two possibilities, as shown in figure 4: (i) *external emission* contributions, in which the additional boson is emitted from one of the external legs, and (ii) *internal emission contributions*, where the boson lands inside the nonradiative amplitude. We now deal with each of these in turn.

[7]The deflection of massless particles with different spins was also considered recently in ref. [88], with the spinless result agreeing with ref. [87].

3.1 External emissions in QCD

As shown in detail in refs. [54, 82], external emission contributions are described by generalised Wilson line operators associated with the hard particle lines. For outgoing boson momentum k, they are given in position space in QCD and gravity by[8]

$$\Phi_{\rm NE}(p_i,z)=\mathcal{P}\exp\left\{-ig_s\mathbf{T}_i\int_0^\infty ds\left[p_{i\mu}A^\mu+\frac{i}{2}\partial_\mu A^\mu+\frac{i}{2}tp_{i\mu}\partial^2A^\mu\right]+\mathcal{O}(g_s^2)\right\} \tag{3.2}$$

and

$$\begin{aligned}\Phi_{g,\rm NE}(p_i,z)=\exp\left\{\frac{i\kappa}{2}\int_0^\infty ds\left[p_{i\mu}p_{i\nu}h^{\mu\nu}+\frac{i}{2}p_{i(\mu}\partial_{\nu)}\left(h^{\mu\nu}-\frac{h}{2}\eta^{\mu\nu}\right)+\frac{i}{2}sp_{i\mu}p_{j\nu}\partial^2h^{\mu\nu}\right]\right.\\ \left.+\mathcal{O}(\kappa^2)\right\},\end{aligned} \tag{3.3}$$

where we have introduced the commonly used notation

$$a^{(\mu}b^{\nu)}=a^\mu b^\nu+a^\nu b^\mu. \tag{3.4}$$

Here p_i is the momentum of the hard emitting particle, whose trajectory is given, as before, by $x_i^\mu=tp_i^\mu+z$ in general. We neglect terms quadratic in the coupling constant here, as we will not need these in the one-loop calculations required for this paper. The first terms in the exponents of eqs. (3.2), (3.3) are the usual eikonal Wilson line exponents of eqs. (2.6). Subsequent terms involve derivatives with respect to the momentum of the gluon or graviton field, and are thus indeed subleading in momentum space. They give rise to *next-to-eikonal Feynman rules* coupling the bosons to the external particle lines, and we will see explicit examples of their use in the following.

Diagrams contributing at next-to-soft level are shown in figures 5 and 6. They can be obtained from the diagrams of figure 3 by replacing at most one eikonal vertex with one of the next-to-soft Feynman rules from eq. (3.2). There are two types, which in Feynman diagram language have two different origins: the second term in eq. (3.2) arises from corrections to the numerators associated with gluon emissions on the external lines, and the third from corrections to the external particle propagator denominators. In fact, the latter does not contribute, which can be seen as follows. When embedded in any of the diagrams of figures 5 and 6, the d'Alembertian acts on the soft gluon propagator to give

$$\partial^2D_{\mu\nu}(x-y)=\eta_{\mu\nu}\delta^d(x-y) \tag{3.5}$$

(i.e. the propagator is a Green's function). The right-hand side implies a non-zero result only if the distance between the two ends of the soft gluon vanishes. Thus, graphs involving the denominator correction can potentially contribute only in the absence of a UV regulator, which acts to remove the short distance region. We will therefore not have to worry about them in what follows. Note that a similar conclusion was reached in ref. [87], which separated denominator correction terms into those containing a single gluon momentum

[8]Note that ref. [54] uses an alternative field definition for the graviton. Here we stick to the canonical choice of eq. (2.17).

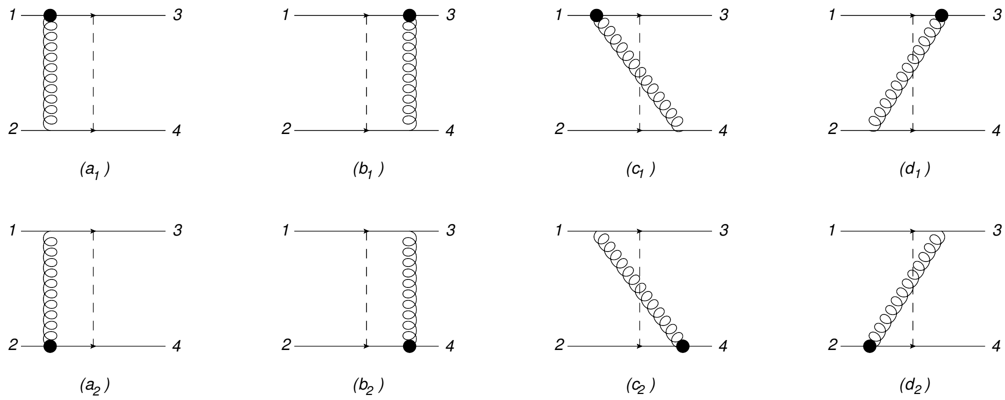

Figure 5. External emission contributions from the generalised Wilson line operator of eq. (3.2), where • represents a next-to-soft vertex, and all other vertices are eikonal.

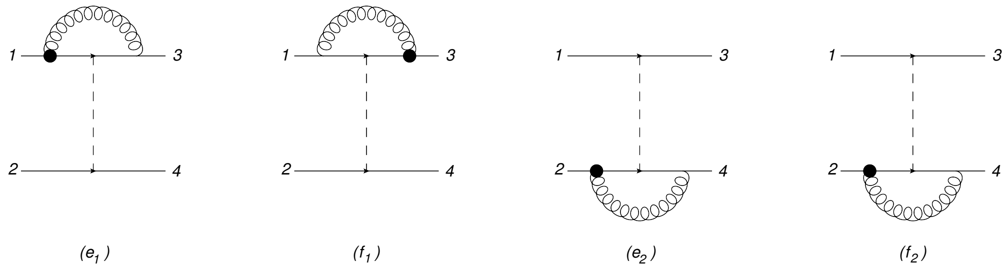

Figure 6. External emission contributions from the generalised Wilson line operator of eq. (3.2), where • represents a next-to-soft vertex, and all other vertices are eikonal.

(corresponding to the Feynman rule in eq. (3.2)), and those involving a pair of gluon momenta. The former were argued to vanish for nonzero impact parameter, as here. The latter are absent in our calculation, as they correspond to effective Feynman rules involving two or more gauge bosons, which are absent at one-loop order in the generalised Wilson line calculation. This corresponds to the fact that such corrections were also found not to affect the next-to-eikonal phase in ref. [87], due to being higher loop order.

It remains to calculate the graphs involving the next-to-soft vertex in the second term of eq. (3.2). As an example, diagram (b_1) is given by

$$\mathcal{A}_{b_1} = -\frac{ig_s^2(\mu^2)^\epsilon}{2}\mathbf{T}_3\cdot\mathbf{T}_4\, p_{4\nu}\int_0^\infty ds_3\int_0^\infty ds_4\frac{\partial}{\partial x_3^\mu}D^{\mu\nu}(x_3-x_4), \tag{3.6}$$

where

$$x_3^\mu = s_3 p_3^\mu + z^\mu, \qquad\qquad x_4^\mu = s_4 p_4^\mu, \tag{3.7}$$

and

$$D_{\mu\nu}(x) = -g_{\mu\nu}\frac{\Gamma(d/2-1)}{4\pi^{d/2}}\left[-x^2+i\varepsilon\right]^{1-d/2} \tag{3.8}$$

is the position space gluon propagator in $d = 4-2\epsilon$ dimensions, such that

$$\frac{\partial}{\partial x_3^\mu}D^{\mu\nu}(x_3-x_4) = -\frac{\Gamma(d/2)}{2\pi^{d/2}}(x_3-x_4)^\nu\left[-(x_3-x_4)^2+i\varepsilon\right]^{-d/2}. \tag{3.9}$$

JHEP01(2017)052

One may then write eq. (3.6) as

$$\mathcal{A}_{b_1} = i g_s^2 \mu^{2\epsilon} \frac{\Gamma(d/2)}{4\pi^{d/2}} \mathbf{T}_3 \cdot \mathbf{T}_4 \, p_{4\mu} \, V^{\mu}_{\rm NE}(p_3, -p_4), \tag{3.10}$$

where we have defined the master integral

$$V^{\mu}_{\rm NE}(\sigma_i p_i, \sigma_j p_j) = \int_0^\infty ds_i \int_0^\infty ds_j \, (\sigma_i \, s_i \, p_i + \sigma_j \, s_j \, p_j + z)^\mu \left[-(\sigma_i p_i + \sigma_j p_j)^2 + \vec{z}^2 + i\varepsilon \right]^{-d/2}, \tag{3.11}$$

and $\sigma_{i,j} = \pm 1$. One can obtain diagram (b_1) by relabelling $p_3 \to -p_1$, $p_4 \to -p_2$ in eq. (3.10). Similarly, diagram (c_1) is given by

$$\mathcal{A}_{c_1} = i g_s^2 \mu^{2\epsilon} \frac{\Gamma(d/2)}{4\pi^{d/2}} \mathbf{T}_1 \cdot \mathbf{T}_4 \, p_{4\mu} \, V^{\mu}_{\rm NE}(p_1, p_4), \tag{3.12}$$

with (d_1) obtained by relabelling $p_1 \to -p_3$, $p_4 \to -p_2$. One may also switch momenta to obtain the diagrams (a_2)–(d_2), and the integral of eq. (3.11) is calculated in appendix A. Combining all diagrams, the total is

$$\begin{aligned} \mathcal{A}_{a-d} &= \frac{g_s^2 \mu^{2\epsilon}}{8\pi^{d/2}} \Gamma\left(\frac{3}{2}\right) \Gamma\left(\frac{d-3}{2}\right) |\vec{z}|^{3-d} \left(\mathbf{T}_1 \cdot \mathbf{T}_2 + \mathbf{T}_3 \cdot \mathbf{T}_4 - \mathbf{T}_1 \cdot \mathbf{T}_4 - \mathbf{T}_2 \cdot \mathbf{T}_3 \right) \left(\frac{1}{m_1} + \frac{1}{m_2} \right) \\ &= -\frac{g_s^2 \mu^{2\epsilon}}{8\pi^{d/2}} \Gamma\left(\frac{3}{2}\right) \Gamma\left(\frac{d-3}{2}\right) |\vec{z}|^{3-d} \left(\frac{1}{m_1} + \frac{1}{m_2} \right) \mathbf{T}_t^2, \end{aligned} \tag{3.13}$$

where we have used the quadratic Casimir operators of eq. (2.10).

There are a number of noteworthy features of this result. Firstly, it is IR finite in $d = 4$, but contains a pole in $d = 3$. The latter is the analogue of the pole in $d = 4$ in the eikonal result of eq. (2.8). In Feynman diagram language, the (next-to)-eikonal approximation amounts to linearising denominator factors. At eikonal level, this introduces a spurious logarithmic UV divergence. Without any additional regulator, all soft integrals are scaleless, and thus vanish in dimensional regularisation. The UV pole in eq. (2.8) is, however, regulated by the impact parameter, leaving a remaining IR pole. At next-to-soft level the story is similar, except for the fact that going to subleading order in the soft momentum means that the spurious UV divergence is linear rather than logarithmic. Without an additional regulator, next-to-soft integrals would be scaleless and thus vanishing in dimensional regularisation. In this case, however, one can understand this cancellation as arising between logarithmic singularities in $d = 3$. Regulating the UV divergence with the impact parameter leaves an (IR) pole in $d = 3$, manifest in eq. (3.13).

Another property of eq. (3.13) is that one cannot take the massless limit $m_i \to 0$ for either of the incoming particles, and the reason for this can again be understood by comparing with the eikonal result of eq. (2.8). If only diagrams (a)–(d) in figure 3 are included, the one-loop amplitude contains logarithms of $s/(m_1 \, m_2)$, rather than the conventional combination $s/(-t)$. The remaining diagrams (e) and (f) are not regulated by the physical impact parameter $\vec{z}$, and vanish in dimensional regularisation. As discussed in ref. [5] and here in section 2, one may choose to also regulate (e) and (f) with the impact parameter, which amounts to using this as a scale at which to remove the UV divergence in these diagrams. Whether or not to include diagrams (e) and (f) thus amounts to a renormalisation scheme choice. The effect of doing so, as can be seen in eq. (2.13), is to shift the logarithms of mass away from the Regge trajectory and into the colour-diagonal terms. The physical interpretation of these terms is that they are collinear singularities associated with the incoming and outgoing particles, where the mass acts as a regulator. The scheme dependence corresponds to the well-known ambiguity as to whether such singularities are part of the Regge trajectory, or absorbed into *impact factors* associated with the upper and lower particle lines (see e.g. ref. [1]).

The above discussion allows us to interpret the behaviour as $m_i \to 0$ of eq. (3.13): the divergence is associated with the virtual next-to-soft gluon becoming collinear with one of the external lines. This divergence is power-like in $d = 4$ but logarithmic in $d = 3$, as expected from a divergence which is both next-to-soft and collinear. Here, as in the eikonal case, we have the option of including the diagrams (e_i) and (f_i) in figure 6, which amounts to a renormalisation scheme choice. We instead take the viewpoint of previous studies [32–36, 87, 93], namely that the impact factor implements a physically motivated cutoff where applicable, and thus only regulate those diagrams in which the gluons straddle both lines.

