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$\mathcal{N} = 2$ heterotic string compactifications on orbifolds of $K3 \times T^2$

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Abstract: We study $\mathcal{N} = 2$ compactifications of $E_8 \times E_8$ heterotic string theory on orbifolds of $K3 \times T^2$ by g' which acts as an $\mathbb{Z}_N$ automorphism of $K3$ together with a $1/N$ shift on a circle of T^2. The orbifold action g' corresponds to the 26 conjugacy classes of the Mathieu group M_{24}. We show that for the standard embedding the new supersymmetric index for these compactifications can always be decomposed into the elliptic genus of $K3$ twisted by g'. The difference in one-loop corrections to the gauge couplings are captured by automorphic forms obtained by the theta lifts of the elliptic genus of $K3$ twisted by g'. We work out in detail the case for which g' belongs to the equivalence class $2B$. We then investigate all the non-standard embeddings for $K3$ realized as a $T^4/\mathbb{Z}_\nu$ orbifold with $\nu = 2, 4$ and g' the $2A$ involution. We show that for non-standard embeddings the new supersymmetric index as well as the difference in one-loop corrections to the gauge couplings are completely characterized by the instanton numbers of the embeddings together with the difference in number of hypermultiplets and vector multiplets in the spectrum.

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Contents

1 Introduction

String compactifications with $\mathcal{N}=2$ supersymmetry has been extensively investigated as an important testing ground for string dualities. The canonical example of such a compactification is the heterotic string on $K3\times T^2$. In the context of string dualities this theory was first investigated in [1]. The various theories studied differed on how the spin connection was embedded in the gauge connection. A simple method of explicitly constructing these compactifications is to realize $K3$ as a $T^4/\mathbb{Z}_\nu$ orbifold with $\nu = 2,3,4,6$. A comprehensive list of these orbifold compactifications together with all possible embeddings of the spin connection in the gauge connection is given in [2, 3]. Supersymmetric observables like the new supersymmetric index or the difference in one loop gauge threshold corrections can be shown to be independent of the orbifold realization [2, 4, 5].

An important observable in these compactifications is the new supersymmetric index [4–9] which is defined by

$$\mathcal{Z}_{\rm new}(q,\bar{q}) = \frac{1}{\eta^2(\tau)}\mathrm{Tr}_R\left(Fe^{i\pi F}q^{L_0-\frac{c}{24}}\bar{q}^{\bar{L}_0-\frac{\bar{c}}{24}}\right) \ . \tag{1.1}$$

Here the trace is performed over the Ramond sector in the internal CFT with central charges $(c,\bar{c}) = (22,9)$. F refers to the world sheet fermion number of the right moving $\mathcal{N}=2$ supersymmetric internal CFT. Recently it has been observed that the new supersymmetric index of $K3\times T^2$ which enumerates BPS states in these compactifications admits Mathieu moonshine symmetry [10], see [11] for a review of aspects of moonshine. This observation was generalized in [12] which considered orbifolds of $K3\times T^2$ by g' acted as a $\mathbb{Z}_N$ automorphism in $K3$ and and $1/N$ shift on one of the circles of T^2. It was observed that for the standard embedding the new supersymmetric index admits a decomposition in terms the elliptic genus of $K3$ twisted by g'. This ensures that the new supersymmetric index admits an expansion in terms of the McKay Thompson series associated with g' embedded in the Mathieu group M_{24}. It was also observed in [12] that the difference in one loop gauge corrections to gauge couplings with Wilson lines for these compactifications can be written in terms of Siegel modular forms corresponding to the theta lift of the twisted elliptic genus of $K3$.

The g' considered in these compactifications of [12] were restricted in the conjugacy class pA of M_{24} with $p=2,3,5,7$. In fact only the class $2A$ was explicitly constructed,[1] and the analysis was restricted to the standard embedding. In this paper we study compactifications of the $E_8\times E_8$ heterotic string theory on orbifolds of $K3\times T^2$ by g' in more detail.

We show that for all g' corresponding to the 26 conjugacy classes of M_{24} and for compactifications which involve the standard embedding of the spin connection of $K3$ into one of the E_8's the resultant new supersymmetric index always can be written in terms of the elliptic genus of $K3$ but twisted by g'. The standard embedding breaks the gauge group to $E_7\times \mathrm{U}(1)\times E_8$. The difference in one loop corrections of the gauge groups E_7 and E_8 are automorphic forms of $\mathrm{SO}(2+s,s;\mathbb{Z})$ with $s=0,1$. For $s=0$, the automorphic forms are functions of Kähler, complex structure of the torus T^2 while for $s=1$ they are also functions of the Wilson line embedding in either of the gauge groups. We show that these automorphic forms are obtained as theta lifts of the elliptic genus of $K3$ twisted by g'. We demonstrate these statements explicitly for 2 examples. We first consider the situation when $K3$ is realized as $T^4/\mathbb{Z}_4$ and then construct the corresponding g' action corresponding to the $2A$ conjugacy class. We show the new supersymmetric index is determined by the corresponding twisted elliptic genus. This result is identical to that obtained in [12] when $K3$ is realized as the orbifold $T^4/\mathbb{Z}_2$ which illustrates that the new supersymmetric index is independent of the realization of $K3$. In the second example we consider the situation when $K3$ is realized as a rational conformal field theory based on the affine algebra $su(2)^6$

[1] We use the ATLAS naming for the conjugacy classes of M_{24} see [13].

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and for g' belonging to the conjugacy class $2B$ studied in [14]. For this situation we show that that the new supersymmetric index is determined by the elliptic genus of $K3$ twisted by the $2B$ action.

We then examine non-standard embeddings of $K3 \times T^2$ compactifications. This is done by considering all the non-standard embeddings in which $K3$ is realized as a T^4/Z_2 as well as T^4/Z_4 orbifold and the action of g' in the conjugacy class $2A$. We study the spectrum and then evaluate the new supersymmetric index for these compactifications. The results for the spectrum are summarized in tables 6, 7, 8, 9, 10. We show that the new supersymmetric index classifies all the models into 4 distinct types depending on the difference of the number of hypermultiplets and vector multiplets, $N_h - N_v$ of the model. The result can be read off using the table 13 and equation (4.7) In each case we see that the new supersymmetric index again admits a decomposition in terms of the elliptic genus of $K3$ twisted by g'. However there is also a dependence in $N_h - N_v$. We then evaluate the difference in one loop gauge coupling corrections for all these models with the Wilson line and show that they result in $\mathrm{SO}(3,2;\mathbb{Z})$ automorphic forms. The automorphic forms for all the models are entirely determined by the instanton numbers of the embeddings as well as $N_h - N_v$ of these models. The result can be read off using the tables 14, 15 and equation (4.19).

The organization of the paper is as follows. In section 2 we prove that for the standard embedding , compactifications on orbifolds of $K3 \times T^2$ result in a new supersymmetric which can always be written in terms of the elliptic genus of $K3$ twisted by g'. Section 3 works out in detail for the situation when $K3$ is realized as $T^4/\mathbb{Z}_4$ with $g' \in 2A$ and when $K3$ is realized as a rational conformal field theory based on the $su(2)^6$ affine algebra with $g' \in 2B$. In section 4 we first introduce all the embeddings in which $K3$ is realized as a $T^4/\mathbb{Z}_\nu$ orbifold with $\nu = 2, 4$ and $g' \in 2A$ and evaluate the spectrum, the new supersymmetric index and the difference in one loop gauge thresholds. Section 5 contains our conclusions. Appendix A contains the notations, conventions and a list of identities used in the paper, appendix B contains the details of evaluating one loop threshold integrals. Finally the appendix C summarises the content of mathematica files which were used to arrive at some of the results in the paper.

2 Standard embedding

In this section we first define $\mathcal{N} = 2$ supersymmetric compactifications of the $E_8 \times E_8$ heterotic string theory on orbifolds of $K3 \times T^2$ by g' in which the spin connection of $K3$ is embedded in one of the E_8's in the standard manner. g' acts as a $\mathbb{Z}_N$ automorphism of $K3$ together with a $1/N$ shift along one of the circles of T^2. The automorphism g' corresponds to any of the 26 conjugacy classes associated with the Mathieu group M_{24} by which one can twist the elliptic genus of $K3$ [15–17].

We define the standard embedding as follows. Let the current algebra of one of the E_8's be realized in terms of left moving fermions $\lambda^I, I = 1, \cdots 16$. The other E_8 can be realized in terms of its bosonic lattice or the fermions λ'^I. The gauge connection is assumed to have the structure

$$\mathcal{G} = \sum_{I,J=1}^{4} \lambda^I B_a^{IJ} \partial X^a \lambda^J + \sum_{I,J=5}^{16} \lambda^I A_i^{IJ} \partial X^i \lambda^J + \sum_{I,J=1}^{16} \lambda'^I A_i'^{IJ} \partial X^i \lambda'^J \,. \tag{2.1}$$

Here A_i, A_i' is the flat connection on the T^2. B_a refers to the SU(2) spin connection of $K3$. Thus we have embedded the spin connection in one of the SU(2)'s of the E_8. This E_8 lattice splits into a $D2$ which is coupled to the spin connection of $K3$ and a free $D6$ lattice. The $D6$ lattice and the second E_8 lattice which can contain the flat connections A_i, A_i' on T^2 are free. Thus we have the $16 - 4 = 12$ free Majorana-Weyl fermions of the $D6$ lattice coupled to the flat connection on the T^2 and 4 interacting Majorana-Weyl fermions coupled to the spin connection of the $K3$. These left moving fermions with the left moving bosons of the $K3$ as well as the right moving supersymmetric sector of $K3$ form a $(6,6)$ conformal field theory. Thus the internal CFT of the heterotic string in the standard embedding splits as

$$\mathcal{H}^{\text{internal}} = \mathcal{H}^{(6,6)}_{D2K3} \otimes \mathcal{H}^{(6,0)}_{D6} \otimes \mathcal{H}^{(8,0)}_{E_8} \otimes \mathcal{H}^{(2,3)}_{T^2} \,. \tag{2.2}$$

Here the second and third Hilbert spaces refer to the $D6$ lattice and the E_8 lattice respectively and the the last refer to the CFT on T^2. With this decomposition, we can now specify the action of g'. The g' acts as a $\mathcal{Z}_N$ automorphism on the $(6,6)$ CFT $\mathcal{H}$ together with a $1/N$ shift on one of the circles in $\mathcal{H}^{(2,3)}_{T^2}$.

2.1 New supersymmetric index and twisted elliptic genus of $K3$

Let us now evaluate the new supersymmetric index on the internal CFT given in (2.2).

$$\mathcal{Z}_{\text{new}} = \frac{1}{\eta^2} \text{Tr}_R \left((-1)^F F q^{L_0 - c/24} \bar{q}^{\bar{L}_0 - \bar{c}/24} \right) . \tag{2.3}$$

The right moving Fermion number F can be written as the sum of the Fermion number on T^2 together with the Fermion number on $K3$

$$F = F^{T^2} + F^{K3}. \tag{2.4}$$

Then it is easy to see that because of the right moving Fermion zero modes on T^2, the only contribution to the index arises from

$$\mathcal{Z}_{\text{new}} = \frac{1}{\eta^2} \text{Tr}_R \left(F^{T^2} e^{i\pi(F^{T^2} + F^{K3})} q^{L_0 - c/24} \bar{q}^{\bar{L}_0 - \bar{c}/24} \right) . \tag{2.5}$$

Again examining the trace we can see that the contributions from left moving bosonic and fermionic oscillators on T^2 cancel. Thus it is only the zero modes on T^2 and the left moving bosonic oscillators on T^2 which contribute to the index. With these arguments we see that the trace reduces to

$$\mathcal{Z}_{\text{new}} = \frac{1}{\eta^2(\tau)} \frac{\Gamma^{(r,s)}_{2,2}(q,\bar{q})}{\eta^2(\tau)} \left[\frac{\theta_2^6(\tau)}{\eta^6(\tau)} \Phi^{(r,s)}_R + \frac{\theta_3^6(\tau)}{\eta^6(\tau)} \Phi^{(r,s)}_{NS+} - \frac{\theta_4^6(\tau)}{\eta^6(\tau)} \Phi^{(r,s)}_{NS-} \right] \frac{E_4(q)}{\eta^8(\tau)} . \tag{2.6}$$

The sum over the sectors (r,s) is implied and r,s run from 0 to $N-1$. The origin and the definition of each term in the index is as follows.

1. The term $\frac{\Gamma_{2,2}^{(r,s)}}{\eta^2}$ arises from the lattice sum on T^2 together with the left moving bosonic oscillators. The lattice sum is defined as

$$\Gamma_{2,2}^{(r,s)}(q,\bar{q}) = \sum_{\substack{m_1,m_2,n_2\in\mathbb{Z},\\ n_1=\mathbb{Z}+\frac{r}{N}}} q^{\frac{p_L^2}{2}}\bar{q}^{\frac{p_R^2}{2}}e^{2\pi i m_1 s/N}, \tag{2.7}$$
$$\frac{1}{2}p_R^2 = \frac{1}{2T_2U_2}|-m_1U+m_2+n_1T+n_2TU|^2,$$
$$\frac{1}{2}p_L^2 = \frac{1}{2}p_R^2+m_1n_1+m_2n_2\,.$$

T, U are the Kähler and complex structure of the T^2. Note that the lattice sum is the only part of the index that contains anti-holomorphic dependence. Furthermore the insertion of g' and the twisted sectors of g' are taken care of by the phase $e^{2\pi i m_1 s/N}$ and the fact the winding modes are shifted from integers by $\frac{r}{N}$.

2. The terms in the square bracket arises from evaluating the index on the lattice $D6$ together with the combined $D2K3$. Note that the partition function on the $D6$ lattice in the various sectors are given by

$$\mathcal{Z}_R(D6;q) = \frac{\theta_2^6}{\eta^6}, \quad \mathcal{Z}_{NS^+}(D6;q) = \frac{\theta_3^6}{\eta^6}, \quad \mathcal{Z}_{NS^-}(D6;q) = \frac{\theta_4^6}{\eta^6}. \tag{2.8}$$

While the indices on the combined $D2K3$, $(6,6)$ conformal field theory are given by

$$\Phi_R^{(r,s)} = \frac{1}{N}\mathrm{Tr}_{R\,R,g^r}\left[g^s(-1)^{F_R}q^{L_0-c/24}\bar{q}^{\bar{L}_0-\bar{c}/24}\right], \tag{2.9}$$
$$\Phi_{NS^+}^{(r,s)} = \frac{1}{N}\mathrm{Tr}_{NS\,R,g^r}\left[g^s(-1)^{F_R}q^{L_0-c/24}\bar{q}^{\bar{L}_0-\bar{c}/24}\right],$$
$$\Phi_{NS^-}^{(r,s)} = \frac{1}{N}\mathrm{Tr}_{NS\,R,g^r}\left[g^s(-1)^{F_R+F_L}q^{L_0-c/24}\bar{q}^{\bar{L}_0-\bar{c}/24}\right].$$

We will relate them to the twisted elliptic genus of $K3$ below.

3. Finally the term $\frac{E_4(q)}{\eta^8(\tau)}$ arises from the partition function of the second E_8 which is untouched in the standard embedding. E_4 is the Eisenstein series of weight 4.