The power of the generalised Wilson line approach is that, just as in the eikonal calculation of refs. [2, 3, 5], the one-loop amplitude formally exponentiates [54, 82]. Keeping only diagrams (a)–(d) in the eikonal calculation, one may thus write the generalised Wilson line amplitude as

$$\mathcal{A}_{\mathrm{E+NE}} = \exp\left\{\frac{g^2}{8\pi^{2-\epsilon}}(\mu^2\vec{z}^2)^\epsilon\left[\frac{\Gamma(1-\epsilon)}{\epsilon}\left(i\pi(\mathbf{T}_s^2 - C_1 - C_2) + \mathbf{T}_t^2\log\left(\frac{s}{m_1 m_2}\right)\right)\right.\right.$$
$$\left.\left. -\frac{\pi}{2}\frac{\mathbf{T}_t^2}{|\vec{z}|}\left(\frac{1}{m_1}+\frac{1}{m_2}\right) + \mathcal{O}(s^{-1})\right]\right\} \tag{3.14}$$

The colour non-diagonal terms in the eikonal piece (first line) contain an imaginary piece $\propto \mathbf{T}_s^2$, and a real part $\propto \mathbf{T}_t^2$. As discussed above, the latter corresponds to the Reggeisation of the gluon, and the former to the eikonal phase (leading to s-channel bound states). In eq. (3.14) we see that at next-to-soft level (second line), there is no imaginary piece, and thus no next-to-soft correction to the eikonal phase from external emission contributions. Instead, there is a power-suppressed correction to the Regge trajectory. This takes the form of pure collinearly divergent terms, which can be absorbed in the impact factors associated with the upper and lower lines.

Having examined the external emission contributions in QCD, we now turn to their calculation in gravity.

3.2 External emissions in gravity

The diagrams needed for the gravity calculation are again those of figures 5 and 6, where now we must use the generalised Wilson line operator of eq. (3.3). As in the QCD case, the third term involving the d'Alembertian operator would contribute only at zero impact parameter, and thus can be neglected. It is convenient to rewrite the remaining next-to-soft term via

$$\frac{i\kappa}{2}\int_0^\infty ds\frac{i}{2}p_{i(\mu}\partial_{\nu)}\left(h^{\mu\nu}-\frac{h}{2}\eta^{\mu\nu}\right)\to\frac{i\kappa}{2}\int_0^\infty ds\frac{i}{2}p_{i\mu}\partial_\nu\left(\eta^{\mu\alpha}\eta^{\nu\beta}+\eta^{\mu\beta}\eta^{\nu\alpha}-\eta^{\mu\nu}\eta^{\alpha\beta}\right)h_{\alpha\beta},\tag{3.15}$$

where we have used the symmetry of the graviton $h^{\alpha\beta}=h^{\beta\alpha}$. Diagram ($\mathrm{b}_1$) then gives

$$\begin{aligned}\mathcal{M}_{b_1}&=-\frac{i}{2}\left(\frac{\kappa}{2}\right)^2\mu^{2\epsilon}p_4^\alpha p_4^\beta p_{3\mu}\left(\eta^{\mu\sigma}\eta^{\nu\tau}+\eta^{\mu\tau}\eta^{\nu\sigma}-\eta^{\mu\nu}\eta^{\alpha\beta}\right)\int_0^\infty ds_3\int_0^\infty ds_4\frac{\partial}{\partial x_3^\nu}\left\langle h_{\sigma\tau}(x_3)h_{\alpha\beta}(x_4)\right\rangle\\&=-i\mu^{2\epsilon}\left(\frac{\kappa}{2}\right)^2\frac{\Gamma(d/2)}{4\pi^{d/2}}(2p_3\cdot p_4)p_{4\mu}V^\mu_{\mathrm{NE}}(p_3,-p_4),\end{aligned}\tag{3.16}$$

where we have used the position-space de Donder gauge graviton propagator

$$\begin{aligned}\langle h_{\sigma\tau}(x)h_{\alpha\beta}(y)\rangle&=P_{\sigma\tau\alpha\beta}\frac{\Gamma(\frac{d}{2}-1)}{4\pi^{d/2}}\left[-(x-y)^2+i\varepsilon\right]^{1-d/2},\\P_{\sigma\tau\alpha\beta}&=\frac{1}{2}\left(\eta_{\sigma\alpha}\eta_{\tau\beta}+\eta_{\sigma\beta}\eta_{\tau\alpha}-\frac{2}{d-2}\eta_{\sigma\tau}\eta_{\alpha\beta}\right),\end{aligned}\tag{3.17}$$

as well as the master integral of eq. (3.11). The form of eq. (3.16) is extremely similar to the QCD result of eq. (3.10), and can be obtained from the latter by making the replacements

$$g_s\to\frac{\kappa}{2},\qquad\qquad \mathbf{T}_i^a\to p^\mu,\tag{3.18}$$

as well as including an additional factor of 2. As in the eikonal case, this is precisely consistent with the double copy [46–48]. The additional factor is combinatorial in nature, and follows from the fact that numerators of gravitational integrands result from combining two copies of a gauge theory numerator. In a given diagram i in which an additional virtual gluon dresses the Born amplitude (where the latter may be taken to already be in double copy form), one may expand the extra contribution to the numerator in the momentum k of the virtual gluon:

$$n_i=n_i^{(0)}+n_i^{(1)}+\mathcal{O}(k^2),\tag{3.19}$$

where $n_i^{(m)}$ is the contribution to the numerator at $\mathcal{O}(k^m)$. The gravity numerator for the same graph is then given by

$$n_in_i=n_i^{(0)}n_i^{(0)}+\left(n_i^{(0)}n_i^{(1)}+n_i^{(1)}n_i^{(0)}\right)+\mathcal{O}(k^2),\tag{3.20}$$

and the fact that there are two terms in the $\mathcal{O}(k)$ contribution is the origin of the additional factor of 2 in eq. (3.16) relative to the QCD case. One also sees that no additional factor is present in the leading (eikonal) term, consistent with the results of ref. [5].

The remaining diagrams can be obtained by relabelling eq. (3.16), or by making the replacements of eq. (3.18) and including the above noted factor of 2. The sum of diagrams (a_i)–(d_i) is then

$$\mathcal{M}_{a-d}=\frac{\mu^{2\epsilon}}{4\pi^{d/2}}\left(\frac{\kappa}{2}\right)^2\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{d-3}{2}\right)|\vec{z}|^{3-d}\left(\frac{1}{m_1}+\frac{1}{m_2}\right)t.\tag{3.21}$$

Combining this with the eikonal result and exponentiating gives (cf. eq. (3.14))

$$\mathcal{M}_{\rm E+NE}=\exp\left\{-\left(\frac{\kappa}{2}\right)^2\frac{(\mu^2\vec{z}^2)^\epsilon}{8\pi^{2-\epsilon}}\left[\frac{\Gamma(1-\epsilon)}{\epsilon}\left(i\pi s+t\log\left(\frac{s}{m_1m_2}\right)\right)-\frac{\pi t}{|\vec{z}|}\left(\frac{1}{m_1}+\frac{1}{m_2}\right)\right]\right\}. \tag{3.22}$$

The effect of the individual colour matrix replacements of eq. (3.18) is to replace the t-channel quadratic Casimir appearing in eq. (3.13) with the Mandelstam invariant t, as in the previously found eikonal replacements of eq. (2.19). Similarly to the QCD calculation of the previous section, one finds a next-to-soft correction to the Regge trajectory only which, being kinematically subleading in gravity, can be neglected in the Regge limit. This is consistent with the fact that external emission contributions (in the present terminology) could be ignored in ref. [87], owing to their being doubly suppressed in mass and momentum transfer.

3.3 Off-shell internal emissions

Having calculated the external emission contributions in both QCD and gravity, we now turn to those soft gluons and gravitons that arise from inside the hard interaction. For on-shell bosons, these are given respectively in QCD and gravity by [54, 79–82][9]

$$\mathcal{A}^\nu_{\rm int.}=g_s\sum_i\mathbf{T}_i\left(\eta^{\alpha\nu}-\frac{\eta_ip_i^\nu k^\alpha}{\eta_ip_i\cdot k+i\varepsilon}\right)\frac{\partial\mathcal{A}_n(\{p_i\})}{\partial p_i^\alpha}=ig_s\sum_i\mathbf{T}_i^a\frac{L^{(i)}_{\mu\nu}}{p_i\cdot k}\mathcal{A}_n(\{p_i\}) \tag{3.23}$$

and

$$\mathcal{M}^{\mu\nu}_{\rm int.}=-\frac{\kappa}{2}p_i^\mu\sum_i\left(\eta^{\alpha\nu}-\frac{\eta_ip_i^\nu k^\alpha}{\eta_ip_i\cdot k+i\varepsilon}\right)\frac{\partial\mathcal{M}_n(\{p_n\})}{\partial p_i^\alpha}=-\frac{i\kappa}{2}\sum_i\frac{p_{i\mu}k^\rho L^{(j)}_{\rho\nu}}{p_j\cdot k}\mathcal{M}_n(\{p_i\}), \tag{3.24}$$

where $\eta_i=\pm1$ according to whether line i is outgoing or incoming, and we have recognised the orbital angular momentum generator associated with line i:

$$L^{(i)}_{\mu\nu}=x_{i\mu}p_{i\nu}-x_{i\nu}p_{i\mu}=i\left(p_{i\mu}\frac{\partial}{\partial p_i^\nu}-p_{i\nu}\frac{\partial}{\partial p_i^\mu}\right). \tag{3.25}$$

This is the same as the total angular momentum for scalar external particles, and thus eqs. (3.23), (3.24) form a special case of the recently studied *next-to-soft theorems* [57–77, 98], as pointed out in more detail in ref. [78]. In the present work, all emitted soft bosons are virtual, and thus off-shell. For the external emission contributions, this is not a problem, as the generalised Wilson line operators of eqs. (3.2), (3.3) are derived fully generally. Equations (3.23), (3.24), however, are not guaranteed to work for off-shell bosons. The aim of this section is to demonstrate that the next-to-soft theorems are indeed broken by off-shell effects, and to present an alternative way to calculate the internal emission contributions, motivated by ref. [87].

[9] Our sign in the QCD result matches our convention for the scalar-scalar-gluon vertex (see eq. (3.36)).

JHEP01(2017)052

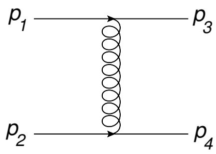

Figure 7. Born diagram for $2 \to 2$ scattering in QCD.

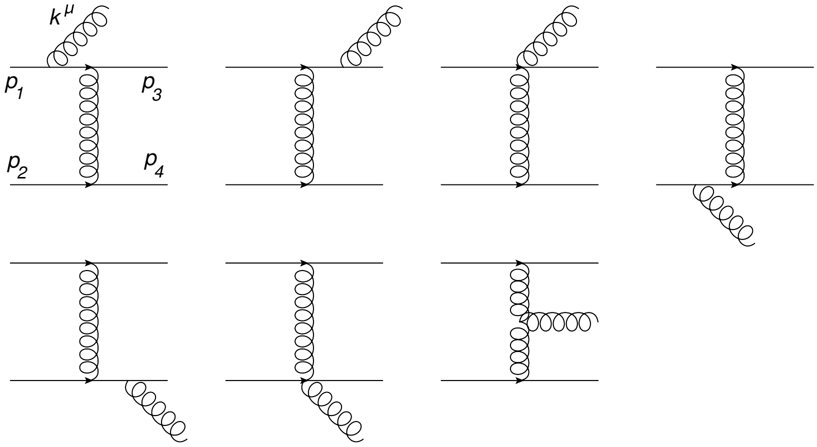

Figure 8. NLO corrections to the Born interaction of figure 7.

Let us begin by considering the QCD Born interaction for $2 \to 2$ scattering of figure 7, in which a hard gluon exchange provides the separation between the incoming particles that gives rise to the impact factor $\vec{z}$ in the Regge limit. It is given by

$$\tilde{\mathcal{A}}_{\rm LO} = i g_s^2 \mathbf{T}_U^a\, \mathbf{T}_L^a \frac{(p_1+p_3)\cdot(p_2+p_4)}{(p_1-p_3)^2}, \tag{3.26}$$

where $\mathbf{T}^a_{U,L}$ is a colour generator on the upper or lower line respectively, and the tilde denotes a momentum space expression. One may now add an additional off-shell gluon emission which, if not working in an effective next-to-soft approach, involves the diagrams of figure 8. These can be evaluated to give[10]

$$\begin{aligned}
\tilde{\mathcal{A}}_{\rm NLO} =& -i g_s^3 \left\{ \frac{1}{(p_1-p_3-k)^2} \left[\mathbf{T}_U^a\, \mathbf{T}_U^b\, \mathbf{T}_L^a \frac{(2p_1-k)^\nu}{-2p_1\cdot k + k^2}(p_1+p_3-k)\cdot(p_2+p_4) \right.\right.\\
&+ \mathbf{T}_U^b\, \mathbf{T}_U^a\, \mathbf{T}_L^a \frac{(2p_3+k)^\nu}{2p_3\cdot k + k^2}(p_1+p_3+k)\cdot(p_2+p_4) - \left.\{\mathbf{T}_U^a, \mathbf{T}_U^b\}\mathbf{T}_L^a (p_2+p_4)^\nu \right]\\
&+ \frac{1}{(p_1-p_3)^2}\left[\mathbf{T}_U^a\, \mathbf{T}_L^a\, \mathbf{T}_L^b \frac{(2p_2-k)^\nu}{-2p_2\cdot k + k^2}(p_1+p_3)\cdot(p_2+p_4-k)\right.\\
&+ \mathbf{T}_U^a \mathbf{T}_L^b \mathbf{T}_L^a \frac{(2p_4+k)^\nu}{2p_4\cdot k + k^2}(p_1+p_3)\cdot(p_2+p_4+k) - \left.\mathbf{T}_U^a\{\mathbf{T}_L^b, \mathbf{T}_L^a\}(p_1+p_3)^\nu \right]\\
&+ i f^{cba} \mathbf{T}_U^c\, \mathbf{T}_L^a \frac{(p_1+p_3)_\mu (p_2+p_4)_\rho}{(p_1-p_3)^2(p_1-p_3-k)^2}\Big((p_1-p_3+k)^\rho \eta^{\mu\nu} + (-2k+p_1-p_3)^\mu \eta^{\nu\rho}\\
&\left. + (-2p_1+2p_3+k)^\nu \eta^{\mu\rho} \Big) \right\}.
\end{aligned} \tag{3.27}$$

[10]We have here suppressed the Feynman $i\varepsilon$ prescription for brevity.