We now show that the indices in (2.9) are related to the twisted elliptic genus of $K3$ by g'. In indices given in (2.9) note that the spin connection of the $K3$ is coupled to the fermions in $D2$ conformal field theory and therefore trace can be thought of as a trace in the $K3$ super conformal field theory with central charge $(6,6)$. Let us examine the twisted elliptic genus of $K3$ which is defined as

$$F^{(r,s)}(\tau,z) = \frac{1}{N}\mathrm{Tr}_{R\,Rg'^r}\left[(-1)^{F_{K3}+\bar{F}_{K3}}g'^s e^{2\pi i z F_{K3}}q^{L_0-c/24}\bar{q}^{\bar{L}_0-\bar{c}/24}\right]. \tag{2.10}$$

Here g' belongs to automorphism related to the 26 conjugacy classes of M_{24}. Since this theory admits a $\mathcal{N}=2$ spectral flow we can relate the trace over the various sectors in (2.9) by the following equations

$$\begin{aligned}\Phi_R^{(r,s)} &= F^{(r,s)}\left(\tau, \frac{1}{2}\right), \\ \Phi_{NS+}^{(r,s)} &= q^{1/4}F^{(r,s)}\left(\tau, \frac{\tau+1}{2}\right), \\ \Phi_{NS-}^{(r,s)} &= q^{1/4}F^{(r,s)}\left(\tau, \frac{\tau}{2}\right).\end{aligned} \tag{2.11}$$

From (2.6) and (2.11) we see that the new supersymmetric index for compactifications which involve the standard embedding admits a decomposition in terms of the elliptic genus of $K3$ twisted by g'. This decomposition then can be used to show that the new supersymmetric index can be expanded in terms of the MacKay-Thompson associated with g' embedded in M_{24} following the arguments of [10, 12].

New supersymmetric index in terms Eisenstein series. Let us further simplify the expression the expression for the new supersymmetric index for the standard embedding. The elliptic genus of $K3$ twisted by g' in general can be written as

$$\begin{aligned}F^{(0,0)}(\tau,z) &= \alpha_{g'}^{(0,0)}A(\tau,z), \\ F^{(0,1)}(\tau,z) &= \alpha_{g'}^{(0,1)}A(\tau,z) + \beta_{g'}^{(0,1)}f_{g'}^{(0,1)}(\tau)B(\tau,z),\end{aligned} \tag{2.12}$$

where the Jacobi forms $A(\tau,z)$ and $B(\tau,z)$ are given by

$$A(\tau,z) = \frac{\theta_2^2(\tau,z)}{\theta_2^2(\tau,0)} + \frac{\theta_3^2(\tau,z)}{\theta_3^2(\tau,0)} + \frac{\theta_4^2(\tau,z)}{\theta_4^2(\tau,0)}, \qquad B(\tau,z) = \frac{\theta_1^2(\tau,z)}{\eta^6(\tau)}. \tag{2.13}$$

The numerical coefficients $\alpha_{g'}, \beta_{g'}$ and the form $f_{g'}^{(0,1)}(\tau)$ depend on the twist g'. For example, for the conjugacy class pA with $p = 2, 3, 5, 7$ of M_{24} we find

$$\alpha_{pA}^{(0,0)} = \frac{8}{p}, \qquad \alpha_{pA}^{(0,1)} = \frac{8}{p(p+1)}, \qquad \beta_{pA}^{(0,1)} = -\frac{2}{p+1}, \tag{2.14}$$

and

$$f_{g'}^{(0,1)}(\tau) = \mathcal{E}_p(\tau) = \frac{12i}{\pi(p-1)}\partial_\tau \log\frac{\eta(\tau)}{\eta(p\tau)}. \tag{2.15}$$

A comprehensive list of the twisted elliptic genus for all the 26 conjugacy classes of M_{24} can be found in [16]. All the remaining elements of the twisted elliptic genus $F^{(r,s)}(\tau,z)$ can be obtained by modular transformations using the relation

$$F^{(r,s)}\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right) = \exp\left(2\pi i\frac{cz^2}{c\tau+d}\right)F^{(cs+ar,ds+br)}(\tau,z), \tag{2.16}$$

with

$$a,b,c,d \in \mathbb{Z}, \qquad ad - bc = 1. \tag{2.17}$$

In (2.16) the indices $cs + ar$ and $ds + br$ are taken to be mod N where N is the order of g'. Using this information of the twisted elliptic genus we can write the new supersymmetric

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index for the standard embedding given in (2.6) in terms of Eisenstein series. Substituting the following identities

$$\begin{aligned} A\left(\tau,\frac{1}{2}\right) &= \frac{(\theta_4^4\theta_2^2+\theta_3^4\theta_2^2)}{4\eta^6}, & B\left(\tau,\frac{1}{2}\right) &= \frac{\theta_2^2}{\eta^6}, \\ A\left(\tau,\frac{\tau+1}{2}\right) &= \frac{q^{-1/4}\left(-\theta_4^4\theta_3^2+\theta_2^4\theta_3^2\right)}{4\eta^6}, & B\left(\tau,\frac{\tau+1}{2}\right) &= \frac{q^{-1/4}\theta_3^2}{\eta^6}, \\ A\left(\tau,\frac{\tau}{2}\right) &= \frac{q^{-1/4}\left(\theta_3^4\theta_4^2+\theta_2^4\theta_4^2\right)}{4\eta^6}, & B\left(\tau,\frac{\tau}{2}\right) &= -\frac{q^{-1/4}\theta_4^2}{\eta^6}. \end{aligned} \tag{2.18}$$

in (2.6) and using (2.11) we obtain

$$\mathcal{Z}_{\rm new}(q,\bar q) = -2\frac{1}{\eta^{24}}\Gamma^{(r,s)}_{2,2}E_4\left[\frac{1}{4}\alpha^{(r,s)}_{g'}E_6-\beta^{(r,s)}_{g'}f^{(r,s)}_{g'}E_4\right]. \tag{2.19}$$

Recall that only the lattice sum is dependent on both $(\tau,\bar\tau)$ while the Eisenstein series E_6, E_4 as well as $f^{(r,s)}$ are holomorphic in τ. Furthermore in the (2.19) sum over r,s from $0,\cdots N-1$ is understood.

2.2 Difference of one loop gauge thresholds

Now let us evaluate the gauge threshold corrections with Wilson line turned on in the untouched E_8 lattice, we call this gauge group G and the broken E_8, G'. From the discussion in [2, 5] and [12], we see that the new supersymmetric index with Wilson line becomes

$$\mathcal{Z}_{\rm new}(q,\bar q) = -2\frac{1}{\eta^{24}}\Gamma^{(r,s)}_{3,2}\otimes E_{4,1}\left[\frac{1}{4}\alpha^{(r,s)}_{g'}E_6-\beta^{(r,s)}_{g'}f^{(r,s)}_{g'}E_4\right]. \tag{2.20}$$

The presence of the Wilson line introduces an additional moduli V and with T,U. The lattices sums now are given by

$$\begin{aligned} \Gamma^{(r,s)}_{3,2}(q,\bar q) &= \sum_{\substack{m_1,m_2,n_2,b\in\mathbb{Z},\\ n_1=\mathbb{Z}+\frac{r}{N}}} q^{\frac{p_L^2}{2}}\bar q^{\frac{p_R^2}{2}}e^{2\pi i m_1 s/N}, \\ \frac{p_R^2}{2} &= \frac{1}{4\,{\rm det Im}\Omega}\left|-m_1U+m_2+n_1T+n_2(TU-V^2)+bV\right|^2, \\ \frac{p_L^2}{2} &= \frac{p_R^2}{2}+m_1n_1+m_2n_2+\frac{1}{4}b^2, \\ \Omega &= \begin{pmatrix} U & V \\ V & T\end{pmatrix}. \end{aligned} \tag{2.21}$$

The product $\otimes$ and function $E_{4,1}$ are defined in the appendix A. The one loop corrections to the gauge coupling G is defined by the following integral over the fundamental domain

$$\Delta(T,U,V) = \int_{\mathcal{F}}\frac{d^2\tau}{\tau_2}(\mathcal{B}_G-b(G)), \tag{2.22}$$

where $\mathcal{B}$ can be written in terms of the new supersymmetric index with the Wilson line as follows

$$\mathcal{B}_G = -\frac{2}{24\eta^{24}}\Gamma^{(r,s)}_{3,2}\otimes\left\{\tilde E_2E_{4,1}-E_{6,1}\right\}\left[\frac{1}{4}\alpha^{(r,s)}_{g'}E_6-\beta^{(r,s)}_{g'}f^{(r,s)}_{g'}E_4\right], \tag{2.23}$$

where

$$\tilde{E}_2 = \left(E_2 - \frac{3}{\pi \tau_2} \right). \tag{2.24}$$

The constant $b(G)$ in (2.22) can be fixed by demanding that the integral is well defined in the limit $\tau_2 \to \infty$. The details which are involved in arriving at the integrand (2.23) are given in [12] where the class $2A$ was discussed in detail. Essentially the action of $\mathcal{B}_G$ is to convert the lattice sum with the Wilson line $E_{4,1} \to \tilde{E}_2 E_{4,1} - E_{6,1}$. This occurs because of is summing over the lattice weighted with the charge vectors. Similarly the one loop corrections to the gauge coupling G' is defined by an integral of the same form in (2.22), with the integrand given by

$$\mathcal{B}_{G'} = -\frac{2}{24\eta^{24}} \Gamma^{(r,s)}_{3,2} \otimes E_{4,1} \left[\frac{1}{4} \alpha^{(r,s)}_{g'} \left\{ \tilde{E}_2 E_6 - E_4^2 \right\} - \beta^{(r,s)}_{g'} f^{(r,s)}_{g'} \left\{ \hat{E}_2 E_4 - E_6 \right\} \right]. \tag{2.25}$$

Here note that $E_6 \to \tilde{E}_2 E_6 - E_4^2$. Using the identities

$$\begin{aligned} \frac{1}{\eta^{24}} \left(E_{4,1}(\tau, z) E_6 - E_{6,1}(\tau, z) E_4 \right) &= -144 B(\tau, z), \\ \frac{1}{\eta^{24}} \left(E_{4,1}(\tau, z) E_4^2 - E_{6,1}(\tau, z) E_6 \right) &= 576 A(\tau, z), \end{aligned} \tag{2.26}$$

we evaluate the difference in the one loop thresholds integrands which results in

$$\mathcal{B}_G - \mathcal{B}_{G'} = -12 \Gamma^{(r,s)}_{3,2} \otimes F^{(r,s)}. \tag{2.27}$$

Thus the difference in the one loop corrections to gauge couplings is given by

$$\Delta_G(T, U, V) - \Delta_{G'}(T, U, V) = -12 \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2} \Gamma^{(r,s)}_{3,2} \otimes F^{(r,s)}. \tag{2.28}$$

There is a constant term that we have ignored in the integrand which is necessary to make the integral well defined in the $\tau_2 \to \infty$ limit.

From (2.28) we conclude that for compactifications on the orbifold $(K3 \times T^2)$ by g' involving the standard embedding, the difference in the one loop thresholds is the automorphic form of $\mathrm{SO}(3,2;\mathbb{Z})$ which is obtained by the theta lift of the elliptic genus of $K3$ twisted by g'. To obtain the threshold correction without the Wilson line one can take the limit $V \to 0$ in (2.28). Then the automorphic form $\mathrm{SO}(3,2;\mathbb{Z})$ reduces to $\mathrm{SO}(2,2;\mathbb{Z})$ modular forms.

3 Standard embedding: 2 examples

In this section we will discuss in detail 2 examples that demonstrate the for standard embeddings, the new supersymmetric index can be written in terms of the twisted elliptic index. The first example deals with the $2A$ orbifold of $K3$ in which $K3$ is at its $T^4/\mathbb{Z}_4$ limit. The second example deals with the recent construction of the $2B$ orbifold of $K3$ [14].

3.1 The $2A$ orbifold from $K3$ as $T^4/\mathbb{Z}_4$

In this section we will construct the orbifold of $K3$ by g' where g' belongs to the class $2A$. The well studied method of obtaining this orbifold is to realize the $K3$ CFT as a $T^4/\mathbb{Z}_2$ orbifold as discussed in [18]. Here we will consider the $2A$ orbifold when $K3$ is at the orbifold limit $T^4/\mathbb{Z}_4$. As far as we are aware the construction is new. This will enable us to investigate the spectrum and the threshold corrections of all the non-standard embeddings of heterotic string at the orbifold $T^4/\mathbb{Z}_4$ discussed in [2] after the g' action.

We define the orbifold of $K3$ by g' as follows. Let us first consider $T^4 \times T^2$ with co-ordinates x_1, x_2 parameterizing T^2 and y_1, y_2, y_3, y_4 labelling T^4. Then $K3$ is realized by the $\mathbb{Z}^4$ which is action given by

$$\begin{aligned} g^s: \; (x_1, x_2, y_1 + iy_2, y_3, +iy_4) \sim (x_1, x_2, e^{2\pi i\, s/4}(y_1 + iy_2), e^{-2\pi i\, s/4}(y_3 + iy_4)), \\ s = 0, 1, 2, 3. \end{aligned} \tag{3.1}$$

This orbifold limit of $K3$ is well known and discussed in [19]. We now consider the g' orbifold which is a $\mathbb{Z}_2$ action given by

$$g': \; (x_1, x_2, y_1, y_2, y_3, y_4) \sim (x_1 + \pi, x_2, y_1 + \pi, y_2 + \pi, y_3 + \pi, y_4 + \pi). \tag{3.2}$$

We will first show that the twisted elliptic genus remains the same as that when $K3$ is realized as a $T^4/\mathbb{Z}_2$ orbifold. This result in fact a test that the orbifold action given in (3.1) and (3.2) in fact $K3$ twisted by the element $2A$. We will then evaluate the spectrum of heterotic string compactified on this orbifold $K3 \times T^2$ for the standard embedding. Using the orbifold action we will explicitly show that the new supersymmetric index admits a decomposition in terms of the twisted elliptic genus. Therefore this is a verification of the result in the previous section that the new supersymmetric index for compactifications on orbifolds of $K3$ in any standard embedding just depends on the twisted elliptic genus of $K3$. We then evaluate the difference in one loop gauge thresholds and show that indeed the resulting modular form is the theta lift of the elliptic genus of $K3$ twisted by the element $2A$.

3.1.1 Twisted elliptic genus

The twisted elliptic genus under under the orbifold (3.1) and (3.2) is given by the index

$$F^{(r,s)}(\tau, z) = \frac{1}{8} \sum_{a,b=0}^{3} Tr_{g^a, g'^r} \left((-1)^{F_L + \bar{F_R}} g^b g'^s e^{2\pi i z F_L} q^{L_0} \bar{q}^{\bar{L}_0} \right).$$

Here the trace is taken over theory of 4 free bosonic coordinates y_1, y_2, y_3, y_4 and 4 free fermions which form their superpartners, F_L, F_R are the left and right moving fermion numbers respectively. We have suppressed the shifts $L_0 - 1/4$, $\bar{L}_0 - 1/4$ in the definition of the index. Let us further define the trace

$$\mathcal{F}(a, r; b, s) = \frac{1}{8} Tr_{g^a, g'^r} \left((-1)^{F_L + \bar{F_R}} g^b g'^s e^{2\pi i z F_L} q^{L_0} \bar{q}^{\bar{L}_0} \right). \tag{3.3}$$

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	Fixed points	g'	g	g^2	g^3	$g'g$	$g'g^2$	$g'g^3$
g	$0,\ \frac{(1+i)}{2}$	×	✓	✓	✓	×	×	×
g^2	$0,\ \frac{(1+i)}{2}$	×	✓	✓	✓	×	×	×
	$\frac{1}{2},\ \frac{i}{2}$	×	×	✓	×	✓	×	✓
g^3	$0,\ \frac{(1+i)}{2}$	×	✓	✓	✓	×	×	×
gg'	$\frac{1}{2},\ \frac{i}{2}$	×	×	✓	×	✓	×	✓
g^2g'	$\frac{1+i}{4},\ \frac{-1-i}{4}$	×	×	×	×	×	✓	×
	$\frac{1-i}{4},\ \frac{-1+i}{4}$	×	×	×	×	×	✓	×
g^3g'	$\frac{1}{2},\ \frac{i}{2}$	×	×	✓	×	✓	×	✓

Table 1. Each row lists the property of fixed points along the y_1, y_2 direction under actions of powers of g, g'. $\times$ indicates that the fixed point moves, while the ✓ indicates the fixed point is invariant. Positions are in units of 2π An identical table exists for the y_3, y_4 direction.