Expanding in the additional gluon momentum k up to next-to-soft level yields

$$
\begin{aligned}
\tilde{\mathcal{A}}_{\rm NLO} = -ig_s^3 \Bigg\{ & \mathbf{T}_U^a\, \mathbf{T}_U^b\, \mathbf{T}_L^a \left[\frac{(p_1+p_3)\cdot(p_2+p_4)}{(p_1-p_3)^2} \left(-\frac{p_1^\nu}{p_1\cdot k} + \frac{k^\nu}{2p_1\cdot k} - \frac{p_1^\nu k^2}{2(p_1\cdot k)^2} \right) \right. \\
& \left. + \frac{p_1^\nu}{p_1\cdot k} \left(\frac{k\cdot(p_2+p_4)}{(p_1-p_3)^2} - \frac{2k\cdot(p_1-p_3)(p_1+p_3)\cdot(p_2+p_4)}{(p_1-p_3)^4} \right) \right] \\
& + \mathbf{T}_U^b\, \mathbf{T}_U^a\, \mathbf{T}_L^a \left[\frac{(p_1+p_3)\cdot(p_2+p_4)}{(p_1-p_3)^2} \left(\frac{p_3^\nu}{p_3\cdot k} + \frac{k^\nu}{2p_3\cdot k} - \frac{p_3^\nu k^2}{2(p_3\cdot k)^2} \right) \right. \\
& \left. + \frac{p_3^\nu}{p_3\cdot k} \left(\frac{k\cdot(p_2+p_4)}{(p_1-p_3)^2} + \frac{2k\cdot(p_1-p_3)(p_1+p_3)\cdot(p_2+p_4)}{(p_1-p_3)^4} \right) \right] \\
& + \mathbf{T}_U^a\, \mathbf{T}_L^a\, \mathbf{T}_L^b \left[\frac{(p_1+p_3)\cdot(p_2+p_4)}{(p_1-p_3)^2} \left(-\frac{p_2^\nu}{p_2\cdot k} + \frac{k^\nu}{2p_2\cdot k} - \frac{p_2^\nu k^2}{2(p_2\cdot k)^2} \right) + \frac{p_2^\nu}{p_2\cdot k} \frac{k\cdot(p_1+p_3)}{(p_1-p_3)^2} \right] \\
& + \mathbf{T}_U^a\, \mathbf{T}_L^b\, \mathbf{T}_L^a \left[\frac{(p_1+p_3)\cdot(p_2+p_4)}{(p_1-p_3)^2} \left(\frac{p_4^\nu}{p_4\cdot k} + \frac{k^\nu}{2p_4\cdot k} - \frac{p_4^\nu k^2}{2(p_4\cdot k)^2} \right) + \frac{p_4^\nu}{p_4\cdot k} \frac{k\cdot(p_1+p_3)}{(p_1-p_3)^2} \right] \\
& - \{\mathbf{T}_U^a, \mathbf{T}_U^b\} \mathbf{T}_L^a \frac{(p_2+p_4)^\nu}{(p_1-p_3)^2} - \mathbf{T}_U^a \{\mathbf{T}_L^a, \mathbf{T}_L^b\} \frac{(p_1+p_3)^\nu}{(p_1-p_3)^2} \\
& - 2if^{cba} \mathbf{T}_U^c\, \mathbf{T}_L^a \frac{(p_1+p_3)\cdot(p_2+p_4)}{(p_1-p_3)^4} (p_1-p_3)^\nu \Bigg\} . \qquad (3.28)
\end{aligned}
$$

We can recognise some of the terms in this expression (the first group of terms in each square bracket) as the Born amplitude of eq. (3.26), dressed by eikonal and next-to-eikonal Feynman rules obtained by Fourier transforming the exponent of eq. (3.2) to momentum space. Thus, these are external emission contributions, so that the remaining contributions must correspond to internal emissions. One may then directly check whether or not they are reproduced from eq. (3.23): an explicit calculation of the latter gives

$$
\begin{aligned}
\tilde{\mathcal{A}}^{b\,\nu}_{\rm int.} = -ig_s^3 \Bigg\{ & \mathbf{T}_U^a\, \mathbf{T}_U^b\, \mathbf{T}_L^a \frac{p_1^\nu}{p_1\cdot k} \left(\frac{k\cdot(p_2+p_4)}{(p_1-p_3)^2} - \frac{2k\cdot(p_1-p_3)(p_1+p_3)\cdot(p_2+p_4)}{(p_1-p_3)^4} \right) \\
& + \mathbf{T}_U^b\, \mathbf{T}_U^a\, \mathbf{T}_L^a \frac{p_3^\nu}{p_3\cdot k} \left(\frac{k\cdot(p_2+p_4)}{(p_1-p_3)^2} + \frac{2k\cdot(p_1-p_3)(p_1+p_3)\cdot(p_2+p_4)}{(p_1-p_3)^4} \right) \\
& + \mathbf{T}_U^a\, \mathbf{T}_L^a\, \mathbf{T}_L^b \frac{p_2^\nu}{p_2\cdot k} \frac{k\cdot(p_1+p_3)}{(p_1-p_3)^2} + \mathbf{T}_U^a\, \mathbf{T}_L^b\, \mathbf{T}_L^a \frac{p_4^\nu}{p_4\cdot k} \frac{k\cdot(p_1+p_3)}{(p_1-p_3)^2} - \mathbf{T}_U^a \{\mathbf{T}_L^a, \mathbf{T}_L^b\} \frac{(p_1+p_3)^\nu}{(p_1-p_3)^2} \\
& - \{\mathbf{T}_U^a, \mathbf{T}_U^b\} \mathbf{T}_L^a \frac{(p_2+p_4)^\nu}{(p_1-p_3)^2} - [\mathbf{T}_U^b, \mathbf{T}_U^a] \mathbf{T}_L^a \frac{2(p_1-p_3)^\nu (p_1+p_3)\cdot(p_2+p_4)}{(p_1-p_3)^4} \Bigg\} . \qquad (3.29)
\end{aligned}
$$

After using the relation

$$
[\mathbf{T}_U^b, \mathbf{T}_U^a] = if^{bac} \mathbf{T}_U^c, \qquad (3.30)
$$

eq. (3.29) precisely reproduces the internal emission terms in eq. (3.28), regardless of the fact that eq. (3.29) is manifestly derived for on-shell gluons. It is instructive to classify the anatomy of this result in more detail. The final three terms in eq. (3.29) (those with no explicit dependence on k) originate from the first term in eq. (3.23), as must be the case given that the latter also has no explicit k dependence. In the full NLO calculation, these correspond to the seagull and three-gluon vertex graphs, evaluated with $k \to 0$. The remaining terms in eq. (3.29) then correspond to the second term in eq. (3.23). Comparison with the full NLO calculation shows that they have the form of eikonal Feynman rules dressing terms obtained from the Born interaction by shifting the external momenta in accordance with the extra gluon emission. This interpretation also follows directly from the form of eq. (3.23), and we will therefore refer to these contributions as *momentum-shift terms* in what follows.

One may carry out a similar analysis for gravity, in which the gluons in figures 7 and 8 are replaced with gravitons, and where the Born interaction is now given by eq. (2.21). The gravitational Feynman rules, including the three-graviton vertex, may be found in e.g. ref. [99] (see also refs. [100, 101]). Due to the cumbersome nature of these rules, the full result for the NLO amplitude, even truncated to next-to-soft order in k, is rather lengthy. We focus only on the non-momentum shift contributions, stemming from the seagull and three-graviton vertex graphs in figure 8. The sum of these contributions as $k \to 0$ is given by

$$\begin{aligned}\tilde{\mathcal{M}}^{\mu\nu} = &-\frac{i\kappa^3}{8t}\Big[(m_1^2+m_2^2-s)\left(p_1^{(\mu}p_2^{\nu)}+p_3^{(\mu}p_4^{\nu)}\right)+(m_1^2+m_2^2-s-t)\left(p_1^{(\mu}p_4^{\nu)}+p_2^{(\mu}p_3^{\nu)}\right)\\ &-t(p_1^{(\mu}p_3^{\nu)}+p_2^{(\mu}p_4^{\nu)})\Big]+\kappa\left[\frac{(p_1-p_3)^\mu(p_1-p_3)^\nu}{(p_1-p_3)^2}+\frac{\eta^{\mu\nu}}{2}\right]\tilde{\mathcal{M}}_{\rm LO}.\end{aligned}\tag{3.31}$$

As in the QCD case, this should be compared with the first term of eq. (3.24), and the result is

$$\begin{aligned}\tilde{\mathcal{M}}^{\mu\nu} = &-\frac{i\kappa^3}{8t}\Big[(m_1^2+m_2^2-s)\left(p_1^{(\mu}p_2^{\nu)}+p_3^{(\mu}p_4^{\nu)}\right)+(m_1^2+m_2^2-s-t)\left(p_1^{(\mu}p_4^{\nu)}+p_2^{(\mu}p_3^{\nu)}\right)\\ &-t(p_1^{(\mu}p_3^{\nu)}+p_2^{(\mu}p_4^{\nu)})\Big]+\kappa\left[\frac{(p_1-p_3)^\mu(p_1-p_3)^\nu}{(p_1-p_3)^2}\right]\tilde{\mathcal{M}}_{\rm LO},\end{aligned}\tag{3.32}$$

which agrees with eq. (3.31) apart from a term involving $\eta^{\mu\nu}$, and proportional to the Born amplitude. This contribution vanishes when contracted with a physical graviton polarisation tensor, and hence eq. (3.24) indeed reproduces all internal emission contributions provided the additional graviton emission is on-shell. For off-shell gravitons, however, it constitutes an explicit breaking of the next-to-soft theorem. The absence of this breaking in the QCD case is perhaps not surprising — there is no invariant tensor with one index that could contribute such a term in a vector theory.

3.4 Seagull and vertex contributions in QCD

The above analysis implies that we must calculate internal emission effects by a more direct method. To this end it is useful, as in the above discussion, to separate the contributions from the seagull and three-boson vertex graphs, from the momentum-shift contributions obtained by dressing the shifted Born amplitude with eikonal Feynman rules. For on-shell emissions, these two types of internal emission correspond exactly to the first and second terms in eqs. (3.23), (3.24) respectively, and we begin by examining the former. The relevant Feynman diagrams are shown in figure 9, and we may write the first of these as

$$\tilde{\mathcal{A}}_{(A)} = \int\frac{d^dk_1}{(2\pi)^d}\frac{c_A\, n_A(\{p_i\},k_1)}{(k_1^2+i\varepsilon)[(p_1-p_3)^2+i\varepsilon][(p_2+k_1)^2-m_2^2+i\varepsilon][(p_1-p_3-k_1)^2+i\varepsilon]},\tag{3.33}$$

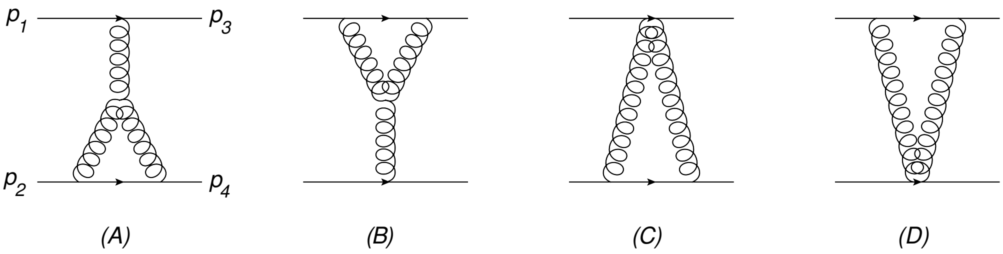

Figure 9. Seagull and triangle diagrams entering the QCD internal emission corrections.