To evaluate each sector of the above twisted elliptic genus we will need the fixed point under the elements $g^a g'^r$ and what elements preserve these fixed points. This information is summarized in table 1.

Let us discuss the twisted elliptic genus for each of the sectors. The sector $(0,0)$ is easiest to deal with. Since there are no twists in g' or insertions of g' to deal with we see that the trace reduces to

$$F^{0,0}(\tau, z) = \frac{1}{2} Z_{K3}(\tau, z) = 4A(\tau, z). \tag{3.4}$$

where Z_{K3} is the elliptic genus of $K3$.

Let us now examine the sector $(0,1)$. We see from table 1, that a single insertion of g' does not preserve any of the fixed points. Thus we have

$$\mathcal{F}(a, 0; b, 1) = 0, \qquad \text{for } a = 1, 3. \tag{3.5}$$

Therefore we need to look at $\mathcal{F}(0,0;b,1)$ and $\mathcal{F}(2,0;b,1)$. Evaluating the trace in the untwisted sector we see the contributions are

$$\begin{aligned}
\mathcal{F}(0,0;0,1) &= 0, \\
\mathcal{F}(0,0;1,1) &= \frac{1}{2}\frac{\theta_1\left(z+\frac{1}{4},\tau\right)\theta_1\left(-z+\frac{1}{4}\right)}{\theta_1^2\left(\frac{1}{4},\tau\right)}, \\
\mathcal{F}(0,0;2,1) &= 2\frac{\theta_1\left(z+\frac{1}{2},\tau\right)\theta_1\left(-z+\frac{1}{2}\right)}{\theta_1^2\left(\frac{1}{2},\tau\right)}, \\
\mathcal{F}(0,0;3,1) &= \frac{1}{2}\frac{\theta_1\left(z+\frac{3}{4},\tau\right)\theta_1\left(-z+\frac{3}{4}\right)}{\theta_1^2\left(\frac{3}{4},\tau\right)}.
\end{aligned} \tag{3.6}$$

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The numerical coefficients in each of the traces occur due to the contribution of the Fermionic zero modes. There are 4 Fermionic right moving zero modes when g^2 is inserted in the trace while there are 2 right moving zero modes for the g and g^3 insertions. Evaluating the contributions to $\mathcal{F}(2,0;b,1)$ we obtain

$$\begin{aligned}
\mathcal{F}(2,0;0,1) &= 0, \qquad \mathcal{F}(2,0;2,1) = 0, \\
\mathcal{F}(2,0;1,1) &= \frac{1}{2}\frac{\theta_1\left(z+\frac{2\tau+1}{4},\tau\right)\theta_1\left(-z+\frac{2\tau+1}{4}\right)}{\theta_1^2\left(\frac{2\tau+1}{4},\tau\right)}, \\
\mathcal{F}(2,0;3,1) &= \frac{1}{2}\frac{\theta_1\left(z+\frac{2\tau+3}{4},\tau\right)\theta_1\left(-z+\frac{2\tau+3}{4}\right)}{\theta_1^2\left(\frac{2\tau+3}{4},\tau\right)}.
\end{aligned} \tag{3.7}$$

The vanishing of the first set of equations in (3.7) is due to the fact that the fixed points in the relevant traces are not invariant under g' or g^2g' insertions as can be seen from the table 1. The numerical factors in the last line equations in (3.7) is due to presence of 4 fixed points in these twisted sectors. Now summing up the contributions we obtain

$$\begin{aligned}
F^{(0,1)}(\tau,z) &= \mathcal{F}(0,0;1,1)+\mathcal{F}(0,0;2,1)+\mathcal{F}(0,0;3,1)+\mathcal{F}(2,0;1,1)+\mathcal{F}(2,0;3,1), \\
&= 4\frac{\theta_2^2(z,\tau)}{\theta_2^2(0,\tau)}, \\
&= \frac{4}{3}A(\tau,z)-\frac{2}{3}\mathcal{E}_2(\tau)B(\tau,z).
\end{aligned} \tag{3.8}$$

The equality in the second line of the above equation is due to identities involving the theta functions. Thus we see that the twisted elliptic genus of the orbifold given in (3.1), (3.2) belongs to the class $2A$.

Though the other sectors of the twisted elliptic genus can be obtained by modular transformations, for completeness we provide some of the details. Lets examine contributions to $F^{(1,0)}$. Due to the presence of right moving Fermionic zero modes we obtain $\mathcal{F}(0,1;0,0)=0$. Now the following vanish

$$\mathcal{F}(0,1,a,0)=0, \qquad \text{for } a=1,2,3, \tag{3.9}$$

This is because due to the insertions of powers of g the trace can contribute only if there are zero modes in the winding sector. However since this sector is twisted in g', the winding modes are all half integer modded and cannot vanish. The only non-trivial contributions arise from the following

$$\begin{aligned}
\mathcal{F}(a,1;b,0) &= \frac{1}{2}\frac{\theta_1\left(z+\frac{b+a\tau}{4}\right)\theta_1\left(-z+\frac{b+a\tau}{4}\right)}{\theta_1^2\left(\frac{b+a\tau}{4},z\right)}, \qquad \text{for } a=1,3,\ b=0,2, \\
\mathcal{F}(2,1;0,0) &= 2\frac{\theta_1\left(z+\frac{\tau}{2},\tau\right)\theta_1\left(-z+\frac{\tau}{2},\tau\right)}{\theta_1^2\left(\frac{\tau}{2},\tau\right)}.
\end{aligned} \tag{3.10}$$

The rest of the indices vanish due to the fact that the fixed points in those sectors are not invariant with the relevant insertions of g, g' in the trace. Summing up the contributions it can be seen that

$$F^{(1,0)} = 4\frac{\theta_1\left(z+\frac{\tau}{2},\tau\right)\theta_1\left(-z+\frac{\tau}{2},\tau\right)}{\theta_1\left(\frac{\tau}{2},\tau\right)^2} \tag{3.11}$$
$$= 4\frac{\theta_4(z,\tau)^2}{\theta_4(0,\tau)^2}.$$

Finally due to the same reasons we see that the only contributions to $F^{(1,1)}$ arise from

$$\mathcal{F}(a,1;b,1) = \frac{1}{2}\frac{\theta_1\left(z+\frac{b+a\tau}{4}\right)\theta_1\left(-z+\frac{b+a\tau}{4}\right)}{\theta_1^2\left(\frac{b+a\tau}{4},z\right)}, \qquad \text{for } a=1,3,\ b=1,3, \tag{3.12}$$
$$\mathcal{F}(2,1;2,1) = 2\frac{\theta_1\left(z+\frac{1+1\tau}{4}\right)\theta_1\left(-z+\frac{1+1\tau}{4}\right)}{\theta_1^2\left(\frac{1+1\tau}{4},z\right)}.$$

Again summing up the contributions leads to

$$F^{(1,1)} = 4\frac{\theta_3^2(z,\tau)}{\theta_3^2(0,\tau)^2}\,. \tag{3.13}$$

To conclude, from (3.4), (3.8), (3.11) and (3.13) we see that the twisted elliptic genus is identical to the class $2A$ first evaluated in [18] using $K3$ in the $T^4/\mathbb{Z}_2$ orbifold limit.

3.1.2 Massless spectrum

In this section we will derive the massless spectrum of heterotic string theory compactified on the orbifold given in g in (3.1) and g' (3.2) with standard embedding. In orbifold language the standard embedding of is achieved by accompanying the $\mathbb{Z}_4$ action (3.1) together with the shift

$$V = \frac{1}{4}\left(1,-1,0^6;0^8\right), \tag{3.14}$$

in the $E_8\times E_8$ lattice. The spectrum of the $T^4/\mathbb{Z}_4$ with the standard shift was first studied in [20]. We will follow the discussion of [21] which set up the general discussion for studying orbifold compactifications of heterotic string theory which preserve $\mathcal{N}=2$ supersymmetry. The orbifold action g' (3.2) does not produce any fixed points and therefore preserves $\mathcal{N}=2$ supersymmetry. Thus the massless spectrum organizes into the 4 dimensional $\mathcal{N}=2$ gravity multiplet coupled to N_v vectors and N_h hypers. The massless states of the theory in the g^n twisted sector is determined by setting left and right masses to zero

$$m_L^2 = N_L + \frac{1}{2}(P+nV)^2 + E_n - 1 = 0, \tag{3.15}$$
$$m_R^2 = N_R + \frac{1}{2}(r+nv)^2 + E_n - \frac{1}{2} = 0. \tag{3.16}$$

Here P is the $E_8\times E_8$ lattice vector which is generically of the form

$$P = \left(P_{E_8};P_{E_8'}\right). \tag{3.17}$$

The 8 dimensional lattice vector P_{E_8} can belong to either the vector or the spinor conjugacy class which we denote by

$$\lambda_A = (n_1, n_2 .... n_8) \qquad \lambda_B = \left(n_1 + \frac{1}{2}, n_2 + \frac{1}{2}, \cdots, n_8 + \frac{1}{2}\right), \tag{3.18}$$

$$\text{with} \sum_{i=1}^{8} n_i = \text{even integer}. \tag{3.19}$$

E_n is the shift in the zero point energy on the ground state due to the twisting and is given by

$$E_n = \frac{1}{4^2} n(\nu - n), \tag{3.20}$$

where $\nu = 4$ for the $T^4/\mathbb{Z}_4$ orbifold and $n = 0, 1, 3, 4$. r is a SO(8) weight vector with

$$\sum_{i=1}^{4} r_i = \text{ odd}, \tag{3.21}$$

v is a 4 dimensional vector given by

$$v = \frac{1}{4}(0, 0, 1, 1). \tag{3.22}$$

Further conditions on r, v, P so that we obtain massless states $m_L = m_R = 0$ will be discussed below. The degeneracy of the massless states can be obtained from [21]

$$D(n) = \frac{1}{4} \sum_{m=0}^{3} \chi(n, m) \Delta(n, m), \tag{3.23}$$

$$\Delta(n, m) = \exp\left\{2\pi i \left[(r + nv)mv - (P + nV)mV + \frac{1}{2}mn\left(V^2 - v^2\right) + m\rho\right]\right\},$$

and $\chi(n, m)$ refers to the number of fixed points in the g^n twisted sector which are invariant under the action of g^m. ρ is the phase by which the oscillators in the T^4 are rotated by the $\mathbb{Z}_4$ action. In the untwisted sector $n = 0$ we have

$$\chi(0, m) = 1, \tag{3.24}$$

and the phases in $D(0)$ simply implement the projection of the spectrum under the action of g^m. From table 1 we see that

$$\begin{aligned} &\chi(1, m) = \chi(3, m) = 4, \\ &\chi(2, 0) = 16, \qquad \chi(2, 1) = 4, \qquad \chi(2, 2) = 16, \qquad \chi(2, 3) = 4. \end{aligned} \tag{3.25}$$

Our goal is to obtain the spectrum when there is a further action by the $\mathbb{Z}_2$ group g' given in (3.2). The first thing to note is that there are no massless states arising from the twisted sectors of g'. This is because all these states have half integer Kaluza-Klein modes on T^4 and therefore they are massive. Thus the only change in obtaining the massless spectrum is that the degeneracy given in (3.23) changes to

$$D(n;g') = \frac{1}{4}\sum_{m=0}^{3}\frac{1}{2}\left[\chi(n,m)+\chi^{(g')}(n,m)\right]\Delta(n,m), \tag{3.26}$$

where $\chi^{(g')}$ is the number for fixed points in the g^n twisted sector invariant under the action of $g^m g'$. Essentially we have inserted the projection over g'. In the untwisted sector

$$\chi^{(g')}(0,m) = \chi(0,m) = 1, \tag{3.27}$$

and again the phases in (3.26) just implement the projection of the spectrum under g^m. For the twisted sector, from the tabel 1 we obtain

$$\begin{aligned} &\chi^{(g')}(1,m) = \chi^{(g')}(3,m) = 0, \\ &\chi(2,0)^{(g')} = 0, \qquad \chi(2,1)^{(g')} = 4, \qquad \chi(2,2)^{(g')} = 0, \qquad \chi(2,3)^{(g')} = 4. \end{aligned} \tag{3.28}$$

We are now ready to obtain the spectrum of the model.

Untwisted sector. It is clear from (3.24), (3.27) and (3.26) we see that there is no change in the spectrum for the untwisted sector. Thus the untwisted sector remains the same as that worked out earlier in [21]. This sector contains the $\mathcal{N}=2$ gravity multiplet and the $\mathcal{N}=2$ vectors. The gauge group breaks from $E_8\times E_8$ to $E_7\times \mathrm{U}(1)\times E_8$.[2] Thus the Non-Abelian $\mathcal{N}=2$ vector multiplets are in the $\mathbf{133}$ of E_7 and the $\mathbf{248}$ of E_8. In the untwisted sector there are 2 singlet hypers under $E_7\times E_8$ which we denote as $(\mathbf{1},\mathbf{1})$ and 2 hypers charged as $(\mathbf{56},\mathbf{1})$.

The twisted sector consists of only hypermultiplets

Twisted by g and g^3. From (3.25), (3.28) and (3.26) we see that the degeneracies in the g^2 and g^3 twisted sector becomes half of the theory on the orbifold $(T^4/\mathbb{Z}_2)\times T^2$ worked out in [21]. In fact the states in the g^3 twisted sector form the anti-particles of the states in the g twisted sector. The hypers for the g' orbifold are $2(\mathbf{56},\mathbf{1})+\mathbf{16}(\mathbf{1},\mathbf{1})$.[3]

Twisted by g^2. It in only in this sector we really need to explicitly work out the details of the states and using the formula (3.26). For massless states in the twisted sector we have the conditions

$$r^2 = 1, \qquad r\cdot v = -\frac{1}{4}. \tag{3.29}$$

Using the equations (3.20), (3.22 and (3.29) we see that p_R given in (3.15) indeed vanishes for $N_R=0$. Lets examine the condition $p_L=0$.

1. For $N_L=0$ in the g^2 twisted sector we see $p_L=0$ results in the condition

$$(P+2V)^2 = 3/2. \tag{3.30}$$

[2] We are ignoring the 2 vector multiplets from the one cycles of the T^2.

[3] We are not keeping track of the U(1) charges in our discussion.

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This condition can only be satisfied by two ways. Firstly we can take the lattice vectors in both the E_8's in the vector conjugacy class. Thus we have

$$\left(n_1 + \frac{1}{2}\right)^2 + \left(n_2 - \frac{1}{2}\right)^2 + \sum_{j=3}^{16} n_j^2 = \frac{3}{2}, \tag{3.31}$$

which in turn can be satisfied by $n_1 = 0, n_2 = 1$ or $n_1 = -1, n_2 = 0$ with one of the $n_j = \pm 1, j = 3, 4, 5, 6, 7, 8$. The restriction that these are in the first lattice comes from the condition in the last line of (3.18). All together this results in 24 solutions. Now the second choice of lattice vectors is, in which we have the spinor conjugacy class in the first E_8 and the vector class in the second E_8. Therefore (3.30) reduces to

$$\left(n_1 + \frac{1}{2} + \frac{1}{2}\right)^2 + \left(n_2 + \frac{1}{2} - \frac{1}{2}\right)^2 + \sum_{j=3}^{8}\left(n_j + \frac{1}{2}\right)^2 + \sum_{k=9}^{16} n_k^2 = \frac{3}{2}. \tag{3.32}$$

Here we can have $n_1 = -1$, $n_2 = 0$ and any odd number of the 6 $n_j's$ as 0 or -1 which can be achieved by 32 ways $({}^6C_1 +^6 C_3 +^6 C_5 = 32)$. The $24 + 32 = 56$ solutions of (3.31) and (3.32) form the $(\mathbf{56}, \mathbf{1})$ dimensional representation of $E_7 \times E_8$. Let us now evaluate the degeneracy of these states. They are solutions to the mass shell condition and satisfy $P \cdot V = -1/4$, and have $\rho = 0$. Using (3.29) and the values of v and V from (3.22) and (3.14) respectively We find that $\Delta(2, 1) = 1$. Then from (3.26) we see that the degeneracy of these states is $D(2, g') = 3$, where we need to divide by 2 to account for the anti-particles. Thus we have $3(\mathbf{56}, \mathbf{1})$ hypers.[4]

2. Now lets look at the case of $N_L = 1/2$, where the oscillators along the T^4 are excited. For these states there is a pair of oscillators each with $\rho = \pm 1/4$. The $m_L = 0$ condition reduces to

$$(P + 2V)^2 = 1/2. \tag{3.33}$$

This can be satisfied only when both the E_8 lattice vectors are chosen in the vector conjugacy class leading to

$$\left(n_1 + \frac{1}{2}\right)^2 + \left(n_2 - \frac{1}{2}\right)^2 + \sum_{j=3}^{16} n_j^2 = \frac{1}{2}. \tag{3.34}$$

This equation admits two solutions: $n_1 = n_2 = n_j = 0$ and $n_1 = -1$, $n_2 = 1$, $n_j = 0$ which have $P \cdot V = 0$. Evaluating the phase $\Delta(2, 1)$ for $\rho = \pm 1/4$ we obtain $\Delta(2, 1) = \pm 1$. The degeneracy from (3.26) for these states is given by $2 \times (3 + 1) = 8$, here again we are not counting anti-particles. The 2 factor arises due to the 2 solutions for (3.34) Finally since we have two pairs of oscillators with $\rho = \pm 1/4$ the total number of states is given by have $2 \times 8 = 16$ These states are singlets with respect to the $E_7 \times E_8$, therefore.[5]

[4] For the model just on $T^4/\mathbb{Z}_4 \times T^2$ we have $D(2) = 5$ for these states

[5] For the model without the g' orbifold the number of such states is 32.

Model	Shift	Sector	Matter	$N_h - N_v$
		g^0	$(\mathbf{56},\mathbf{1})+\mathbf{2}(\mathbf{1},\mathbf{1})$	-12
$(T^4/\mathbb{Z}_4 \times T^2)/g'$	$E_7 \times \mathrm{U}(1) \times E_8$	$g+g^3$	$2(\mathbf{56},\mathbf{1})+\mathbf{16}(\mathbf{1},\mathbf{1})$	
	$\frac{1}{4}(1,-1,0^6;0^8)$	g^2	$3(\mathbf{56},\mathbf{1})+\mathbf{16}(\mathbf{1},\mathbf{1})$	

Table 2. Hypermultiplet content of the g' orbifold of $T^4/\mathbb{Z}_4 \times T^2$ with the standard embedding.

Model	Shift	Sector	Matter	$N_h - N_v$
		g^0	$(\mathbf{56},\mathbf{1})+\mathbf{2}(\mathbf{1},\mathbf{1})$	+244
$T^4/\mathbb{Z}_4 \times T^2$	$E_7 \times \mathrm{U}(1) \times E_8$	$g+g^3$	$4(\mathbf{56},\mathbf{1})+\mathbf{32}(\mathbf{1},\mathbf{1})$	
	$\frac{1}{4}(1,-1,0^6;0^8)$	g^2	$5(\mathbf{56},\mathbf{1})+\mathbf{32}(\mathbf{1},\mathbf{1})$	

Table 3. Hypermultiplet content of $T^4/\mathbb{Z}_4 \times T^2$ with the standard embedding.

To summarize the spectrum of the g' orbifold of $T^4/\mathbb{Z}_4$ with the standard shift of (3.14) consists of a $\mathcal{N}=2$ gravity multiplet with a gauge multiplet in the $(\mathbf{133},\mathbf{1}) \oplus (\mathbf{1},\mathbf{248})$ of $E_7 \times E_8$ and a U(1). The hypermultiplet content is summarized in table 2. Evaluating $N_h - N_v = -12$. For comparison we have also summarized the hypermultiplet content of the same model without the g' model in table 3. The vector multiplet content is the same. $N_h - N_v = -244$ for this model which is dictated by anomaly cancellation since this model admits a lift to a chiral $6d$ theory unlike the g' orbifold. This phenomenon of the vector multiplet being invariant but the reduction of the number of hypers by the action of g' was also observed in [12]. In the subsequent section we will verify that the $N_h - N_v = -12$ for the g' orbifold by evaluating the new supersymmetric index.

3.1.3 The new supersymmetric index

In this section we will evaluate the new supersymmetric index for the orbifold defined by the actions (3.1), (3.2) with the shift in (3.14) in $E_8 \times E_8$. We adapt the method developed in [2] to incorporate the additional g' orbifolding action. Evaluating the trace, the new supersymmetric index given in (2.3) splits into the following sectors

$$\mathcal{Z}_{\rm new}(q,\bar{q}) = -\frac{1}{2\eta^{20}(\tau)} \sum_{a,b=0}^{3} \sum_{r,s=0}^{1} e^{-\frac{2\pi i ab}{16}} Z_{E_8}^{(a,b)}(\tau) \times E_4(q) \times \frac{1}{8} F(a,r,b,s;q) \Gamma_{2,2}^{(r,s)}(q,\bar{q}). \tag{3.35}$$

First note that the anti-holomorphic dependence in q occurs only in the lattice sum $\Gamma_{2,2}^{(r,s)}(q,\bar{q})$ Let us define each of the component in (3.35). The trace over the T^4 directions is given by

$$F(a,r,b,s;q) = \mathrm{Tr}_{g^a\, g'^s R}\left(g^b g'^s e^{i\pi F_R^{T^4}} q^{L_0} \bar{q}^{\bar{L}_0} \right). \tag{3.36}$$

Here the left moving CFT consists of 4 free bosons with $c=4$ and the right movers consists of 4 free bosons and 4 free Fermions which is in the Ramond sector. The F_R is the fermion

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number of the right moving states. The explicit expressions for this trace using the orbifold action in (3.1), (3.2) is given by

$$F(a,r,b,s;q) = k^{(a,r,b,s)}\eta^2(\tau)q^{\frac{-a^2}{16}}\frac{1}{\theta_1^2\left(\frac{a\tau+b}{4},\tau\right)}. \tag{3.37}$$

The coefficients $k^{(a,r,b,s)}$ for the various values of (r,s) are given by the following matrices

$$\begin{aligned} k^{(a,0,b,0)} &= 16\begin{pmatrix}0&1&4&1\\1&1&1&1\\4&1&4&1\\1&1&1&1\end{pmatrix}, \quad k^{(a,0,b,1)} = 16\begin{pmatrix}0&1&4&1\\0&0&0&0\\0&1&0&1\\0&0&0&0\end{pmatrix}, \\ k^{(a,1,b,0)} &= 16\begin{pmatrix}0&0&0&0\\1&0&1&0\\4&1&0&0\\1&0&1&0\end{pmatrix}, \quad k^{(a,1,b,1)} = 16\begin{pmatrix}0&0&0&0\\0&1&0&1\\0&0&4&0\\0&1&0&1\end{pmatrix}. \end{aligned} \tag{3.38}$$

Note that rows and columns are labelled by a and b respectively. The coefficients for $(r,s) = (0,0)$ are identical to the situation without the g' orbifolding. The remaining coefficients can be easily obtained by using the same arguments discussed in section while evaluating the twisted elliptic genus of this orbifold. The Eisenstein series $E_4(q)$ in (3.35) results from the partition function of the untouched E_8 lattice which is not coupled to the spin connection of $K3$. The partition function of the first E_8 lattice with the shifts are given by

$$\begin{aligned}
Z_{E_8}^{(0,1)} &= \frac{1}{2}\left\{\theta_3^6\theta\begin{bmatrix}0\\1/2\end{bmatrix}\theta\begin{bmatrix}0\\-1/2\end{bmatrix} + \theta_2^6\theta\begin{bmatrix}1\\1/2\end{bmatrix}\theta\begin{bmatrix}1\\-1/2\end{bmatrix} + \theta_4^6\theta\begin{bmatrix}0\\3/2\end{bmatrix}\theta\begin{bmatrix}0\\-1/2\end{bmatrix}\right\} \\
&= Z_{E_8}^{(0,3)}, \\
Z_{E_8}^{(1,0)} &= \frac{1}{2}\left(\theta_3^6\theta\begin{bmatrix}1/2\\0\end{bmatrix}\theta\begin{bmatrix}-1/2\\0\end{bmatrix} + \theta_2^6\theta\begin{bmatrix}3/2\\0\end{bmatrix}\theta\begin{bmatrix}-1/2\\0\end{bmatrix} + \theta_4^6\theta\begin{bmatrix}1/2\\1\end{bmatrix}\theta\begin{bmatrix}-1/2\\1\end{bmatrix}\right) \\
&= Z_{E_8}^{(3,0)}, \\
Z_{E_8}^{(1,1)} &= \frac{1}{2}\left(\theta_3^6\theta\begin{bmatrix}1/2\\1/2\end{bmatrix}\theta\begin{bmatrix}-1/2\\-1/2\end{bmatrix} + \theta_2^6\theta\begin{bmatrix}3/2\\1/2\end{bmatrix}\theta\begin{bmatrix}1/2\\-1/2\end{bmatrix} + \theta_4^6\theta\begin{bmatrix}1/2\\3/2\end{bmatrix}\theta\begin{bmatrix}-1/2\\1/2\end{bmatrix}\right) \\
&= -Z_{E_8}^{(3,3)}, \\
Z_{E_8}^{(1,2)} &= \frac{1}{2}\left(\theta_3^6\theta\begin{bmatrix}1/2\\1\end{bmatrix}\theta\begin{bmatrix}-1/2\\-1\end{bmatrix} + \theta_2^6\theta\begin{bmatrix}3/2\\1\end{bmatrix}\theta\begin{bmatrix}1/2\\-1\end{bmatrix} + \theta_4^6\theta\begin{bmatrix}1/2\\2\end{bmatrix}\theta\begin{bmatrix}-1/2\\0\end{bmatrix}\right) \\
&= -Z_{E_8}^{(3,2)}, \\
Z_{E_8}^{(1,3)} &= \frac{1}{2}\left(\theta_3^6\theta\begin{bmatrix}1/2\\3/2\end{bmatrix}\theta\begin{bmatrix}-1/2\\-3/2\end{bmatrix} + \theta_2^6\theta\begin{bmatrix}3/2\\3/2\end{bmatrix}\theta\begin{bmatrix}1/2\\-3/2\end{bmatrix} + \theta_4^6\theta\begin{bmatrix}1/2\\5/2\end{bmatrix}\theta\begin{bmatrix}-1/2\\-1/2\end{bmatrix}\right) \\
&= -Z_{E_8}^{(3,1)}, \\
Z_{E_8}^{(2,1)} &= \frac{1}{2}\left(\theta_3^6\theta\begin{bmatrix}1\\1/2\end{bmatrix}\theta\begin{bmatrix}-1\\-1/2\end{bmatrix} + \theta_2^6\theta\begin{bmatrix}2\\1/2\end{bmatrix}\theta\begin{bmatrix}0\\-1/2\end{bmatrix} + \theta_4^6\theta\begin{bmatrix}1\\3/2\end{bmatrix}\theta\begin{bmatrix}-1\\-1/2\end{bmatrix}\right) \\
&= Z_{E_8}^{(2,3)}.
\end{aligned} \tag{3.39}$$

Also in the $\mathbb{Z}_2$ subgroup sector we have

$$\begin{aligned} Z_{E_8}^{(0,2)} &= \frac{1}{2}\left(\theta_3^6\theta\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]\theta\left[\begin{smallmatrix}0\\-1\end{smallmatrix}\right]+\theta_4^6\theta\left[\begin{smallmatrix}0\\2\end{smallmatrix}\right]\theta\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]\right) \\ &= \frac{1}{2}\left(\theta_3^6\theta_4^2+\theta_4^6\theta_3^2\right), \\ Z_{E_8}^{(2,0)} &= \frac{1}{2}\left(\theta_3^6\theta\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]\theta\left[\begin{smallmatrix}-1\\0\end{smallmatrix}\right]+\theta_2^6\theta\left[\begin{smallmatrix}2\\0\end{smallmatrix}\right]\theta\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]\right) \\ &= \frac{1}{2}\left(\theta_3^6\theta_2^2+\theta_2^6\theta_3^2\right), \\ Z_{E_8}^{(2,2)} &= \frac{1}{2}\left(\theta_4^6\theta\left[\begin{smallmatrix}1\\2\end{smallmatrix}\right]\theta\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]+\theta_2^6\theta\left[\begin{smallmatrix}2\\1\end{smallmatrix}\right]\theta\left[\begin{smallmatrix}0\\-1\end{smallmatrix}\right]\right) \\ &= \frac{1}{2}\left(-\theta_4^6\theta_2^2+\theta_2^6\theta_4^2\right). \end{aligned} \tag{3.40}$$

The definition of the generalized Jacobi theta functions is given by

$$\theta\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right](\tau,z)=\sum_{k\in\mathbb{Z}}q^{\pi i(k+\frac{a}{2})^2}e^{\pi i(k+\frac{a}{2})b}e^{2\pi i z(k+\frac{a}{2})}. \tag{3.41}$$

Note that $\theta_1(\tau,z)=\theta\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right](\tau,z)$ In the above equation when the argument of the θ-function is not explicitly mentioned, it is understood that it is evaluated at $z=0$ and at τ.

We can now sum over (a,b) in the equation (3.35). After using (3.36) and (3.39) we obtain the expected results

$$\begin{aligned} \mathcal{Z}_{\rm new}(q,\bar q) &= -\frac{2}{\eta^{24}(\tau)}\sum_{r,s=0}^{1}\Gamma_{2,2}^{(r,s)}E_4\left[\frac{1}{4}\alpha_{2A}^{(r,s)}E_6-\beta_{2A}^{r,s}f_{2A}^{(r,s)}(\tau)E_4\right], \\ \alpha_{2A}^{(0,0)} &= 4, \qquad \beta_{2A}^{(0,0)}=0, \\ \alpha_{2A}^{(0,1)} &= \frac{4}{3}, \qquad \beta_{2A}^{(0,1)}=-\frac{2}{3}, \\ \alpha_{2A}^{(1,0)} &= \alpha_{2A}^{(1,1)}=\frac{4}{3}, \qquad \beta_{2A}^{(1,0)}=\beta_{2A}^{(1,1)}=\frac{1}{3}, \\ f_{2A}^{(0,1)}(\tau) &= \mathcal{E}_2(\tau), \quad f_{2A}^{(1,0)}(\tau)=\mathcal{E}_2\left(\frac{\tau}{2}\right), \quad f_{2A}^{(1,1)}(\tau)=\mathcal{E}_2\left(\frac{\tau+1}{2}\right). \end{aligned} \tag{3.42}$$

We performed the sum over (a,b) in (3.35) for each of the (r,s) sectors using Mathematica to arrive at the result (3.42).