where the colour factor and kinematic numerator are

$$c_A = f^{abc}\,\mathbf{T}_U^a\,\mathbf{T}_L^b\,\mathbf{T}_L^c \tag{3.34}$$

and

$$\begin{aligned} n_A(\{p_i\},k_1) = iV_{\phi\phi g}^{\alpha_1}(p_1,-p_3)\,P_{\alpha_1\alpha_2}\,V_{ggg}^{\alpha_2\beta_2\gamma_2}[p_1-p_3,-(p_1-p_3-k_1),-k_1] \\ \times P_{\gamma_2\gamma_1}\,P_{\beta_2\beta_1}\,V_{\phi\phi g}^{\gamma_1}[p_2,-(p_2+k_1)]\,V_{\phi\phi g}^{\beta_1}(p_2+k_1,-p_4) \end{aligned} \tag{3.35}$$

respectively. Here

$$V_{\phi\phi g}^{\mu}(p_1,p_2) = ig_s(p_1^\mu - p_2^\mu) \tag{3.36}$$

and

$$V_{ggg}^{\alpha\beta\gamma}(p_1,p_2,p_3) = g_s\left[\eta^{\alpha\beta}(p_1-p_2)^\gamma + \eta^{\beta\gamma}(p_2-p_3)^\alpha + \eta^{\alpha\gamma}(p_3-p_1)^\beta\right] \tag{3.37}$$

are the scalar-scalar-gluon and three-gluon vertices with all momenta incoming, and we have defined

$$P_{\alpha\beta} = -i\eta_{\alpha\beta} \tag{3.38}$$

to be the numerator of the Feynman gauge gluon propagator. To extract the next-to-soft contribution from eq. (3.33), one may introduce an additional delta function as in ref. [87] to rewrite this as

$$\tilde{\mathcal{A}}_{(A)} = (2\pi)^d \int \frac{d^dk_1}{(2\pi)^d}\int\frac{d^dk_2}{(2\pi)^d}\frac{\delta^{(d)}(k_1+k_2-q)\,c_A\,n_A}{(k_1^2+i\varepsilon)(k_2^2+i\varepsilon)[(k_1+k_2)^2+i\varepsilon][(p_2+k_1)^2-m_2^2+i\varepsilon]}, \tag{3.39}$$

such that k_1 and k_2 are now the momenta of the lower two gluons in figure 9(A), and we have introduced the momentum transfer 4-vector (conjugate to z^μ)

$$q^\mu = (p_1 - p_3)^\mu. \tag{3.40}$$

The momenta k_1 and k_2 are on an equal footing, so that to isolate next-to-soft contributions, one must expand in both of these momenta. Returning to the original integral of eq. (3.33), this can be achieved by writing

$$p_3 = p_1 - q, \qquad\qquad p_4 = p_2 + q, \tag{3.41}$$

before scaling

$$q \to \lambda q, \qquad\qquad k_1 \to \lambda k_1, \tag{3.42}$$

and expanding to next-to-soft order in λ. Finally, one may set $\lambda \to 1$. The result may be written

$$\begin{aligned}\tilde{\mathcal{A}}_{(A)} = &-\frac{4ig_s^4\mu^{4\epsilon}c_A}{q^2}\Big\{q^2(s-m_1^2-m_2^2)S(p_2) + \left[2p_2^\mu(s-m_1^2-m_2^2) - 4m_2^2p_1^\mu + 2m_2^2q^\mu\right]V_\mu(p_2)\\ &+[-2p_1^\mu p_2^\nu + (s-m_1^2-m_2^2)\eta^{\mu\nu}]T_{\mu\nu}(p_2)\Big\},\end{aligned} \tag{3.43}$$

where we have defined the scalar, vector and tensor integrals

$$\begin{aligned} S(p_i) &= \int \frac{d^dk}{(2\pi)^d}\frac{1}{(k^2+i\varepsilon)[(q-k)^2+i\varepsilon](2p_i\cdot k+i\varepsilon)};\\ V^\mu(p_i) &= \int \frac{d^dk}{(2\pi)^d}\frac{k^\mu}{(k^2+i\varepsilon)[(q-k)^2+i\varepsilon](2p_i\cdot k+i\varepsilon)};\\ T^{\mu\nu}(p_i) &= \int \frac{d^dk}{(2\pi)^d}\frac{k^\mu k^\nu}{(k^2+i\varepsilon)[(q-k)^2+i\varepsilon](2p_i\cdot k+i\varepsilon)}.\end{aligned} \tag{3.44}$$

We calculate these in appendix B, and the final result for diagram (A) is

$$\tilde{\mathcal{A}}_{(A)} = \frac{g_s^4 c_A(m_1^2+m_2^2-s)}{16m_2|\vec{q}|} + \mathcal{O}(\epsilon), \tag{3.45}$$

where $\vec{q}$ is the (two-dimensional) momentum transfer defined in eq. (2.22). Diagram (B) can be obtained by flipping diagram (A), yielding

$$\tilde{\mathcal{A}}_{(B)} = \frac{g_s^4 c_B(m_1^2+m_2^2-s)}{16m_1|\vec{q}|} + \mathcal{O}(\epsilon), \qquad\qquad c_B = f^{abc}\mathbf{T}_U^b\mathbf{T}_U^c\mathbf{T}_L^a. \tag{3.46}$$

Next, one has the seagull graph of figure 9(C). We may write this as

$$\tilde{\mathcal{A}}_{(C)} = \int\frac{d^dk_1}{(2\pi)^d}\frac{c_C\, n_C(\{p_i\},k_1)}{(k_1^2+i\varepsilon)[(p_1-p_3-k_1)^2+i\varepsilon][(p_2+k_1)^2-m_2^2+i\varepsilon]}, \tag{3.47}$$

where the colour factor and kinematic numerator are

$$c_C = \{\mathbf{T}_U^a, \mathbf{T}_U^b\}\mathbf{T}_L^a\mathbf{T}_L^b \tag{3.48}$$

and

$$n_C(\{p_i\},k_1) = iV_{\phi\phi gg}^{\alpha_1\beta_1}P_{\alpha_1\alpha_2}P_{\beta_1\beta_2}V_{\phi\phi g}^{\alpha_2}[p_2,-(p_2+k_1)]V_{\phi\phi g}^{\beta_2}(p_2+k_1,-p_4), \tag{3.49}$$

where

$$V_{\phi\phi gg}^{\mu\nu} = ig_s^2\eta^{\mu\nu} \tag{3.50}$$

is the kinematic part of the seagull vertex. One may expand this according to the procedure of eqs. (3.41) and (3.42), and the result is

$$\tilde{\mathcal{A}}_{(C)} = -4g_s^4\, c_C\, m_2^2\, S(p_2) = \frac{ig_s^4\, c_C\, m_2}{8|\vec{q}|} + \mathcal{O}(\epsilon). \tag{3.51}$$

Likewise, one has

$$\tilde{\mathcal{A}}_{(D)} = \frac{i g_s^4\, c_D\, m_1}{8|\vec{q}|} + \mathcal{O}(\epsilon), \qquad\qquad c_D = \mathbf{T}_U^a\, \mathbf{T}_U^b \{\mathbf{T}_L^a, \mathbf{T}_L^b\}. \tag{3.52}$$

In order to further interpret these results, it is useful to rewrite the colour factors in terms of the Born colour factor $\mathbf{T}_U^a \mathbf{T}_L^a$, and the quadratic Casimir operators of eq. (2.10). One has

$$\begin{aligned}
\mathbf{T}_s^2 \mathbf{T}_U^a \mathbf{T}_L^a &= (C_1 + C_2)\mathbf{T}_U^a \mathbf{T}_L^a + 2\mathbf{T}_U^b \mathbf{T}_U^a \mathbf{T}_L^b \mathbf{T}_L^a;\\
\mathbf{T}_u^2 \mathbf{T}_U^a \mathbf{T}_L^a &= (C_1 + C_2)\mathbf{T}_U^a \mathbf{T}_L^a - 2\mathbf{T}_U^b \mathbf{T}_U^a \mathbf{T}_L^a \mathbf{T}_L^b;\\
\mathbf{T}_t^2 \mathbf{T}_U^a \mathbf{T}_L^a &= C_A \mathbf{T}_U^a \mathbf{T}_L^a,
\end{aligned} \tag{3.53}$$

such that the various colour factors above can be written

$$c_A = c_B = \frac{i}{2}\mathbf{T}_t^2 \mathbf{T}_U^a \mathbf{T}_L^a, \qquad\qquad c_C = c_D = \frac{(\mathbf{T}_s^2 - \mathbf{T}_u^2)}{2}\mathbf{T}_U^a \mathbf{T}_L^a. \tag{3.54}$$

The total contribution from the diagrams of figure 9 is then

$$\begin{aligned}
\tilde{\mathcal{A}}_{A-D} &= \frac{i g_s^4}{32|\vec{q}|}\left[(m_1^2 + m_2^2 - s)\left(\frac{1}{m_1} + \frac{1}{m_2}\right)\mathbf{T}_t^2 + 2(m_1 + m_2)(\mathbf{T}_s^2 - \mathbf{T}_u^2)\right]\mathbf{T}_U^a \mathbf{T}_L^a + \mathcal{O}(\epsilon)\\
&\to -\frac{i g_s^4}{|\vec{q}|}\frac{s}{32}\left(\frac{1}{m_1} + \frac{1}{m_2}\right)\mathbf{T}_t^2 \mathbf{T}_U^a \mathbf{T}_L^a + \mathcal{O}(\epsilon),
\end{aligned} \tag{3.55}$$

where we have taken the Regge limit in the second line. One may Fourier transform this result back to impact parameter space, where it becomes

$$\mathcal{A}_{A-D} = -\frac{i g_s^4}{|\vec{z}|}\frac{s}{64\pi}\left(\frac{1}{m_1} + \frac{1}{m_2}\right)\mathbf{T}_t^2 \mathbf{T}_U^a \mathbf{T}_L^a + \mathcal{O}(\epsilon). \tag{3.56}$$

Comparing this with eq. (3.14), we see that the form of eq. (3.56) is the same as that of the external emission correction, namely a t-channel Casimir acting on the Born colour factor, with a real coefficient. Were one able to exponentiate eq. (3.56), it would thus correspond to a power-suppressed correction to the Regge trajectory, rather than the eikonal phase. For the external emission contributions, exponentiation follows immediately from the fact that such terms are described by generalised Wilson line operators [54, 82]. For the internal emission contributions, there is no such argument for exponentiation. However, one can still choose to exponentiate them: expanding the exponential will result in higher powers of next-to-soft terms, which are then higher order in the momentum expansion, and thus of the same formal accuracy as the non-exponentiated result.

3.5 Seagull and vertex contributions in gravity

We may repeat the above analysis for gravity, by replacing the gluons in figure 9 with gravitons. Given that intermediate results are a great deal more cumbersome, we here report the final results only. Diagrams (A) and (C) are found to be given in momentum space by

$$\begin{aligned}
\tilde{\mathcal{M}}_{(A)} &= -\frac{i\kappa^4 m_2}{2048|\vec{q}|}\left[(m_1^2 + m_2^2 - s)^2 + 12 m_1^2 m_2^2\right] + \mathcal{O}(\epsilon);\\
\tilde{\mathcal{M}}_{(C)} &= \frac{i\kappa^4 m_2}{128|\vec{q}|}\left[m_1^2 + m_2^2 - s\right]^2 + \mathcal{O}(\epsilon).
\end{aligned} \tag{3.57}$$

As before, diagrams (B) and (D) can be obtained by relabelling $m_1 \leftrightarrow m_2$. The sum of all contributions is then

$$\begin{aligned}\tilde{\mathcal{M}}_{A-D} &= \frac{i\kappa^4(m_1+m_2)}{2048|\vec{q}|}\left[15(m_1^2+m_2^2-s)^2-12m_1^2m_2^2\right]+\mathcal{O}(\epsilon)\\ &\to \frac{15i\kappa^4s^2(m_1+m_2)}{2048|\vec{q}|}+\mathcal{O}(\epsilon),\end{aligned} \tag{3.58}$$

where we have taken the Regge limit in the second line.

It is interesting to compare eq. (3.58) with its counterpart in QCD, eq. (3.55). Up to colour diagonal terms, the QCD result has a term involving a t-channel Casimir that dominates in the Regge limit, and a suppressed contribution involving the s-channel Casimir (n.b. one may eliminate $\mathbf{T}_u^2$ in eq. (3.55) using eq. (2.11)). One expects something like the replacements of eq. (2.19) in moving to the gravity result, so that the t-channel result is subleading, and the s-channel term dominant. Indeed the form of the second term in the brackets of eq. (3.55) is qualitatively the same as eq. (3.58) under eq. (2.19), together with the additional replacements

$$\mathbf{T}^a_{U,L} \to p^\mu_{1,2}, \tag{3.59}$$

consistent with colour generators on the upper and lower lines corresponding to momenta of these lines in gravity (n.b. one may equally choose p_3 and p_4 in this correspondence, given that $p_1 \simeq p_3$ and $p_2 \simeq p_4$ up to subleading corrections). This is analogous to how, at eikonal level, Reggeisation is the leading effect in QCD, whereas the eikonal phase is more important in gravity. Note that the coefficient of the s-channel term in QCD is not simply related to that in gravity, which naïvely suggests that there is no double copy relationship between these quantities. This is misleading for a number of reasons. Firstly, the double copy only formally applies at integrand level, rather than after integrating over the loop momentum. Secondly, for the double copy to work for the seagull and vertex contributions, one must choose a (generalised) gauge such that BCJ duality is manifest in QCD. Here we have used the Feynman and de Donder gauges in QCD and gravity respectively, which may obscure a direct double copy. That a double copy is possible for these graphs, however, follows from the results of ref. [102].

3.6 Momentum shift contributions

According to the discussion of section 3.3, the remaining internal emission contributions comprise the Born interaction evaluated with shifted momentum, dressed by an additional eikonal emission. Again regarding as nonzero only those diagrams which are regulated by the impact factor, the relevant diagrams are those of figure 3(a)–(d), where the Born amplitude is shifted appropriately.