From (2.14) we see that the new supersymmetric index of the orbifold of $T^4/\mathbb{Z}_4\times T^2$ by g' agrees with that of the $2A$ orbifold of $K3\times T^2$. This result was expected since we have seen in section 3.1.1, that the twisted elliptic genus of the orbifold in (3.1), (3.2) agrees with the $2A$ class. Then the general arguments in section 2.1 show that for standard embeddings the new supersymmetric index can be written in terms of the twisted elliptic genus. However it is indeed nice to see this using explicit computations.

As a consistency check of our calculations we will evaluate the N_h-N_v from the new supersymmetric index. From the general arguments of [4] the $q^{1/6}$ coefficient of the following expression which is related to the new supersymmetric index evaluates $N_h - N_v$.

$$N_h - N_v = \frac{1}{4}\eta^4 \left(\sum_{s=0}^{N-1} \mathcal{Z}_{\rm new}^{(0,s)} \right)\Bigg|_{q^{1/6}}, \tag{3.43}$$

where $\mathcal{Z}_{\rm new}^{(0,s)}$ is the corresponding sector of the new supersymmetric index without the lattice factor $\Gamma_{2,2}^{(0,s)}$. We focus on these terms to extract out the massless states contributing to the new supersymmetric index. The $\frac{1}{4}$ factor is introduced to take into account the normalizations of the new supersymmetric index used in this paper. Substituting the new supersymmetric index for the standard embedding of the $2A$ orbifold of $K3 \times T^2$ evaluated in (3.42) we obtain

$$(N_h - N_v)|_{2A} = -12. \tag{3.44}$$

Note that this agrees with the explicit computation of the spectrum in table 2.[6]

Now turning on Wilson line in the unbroken E_8 and evaluating the thresholds proceeds identically to that discussed in section 2.2. We thus obtain the result that the difference in one loop gauge thresholds for this orbifold compactification is the theta lift of the twisted elliptic genus of $K3$ belonging to the class $2A$.

3.2 The $2B$ orbifold from $K3$ based on su$(2)^6$

Recently in [14], the $K3$ sigma model has been studied in terms of a rational conformal field theory based on the affine algebra su$(2)^6$. In this model of $K3$ the action of g',[7] an element of order 4, which belongs to the conjugacy class $2B$ of M_{24} was explicitly constructed and the twisted elliptic genus was evaluated. In this section we will use this realization of $K3$ to evaluate the new supersymmetric index of heterotic compactified on $K3 \times T^2$ orbifolded by the order 4 element g'. We will show that indeed as demonstrated by the general analysis of section 2.1, that new supersymmetric index can be written in terms of the twisted elliptic genus of $K3$ twisted by g'. Furthermore as discussed in section 2.2, this implies that the difference in one loop gauge thresholds is determined by the theta lift of the corresponding twisted elliptic genus.

3.2.1 Twisted elliptic genus

Let us evaluate the twisted elliptic genus as defined by the trace in (2.10). From the definition of the trace we need the characters of the su$(2)^6$ model in the Ramond section. These were listed in [14], here we present them in the table 4. su$(2)_k$ characters of the highest weight representation $[a]$ with $a = 0, \ldots k$ are given by

$$\mathrm{ch}_{k,\frac{a}{2}}(\tau, z) = \mathrm{Tr}_{[a]_k} q^{L_0 - c/24} e^{2\pi i z J_0}\,. \tag{3.45}$$

[6] We have evaluated $(N_h - N_v)$ from the new supersymmetric index for all the pA orbifolds of $K3 \times T^2$ with $p = 3, 5, 7, 11$. We obtain $-134, -256, -317, -376$ respectively which indicates that the number of hypers is reduced by this orbifolding. It is also an important check on the compactification that we obtain integers in all these situations.

[7] In [14], g' was referred to as g, see section 6.1.

R^-	[10 00 00, 10 00 00]	-[01 11 11, 01 00 00]
	[01 00 00, 01 00 00]	-[10 11 11, 10 00 00]
	[00 10 00, 00 10 00]	-[11 01 11, 00 10 00]
	[00 01 00, 00 01 00]	-[11 10 11, 00 01 00]
	[00 00 10, 00 00 10]	-[11 11 01, 00 00 10]
	[00 00 01, 00 00 01]	-[11 11 10, 00 00 01]

Table 4. su$(2)^6$ characters in the Ramond sector with the sign $(-1)^{F_L+F_R}$.

Thus 0 in table 4 represents the su(2) character at level 1

$$\mathrm{ch}_{1,0} = \frac{\theta_3(2\tau, 2z)}{\eta(\tau)}, \tag{3.46}$$

while 1 represent the spinorial su(2) character given by

$$\mathrm{ch}_{1,\frac{1}{2}} = \frac{\theta_2(2\tau, 2z)}{\eta(\tau)}. \tag{3.47}$$

The comma in the list of table 4 separates the left moving su(2) characters and the right moving ones. The $\mathrm{SU}(2)_L\times\mathrm{SU}(2)_R$ R-symmetry of $K3$ is carried by the first su(2) character among the left and right moving characters respectively. As shown in [14], the elliptic genus with the characters given in the table reduces to that of $K3$.

The g' orbifold on $K3$ is implemented by the action

$$g' = \rho_L \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \right]. \tag{3.48}$$

Where ρ_L refers to the fact that the action of g' is restricted to the left moving characters. The SU(2) rotation matrices of g' on the su(2) characters is given by

$$\begin{aligned} \mathrm{Tr}_{[0]}\left[\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} q^{L_0-\frac{1}{24}} e^{2\pi i J_0} \right] &= \frac{\theta_3(2\tau, 2z)}{\eta(\tau)}, \\ \mathrm{Tr}_{[1]}\left[\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} q^{L_0-\frac{1}{24}} e^{2\pi i J_0} \right] &= -\frac{\theta_2(2\tau, 2z)}{\eta(\tau)}, \\ \mathrm{Tr}_{[0]}\left[\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} q^{L_0-\frac{1}{24}} e^{2\pi i J_0} \right] &= \frac{\theta_4(2\tau, 2z)}{\eta(\tau)}, \\ \mathrm{Tr}_{[1]}\left[\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} q^{L_0-\frac{1}{24}} e^{2\pi i J_0} \right] &= -\frac{\theta_1(2\tau, 2z)}{\eta(\tau)}. \end{aligned} \tag{3.49}$$

The $F^{(0,0)}$ component of the elliptic genus is easy to evaluate and we see that it is given by

$$\begin{aligned} F^{0,0}(\tau,z) &= \frac{1}{2\eta^6(\tau)}\Big[\theta_2(2\tau,2z)\theta_3(2\tau)^5 - \theta_3(2\tau,2z)\theta_2(2\tau)^5 \\ &\quad +5\theta_3(2\tau,2z)\theta_2(2\tau)\theta_3(2\tau)^4 - 5\theta_2(2\tau,2z)\theta_3(2\tau)\theta_2(2\tau)^4\Big] \\ &= 2A(\tau,z). \end{aligned} \tag{3.50}$$

On evaluating the trace, the right movers contribute a factor of 2 since the zero modes form a SU(2) doublet. Note that the $F^{(0,0)}$, component differs from the elliptic genus of $K3$ by a 1/4 factor. Using the action of g' on the characters we evaluate the following components of the twisted elliptic genus to be

$$\begin{aligned} F^{(0,1)}(\tau,z) &= \frac{1}{2\eta^6(\tau)}\Big[\theta_2(2\tau,2z)\theta_3(2\tau)\theta_4(2\tau)^4 - \theta_3(2\tau,2z)\theta_2(2\tau)\theta_4(2\tau)^4\Big] \\ &= \frac{1}{2}\left[\mathcal{E}_2(\tau) - 2\mathcal{E}_4(\tau)\right]B(\tau,z), \\ F^{(0,2)}(\tau,z) &= \frac{1}{2\eta^6(\tau)}\Big[\theta_2(2\tau,2z)\theta_3(2\tau)^5 - \theta_3(2\tau,2z)\theta_2(2\tau)^5 \\ &\quad -3\theta_3(2\tau,2z)\theta_2(2\tau)\theta_3(2\tau)^4 + 3\theta_2(2\tau,2z)\theta_3(2\tau)\theta_2(2\tau)^4\Big] \\ &= -\frac{2}{3}\left[A(\tau,z) + \mathcal{E}_2(\tau)B(\tau,z)\right]. \end{aligned} \tag{3.51}$$

All the remaining components of the twisted elliptic genus can be obtained from modular transform given in (2.16). Note that the twisted elliptic genus falls into the form given in (2.12) with the identifications

$$\begin{aligned} &\alpha_{2B}^{(0,0)} = 2, && \alpha_{2B}^{(0,1)} = 0, && \alpha_{2B}^{(0,2)} = -\frac{2}{3}, \\ &\beta_{2B}^{(0,1)} = \frac{1}{2}, && f_{2B}^{(0,1)} = \mathcal{E}_2(\tau) - 2\mathcal{E}_4(\tau), && \\ &\beta_{2B}^{(0,2)} = -\frac{2}{3}, && f_{2B}^{(0,2)} = \mathcal{E}_2(\tau). && \end{aligned} \tag{3.52}$$

3.2.2 New supersymmetric index

From the discussion in section 3.2.1 in which $K3$ is realized as a rational su$(2)^6$ rational conformal field theory we see that the R symmetry of the model is carried by the first character among both the left and right movers. The new supersymmetric index given in (2.3) involves the trace in which the right movers are always in the Ramond sector with a $(-1)^{F_R}$. The right moving characters listed in the table 4 are indeed in the R^- sector. The standard embedding identifies R symmetry of the left movers carried by the first character of in the su$(2)^6$ model with the fermions of the $D2$ lattice in the first E_8. Now from the expression of the new supersymmetric index in (2.6) we see one needs this first character in the R^+, NS^+ and NS^- sectors. These sectors couple to the corresponding sectors of the $D6$ lattice realized in terms of fermions. Table 5 lists the characters the R^+, NS^+ and NS^- of the su$(2)^6$ CFT. Comparing tables (5) and (4) we can see how the

R^+	-[10 00 00, 10 00 00]	-[01 11 11, 01 00 00]
	[01 00 00, 01 00 00]	[10 11 11, 10 00 00]
	[00 10 00, 00 10 00]	[11 01 11, 00 10 00]
	[00 01 00, 00 01 00]	[11 10 11, 00 01 00]
	[00 00 10, 00 00 10]	[11 11 01, 00 00 10]
	[00 00 01, 00 00 01]	[11 11 10, 00 00 01]
NS^-	[00 00 00, 10 00 00]	-[11 11 11, 01 00 00]
	[11 00 00, 01 00 00]	-[00 11 11, 10 00 00]
	[10 10 00, 00 10 00]	-[01 01 11, 00 10 00]
	[10 01 00, 00 01 00]	-[01 10 11, 00 01 00]
	[10 00 10, 00 00 10]	-[01 11 01, 00 00 10]
	[10 00 01, 00 00 01]	-[01 11 10, 00 00 01]
NS^+	-[00 00 00, 10 00 00]	-[11 11 11, 01 00 00]
	[11 00 00, 01 00 00]	[00 11 11, 10 00 00]
	[10 10 00, 00 10 00]	[01 01 11, 00 10 00]
	[10 01 00, 00 01 00]	[01 10 11, 00 01 00]
	[10 00 10, 00 00 10]	[01 11 01, 00 00 10]
	[10 00 01, 00 00 01]	[01 11 10, 00 00 01]

Table 5. $\widehat{\rm su}(2)^6$ characters in sectors relevant of evaluating $Z_{\rm new}$.

spinor representations of the first character in the left moving sector has become a scalar character when the Ramond sector flows to the Neveu-Schwarz sector.

Let us first evaluate the component $\Phi^{(0,0)}$ in various sectors. Using the character table 5 and the rules in (3.46) and (3.47) we obtain

$$\begin{aligned}
\Phi^{(0,0)}_{R^+} &= \frac{1}{2\eta(\tau)^6}\left(4\theta_3^5(2\tau)\theta_2(2\tau)+4\theta_2^5(2\tau)\theta_3(2\tau)\right), \\
&= \frac{1}{2}\left[\frac{\theta_2^2}{\eta^6}(\theta_3^4+\theta_4^4)\right], \\
\Phi^{(0,0)}_{NS^-} &= \frac{1}{2\eta(\tau)^6}\left[5\theta_2^2(2\tau)\theta_3^4(2\tau)-5\theta_3^2(2\tau)\theta_2^4(2\tau)+\theta_3^6(2\tau)-\theta_2^6(2\tau)\right], \\
&= \frac{1}{2}\left[\frac{\theta_4^2}{\eta^6}(\theta_3^4+\theta_2^4)\right], \\
\Phi^{(0,0)}_{NS^+} &= \frac{1}{2\eta(\tau)^6}\left[5\theta_2^2(2\tau)\theta_3^4(2\tau)+5\theta_3^2(2\tau)\theta_2^4(2\tau)-\theta_3^6(2\tau)-\theta_2(2\tau)^6\right], \\
&= \frac{1}{2}\left[\frac{\theta_3^2}{\eta^6}(\theta_2^4-\theta_4^4)\right].
\end{aligned} \tag{3.53}$$

Here we have used Riemann's bilinear identities to simplify the resulting expressions and obtain the result in terms of theta functions with argument τ. We can now multiply these along with the characters of the $D6$ lattice in the corresponding sectors as given in (2.6) and we obtain the following result for the $(0,0)$ sector of the new supersymmetric index

$$\mathcal{Z}_{\rm new}|_{(0,0)} = -2\frac{1}{\eta^{24}(\tau)}\Gamma^{(0,0)}_{2,2} \times \frac{2}{4}E_4 E_6\,. \tag{3.54}$$

Note that this is $\frac{1}{4}$ of the result expected for compactifications of heterotic on $K3 \times T^2$. Lets move now to the $(0,1)$ sector which represents a single insertion of g'. For $\Phi^{(0,1)}_{R^+}$ using the results in (3.49) for the characters with a single insertion of g' we see that the only characters which survive are $-[100000, 100000]$ and $[010000, 010000]$. This results in

$$\Phi^{(0,1)}_{R^+} = \frac{1}{2\eta^6(\tau)}\left(-2\theta_2(2\tau)\theta_3(2\tau)\theta_4^4(2\tau)\right) = -\frac{1}{2\eta^6(\tau)}\theta_2^2(\tau)\theta_4^4(2\tau). \tag{3.55}$$

In the $\Phi^{(0,1)}_{NS^-}$ sector the characters which are present are $[000000, 100000]$ and $[110000, 010000]$ lead to

$$\begin{aligned}\Phi^{(0,1)}_{NS^-} &= \frac{1}{2\eta^6(\tau)}\left(\theta_3^2(2\tau) - \theta_2^2(2\tau)\right)\theta_4^4(2\tau), \\ &= \frac{1}{2\eta^6(\tau)}\theta_4^2(\tau)\theta_4^4(2\tau).\end{aligned} \tag{3.56}$$

Finally the characters which survive the g' insertion in $\Phi^{(0,1)}_{NS^-}$ are $-[000000, 100000]$ and $[110000, 010000]$ giving rise to

$$\begin{aligned}\Phi^{(0,1)}_{NS^+} &= -\frac{1}{2\eta^6(\tau)}\left(\theta_3^2(2\tau) + \theta_2^2(2\tau)\right)\theta_4^4(2\tau), \\ &= -\frac{1}{2\eta^6(\tau)}\theta_3^2(\tau)\theta_4^4(2\tau).\end{aligned} \tag{3.57}$$