Focusing first on the case of QCD, the momentum shift contribution from diagram (a) is given by

$$\tilde{\mathcal{A}}_a^{\text{mom.}} = -ig_s^2\mu^{2\epsilon}\mathbf{T}_1\cdot\mathbf{T}_2 p_1\cdot p_2\int\frac{d^dk}{(2\pi)^d}\frac{\tilde{\mathcal{A}}_{\rm LO}(p_1-k,p_2+k)}{(k^2+i\varepsilon)(-p_1\cdot k+i\varepsilon)(p_2\cdot k+i\varepsilon)}\bigg|_{\mathcal{O}(k)}, \tag{3.60}$$

where the numerator contains the Born amplitude of eq. (3.26), and taking the $\mathcal{O}(k)$ piece isolates the effect of including a single momentum shift (i.e. terms $\mathcal{O}(k^2)$ are next-to-next-to-soft). Substituting eq. (3.26) into eq. (3.60), the latter becomes

$$\tilde{\mathcal{A}}_a^{\text{mom.}} = \frac{g_s^4 \mu^{4\epsilon}}{2} [\mathbf{T}_1 \cdot \mathbf{T}_2 \mathbf{T}_U^a \mathbf{T}_L^a] \; p_1 \cdot p_2 \; (p_1 + p_3 - p_2 - p_4)_\mu V_{\text{box}}^\mu(-p_1, p_2), \tag{3.61}$$

where we have defined the vector box integral

$$V_{\text{box}}^\mu = \int \frac{d^d k}{(2\pi)^d} \frac{k^\mu}{(k^2 + i\varepsilon)[(k-q)^2 + i\varepsilon](\sigma_i p_i \cdot k + i\varepsilon)(\sigma_j p_j \cdot k + i\varepsilon)}. \tag{3.62}$$

We calculate this integral in appendix B, and the result for diagram (a) is (taking the Regge limit)

$$\tilde{\mathcal{A}}_a^{\text{mom.}} = \frac{i g_s^4 s [\mathbf{T}_1 \cdot \mathbf{T}_2 \mathbf{T}_U^a \mathbf{T}_L^a](m_1 + m_2)}{16 |\vec{q}| m_1 m_2}. \tag{3.63}$$

A similar analysis for diagram (c) yields

$$\tilde{\mathcal{A}}_b^{\text{mom.}} = -\frac{i g_s^4 s [\mathbf{T}_1 \cdot \mathbf{T}_4 \mathbf{T}_U^a \mathbf{T}_L^a](m_1 + m_2)}{16 |\vec{q}| m_1 m_2}, \tag{3.64}$$

where we have expanded about $d = 4$. Diagrams (b) and (d) are equal to (a) and (c) respectively (n.b. they can be simply obtained by relabelling masses and colour generators), so that the final result for the sum of all diagrams is

$$\tilde{\mathcal{A}}_{a-d}^{\text{mom.}} = -\frac{i g_s^4 s (m_1 + m_2)}{16 |\vec{q}| m_1 m_2} \mathbf{T}_t^2 \mathbf{T}_U^a \mathbf{T}_L^a. \tag{3.65}$$

It is straightforward to carry the above analysis over to gravity e.g. in eq. (3.60) one simply replaces the prefactors with those arising from the gravitational eikonal Feynman rules, and the Born amplitude in the integrand with that of eq. (2.21). The final result for the momentum shift contribution upon summing all diagrams is

$$\tilde{\mathcal{M}}_{a-d}^{\text{mom.}} = -\frac{i \kappa^4 s^2 t (m_1 + m_2)}{256 |\vec{q}| m_1 m_2}. \tag{3.66}$$

Similarly to the external emisson contributions in section 3.2, this result can be obtained from the QCD expression by the replacements of eq. (3.18), (3.59). There is an additional factor of 2 in eq. (3.66) relative to eq. (3.65) after making the replacements, which factor has also been explained in section 3.2.

In both QCD and gravity, the momentum shift contributions contain a t-channel Casimir, and thus correspond to shifts in the Regge trajectory of the gluon / graviton. In gravity, this contribution is subleading in t and can be discarded. In QCD, the result involves power-like collinear divergences which can be absorbed into the impact factors coupling the incoming particles to the Reggeised gluon. That the momentum shift contributions have the same form as the external emission contributions of section 3.2 is not surprising. Here we have drawn a distinction between the gluon entering the Born amplitude, and the external gluons described by generalised Wilson line operators. Another

JHEP01(2017)052

approach is to consider all exchanged gluons symmetrically, in which case the momentum shift and external emission contributions are on an equal footing. The latter approach is taken in ref. [87], which indeed neglects the momentum shift contributions in gravity as being subleading.

This now completes our calculation of all contributions to $2 \to 2$ scattering in the high energy limits of QCD and gravity that are of first subleading order in the momentum transfer. The more detailed interpretation of these results is the subject of the following section.

4 Discussion

In this section, our aim is to draw together the various results of this paper and discuss their implications in more detail, making contact with previous calculations in the literature. The complete next-to-soft corrections in either QCD or gravity are obtained by summing the external and internal emission contributions. As discussed above and in refs. [54, 82], the former formally exponentiate, as a direct consequence of being described by generalised Wilson line operators. The internal emission contributions can be chosen to exponentiate, given that higher order terms generated by the exponentiation are progressively subleading in the impact factor expansion. Upon doing so, all of the QCD contributions in eqs. (3.13), (3.56), (3.65) correspond to subleading corrections to the Regge trajectory of the gluon. As already noted in section 3, this correction consists of purely singular terms as $m_i \to 0$, associated with the exchanged gluons becoming collinear with one of the external lines. These divergences are not problematic in practice, as according to the Regge limit of eq. (2.3), one cannot take $m_i \to 0$ whilst keeping t fixed. One way around this is to consider the alternative Regge limit

$$s \gg -t \gg m_i^2, \tag{4.1}$$

and to include diagrams such as figure 3(e) and (f), with a suitable regulator to remove the short-distance singularity. In the eikonal calculation of ref. [5] (reviewed here in section 2), the inclusion of the additional diagrams explicitly removes collinear singularities from the Regge trajectory, such that they can be absorbed in so-called *impact factors* associated with the external lines. Their inclusion in the Regge trajectory is then a rather unphysical scheme choice, and thus there is little merit in interpreting the QCD calculation further.

The situation in gravity is more interesting. As already remarked in sections 3.2 and 3.6, the external emission and momentum shift contributions are kinematically subleading, mimicking the suppression of the Regge trajectory at eikonal level. The only surviving contribution then comes from the seagull and vertex graphs, and is given in eq. (3.58). Combining this with the eikonal amplitude of eq. (2.25), one may write [87]

$$\begin{aligned}\mathcal{M}(\vec{z}) &= 2s\left[e^{i\chi_{\rm E}(\vec{z})}\left(1+i\chi_{\rm NE}(\vec{z})\right)-1\right]\\ &= 2s\left[e^{i(\chi_{\rm E}(\vec{z})-i\ln[1+i\chi_{\rm NE}(\vec{z})])}-1\right],\end{aligned} \tag{4.2}$$

where we have defined

$$\chi_{\rm NE} = \frac{15\kappa^4 s^2(m_1+m_2)}{4096\pi|\vec{z}|}, \tag{4.3}$$

obtained by Fourier transforming eq. (3.58) to position space. In the second line of eq. (4.2) we have written the NE contribution as the exponential of its own logarithm. Provided that $\chi_{\rm NE}$ is small, however, one may expand the logarithm so that the amplitude assumes the simpler form of eq. (2.25), but with a total phase

$$\chi = \chi_{\rm E} + \chi_{\rm NE} = G_N s \mu^{2\epsilon}\left[-\frac{|\vec{z}|^{2\epsilon}}{\epsilon} + \frac{15\pi G_N(m_1+m_2)}{8|\vec{z}|}\right]. \tag{4.4}$$

This approximation is valid provided the impact parameter is large, or conversely if the momentum transfer is small relative to the centre of mass energy. This is precisely the Regge limit of eq. (2.3). One may now consider the momentum space amplitude

$$\tilde{\mathcal{M}}(\vec{q}) = \int d^{d-2}\vec{z}\, e^{-i\vec{z}\cdot\vec{q}}\,\mathcal{M}(\vec{z}), \tag{4.5}$$

where the exponential integral will be dominated by the saddle point, leading to the stationary phase condition

$$\vec{q} = \frac{\partial\chi}{\partial|\vec{z}|}\frac{\vec{z}}{|\vec{z}|}. \tag{4.6}$$

To interpret this result, let us first consider the case that $m_2 \gg m_1$. This is the situation considered in ref. [87], and one may then parametrise

$$p_1^\mu = E_1(1,0,0,1), \qquad p_2^\mu = (m_2,0,0,0), \qquad z^\mu = (0,0,|\vec{z}|,0). \tag{4.7}$$

The 4-momentum of the first particle after scattering is

$$p_1'^{\,\mu} = E_1(1,0,\sin\theta,\cos\theta), \tag{4.8}$$

where θ is the scattering angle. This in turn implies

$$q^\mu = p_1'^{\,\mu} - p_1^\mu = E_1(0,0,\sin\theta,1-\cos\theta) \quad \Rightarrow \quad \vec{q}\cdot\vec{z} = -E_1|\vec{z}|\sin\theta \simeq -E_1|\vec{z}|\theta, \tag{4.9}$$

with the small angle approximation justified by the Regge limit. Equation (4.6) then gives

$$\theta \;=\; -\frac{1}{E_1}\frac{\partial\chi}{\partial|\vec{z}|} \;=\; \frac{2R_2}{|\vec{z}|} + \frac{15\pi}{16}\left(\frac{R_2}{|\vec{z}|}\right)^2 + \ldots \tag{4.10}$$

where $R_2 = 2G_N m_2$ is the Schwarzschild radius associated with the mass m_2, and we have used $s \simeq 2E_1 m_2$. The ellipsis denotes higher order terms in the inverse impact parameter, which mix with corrections to the next-to-soft approximation and can therefore be neglected. Equation (4.10) does indeed correspond to the classical deflection angle experienced by a light test particle scattering on a black hole (see e.g. ref. [42],[11] and ref. [88] for a recent derivation). Moreover, the simple form of eq. (4.3) is independent of

[11] We are very grateful to Rodolfo Russo for providing unpublished notes relating to the specific case of the Schwarzschild black hole in four dimensions.

whether the mass m_2 is small or asymptotically large relative to s. Thus, it applies equally to the case of a test particle scattering off a black hole, or from a boosted mass, the extremal case of which is an Aichelburg-Sexl shockwave [103]. This can be further understood from the fact that at $\mathcal{O}(G_N)$ one can form two independent dimensionless combinations from m_i, s and $|\vec{z}|$:

$$\frac{G_N m_i}{|\vec{z}|}, \qquad \frac{m_i^2}{s}, \tag{4.11}$$

where the first is fixed by the requirement that one expands to next-to-soft level in the impact parameter only. In the Regge limit, the second combination is zero, which uniquely fixes the next-to-eikonal phase to be linear in the mass of each particle. The symmetry of eq. (4.3) under interchange of the two masses shows that the same deflection angle would be experienced by particle 2 treated as a test particle scattering off particle 1. Thus, the ultimate interpretation of our general next-to-soft calculation is that it reproduces the two independent classical deflections experienced by each incoming particle, treated as a test particle in the field of the other particle.

The above discussion relates directly to the investigation of ref. [36], which reconsidered transplanckian scattering in a variety of supersymmetric extensions of gravity, arguing that additional particle content (and thus the presence or absence of UV renormalisability) is irrelevant at leading power in the transplanckian regime. It was pointed out that the complete geometry corresponding to two colliding shockwaves is not known, and conjectured that at first subleading level in the momentum expansion of exchanged gravitons, each incoming shockwave should experience a classical deflection angle due to the gravitational field of the other shock. The present analysis precisely confirms this view. It is also consistent with the known fact that the scattering angle at eikonal level is the same for a Schwarzschild black hole as for a shockwave (see e.g. [104]), and indeed generalises this result to subleading order in the impact parameter.

Some further comments are in order regarding the fact that we have expanded the logarithm in eq. (4.2). This approximation is justified when the impact parameter is large, and amounts to exponentiating the full NE phase. This has been argued to be correct even for smaller impact parameters, given that at sufficiently large s the NE correction to the fixed order scattering amplitude violates unitarity [42]. Reference [87] suggested that the seagull and vertex graphs formed part of the gravitational Wilson line operator, and thus could be exponentiated. This is not immediately borne out in our approach. However, it may well be that the $\mathcal{O}(\kappa^2)$ terms in the generalised Wilson operator of eq. (3.3) generate multiple copies of the seagull and vertex graphs, in which case a full exponentiation of these contributions could be formally proven.

5 Conclusion

In this paper, we have examined the high energy (Regge) limit of $2 \to 2$ scattering in QCD and gravity, extending previous results to include corrections subleading by a single power of the impact factor. This generalises previous gravity results for massless particles [32–35], and for the case in which only one particle is taken to be highly massive [87, 88]. To the best of our knowledge, no analogous calculations have been carried out in QCD.

Our calculational approach builds upon a well-known description of the Regge limit (at eikonal level) as two Wilson lines separated by a transverse distance, developed for QCD in refs. [2, 3], and applied to gravity in ref. [5]. The generalisation to next-to-soft level uses the generalised Wilson line approach of refs. [54, 82], which has a number of significant advantages. Firstly, vacuum expectation values of generalised Wilson line operators automatically exponentiate, completely circumventing the combinatorial complexities of diagrammatic analyses such as that of ref. [87] (although, of course, the latter approach remains useful in its own right). Secondly, the language of generalised Wilson lines reveals that the calculations in QCD and gravity are extremely similar, even if the physical interpretation of the results is completely different. This hints at a deeper underlying relationship between the two theories, and indeed our results (as discussed in detail throughout) are entirely consistent with the double copy of refs. [46–48].

In QCD, we have found a correction to the Regge trajectory of the gluon, suppressed by a power of the impact parameter, and which is also purely collinearly singular. This can be removed from the Regge trajectory by absorbing this correction into impact factors associated with the incoming particles. However, it would be interesting to see whether similar methods to those in this paper could be used to study further power-suppressed terms (in t/s) in the Regge limit of supergravity theories, whose classification remains elusive (see ref. [5] for a recent discussion).

In gravity, we have found a general correction to the eikonal phase, valid for arbitrary masses of the incoming particles. The interpretation of this correction is that it describes the deflection angle associated by each particle, considered as a test particle in the gravitational field of the other. This precisely confirms the picture conjectured recently in ref. [36], which discussed possible interpretations of corrections to eikonal scattering in supergravity theories.

In calculating contributions stemming from soft gluons or gravitons emanating from inside the hard interaction, we have found that the gravity next-to-soft theorem of eq. (3.24) is not sufficient, but must be supplemented by an additional term proportional to the metric tensor (and which would vanish upon contraction with a physical polarisation tensor). This seems at odds with the fact that the result of our gravity calculation is to reproduce a purely classical effect. It may be that the correction term is a purely gauge-dependent artifact, but in any case the generalisation of next-to-soft theorems for off-shell gauge bosons perhaps deserves further study.

Finally, we hope that our paper motivates the further use of (generalised) gravitational Wilson lines, which have been relatively unexplored. We believe that they provide an elegant, and panoramic insight into non-abelian gauge theories and gravity, and our investigation of further applications is in progress.

Acknowledgments

We are very grateful to Pierre Vanhove and Emil Bjerrum-Bohr for clarification regarding the three-graviton vertex, and to Ratin Akhoury and Rodolfo Russo for numerous discussions and comparisons. CDW further thanks Domenico Bonocore, Einan Gardi, Eric

Laenen, Lorenzo Magnea, and Leonardo Vernazza for conversations and collaboration on related projects. AL is supported by a Conacyt studentship, and thanks the Centre for Research in String Theory at Queen Mary University of London for hospitality. SM and CDW are supported by the U.K. Science and Technology Facilities Council (STFC). SGN is supported by the National Science Foundation under Grant No. PHY14-16123. SGN also gratefully acknowledges sabbatical support from the Simons Foundation (Grant No. 342554 to Stephen Naculich). The authors also thank the Michigan Center for Theoretical Physics (University of Michigan) and the Higgs Centre for Theoretical Physics (University of Edinburgh), where part of this research was carried out.

A Calculation of the master integral $V^{\mu}_{\rm NE}$

In this appendix, we calculate the integral of eq. (3.11). One may first set

$$s_i = \frac{\sqrt{\vec{z}^2}}{m_i} st, \quad s_j = \frac{\sqrt{\vec{z}^2}}{m_j} s, \tag{A.1}$$

so that eq. (3.11) becomes

$$\begin{aligned} V^{\mu}_{\rm NE}(\sigma_i p_i, \sigma_j p_j) = \frac{|\vec{z}|^{3-d}}{m_i m_j} \int_0^\infty dt \int_0^\infty ds\, s \left(\hat{z}^\mu + st\sigma_i \frac{p_i^\mu}{m_i} + s\sigma_j \frac{p_j^\mu}{m_j} \right) \\ \times \left[1 - s^2(t^2 + 2\sigma t \cosh\gamma_{ij} + 1 - i\varepsilon) + i\varepsilon\right]^{-d/2}, \end{aligned} \tag{A.2}$$

where $\hat{z}^\mu = z^\mu/|\vec{z}|$ and $\sigma = \sigma_i\sigma_j$. For convenience, let us now rewrite this as

$$V^{\mu}_{\rm NE}(\sigma_i p_i, \sigma_j p_j) = \frac{|\vec{z}|^{3-d}}{m_i m_j} \left[\hat{z}^\mu V_z + \frac{\sigma_i p_i^\mu}{m_i} V_i + \frac{\sigma_j p_j^\mu}{m_j} V_j \right]. \tag{A.3}$$

We will not need to calculate the coefficient V_z, due to the fact that the master integral is only ever contracted with one of the external lines, and $p_i \cdot z = 0$. The coefficient V_j is given by

$$\begin{aligned} V_j &= \int_0^\infty dt \int_0^\infty ds\, s^2 \left[1 - s^2(t^2 + 2\sigma t \cosh\gamma_{ij} + 1 - i\varepsilon) + i\varepsilon\right]^{-d/2} \\ &= \sinh\gamma_{ij} \int_{\sigma \coth\gamma_{ij}}^\infty dx \int_0^\infty ds\, s^2 \left[1 - s^2 \sinh^2\gamma_{ij}(x^2 - 1 - i\varepsilon) + i\varepsilon\right]^{-d/2}, \end{aligned} \tag{A.4}$$

where we have set $t = x\sinh\gamma_{ij} - \sigma\cosh\gamma_{ij}$ in the second line. Upon making the substitution

$$s = \sqrt{\frac{u}{1-u}}, \quad ds = \frac{1}{2}\sqrt{\frac{1}{u(1-u)^3}}, \tag{A.5}$$

the s integral in eq. (A.4) becomes

$$\begin{aligned} &\frac{1}{2}\int_0^1 u^{1/2}(1-u)^{(d-5)/2} \left[1 - u(1 + \sinh^2\gamma_{ij}(x^2 - 1 - i\varepsilon) + i\varepsilon)\right]^{-d/2} \\ &= \frac{\Gamma(\frac{3}{2})\Gamma\left(\frac{d}{2} - \frac{3}{2}\right)}{2\Gamma\left(\frac{d}{2}\right)} {}_2F_1\left(\frac{d}{2}, \frac{3}{2}; \frac{d}{2}; 1 + \sinh^2\gamma_{ij}(x^2 - 1 - i\varepsilon) + i\varepsilon\right) \\ &= \frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{d}{2} - \frac{3}{2}\right)}{2\Gamma\left(\frac{d}{2}\right)} \frac{1}{[-\sinh^2\gamma_{ij}(x^2 - 1 - i\varepsilon) + i\varepsilon]^{3/2}}, \end{aligned} \tag{A.6}$$

JHEP01(2017)052

where we have used the identity

$$ {}_2F_1(a,b;a;z) = (1-z)^{-b}. \tag{A.7} $$

Equation (A.4) now becomes

$$ V_j = \frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{d}{2}-\frac{3}{2}\right)}{2\Gamma\left(\frac{d}{2}\right)\sinh^2\gamma_{ij}} \int_{\sigma\coth\gamma_{ij}}^{\infty} \frac{dx}{(1-x^2+i\varepsilon)^{3/2}}. \tag{A.8} $$

A careful contour integration gives

$$ \int_{\sigma\coth\gamma_{ij}}^{\infty} \frac{dx}{(1-x^2+i\varepsilon)^{3/2}} = i(\sigma\cosh\gamma_{ij}-1), \tag{A.9} $$

so that

$$ V_j = \frac{i\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{d}{2}-\frac{3}{2}\right)}{2\Gamma\left(\frac{d}{2}\right)(1+\sigma\cosh\gamma_{ij})}. \tag{A.10} $$

Symmetry of eq. (A.2) under $i \leftrightarrow j$ implies that $V_i = V_j$ in eq. (A.3) (n.b. we have also confirmed this by explicit calculation). One thus finally obtains

$$ V^{\mu}_{\mathrm{NE}}(\sigma_i p_i, \sigma_j p_j) = \frac{i\Gamma\left(\frac{d}{2}-\frac{3}{2}\right)}{8\pi^{(d-1)/2}} \frac{|\vec{z}|^{3-d}}{m_i m_j} \left(\frac{\sigma_i p_i^{\mu}}{m_i} + \frac{\sigma_j p_j^{\mu}}{m_j} + \dots \right) \frac{1}{(1+\sigma\cosh\gamma_{ij})}. \tag{A.11} $$

where the ellipsis denotes terms $\propto z^{\mu}$.

B Calculation of internal emission integrals

In this appendix, we calculate the scalar, vector and tensor integrals of eq. (3.44), and the vector box integral of eq. (3.62). Beginning with the scalar case, one may introduce Schwinger parameters according to

$$ \int_0^{\infty} ds e^{is(x+i\varepsilon)} = \frac{i}{x+i\varepsilon}, \tag{B.1} $$

yielding[12]

$$ S(p_i) = i\int \frac{d^d\tilde{k}}{(2\pi)^d} \int_0^{\infty} d\alpha_1 \int_0^{\infty} d\alpha_2 \int_0^{\infty} d\alpha_3 \exp\left[i\left((\alpha_1+\alpha_2)\tilde{k}^2 - \frac{\alpha_3^2 m_i^2}{\alpha_1+\alpha_2} + \frac{\alpha_1\alpha_2 q^2}{\alpha_1+\alpha_2} \right) \right], \tag{B.2} $$

where we have also shifted the momentum variable according to

$$ \tilde{k}^{\mu} = k^{\mu} + \frac{(\alpha_3 p_i - \alpha_2 q)^{\mu}}{\alpha_1+\alpha_2}. \tag{B.3} $$

[12] Note that we have ignored a term $\sim p_i \cdot q$ in the exponent of eq. (B.2). Keeping this term introduces corrections subleading by two powers of $|\vec{q}|$ in the final result, which can therefore be neglected.

Carrying out the momentum integral gives

$$S(p_i) = \; - \; \frac{1}{(4\pi i)^{d/2}} I\left(\frac{d}{2}, 0, 0\right), \tag{B.4}$$

where

$$I(l,m,n) = \int_0^\infty d\alpha_1 \int_0^\infty d\alpha_2 \int_0^\infty d\alpha_3 (\alpha_1+\alpha_2)^{-l} \alpha_2^m \alpha_3^n \exp\left[-\frac{i}{\alpha_1+\alpha_2}\left(\alpha_3^2 m_i^2 - \alpha_1\alpha_2 q^2\right)\right] \tag{B.5}$$

is a master integral that will be convenient in what follows. The α_3 integral is Gaussian, and can be carried out to give

$$I(l,m,n) = \frac{i^{-(n+1)/2}}{2m_i^{n+1}} \Gamma\left(\frac{1+n}{2}\right) \int_0^\infty d\alpha_1 \int_0^\infty d\alpha_2\, \alpha_2^m (\alpha_1+\alpha_2)^{-l+(n+1)/2} \exp\left[\frac{i\alpha_1\alpha_2 q^2}{\alpha_1+\alpha_2}\right]. \tag{B.6}$$

One may now transform

$$\alpha_1 = \alpha x, \quad \alpha_2 = \alpha(1-x), \quad d\alpha_1 d\alpha_2 = \alpha d\alpha dx, \tag{B.7}$$

followed by

$$\alpha = \frac{i\beta}{x(1-x)q^2} \tag{B.8}$$

to get

$$\begin{aligned} I(l,m,n) =& \frac{i^{-n+l-m-3}}{2m_i^{n+1}} \Gamma\left(\frac{1+n}{2}\right) (-q^2)^{l-(n+1)/2-m-2} \int_0^\infty d\beta \beta^{m+1-l+(n+1)/2} e^{-\beta} \\ &\times \int_0^1 dx x^{l-(n+1)/2-m-2} (1-x)^{l-(n+1)/2-2} \\ =& \frac{i^{-n+l-m-3}}{2m_i^{n+1}} \frac{\Gamma\left(\frac{1+n}{2}\right)\Gamma\left(m-l+\frac{n}{2}+\frac{5}{2}\right)\Gamma\left(l-\frac{n}{2}-\frac{3}{2}\right)\Gamma\left(l-\frac{n}{2}-m-\frac{3}{2}\right)}{\Gamma(2l-n-m-3)} |\vec{q}|^{2l-n-2m-5}, \end{aligned} \tag{B.9}$$

where we have defined the square of the two-dimensional momentum transfer via (cf. eq. (2.22))

$$q^2 \simeq -\vec{q}^2. \tag{B.10}$$

Substituting eq. (B.9) into eq. (B.4), the final result for the scalar integral is

$$S(p_i) = -\frac{i\sqrt{\pi}}{2(4\pi)^{d/2}} \frac{|\vec{q}|^{d-5}}{m_i} \frac{\Gamma\left(\frac{5-d}{2}\right)\Gamma^2\left(\frac{d-3}{2}\right)}{\Gamma(d-3)}. \tag{B.11}$$

One may carry out the momentum integrals for the vector and tensor cases in a similar manner. They are given in terms of the master integral of eq. (B.5) as follows:

$$\begin{aligned} V^\mu(p_i) &= -\frac{1}{(4\pi i)^{d/2}} \left[-p_i^\mu I\left(\frac{d}{2}+1,0,1\right) + q^\mu I\left(\frac{d}{2}+1,1,0\right)\right]; \\ T^{\mu\nu}(p_i) &= -\frac{1}{(4\pi i)^{d/2}} \left[p_i^\mu p_i^\nu I\left(\frac{d}{2}+2,0,2\right) - q^{(\mu} p_i^{\nu)} I\left(\frac{d}{2}+2,1,1\right)\right. \\ &\quad \left. + q^\mu q^\nu I\left(\frac{d}{2}+2,2,0\right) + \frac{i}{2}\eta^{\mu\nu} I\left(\frac{d}{2}+1,0,0\right)\right]. \end{aligned} \tag{B.12}$$