Now combining this along with the corresponding $D6$ characters as in (2.6) we obtain

$$\mathcal{Z}_{\rm new}|_{(0,1)} = -2\frac{1}{\eta^{24}(\tau)}\Gamma^{(0,1)}_{2,2} \times E_4\left[-\frac{1}{2}\left(\mathcal{E}_2(\tau) - 2\mathcal{E}_4(\tau)\right).\right]E_4 \tag{3.58}$$

Here there we have used identities which relate the θ functions to Eisenstein series which are provided in the appendix. Using the action of g'^2 which is given by

$$(g')^2 = \rho_L\left[\begin{pmatrix}1&0\\0&1\end{pmatrix}\begin{pmatrix}1&0\\0&1\end{pmatrix}\begin{pmatrix}-1&0\\0&-1\end{pmatrix}\begin{pmatrix}-1&0\\0&-1\end{pmatrix}\begin{pmatrix}-1&0\\0&-1\end{pmatrix}\begin{pmatrix}-1&0\\0&-1\end{pmatrix}\right], \tag{3.59}$$

and the character list in table 5 the contributions for the $\Phi^{(0,2)}$ are evaluated. This results in

$$\begin{aligned}\Phi^{(0,2)}_{R^+} &= -\frac{1}{2\eta^6(\tau)}4\left(\theta_2^5(2\tau)\theta_3(2\tau) + \theta_3^5(2\tau)\theta_2(2\tau)\right) = -\frac{1}{2\eta^6(\tau)}\theta_2^2\left(\theta_3^4 + \theta_4^4\right), \\ \Phi^{(0,2)}_{NS^-} &= \frac{1}{2\eta^6(\tau)}\left(\theta_3^6(2\tau) - \theta_2^6(2\tau) - 3\theta_2^2(2\tau)\theta_3^4(2\tau) + 3\theta_2^4(2\tau)\theta_3^2(2\tau)\right), \\ \Phi^{(0,2)}_{NS^+} &= \frac{1}{2\eta^6(\tau)}\left(-\theta_3^6(2\tau) - \theta_2^6(2\tau) - 3\theta_2^2(2\tau)\theta_3^4(2\tau) - 3\theta_2^4(2\tau)\theta_3^2(2\tau)\right).\end{aligned} \tag{3.60}$$

Again combining these with the corresponding $D6$ characters and after using identities (A.21) which relate the theta functions to Eisenstein series we obtain

$$\mathcal{Z}_{\rm new}|_{(0,2)} = -2\frac{1}{\eta^{24}(\tau)}\Gamma_{2,2}^{(0,2)} \times E_4 \times \left(-\frac{1}{6}E_6 + \frac{2}{3}\mathcal{E}_2(\tau)E_4\right). \tag{3.61}$$

All the remaining terms in the new supersymmetric index can be obtained by performing modular transformations.

On comparing the coefficients of the twisted elliptic genus of the $2B$ orbifold given in (3.52) with new supersymmetric index given in (3.54), (3.58), (3.61) we see that it agrees with the expression derived in (2.19) using general arguments for the standard embedding. It is important to realize that this agreement was due to non-trivial identities relating the theta functions to Eisenstein series together with the function $\mathcal{E}_2$ and $\mathcal{E}_4$. Using the expression (3.43) we obtain $N_h - N_v = -380$ for this model.

Now that we have shown the new supersymmetric index admits a decomposition in terms of the twisted elliptic genus for standard embeddings, the rest of the analysis in section 2.2 can be applied. Therefore we conclude that the difference in one loop gauge thresholds when the Wilson line is embedded in the unbroken E_8 is the theta lift of twisted elliptic genus.

4 Non-standard embeddings

In this section we study the non-standard embeddings of heterotic compactifications of $K3 \times T^2$ orbifolded by g' belonging to the conjugacy class $2A$. We first realize $K3$ as the $\mathbb{Z}_2$ orbifold of T^4 and consider the 2 non-standard embedding studied in [2]. We then move one to the situation in which $K3$ is realized as the $\mathbb{Z}_4$ orbifold of T^4 and g' is implemented as given in equations (3.1) and (3.2). We consider all the 12 non-standard embeddings studied in [2]. In these orbifold limits, the various embeddings are implemented by different lattice shifts in the $E_8 \times E_8$. From the spectrum of these embeddings we show that the they can be organized into 4 types depending on the difference $N_h - N_v$ which take values $-12, 52, 84, 116$ for these types. The value -12 as we have seen corresponds to the standard type. The new supersymmetric index for all the embeddings also depends only on $N_h - N_v$. After turning on the Wilson line we show that the new supersymmetric index as well as the difference in one loop gauge thresholds depends on $N_h - N_v$ and the instanton numbers of the embedding.

4.1 Massless spectrum

We can evaluate the massless spectrum of the non-standard embeddings by following the same method as discussed in section 3.1.2. The spectrum for various non-standard embeddings of $K3\times T^2$ without the g' orbifold were obtained in [3]. Essentially the orbifold by g' changes the degeneracy formula given in (3.23) by changing the number of fixed points of the various twisted sectors as discussed around (3.26) for the orbifold in (3.1), (3.2). The various embeddings are determined by the lattice shifts in $E_8 \times E_8$. In table 6, we first

Gauge group, Shift $(\gamma;\tilde{\gamma})$	Sector	Matter
$E_7\times \mathrm{SU}(2)\times E_8$	g^0	$(\mathbf{56};\mathbf{2})+4(\mathbf{1;1})$
$(1,-1,0^6;0^8)$	g^1	$4(\mathbf{56;1})+16(\mathbf{1;2})$
$E_7\times \mathrm{SU}(2)\times \mathrm{SO}(16)$	g^0	$(\mathbf{56,2;1})+4(\mathbf{1,1;1})$
		$+(\mathbf{1,1;128})$
$(1^2,0^6;2,0^7)$	g^1	$4(\mathbf{1,2;16})$

Table 6. spectrum for different embeddings with K_3 as T^4/Z_2. The first shift realizes $N_h - N_v = -12$, while the second shift realizes $N_h - N_v = 116$.

Gauge group, Shift $(\gamma;\tilde{\gamma})$	Sector	Matter
$E_7\times \mathrm{U}(1)\times E_8$	g^0	$(\mathbf{56};\mathbf{1})+2(\mathbf{1};\mathbf{1})$
	g^1+g^3	$2(\mathbf{56};\mathbf{1})+4(\mathbf{1};\mathbf{1})+12(\mathbf{1};\mathbf{1})$
$(1,1,0^6;0^8)$	g^2	$3(\mathbf{56};\mathbf{1})+16(\mathbf{1};\mathbf{1})$
$E_7\times \mathrm{U}(1)\times E_7\times \mathrm{SU}(2)$	g^0	$(\mathbf{56};\mathbf{1},\mathbf{1})+2(\mathbf{1};\mathbf{1},\mathbf{1})$
	g^1+g^3	$6(\mathbf{1};\mathbf{1},\mathbf{2})+2(\mathbf{1};\mathbf{1},\mathbf{2})+2(\mathbf{1};\mathbf{56},\mathbf{1})$
$(1,1,0^6;2,2,0^6)$	g^2	$1(\mathbf{56};\mathbf{1},\mathbf{1})+16(\mathbf{1};\mathbf{1},\mathbf{1})$
$\mathrm{SO}(12)\times \mathrm{SU}(2)\times \mathrm{U}(1)\times E_8$	g^0	$(\mathbf{12},\mathbf{2};\mathbf{1})+(\mathbf{32},\mathbf{1};\mathbf{1})+2(\mathbf{1},\mathbf{1};\mathbf{1})$
	g^1+g^3	$6(\mathbf{1},\mathbf{2};\mathbf{1})+4(\mathbf{12},\mathbf{1};\mathbf{1})$
$(3,1,0^6;0^8)$		$2(\mathbf{1},\mathbf{2};\mathbf{1})+2(\mathbf{32},\mathbf{1};\mathbf{1})$
	g^2	$16(\mathbf{1},\mathbf{1};\mathbf{1})+3(\mathbf{12},\mathbf{2};\mathbf{1})+(\mathbf{32},\mathbf{1};\mathbf{1})$

Table 7. Spectrum of 2A orbifold of $K3\times T^2$ for different embeddings belonging to type 0 for K_3 as $T^4/\mathbb{Z}_4$ with $N_h - N_v = -12$.

tabulate the spectrum for embeddings when $K3$ is realized as the $T^4/\mathbb{Z}_2$ orbifold and g' as half shift given by following orbifold actions

$$\begin{aligned} g:&\quad (x_1,x_2,y_1,y_2,y_3,y_4)\sim(x_1,x_2,-y_1,-y_2,-y_3,-y_4),\\ g':&\quad (x_1,x_2,y_1,y_2,y_3)\sim(x_1+\pi,x_2,y_1+\pi,y_2,y_3,y_4). \end{aligned} \tag{4.1}$$

The spectrum for the 12 non-standard embeddings when for $K3$ is at the $T^4/\mathbb{Z}_4$ orbifold limit with g' as shifts given in (3.2) are listed in tables 7, 8 9 and 10. In these tables the shifts are denoted by $(\gamma;\tilde{\gamma})$ where $\gamma,\tilde{\gamma}$ are 8 dimensional vectors in $E_8\times E_8$. We observe from these tables that the the orbifold by g' results in only 4 distinct values of N_h-N_v given by $-12, 52, 84, 116$, the value -12 corresponds to the standard embedding. We classify these embeddings as type 0, type 1, type 2 and type 3 respectively.

Finally in table 11 and 12 we group the shifts according to the type based on the value of $N_h - N_v$.

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Gauge group, Shift $(\gamma;\tilde{\gamma})$	Sector	Matter
$E_7 \times \mathrm{U}(1) \times \mathrm{SO}(16)$	g^0	$(\mathbf{56};\mathbf{1}) + 2(\mathbf{1};\mathbf{1})$
	$g^1 + g^3$	$8(\mathbf{1};\mathbf{16})$
$(1,1,0^6;4,0^7)$	g^2	$3(\mathbf{56};\mathbf{1}) + 16(\mathbf{1};\mathbf{1})$
$\mathrm{SO}(12) \times \mathrm{SU}(2) \times \mathrm{U}(1) \times E_7 \times \mathrm{SU}(2)$	g^0	$(\mathbf{12},\mathbf{2};\mathbf{1},\mathbf{1}) + (\mathbf{32},\mathbf{1};\mathbf{1},\mathbf{1}) + 2(\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{1})$
	$g^1 + g^3$	$4(\mathbf{1},\mathbf{2};\mathbf{1},\mathbf{2}) + 2(\mathbf{12},\mathbf{1};\mathbf{1},\mathbf{2})$
$(3,1,0^6;2,2,0^6)$	g^2	$16(\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{1}) + (\mathbf{12},\mathbf{2};\mathbf{1},\mathbf{1}) + 3(\mathbf{32},\mathbf{1};\mathbf{1},\mathbf{1})$
$\mathrm{SO}(12) \times \mathrm{SU}(2) \times \mathrm{U}(1) \times \mathrm{SO}(16)$	g^0	$(\mathbf{12},\mathbf{2};\mathbf{1}) + (\mathbf{32},\mathbf{1};\mathbf{1}) + 2(\mathbf{1},\mathbf{1};\mathbf{1})$
	$g^1 + g^3$	$2(\mathbf{1},\mathbf{2};\mathbf{16})$
$(3,1,0^6;4,0^7)$	g^2	$16(\mathbf{1},\mathbf{1};\mathbf{1}) + 3(\mathbf{12},\mathbf{2};\mathbf{10} + (\mathbf{32},\mathbf{1};\mathbf{1})$

Table 8. Spectrum of 2A orbifold of $K3 \times T^2$ for different embeddings in type 1 for K_3 as $T^4/\mathbb{Z}_4$ with $N_h - N_v = 52$.

Gauge group, Shift $(\gamma;\tilde{\gamma})$	Sector	Matter
$E_7 \times \mathrm{U}(1) \times \mathrm{SU}(8) \times \mathrm{U}(1)$	g^0	$(\mathbf{56};\mathbf{1}) + (\mathbf{1};\mathbf{8}) + (\mathbf{1};\mathbf{56}) + 2(\mathbf{1};\mathbf{1})$
$(1,1,0^6;1^7,-1)$	$g^1 + g^3$	$6(\mathbf{1};\mathbf{1}) + 2(\mathbf{1};\mathbf{1}) + 2(\mathbf{1};\overline{\mathbf{28}})$ $+4(\mathbf{1},\mathbf{8})$
	g^2	$6(\mathbf{1};\mathbf{8}) + 2(\mathbf{1};\mathbf{8})$
$E_6 \times \mathrm{SU}(2) \times \mathrm{U}(1) \times \mathrm{SO}(14) \times \mathrm{U}(1)$	g^0	$(\mathbf{27},\mathbf{2};\mathbf{1}) + (\mathbf{1},\mathbf{2};\mathbf{1}) + (\mathbf{1},\mathbf{1};\mathbf{64})$ $+2(\mathbf{1},\mathbf{1};\mathbf{1})$
$(2,1,1,0^5;2,0^7)$	$g^1 + g^3$	$6(\mathbf{1},\mathbf{1};\mathbf{1}) + 4(\mathbf{1},\mathbf{2};\mathbf{1})$ $+2(\mathbf{27},\mathbf{1};\mathbf{1}) + 2(\mathbf{1},\mathbf{1};\mathbf{14})$
	g^2	$(\mathbf{1},\mathbf{2};\mathbf{14}) + 6(\mathbf{1},\mathbf{2};\mathbf{1})$

Table 9. Spectrum of 2A orbifold of $K3 \times T^2$ for different embeddings in type 2 for K_3 as $T^4/\mathbb{Z}_4$ with $N_h - N_v = 84$.

4.2 New supersymmetric index

In this section we evaluate the new supersymmetric index for all the embeddings discussed in section 4.1. We will show that for the when the Wilson line is not turned on, the index $\mathcal{Z}_{\text{new}}$ for the $2A$ orbifold of $K3 \times T^2$ depends only on the 4 types of the lattice shifts organized in tables 11 and 12. $\mathcal{Z}_{\text{new}}$ is invariant for any lattice shift belonging to a given type. When the Wilson line is turned on, then the index depends both on the type as well as the instanton number corresponding to the lattice shift.