Let us now turn to the vector box integral of eq. (3.62). Introducing Schwinger parameters, this is given by

$$\begin{aligned}
V^\mu_{\rm box} &= 4\int \frac{d^dk}{(2\pi)^d}\int_0^\infty d\alpha_1\int_0^\infty d\alpha_2\int_0^\infty d\alpha_3\int_0^\infty d\alpha_4\, k^\mu \\
&\quad\times \exp\left[i\alpha_1 k^2 + i\alpha_2(q-k)^2 + 2i\sigma_i\alpha_3 p_i\cdot k + 2i\sigma_j\alpha_4 p_j\cdot k - \sum_i \alpha_i\epsilon\right] \\
&= -\frac{4i^{1-d/2}}{(4\pi)^{d/2}}\int_0^\infty d\alpha_1\int_0^\infty d\alpha_2\int_0^\infty d\alpha_3\int_0^\infty d\alpha_4\,(\sigma_i\alpha_3 p_i + \sigma_j\alpha_4 p_j - \alpha_2 q)^\mu\,(\alpha_1+\alpha_2)^{-1-d/2} \\
&\quad\times \exp\left[\frac{i\alpha_1\alpha_2 q^2}{\alpha_1+\alpha_2} + \frac{i(-\alpha_3^2 m_i^2 - \alpha_4 m_j^2 - 2\alpha_3\alpha_4\sigma m_i m_j\cosh\gamma_{ij} + i\epsilon)}{\alpha_1+\alpha_2}\right], \qquad \text{(B.13)}
\end{aligned}$$

where we have carried out the momentum integration in the second equality, defined $\sigma = \sigma_i\sigma_j$, and absorbed positive definite factors into ε where necessary. Here the term in q^μ may be ignored, as it will vanish upon contraction with any external momenta. For the term in p_i^μ, one may rescale $\alpha_3 \to \alpha_3\sqrt{\alpha_1+\alpha_2}/m_i$, $\alpha_4 \to \alpha_4\sqrt{\alpha_1+\alpha_2}/m_j$, then make the transformations of eqs. (B.7), (B.8) to carry out the $(\alpha_1,\,\alpha_2)$ integrals, leaving

$$\begin{aligned}
V^\mu_{\rm box}\big|_{p_i^\mu} &= -\frac{4i^{-3/2}}{(4\pi)^{d/2}}\frac{\sigma_i p_i^\mu}{m_i^2 m_j}\frac{\Gamma\left(\frac{5}{2}-\frac{d}{2}\right)\Gamma^2\left(\frac{d}{2}-\frac{3}{2}\right)}{\Gamma(d-3)}|\vec{q}|^{d-5}\int_0^\infty d\alpha_3\int_0^\infty d\alpha_4\,\alpha_3 \\
&\quad\times \exp\left[i(-\alpha_3^2-\alpha_4^2-2\alpha_3\alpha_4\sigma\cosh\gamma_{ij}+i\varepsilon)\right]. \qquad \text{(B.14)}
\end{aligned}$$

After setting $\alpha_4 \to \alpha_3\alpha_4$, the α_3 integral may be carried out to give

$$V^\mu_{\rm box}\big|_{p_i^\mu} = -\frac{\sqrt{\pi}}{(4\pi)^{d/2}}\frac{\sigma_i p_i^\mu}{m_i^2 m_j}\frac{\Gamma\left(\frac{5}{2}-\frac{d}{2}\right)\Gamma^2\left(\frac{d}{2}-\frac{3}{2}\right)}{\Gamma(d-3)}|\vec{q}|^{d-5}\int_0^\infty\frac{d\alpha_4}{(-1-\alpha_4^2-2\alpha_4\sigma\cosh\gamma_{ij}+i\varepsilon)^{3/2}}, \qquad \text{(B.15)}$$

and the transformation $\alpha_4 = x\sinh\gamma_{ij} - \sigma\cosh\gamma_{ij}$ subsequently yields

$$V^\mu_{\rm box}\big|_{p_i^\mu} = -\frac{\sqrt{\pi}}{(4\pi)^{d/2}}\frac{\sigma_i p_i^\mu}{m_i^2 m_j\sinh^2\gamma_{ij}}\frac{\Gamma\left(\frac{5}{2}-\frac{d}{2}\right)\Gamma^2\left(\frac{d}{2}-\frac{3}{2}\right)}{\Gamma(d-3)}|\vec{q}|^{d-5}\int_{\sigma\coth\gamma_{ij}}^\infty [1-x^2+i\varepsilon]^{-3/2}. \qquad \text{(B.16)}$$

The x integral has already been carried out in eq. (A.9). Furthermore, symmetry allows the coefficient of p_j^μ in eq. (B.13) to be straightforwardly obtained from that of p_i^μ. The final result for the box integral is

$$V^\mu_{\rm box} = -\frac{i\sqrt{\pi}}{(4\pi)^{d/2}}\frac{1}{m_i m_j(1+\sigma\cosh\gamma_{ij})}\frac{\Gamma\left(\frac{5}{2}-\frac{d}{2}\right)\Gamma^2\left(\frac{d}{2}-\frac{3}{2}\right)}{\Gamma(d-3)}|\vec{q}|^{d-5}\left(\frac{\sigma_i p_i^\mu}{m_i}+\frac{\sigma_j p_j^\mu}{m_j}\right)+\dots, \qquad \text{(B.17)}$$

where the ellipsis denotes the term in q^μ that can be ignored.

References

[1] V. Del Duca, *An introduction to the perturbative QCD Pomeron and to jet physics at large rapidities*, hep-ph/9503226 [INSPIRE].

[2] I.A. Korchemskaya and G.P. Korchemsky, *High-energy scattering in QCD and cross singularities of Wilson loops*, Nucl. Phys. **B 437** (1995) 127 [hep-ph/9409446] [INSPIRE].

[3] I.A. Korchemskaya and G.P. Korchemsky, *Evolution equation for gluon Regge trajectory*, Phys. Lett. **B 387** (1996) 346 [hep-ph/9607229] [INSPIRE].

[4] I. Balitsky, *Factorization for high-energy scattering*, Phys. Rev. Lett. **81** (1998) 2024 [hep-ph/9807434] [INSPIRE].

[5] S. Melville, S.G. Naculich, H.J. Schnitzer and C.D. White, *Wilson line approach to gravity in the high energy limit*, Phys. Rev. **D 89** (2014) 025009 [arXiv:1306.6019] [INSPIRE].

[6] S. Caron-Huot, *When does the gluon reggeize?*, JHEP **05** (2015) 093 [arXiv:1309.6521] [INSPIRE].

[7] V. Del Duca, C. Duhr, E. Gardi, L. Magnea and C.D. White, *An infrared approach to Reggeization*, Phys. Rev. **D 85** (2012) 071104 [arXiv:1108.5947] [INSPIRE].

[8] V. Del Duca, C. Duhr, E. Gardi, L. Magnea and C.D. White, *The Infrared structure of gauge theory amplitudes in the high-energy limit*, JHEP **12** (2011) 021 [arXiv:1109.3581] [INSPIRE].

[9] V. Del Duca, G. Falcioni, L. Magnea and L. Vernazza, *High-energy QCD amplitudes at two loops and beyond*, Phys. Lett. **B 732** (2014) 233 [arXiv:1311.0304] [INSPIRE].

[10] V. Del Duca, G. Falcioni, L. Magnea and L. Vernazza, *Analyzing high-energy factorization beyond next-to-leading logarithmic accuracy*, JHEP **02** (2015) 029 [arXiv:1409.8330] [INSPIRE].

[11] I.Z. Rothstein and I.W. Stewart, *An Effective Field Theory for Forward Scattering and Factorization Violation*, JHEP **08** (2016) 025 [arXiv:1601.04695] [INSPIRE].

[12] G. Altarelli, R.D. Ball and S. Forte, *Small x Resummation with Quarks: Deep-Inelastic Scattering*, Nucl. Phys. **B 799** (2008) 199 [arXiv:0802.0032] [INSPIRE].

[13] C.D. White and R.S. Thorne, *A Global Fit to Scattering Data with NLL BFKL Resummations*, Phys. Rev. **D 75** (2007) 034005 [hep-ph/0611204] [INSPIRE].

[14] M. Ciafaloni, D. Colferai, G.P. Salam and A.M. Stasto, *A Matrix formulation for small-x singlet evolution*, JHEP **08** (2007) 046 [arXiv:0707.1453] [INSPIRE].

[15] M. Bonvini, S. Marzani and T. Peraro, *Small-x resummation from HELL*, Eur. Phys. J. **C 76** (2016) 597 [arXiv:1607.02153] [INSPIRE].

[16] J.R. Andersen and C.D. White, *A New Framework for Multijet Predictions and its application to Higgs Boson production at the LHC*, Phys. Rev. **D 78** (2008) 051501 [arXiv:0802.2858] [INSPIRE].

[17] J.R. Andersen, V. Del Duca and C.D. White, *Higgs Boson Production in Association with Multiple Hard Jets*, JHEP **02** (2009) 015 [arXiv:0808.3696] [INSPIRE].

[18] J.R. Andersen and J.M. Smillie, *Constructing All-Order Corrections to Multi-Jet Rates*, JHEP **01** (2010) 039 [arXiv:0908.2786] [INSPIRE].

JHEP01(2017)052

[19] J.R. Andersen and J.M. Smillie, *The Factorisation of the t-channel Pole in quark-gluon Scattering*, Phys. Rev. **D 81** (2010) 114021 [arXiv:0910.5113] [INSPIRE].

[20] J.R. Andersen and J.M. Smillie, *Multiple Jets at the LHC with High Energy Jets*, JHEP **06** (2011) 010 [arXiv:1101.5394] [INSPIRE].

[21] J.R. Andersen, L. Lönnblad and J.M. Smillie, *A Parton Shower for High Energy Jets*, JHEP **07** (2011) 110 [arXiv:1104.1316] [INSPIRE].

[22] J.R. Andersen, T. Hapola and J.M. Smillie, *W Plus Multiple Jets at the LHC with High Energy Jets*, JHEP **09** (2012) 047 [arXiv:1206.6763] [INSPIRE].

[23] J.R. Andersen, J.J. Medley and J.M. Smillie, *Z/γ^* plus multiple hard jets in high energy collisions*, JHEP **05** (2016) 136 [arXiv:1603.05460] [INSPIRE].

[24] M. Deak, F. Hautmann, H. Jung and K. Kutak, *Forward Jet Production at the Large Hadron Collider*, JHEP **09** (2009) 121 [arXiv:0908.0538] [INSPIRE].

[25] F. Caporale, G. Chachamis, B. Murdaca and A. Sabio Vera, *Balitsky-Fadin-Kuraev-Lipatov Predictions for Inclusive Three Jet Production at the LHC*, Phys. Rev. Lett. **116** (2016) 012001 [arXiv:1508.07711] [INSPIRE].

[26] A. Sabio Vera and F. Schwennsen, *Azimuthal decorrelation of forward jets in Deep Inelastic Scattering*, Phys. Rev. **D 77** (2008) 014001 [arXiv:0708.0549] [INSPIRE].

[27] A. Sabio Vera and F. Schwennsen, *The Azimuthal decorrelation of jets widely separated in rapidity as a test of the BFKL kernel*, Nucl. Phys. **B 776** (2007) 170 [hep-ph/0702158] [INSPIRE].

[28] J. Bartels, A. Sabio Vera and F. Schwennsen, *NLO inclusive jet production in k_T-factorization*, JHEP **11** (2006) 051 [hep-ph/0608154] [INSPIRE].

[29] A.H. Mueller, *Parton saturation: An Overview*, hep-ph/0111244 [INSPIRE].

[30] G. 't Hooft, *Graviton Dominance in Ultrahigh-Energy Scattering*, Phys. Lett. **B 198** (1987) 61 [INSPIRE].

[31] H.L. Verlinde and E.P. Verlinde, *Scattering at Planckian energies*, Nucl. Phys. **B 371** (1992) 246 [hep-th/9110017] [INSPIRE].

[32] D. Amati, M. Ciafaloni and G. Veneziano, *Classical and Quantum Gravity Effects from Planckian Energy Superstring Collisions*, Int. J. Mod. Phys. **A 3** (1988) 1615 [INSPIRE].

[33] D. Amati, M. Ciafaloni and G. Veneziano, *Higher order gravitational deflection and soft bremsstrahlung in Planckian energy superstring collisions*, Nucl. Phys. **B 347** (1990) 550 [INSPIRE].

[34] D. Amati, M. Ciafaloni and G. Veneziano, *Planckian scattering beyond the semiclassical approximation*, Phys. Lett. **B 289** (1992) 87 [INSPIRE].

[35] D. Amati, M. Ciafaloni and G. Veneziano, *Effective action and all order gravitational eikonal at Planckian energies*, Nucl. Phys. **B 403** (1993) 707 [INSPIRE].

[36] S.B. Giddings, M. Schmidt-Sommerfeld and J.R. Andersen, *High energy scattering in gravity and supergravity*, Phys. Rev. **D 82** (2010) 104022 [arXiv:1005.5408] [INSPIRE].

[37] S.B. Giddings and R.A. Porto, *The Gravitational S-matrix*, Phys. Rev. **D 81** (2010) 025002 [arXiv:0908.0004] [INSPIRE].

[38] S.B. Giddings, *The gravitational S-matrix: Erice lectures*, Subnucl. Ser. **48** (2013) 93 [arXiv:1105.2036] [INSPIRE].

[39] D. Amati, M. Ciafaloni and G. Veneziano, *Towards an S-matrix description of gravitational collapse*, JHEP **02** (2008) 049 [arXiv:0712.1209] [INSPIRE].

[40] M. Ciafaloni, D. Colferai and G. Veneziano, *Emerging Hawking-Like Radiation from Gravitational Bremsstrahlung Beyond the Planck Scale*, Phys. Rev. Lett. **115** (2015) 171301 [arXiv:1505.06619] [INSPIRE].

[41] M. Ciafaloni, D. Colferai, F. Coradeschi and G. Veneziano, *Unified limiting form of graviton radiation at extreme energies*, Phys. Rev. **D 93** (2016) 044052 [arXiv:1512.00281] [INSPIRE].

[42] G. D'Appollonio, P. Di Vecchia, R. Russo and G. Veneziano, *High-energy string-brane scattering: Leading eikonal and beyond*, JHEP **11** (2010) 100 [arXiv:1008.4773] [INSPIRE].