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Gauge group, Shift $(\gamma;\tilde{\gamma})$	Sector	Matter
$E_6\times\mathrm{SU}(2)\times\mathrm{U}(1);\mathrm{SO}(10)\times\mathrm{SO}(6)$	g^0	$(\mathbf{27},\mathbf{2};\mathbf{1},\mathbf{1})+(\mathbf{1},\mathbf{2};\mathbf{1},\mathbf{1})+(\mathbf{1},\mathbf{1};\mathbf{16},\mathbf{4})$ $+2(\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{1})$
$(2,1,1,0^5;2^3,0^5)$	g^1+g^3	$4(\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{4})+2(\mathbf{1},\mathbf{2};\mathbf{1},\mathbf{4})$ $+2(\mathbf{1},\mathbf{1};\overline{\mathbf{16}},\mathbf{1})$
	g^2	$3(\mathbf{1},\mathbf{2};\mathbf{10},\mathbf{1})+(\mathbf{1},\mathbf{2};\mathbf{1},\mathbf{6})$
$\mathrm{SU}(8)\times\mathrm{SU}(2)\times\mathrm{SO}(10)\times\mathrm{SO}(6)$	g^0	$(\mathbf{28},\mathbf{2};\mathbf{1},\mathbf{1})+(\mathbf{1},\mathbf{1};\mathbf{16},\mathbf{4})+2(\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{1})$
	g^1+g^3	$2(\mathbf{8},\mathbf{1};\mathbf{1},\mathbf{4})$
$(3,1^5,0^2;2^3,0^5)$	g^2	$16(\mathbf{1},\mathbf{1};\mathbf{1})+3(\mathbf{12},\mathbf{2};\mathbf{1},\mathbf{6})+(\mathbf{1},\mathbf{2};\mathbf{10},\mathbf{1})$
$\mathrm{SU}(8)\times\mathrm{SU}(2)\times\mathrm{SO}(14)\times\mathrm{U}(1)$	g^0	$(\mathbf{28},\mathbf{2};\mathbf{1})+(\mathbf{1},\mathbf{1};\mathbf{64})+2(\mathbf{1},\mathbf{1};\mathbf{1})$
	g^1+g^3	$4(\bar{\mathbf{8}},\mathbf{1};\mathbf{1})+2(\mathbf{8},\mathbf{2};\mathbf{1})$
$(3,1^5,0^2;2,0^7)$	g^2	$3(\mathbf{1},\mathbf{2};\mathbf{14})+2(\mathbf{1},\mathbf{2};\mathbf{1})$
$\mathrm{SU}(8)\times\mathrm{U}(1)\times\mathrm{SO}(12)\times\mathrm{SU}(2)\times\mathrm{U}(1)$	g^0	$(\mathbf{8};\mathbf{1},\mathbf{1})+(\mathbf{56};\mathbf{1},\mathbf{1})+(\mathbf{1};\mathbf{12},\mathbf{1})$ $(\mathbf{1};\mathbf{32},\mathbf{1})+2(\mathbf{1};\mathbf{1},\mathbf{1})$
$(1^7,-1;3,1,0)$	g^1+g^3	$4(\mathbf{1};\mathbf{1},\mathbf{2})+2(\mathbf{1};\mathbf{12},\mathbf{1})+2(\mathbf{8};\mathbf{1},\mathbf{2})$
	g^2	$6(\mathbf{8};\mathbf{1},\mathbf{1})+2(\mathbf{8};\mathbf{1},\mathbf{1})$

Table 10. Spectrum of 2A orbifold of $K3\times T^2$ for different embeddings in type 3 for K_3 as $T^4/\mathbb{Z}_4$ with $N_h-N_v=116$.

γ	$\tilde{\gamma}$	Type	N_h-N_v
(1,1,0,0,0,0,0,0)	(0,0,0,0,0,0,0,0)	Type 0	-12
(1,-1,0,0,0,0,0,0)	(2,0,0,0,0,0,0,0)	Type 3	116

Table 11. Lattice shifts in the $2A$ orbifold with $K3=T^4/\mathbb{Z}_2$ and N_h-N_v.

Let us first discuss the case without the Wilson line. Evaluating the trace defined in (2.3) we see that it reduces to

$$\mathcal{Z}_{\mathrm{new}}(q,\bar{q})=-\frac{1}{2\eta^{20}(\tau)}\sum_{a,b=0}^{\nu-1}\sum_{r,s=0}^{1}e^{-\frac{2\pi iab}{\nu^2}}Z_{E_8}^{(a,b)}(\tau)\times Z_{E_8'}^{(a,b)}(\tau)\times\frac{1}{2\nu}F(a,r,b,s;q)\Gamma_{2,2}^{(r,s)}(q,\bar{q}), \tag{4.2}$$

where $\nu=2,4$ depending on the whether $K3$ is realized as a $T^4/\mathbb{Z}_2$ or $T^4/\mathbb{Z}_4$ orbifold. The partition function over the shifted E_8 lattices are defined by

$$Z_{E_8}^{a,b}(q)=\frac{1}{2}\sum_{\alpha,\beta=0}^{1}e^{-i\pi\beta\frac{a}{\nu}\sum_{I=1}^{8}\gamma^I}\prod_{I=1}^{8}\theta\begin{bmatrix}\alpha+2\frac{a}{\nu}\gamma^I\\ \beta+2\frac{b}{\nu}\gamma^I\end{bmatrix}, \tag{4.3}$$

$$Z_{E_8'}^{a,b}(q)=\frac{1}{2}\sum_{\alpha,\beta=0}^{1}e^{-i\pi\beta\frac{a}{\nu}\sum_{I=1}^{8}\tilde{\gamma}^I}\prod_{I=1}^{8}\theta\begin{bmatrix}\alpha+2\frac{a}{\nu}\tilde{\gamma}^I\\ \beta+2\frac{b}{\nu}\tilde{\gamma}^I\end{bmatrix}, \tag{4.4}$$

where $\gamma, \tilde{\gamma}$ are the shifts in the two E_8 lattices. The trace over the T^4 directions is as defined in (3.36). However the g, g' correspond to the actions in (4.1) for the $\mathbb{Z}_2$ orbifold limit of $K3$ and to actions (3.1) and (3.2) for the $\mathbb{Z}_4$ orbifold limit of $K3$. This trace is given by

$$F(a,r,b,s;q) = k_{(\nu)}^{(a,r,b,s)} \eta^2(\tau) q^{\frac{-a^2}{\nu^2}} \frac{1}{\theta_1^2(\frac{a\tau+b}{\nu},\tau)} \,, \tag{4.5}$$

where the k's are read out from the following matrices.

$$\begin{aligned}
k_{(2)}^{(a,0,b,0)} &= 64 \begin{pmatrix} 0 & 1 \\ 1 & e^{-\pi i(2-\Gamma^2)/4} \end{pmatrix}, \qquad k_{(2)}^{(a,0,b,1)} = 64 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \\
k_{(2)}^{(a,1,b,0)} &= 64 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad k_{(2)}^{(a,1,b,1)} = 64 \begin{pmatrix} 0 & 0 \\ 0 & e^{-\pi i(2-\Gamma^2)/4} \end{pmatrix}, \\
k_{(4)}^{(a,0,b,0)} &= 16 \begin{pmatrix} 0 & 1 & 4 & 1 \\ 1 & e^{-\pi i\frac{1}{16}(2-\Gamma^2)} & e^{-\pi i\frac{1}{8}(2-\Gamma^2)} & e^{-\pi i\frac{3}{16}(2-\Gamma^2)} \\ 4 & e^{\pi i\frac{3}{8}(2-\Gamma^2)} & 4e^{-\pi i\frac{1}{4}(2-\Gamma^2)} & e^{\pi i\frac{1}{8}(2-\Gamma^2)} \\ 1 & e^{\pi i\frac{9}{16}(2-\Gamma^2)} & e^{\pi i\frac{1}{8}(2-\Gamma^2)} & e^{-\pi i\frac{9}{16}(2-\Gamma^2)} \end{pmatrix}, \\
k_{(4)}^{(a,0,b,1)} &= 16 \begin{pmatrix} 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & e^{\pi i\frac{3}{8}(2-\Gamma^2)} & 0 & e^{\pi i\frac{3}{8}(2-\Gamma^2)} \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\
k_{(4)}^{(a,1,b,0)} &= 16 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & e^{-\pi i\frac{1}{8}(2-\Gamma^2)} & 0 \\ 4 & 0 & 0 & 0 \\ 1 & 0 & e^{\pi i\frac{1}{8}(2-\Gamma^2)} & 0 \end{pmatrix}, \\
k_{(4)}^{(a,1,b,1)} &= 16 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & e^{-\pi i\frac{1}{16}(2-\Gamma^2)} & 0 & e^{-\pi i\frac{3}{16}(2-\Gamma^2)} \\ 0 & 0 & 4e^{-\pi i\frac{1}{4}(2-\Gamma^2)} & 0 \\ 0 & e^{\pi i\frac{9}{16}(2-\Gamma^2)} & 0 & e^{-\pi i\frac{9}{16}(2-\Gamma^2)} \end{pmatrix},
\end{aligned} \tag{4.6}$$

where $\Gamma^2 = \gamma^2 + \tilde{\gamma}^2$. Using all this inputs we evaluate the new supersymmetric index for the list of lattice shifts given in tables 11 and 12. This results in following general result

$$\begin{aligned}
\mathcal{Z}_{\rm new} = -\frac{1}{\eta^{24}} \Big\{ & 2\Gamma_{2,2}^{(0,0)} E_4 E_6 \\
& + \Gamma_{2,2}^{(0,1)} \left[(E_6 + 2\mathcal{E}_2(\tau) E_4) \left(\hat{b}\mathcal{E}_2^2(\tau) + \left(\frac{2}{3} - \hat{b} \right) E_4 \right) \right] \\
& + \Gamma_{2,2}^{(1,0)} \left[\left(E_6 - \mathcal{E}_2\left(\frac{\tau}{2}\right) E_4 \right) \left(\frac{\hat{b}}{4} \mathcal{E}_2^2\left(\frac{\tau}{2}\right) + \left(\frac{2}{3} - \hat{b} \right) E_4 \right) \right] \\
& + \Gamma_{2,2}^{(1,1)} \left[\left(E_6 - \mathcal{E}_2\left(\frac{\tau+1}{2}\right) E_4 \right) \left(\frac{\hat{b}}{4} \mathcal{E}_2^2\left(\frac{\tau+1}{2}\right) + \left(\frac{2}{3} - \hat{b} \right) E_4 \right) \right] \Big\} .
\end{aligned} \tag{4.7}$$

γ	$\tilde{\gamma}$	Type	$N_h - N_v$
(1,-1,0,0,0,0,0,0)	(0,0,0,0,0,0,0,0)	Type 0	-12
(1,1,0,0,0,0,0,0)	(0,0,0,0,0,0,0,0)		
(1,1,0,0,0,0,0,0)	(2,2,0,0,0,0,0,0)		
(3,1,0,0,0,0,0,0)	(0,0,0,0,0,0,0,0)		
(1,1,0,0,0,0,0,0)	(4,0,0,0,0,0,0,0)	Type 1	52
(3,1,0,0,0,0,0,0)	(4,0,0,0,0,0,0,0)		
(3,1,0,0,0,0,0,0)	(2,2,0,0,0,0,0,0)		
(2,1,1,0,0,0,0,0)	(2,0,0,0,0,0,0,0)	Type 2	84
(1,1,0,0,0,0,0,0)	(1,1,1,1,1,1,1,-1)		
(2,1,1,0,0,0,0,0)	(2,2,2,0,0,0,0,0)	Type 3	116
(3,1,1,1,1,1,0,0)	(2,0,0,0,0,0,0,0)		
(3,1,1,1,1,1,0,0)	(2,2,2,0,0,0,0,0)		
(1,1,1,1,1,1,-1)	(3,1,0,0,0,0,0,0)		

Table 12. Lattice shifts in the $2A$ orbifold with $K3 = T^4/\mathbb{Z}_4$ and $N_h - N_v$.

Type	Type 0	Type 1	Type 2	Type 3
$\hat{b}$	0	$\frac{4}{9}$	$\frac{2}{3}$	$\frac{8}{9}$

Table 13. Value of $\hat{b}$ for each type of lattice shift.

The value of $\hat{b}$ for each of type of embeddings is given in table 13. Thus the values $\hat{b}$ takes are discrete and just depends on the type of embedding or lattice shift. In fact since $N_h - N_v$ remains constant in each type of embedding we can relate it to $\hat{b}$. This relation can be found by using the equation in (3.43) and is given by

$$N_h - N_v = 144\hat{b} - 12. \tag{4.8}$$

Note that standard embedding belongs the case $\hat{b} = 0$, also note that the only non-standard embedding of the $2A$ orbifold when $K3$ is realized as $T^4/\mathbb{Z}_2$ as seen in table 11 belongs to type 3. One important point to emphasize is that the new supersymmetric index in (4.7) still can be decomposed in terms of the twisted elliptic genus of $K3$. Comparing (2.19) for the $2A$ orbifold with (4.7) the only difference is that the lattice sum E_4 has been replaced by $\left(\hat{b}\mathcal{E}_2^2(\tau) + (\frac{2}{3} - \hat{b})E_4\right)$ for the $(0,1)$ sector. The lattice sum $\left(E_6 - \mathcal{E}_2(\frac{\tau}{2})E_4\right)$ associated by the $2A$ orbifold remains the same. Similar statements can be made for all the other sectors.

Let us now turn on the Wilson line in the E_8' lattice and evaluate the new supersymmetric index. To do this we follow the procedure in [2]. First the partition function in the

E_8' lattice is evaluated with a chemical potential along one of U(1) directions. The lattice sum then becomes

$$Z_{E_8'}^{a,b}(\tau,z) = \frac{1}{2}\sum_{\alpha,\beta=0}^{1} e^{-i\pi\beta\frac{a}{4}\sum_{I=1}^{8}\tilde{\gamma}^I}\prod_{I=1}^{6}\theta\begin{bmatrix}\alpha+2\frac{a}{4}\tilde{\gamma}^I\\ \beta+2\frac{b}{4}\tilde{\gamma}^I\end{bmatrix}(\tau)\prod_{I=7}^{8}\theta\begin{bmatrix}\alpha+2\frac{a}{4}\tilde{\gamma}^I\\ \beta+2\frac{b}{4}\tilde{\gamma}^I\end{bmatrix}(\tau,z). \tag{4.9}$$

This modified lattice sum $Z_{E_8'}^{a,b}(\tau,z)$ is then coupled to the $\Gamma_{3,2}$ lattice using the $\otimes$ product defined in the appendix. It was shown in [2] that for all orbifold realizations of $K3$, the new supersymmetric index just depends on instanton numbers of the embedding or the lattice shifts. The result is given by the expression

$$\mathcal{Z}_{\rm new} = -\frac{1}{6\eta^{24}}\Gamma_{3,2}(q,\bar{q})\otimes[n_1E_{4,1}E_6 + n_2E_{6,1}E_4]\,, \tag{4.10}$$

where n_1, n_2 are the instanton numbers of the embedding and $n_1 + n_2 = 24$. For the standard embedding $n_1 = 24, n_2 = 0$. Thus the new supersymmetric index with the Wilson line is sensitive to the the instanton numbers.

For compactifications on $(K3\times T^2)/g'$ with $K3$ realized either by $T^4/\mathbb{Z}_2$ or the $T^4/\mathbb{Z}_4$ and g' in the $2A$ conjugacy class, the new supersymmetric index with the Wilson line depends on $\hat{b}$ which is related to $N_h - N_v$ of the model by (4.8) and also the instanton number of the embedding. The result for the index for all the embeddings can be summarized in the following compact expression

$$\begin{aligned}\mathcal{Z}_{\rm new} = -\frac{1}{\eta^{24}}\Big\{&\Gamma_{3,2}^{(0,0)}\otimes\frac{1}{12}[n_1E_{4,1}E_6+n_2E_{6,1}E_4] \\ &+\Gamma_{3,2}^{(0,1)}\otimes\Big[\hat{a}E_{4,1}(E_6+2\mathcal{E}_2(\tau)E_4)+\hat{b}\mathcal{E}_2(\tau)^2(E_{6,1}+2\mathcal{E}_2(\tau)E_{4,1})+\hat{c}E_4(E_{6,1}+2\mathcal{E}_2(\tau)E_{4,1})\Big] \\ &+\Gamma_{3,2}^{(1,0)}\otimes[\quad\cdot\quad]+\Gamma_{3,2}^{(1,1)}\otimes[\quad\cdot\quad]\Big\}.\end{aligned} \tag{4.11}$$

Here the parameters $\hat{a}, \hat{c}$ depend on the instanton numbers n_1, n_2 of the embedding and the value of $\hat{b}$ by

$$\hat{a} = \frac{n_1}{36} - \frac{\hat{b}}{2}, \qquad \hat{c} = \frac{2}{3} - \hat{a} - \hat{b}\,. \tag{4.12}$$

The $[\quad\cdot\quad]$ denotes the corresponding term obtained by modular transformation of the $(0,1)$ sector. For example in the $(1,0)$ sector, we replace the terms with $\mathcal{E}_2(\tau)$ of the $(0,1)$ sector to $-\frac{1}{2}\mathcal{E}_2(\frac{\tau}{2})$. Similarly in the $(1,1)$ we have $-\frac{1}{2}\mathcal{E}_2(\frac{\tau+1}{2})$. We summarize the values of $\hat{a}, \hat{b}, n_1$ for each of the shifts considered in the tables 14 and 15. Using these tables and equation (4.11), the result for the new supersymmetric index with the Wilson line for these orbifolds can be read out.