[43] G. D'Appollonio, P. Vecchia, R. Russo and G. Veneziano, *Microscopic unitary description of tidal excitations in high-energy string-brane collisions*, JHEP **11** (2013) 126 [arXiv:1310.1254] [INSPIRE].

[44] G. D'Appollonio, P. Di Vecchia, R. Russo and G. Veneziano, *A microscopic description of absorption in high-energy string-brane collisions*, JHEP **03** (2016) 030 [arXiv:1510.03837] [INSPIRE].

[45] G. D'Appollonio, P. Di Vecchia, R. Russo and G. Veneziano, *Regge behavior saves String Theory from causality violations*, JHEP **05** (2015) 144 [arXiv:1502.01254] [INSPIRE].

[46] Z. Bern, J.J.M. Carrasco and H. Johansson, *New Relations for Gauge-Theory Amplitudes*, Phys. Rev. **D 78** (2008) 085011 [arXiv:0805.3993] [INSPIRE].

[47] Z. Bern, J.J.M. Carrasco and H. Johansson, *Perturbative Quantum Gravity as a Double Copy of Gauge Theory*, Phys. Rev. Lett. **105** (2010) 061602 [arXiv:1004.0476] [INSPIRE].

[48] Z. Bern, T. Dennen, Y.-t. Huang and M. Kiermaier, *Gravity as the Square of Gauge Theory*, Phys. Rev. **D 82** (2010) 065003 [arXiv:1004.0693] [INSPIRE].

[49] R. Saotome and R. Akhoury, *Relationship Between Gravity and Gauge Scattering in the High Energy Limit*, JHEP **01** (2013) 123 [arXiv:1210.8111] [INSPIRE].

[50] A. Sabio Vera, E. Serna Campillo and M.A. Vazquez-Mozo, *Color-Kinematics Duality and the Regge Limit of Inelastic Amplitudes*, JHEP **04** (2013) 086 [arXiv:1212.5103] [INSPIRE].

[51] S. Oxburgh and C.D. White, *BCJ duality and the double copy in the soft limit*, JHEP **02** (2013) 127 [arXiv:1210.1110] [INSPIRE].

[52] H. Johansson, A. Sabio Vera, E. Serna Campillo and M.Á. Vázquez-Mozo, *Color-Kinematics Duality in Multi-Regge Kinematics and Dimensional Reduction*, JHEP **10** (2013) 215 [arXiv:1307.3106] [INSPIRE].

[53] S.G. Naculich and H.J. Schnitzer, *Eikonal methods applied to gravitational scattering amplitudes*, JHEP **05** (2011) 087 [arXiv:1101.1524] [INSPIRE].

[54] C.D. White, *Factorization Properties of Soft Graviton Amplitudes*, JHEP **05** (2011) 060 [arXiv:1103.2981] [INSPIRE].

[55] D.J. Miller and C.D. White, *The Gravitational cusp anomalous dimension from AdS space*, Phys. Rev. **D 85** (2012) 104034 [arXiv:1201.2358] [INSPIRE].

JHEP01(2017)052

[56] A. Brandhuber, P. Heslop, A. Nasti, B. Spence and G. Travaglini, *Four-point Amplitudes in $N = 8$ Supergravity and Wilson Loops*, *Nucl. Phys.* **B 807** (2009) 290 [arXiv:0805.2763] [INSPIRE].

[57] F. Cachazo, S. He and E.Y. Yuan, *Scattering of Massless Particles in Arbitrary Dimensions*, *Phys. Rev. Lett.* **113** (2014) 171601 [arXiv:1307.2199] [INSPIRE].

[58] F. Cachazo, S. He and E.Y. Yuan, *Scattering of Massless Particles: Scalars, Gluons and Gravitons*, *JHEP* **07** (2014) 033 [arXiv:1309.0885] [INSPIRE].

[59] A. Strominger, *On BMS Invariance of Gravitational Scattering*, *JHEP* **07** (2014) 152 [arXiv:1312.2229] [INSPIRE].

[60] T. He, V. Lysov, P. Mitra and A. Strominger, *BMS supertranslations and Weinberg's soft graviton theorem*, *JHEP* **05** (2015) 151 [arXiv:1401.7026] [INSPIRE].

[61] F. Cachazo and A. Strominger, *Evidence for a New Soft Graviton Theorem*, arXiv:1404.4091 [INSPIRE].

[62] E. Casali, *Soft sub-leading divergences in Yang-Mills amplitudes*, *JHEP* **08** (2014) 077 [arXiv:1404.5551] [INSPIRE].

[63] B.U.W. Schwab and A. Volovich, *Subleading Soft Theorem in Arbitrary Dimensions from Scattering Equations*, *Phys. Rev. Lett.* **113** (2014) 101601 [arXiv:1404.7749] [INSPIRE].

[64] Z. Bern, S. Davies and J. Nohle, *On Loop Corrections to Subleading Soft Behavior of Gluons and Gravitons*, *Phys. Rev.* **D 90** (2014) 085015 [arXiv:1405.1015] [INSPIRE].

[65] S. He, Y.-t. Huang and C. Wen, *Loop Corrections to Soft Theorems in Gauge Theories and Gravity*, *JHEP* **12** (2014) 115 [arXiv:1405.1410] [INSPIRE].

[66] A.J. Larkoski, *Conformal Invariance of the Subleading Soft Theorem in Gauge Theory*, *Phys. Rev.* **D 90** (2014) 087701 [arXiv:1405.2346] [INSPIRE].

[67] F. Cachazo and E.Y. Yuan, *Are Soft Theorems Renormalized?*, arXiv:1405.3413 [INSPIRE].

[68] N. Afkhami-Jeddi, *Soft Graviton Theorem in Arbitrary Dimensions*, arXiv:1405.3533 [INSPIRE].

[69] T. Adamo, E. Casali and D. Skinner, *Perturbative gravity at null infinity*, *Class. Quant. Grav.* **31** (2014) 225008 [arXiv:1405.5122] [INSPIRE].

[70] M. Bianchi, S. He, Y.-t. Huang and C. Wen, *More on Soft Theorems: Trees, Loops and Strings*, *Phys. Rev.* **D 92** (2015) 065022 [arXiv:1406.5155] [INSPIRE].

[71] Z. Bern, S. Davies, P. Di Vecchia and J. Nohle, *Low-Energy Behavior of Gluons and Gravitons from Gauge Invariance*, *Phys. Rev.* **D 90** (2014) 084035 [arXiv:1406.6987] [INSPIRE].

[72] J. Broedel, M. de Leeuw, J. Plefka and M. Rosso, *Constraining subleading soft gluon and graviton theorems*, *Phys. Rev.* **D 90** (2014) 065024 [arXiv:1406.6574] [INSPIRE].

[73] T. He, P. Mitra, A.P. Porfyriadis and A. Strominger, *New Symmetries of Massless QED*, *JHEP* **10** (2014) 112 [arXiv:1407.3789] [INSPIRE].

[74] M. Zlotnikov, *Sub-sub-leading soft-graviton theorem in arbitrary dimension*, *JHEP* **10** (2014) 148 [arXiv:1407.5936] [INSPIRE].

[75] C. Kalousios and F. Rojas, *Next to subleading soft-graviton theorem in arbitrary dimensions*, *JHEP* **01** (2015) 107 [arXiv:1407.5982] [INSPIRE].

[76] Y.-J. Du, B. Feng, C.-H. Fu and Y. Wang, *Note on Soft Graviton theorem by KLT Relation*, *JHEP* **11** (2014) 090 [arXiv:1408.4179] [INSPIRE].

[77] H. Lüo, P. Mastrolia and W.J. Torres Bobadilla, *Subleading soft behavior of QCD amplitudes*, *Phys. Rev.* **D 91** (2015) 065018 [arXiv:1411.1669] [INSPIRE].

[78] C.D. White, *Diagrammatic insights into next-to-soft corrections*, *Phys. Lett.* **B 737** (2014) 216 [arXiv:1406.7184] [INSPIRE].

[79] F.E. Low, *Bremsstrahlung of very low-energy quanta in elementary particle collisions*, *Phys. Rev.* **110** (1958) 974 [INSPIRE].

[80] T.H. Burnett and N.M. Kroll, *Extension of the low soft photon theorem*, *Phys. Rev. Lett.* **20** (1968) 86 [INSPIRE].

[81] V. Del Duca, *High-energy Bremsstrahlung Theorems for Soft Photons*, *Nucl. Phys.* **B 345** (1990) 369 [INSPIRE].

[82] E. Laenen, G. Stavenga and C.D. White, *Path integral approach to eikonal and next-to-eikonal exponentiation*, *JHEP* **03** (2009) 054 [arXiv:0811.2067] [INSPIRE].

[83] E. Laenen, L. Magnea, G. Stavenga and C.D. White, *Next-to-eikonal corrections to soft gluon radiation: a diagrammatic approach*, *JHEP* **01** (2011) 141 [arXiv:1010.1860] [INSPIRE].

[84] D. Bonocore, E. Laenen, L. Magnea, S. Melville, L. Vernazza and C.D. White, *A factorization approach to next-to-leading-power threshold logarithms*, *JHEP* **06** (2015) 008 [arXiv:1503.05156] [INSPIRE].

[85] D. Bonocore, E. Laenen, L. Magnea, L. Vernazza and C.D. White, *The method of regions and next-to-soft corrections in Drell-Yan production*, *Phys. Lett.* **B 742** (2015) 375 [arXiv:1410.6406] [INSPIRE].

[86] D. Bonocore, E. Laenen, L. Magnea, L. Vernazza and C.D. White, *Non-abelian factorisation for next-to-leading-power threshold logarithms*, *JHEP* **12** (2016) 121 [arXiv:1610.06842] [INSPIRE].

[87] R. Akhoury, R. Saotome and G. Sterman, *High Energy Scattering in Perturbative Quantum Gravity at Next to Leading Power*, arXiv:1308.5204 [INSPIRE].

[88] N.E.J. Bjerrum-Bohr, J.F. Donoghue, B.R. Holstein, L. Plante and P. Vanhove, *Light-like Scattering in Quantum Gravity*, *JHEP* **11** (2016) 117 [arXiv:1609.07477] [INSPIRE].

[89] E. Gardi and L. Magnea, *Infrared singularities in QCD amplitudes*, *Nuovo Cim.* **C32N5-6** (2009) 137 [arXiv:0908.3273] [INSPIRE].

[90] S. Catani and M.H. Seymour, *The Dipole formalism for the calculation of QCD jet cross-sections at next-to-leading order*, *Phys. Lett.* **B 378** (1996) 287 [hep-ph/9602277] [INSPIRE].

[91] S. Catani and M.H. Seymour, *A General algorithm for calculating jet cross-sections in NLO QCD*, *Nucl. Phys.* **B 485** (1997) 291 [*Erratum ibid.* **B 510** (1998) 503] [hep-ph/9605323] [INSPIRE].

[92] E. Brézin, C. Itzykson and J. Zinn-Justin, *Relativistic balmer formula including recoil effects*, *Phys. Rev.* **D 1** (1970) 2349 [INSPIRE].

[93] D.N. Kabat and M. Ortiz, *Eikonal quantum gravity and Planckian scattering*, *Nucl. Phys.* **B 388** (1992) 570 [hep-th/9203082] [INSPIRE].

[94] S. Weinberg, *Infrared photons and gravitons*, *Phys. Rev.* **140** (1965) B516.

[95] R. Akhoury, R. Saotome and G. Sterman, *Collinear and Soft Divergences in Perturbative Quantum Gravity*, *Phys. Rev.* **D 84** (2011) 104040 [arXiv:1109.0270] [INSPIRE].

[96] M. Beneke and G. Kirilin, *Soft-collinear gravity*, *JHEP* **09** (2012) 066 [arXiv:1207.4926] [INSPIRE].

[97] G.C. Gellas, A.I. Karanikas and C.N. Ktorides, *Worldline approach to eikonals for QED and linearized quantum gravity and their off mass shell extensions*, *Phys. Rev.* **D 57** (1998) 3763 [INSPIRE].

[98] D.J. Gross and R. Jackiw, *Low-Energy Theorem for Graviton Scattering*, *Phys. Rev.* **166** (1968) 1287 [INSPIRE].

[99] B.S. DeWitt, *Quantum Theory of Gravity. 3. Applications of the Covariant Theory*, *Phys. Rev.* **162** (1967) 1239 [INSPIRE].

[100] N.E.J. Bjerrum-Bohr, J.F. Donoghue and P. Vanhove, *On-shell Techniques and Universal Results in Quantum Gravity*, *JHEP* **02** (2014) 111 [arXiv:1309.0804] [INSPIRE].

[101] N.E. Bjerrum-Bohr, *Quantum gravity, effective fields and string theory*, Ph.D. Thesis, Bohr Institute, Copenhagen Denmark (2004). [hep-th/0410097] [INSPIRE].

[102] Z. Bern, S. Davies, T. Dennen, Y.-t. Huang and J. Nohle, *Color-Kinematics Duality for Pure Yang-Mills and Gravity at One and Two Loops*, *Phys. Rev.* **D 92** (2015) 045041 [arXiv:1303.6605] [INSPIRE].

[103] P.C. Aichelburg and R.U. Sexl, *On the Gravitational field of a massless particle*, *Gen. Rel. Grav.* **2** (1971) 303 [INSPIRE].

[104] G. Veneziano, *Transplanckian string collisions: An update*, in *Proceedings of 12th Marcel Grossmann Meeting on General Relativity: On recent developments in theoretical and experimental general relativity, astrophysics and relativistic field theories*, Paris France, 12–18 July 2009, pg. 95.

JHEP01(2017)052