4.3 Difference of one loop gauge thresholds

We now evaluate the difference in one loop gauge thresholds for all models whose new supersymmetric index is given by (4.11). The one loop threshold for the group G is given by (2.22). We take the G to be the group the Wilson line is embedded in. Then using (4.11)

Type	γ	$\tilde{\gamma}$	(n_1, n_2)	$\hat{a}$	$\hat{b}$	$\hat{c}$
Type 0	(1,-1,0,0,0,0,0,0)	(0,0,0,0,0,0,0,0)	(24,0)	2/3	0	0
Type 3	(1,-1,0,0,0,0,0,0)	(2,0,0,0,0,0,0,0)	(8,16)	-2/9	8/9	0

Table 14. Lattice shifts for $((T^4/\mathbb{Z}_2) \times T^2)/g'$ and their $\hat{a}, \hat{b}, \hat{c}$ values.

Type	γ	$\tilde{\gamma}$	(n_1, n_2)	$\hat{a}$	$\hat{b}$	$\hat{c}$
Type 0	(1,-1,0,0,0,0,0,0)	(0,0,0,0,0,0,0,0)	(24,0)	2/3	0	0
	(1,1,0,0,0,0,0,0)	(0,0,0,0,0,0,0,0)	(24,0)	2/3	0	0
	(3,1,0,0,0,0,0,0)	(0,0,0,0,0,0,0,0)	(24,0)	2/3	0	0
	(1,1,0,0,0,0,0,0)	(2,2,0,0,0,0,0,0)	(12,12)	1/3	0	1/3
Type 1	(1,1,0,0,0,0,0,0)	(4,0,0,0,0,0,0,0)	(16,8)	2/9	4/9	0
	(3,1,0,0,0,0,0,0)	(4,0,0,0,0,0,0,0)	(16,8)	2/9	4/9	0
	(3,1,0,0,0,0,0,0)	(2,2,0,0,0,0,0,0)	(20,4)	1/3	4/9	-1/9
Type 2	(2,1,1,0,0,0,0,0)	(2,0,0,0,0,0,0,0)	(12,12)	0	2/3	0
	(1,1,0,0,0,0,0,0)	(1,1,1,1,1,1,1,-1)	(6,18)	-1/6	2/3	1/6
Type 3	(2,1,1,0,0,0,0,0)	(2,2,2,0,0,0,0,0)	(12,12)	-2/9	8/9	0
	(3,1,0,0,0,0,0,0)	(1,1,1,1,1,1,1,-1)	(14,10)	-1/18	8/9	-1/6
	(3,1,1,1,1,1,0,0)	(2,0,0,0,0,0,0,0)	(12,12)	-1/9	8/9	-1/9
	(3,1,1,1,1,1,0,0)	(2,2,2,0,0,0,0,0)	(12,12)	-1/9	8/9	-1/9

Table 15. Lattice shifts for $((T^4/\mathbb{Z}_4) \times T^2)/g'$ and their $\hat{a}, \hat{b}, \hat{c}$ values.

we obtain

$$\begin{aligned}\mathcal{B}_G = -\frac{1}{\eta^{24}} \Big\{ &\Gamma_{3,2}^{(0,0)} \otimes \frac{1}{288} \Big[n_1 \Big(\tilde{E}_2 E_{4,1} - E_{6,1} \Big) E_6 + n_2 \Big(\tilde{E}_2 E_{6,1} - E_{4,1} E_4 \Big) E_6 \Big] \\ &+\Gamma_{3,2}^{(0,1)} \otimes \Big[\frac{\hat{a}}{24} \Big(E_{4,1} \tilde{E}_2 - E_{6,1} \Big) (E_6 + 2\mathcal{E}_2(\tau) E_4) \\ &+\frac{\hat{c}}{24} E_4 \Big(E_{6,1} \tilde{E}_2 - E_{4,1} E_4 + 2\mathcal{E}_2(\tau) \Big(E_{4,1} \tilde{E}_2 - E_{6,1} \Big) \Big) \\ &+\frac{\hat{b}}{120} \Big(E_4 + 4E_4(2\tau) \Big) \Big(E_{6,1} \tilde{E}_2 - E_{4,1} E_4 + 2\mathcal{E}_2(\tau) E_{4,1} \tilde{E}_2 - 2\mathcal{E}_2(\tau) E_{6,1} \Big) \Big] \\ &+\Gamma_{3,2}^{(1,0)} \otimes [\quad \cdot \quad] + \Gamma_{3,2}^{(1,1)} \otimes [\quad \cdot \quad] \Big\}. \end{aligned} \tag{4.13}$$

where the terms in the $[\quad \cdot \quad]$ can be obtained by modular transformation from the corresponding term in the $(0,1)$ sector. Note that we have used the identity

$$\mathcal{E}_2^2(\tau) = \frac{1}{5} \left(4E_4(2\tau) + E_4 \right), \tag{4.14}$$

in the terms proportional to $\hat{b}$. Similarly the terms for the gauge group G' we obtain

$$\begin{aligned}\mathcal{B}_{G'} = &-\frac{1}{\eta^{24}}\Bigg\{\Gamma^{(0,0)}_{3,2}\otimes\frac{1}{288}\left[n_1 E_{4,1}\left(\tilde{E}_2E_6-E_4^2\right)+n_2\left(\tilde{E}_2E_4-E_6\right)\right] \\ &+\Gamma^{(0,1)}_{3,2}\otimes\left[\frac{\hat{a}}{24}E_{4,1}\left(E_6\tilde{E}_2-E_4^2+2\mathcal{E}_2(\tau)(E_4\tilde{E}_2-E_6)\right)\right. \\ &+\frac{\hat{c}}{24}\left(E_4\tilde{E}_2-E_6\right)(E_{6,1}+2\mathcal{E}_2(\tau)E_{4,1}) \\ &\left.+\frac{\hat{b}}{120}\left(\tilde{E}_2E_4-E_6+8\left(\tilde{E}_2(2\tau)E_4(2\tau)-E_6(2\tau)\right)\left(E_{6,1}+2\mathcal{E}_2(\tau)E_{4,1}\right)\right)\right] \\ &+\Gamma^{(1,0)}_{3,2}\otimes[\quad\cdot\quad]+\Gamma^{(1,1)}_{3,2}\otimes[\quad\cdot\quad]\Bigg\}.\end{aligned} \tag{4.15}$$

We now evaluate the difference in the threshold integrals. To simplify the expressions we use the following identities

$$\mathcal{E}_2(\tau)=2\tilde{E}_2(2\tau)-\tilde{E}_2,\qquad E_6(2\tau)=\frac{\mathcal{E}_2(\tau)}{8}(11\mathcal{E}_2^2(\tau)-3E_4), \tag{4.16}$$

together with (4.14) and

$$\mathcal{E}_2(\tau)^3=\frac{3}{4}E_4\mathcal{E}_2(\tau)+\frac{1}{4}E_6\,. \tag{4.17}$$

This results in the following expression for the threshold integral

$$\begin{aligned}\Delta_G(T,U,V)-\Delta_{G'}(T,U,V) &= \int_{\mathcal{F}}\frac{d^2\tau}{\tau_2}\{\mathcal{B}_G-\mathcal{B}_{G'}\} \\ &=\int_{\mathcal{F}}\frac{d^2\tau}{\tau_2}\Bigg\{\Gamma^{(0,0)}\otimes 2(n_2-n_1)A(z) \\ &-\Gamma^{0,1}\otimes\left[24A(z)\left(\frac{n_1-12}{18}\right)-12B(z)\mathcal{E}_2(\tau)\left(\frac{2}{3}-\frac{\hat{b}}{2}\right)\right] \\ &-\Gamma^{(1,0)}\otimes\left[24A(z)\left(\frac{n_1-12}{18}\right)+6B(z)\mathcal{E}_2\left(\frac{\tau}{2}\right)\left(\frac{2}{3}-\frac{\hat{b}}{2}\right)\right] \\ &-\Gamma^{(1,1)}\otimes\left[24A(z)\left(\frac{n_1-12}{18}\right)+6B(z)\mathcal{E}_2\left(\frac{\tau+1}{2}\right)\left(\frac{2}{3}-\frac{\hat{b}}{2}\right)\right]\Bigg\},\end{aligned} \tag{4.18}$$

where we have used the relations (2.26). Note that the integrands for all the embeddings in table (14) and (15) just depend on the instanton number and the $\hat{b}$ which is related to the difference N_h-N_v. One simple check of our result is that on setting $b=0, n_1=24$, the equation in (4.18) reduces to the standard embedding result for the $2A$ orbifold of $K3$.

The threshold integral in (4.18) over the fundamental domain can be performed using the methods developed in [22]. The details are provided in the appendix B. Here we quote the final result.

$$\Delta_G(T,U,V) - \Delta_{G'}(T,U,V) = 48\left(\left(\frac{1}{2}-\frac{3\hat{b}}{8}\right)\log(\det(\mathrm{Im}(\Omega))^6\,|\Phi_6(\mathrm{U},\mathrm{T},\mathrm{V})|^2) \right.$$
$$+\left(\frac{n_1}{72}-\frac{1}{3}+\frac{\hat{b}}{8}\right)\log(\det\left(\mathrm{Im}(\Omega)\right)^{10}|\Phi_{10}(\mathrm{U},\mathrm{T},\mathrm{V})|^2\right)$$
$$\left.+\left(\frac{n_1}{72}-\frac{1}{3}+\frac{\hat{b}}{8}\right)\log(\det\left(\mathrm{Im}(\Omega)\right)^{10}|\Phi_{10}(2\mathrm{U},\mathrm{T}/2,\mathrm{V})|^2\right)\right) \tag{4.19}$$

Here Φ_{10} is the unique cusp form of weight 10 under $\mathrm{Sp}(2,\mathbb{Z})$, while Φ_6 is the Siegel modular form of weight 6 which is obtained from the theta lift of the elliptic genus of $K3$ twisted by the $2A$ orbifold action. Φ_6 was first constructed as a theta lift in [18]. As expected for the standard embedding $\hat{b}=0, n_1=24$ the threshold integral reduces to only Φ_6.

5 Conclusions

We have explored $\mathcal{N}=2$ compactifications of heterotic string theory on orbifolds of $K3\times T^2$ by g' which acts as a $\mathbb{Z}_N$ automorphism on $K3$ together with a $1/N$ shift on one of the circles of T^2. g' can correspond to any of the 26 conjugacy classes of the Mathieu group M_{24}. We showed that for the standard embedding of the spin connection in one of the E_8 the new supersymmetric index can be written in terms of the elliptic genus of $K3$ twisted by g'. The difference in gauge thresholds are shown to be theta lifts of the twisted elliptic genus of these compactifications. This generalizes the observation in [12] as well as [23, 24] who observed similar behaviour for non-supersymmetric compactifications.[8] We demonstrated this by explicitly studying 2 examples. The first one considered the $2A$ orbifold of $K3$ when $K3$ is realized as $T^4/\mathbb{Z}_4$. The result is same as that obtained in [12] where the $2A$ orbifold of $K3$ is obtained by taking $K3$ to be $T^4/\mathbb{Z}_2$. We also studied the recently constructed [14] $2B$ orbifold of $K3$ when $K3$ is realized as $\mathrm{su}(2)^6$ rational conformal field theory. Finally we considered non-standard embeddings for the $2A$ orbifold of $K3$ and showed that the new supersymmetric index depends only on the difference $N_h - N_v$ of the model and the gauge threshold correction depends on the instanton number of the embedding as well as $N_h - N_v$. The detailed spectrum of these compactifications has also be obtained.

There are a number of directions which are worth exploring. One is to generalize the study of non-standard embedding to all the orbifold limits of $K3$, here we considered only the limits $T^4/\mathbb{Z}_2$ and $T^4/\mathbb{Z}_4$. Another direction is to study the type II duals of these theories. Not only this will teach us more about S-duality, but it will also involve the study of new Calabi-Yau manifolds. However perhaps the most interesting extrapolation of the observations of this paper is the fact that it is also possible to consider compactifications of string theory of type II on $(K3\times T^2)/g'$ where g' corresponds to any of the 26 conjugacy classes of M_{24}. These compactifications preserve $\mathcal{N}=4$ supersymmetry. The theta lifts of the twisted elliptic genus for all these cases should capture degeneracies of 1/4 BPS dyons.

[8] In the case of non-supersymmetric compactifications, the difference in the gauge threshold integrand was the lattice sum $\Gamma_{2,2}$ folded with a holomorphic function which resembled an index.

The case of g' in the conjugacy class pA, $p = 1, 2, 3, 5, 7$ was studied in [18, 25–31]. It will be certainly interesting to generalize the results regarding dyon partition functions to all the conjugacy classes of M_{24}. This will possibly will teach us about black hole degeneracies in $\mathcal{N} = 4$ string theory and its relation to the symmetry M_{24}.

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A Notations, conventions and identities

In this appendix we summarize the notations and conventions and properties of the modular functions used in this paper. We define the generalized form of Jacobi theta functions as

$$\theta\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right](q,z) = \sum_{k\in\mathbb{Z}} q^{\frac{1}{2}(k+\frac{a}{2})^2} e^{\pi i (k+\frac{a}{2})b} e^{(2\pi i z)(k+\frac{a}{2})}. \tag{A.1}$$

If the variable z is not stated in the argument then it is understood to be the theta function is at $z = 0$. We use $q = e^{2\pi i\tau}$ and τ interchangeably in the arguments of the modular functions. We also define

$$\begin{aligned} \theta_1(\tau,z) &= \theta\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right](\tau,z) & \theta_2(\tau,z) &= \theta\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right](\tau,z), \\ \theta_3(\tau,z) &= \theta\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right](\tau,z) & \theta_4(\tau,z) &= \theta\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right](\tau,z). \end{aligned} \tag{A.2}$$

In various manipulations the following Riemann bi-linear identities are useful

$$\begin{aligned} \theta_1^2(\tau,z) &= \theta_2(2\tau)\theta_3(2\tau,2z) - \theta_3(2\tau)\theta_2(2\tau,2z), \\ \theta_2^2(\tau,z) &= \theta_2(2\tau)\theta_3(2\tau,2z) + \theta_3(2\tau)\theta_2(2\tau,2z), \\ \theta_3^2(\tau,z) &= \theta_3(2\tau)\theta_3(2\tau,2z) + \theta_2(2\tau)\theta_2(2\tau,2z), \\ \theta_4^2(\tau,z) &= \theta_3(2\tau)\theta_3(2\tau,2z) - \theta_2(2\tau)\theta_2(2\tau,2z). \end{aligned} \tag{A.3}$$

At $z = 0$, these identities reduce to

$$\begin{aligned} &\theta_2^2 = 2\theta_2(2\tau)\theta_3(2\tau), \qquad \theta_3^2 = \theta_2^2(2\tau) + \theta_3^2(2\tau), \qquad \theta_4^2 = -\theta_2^2(2\tau) + \theta_3^2(2\tau), \\ &2\theta_2^2(2\tau) = \theta_3^2 - \theta_4^2, \qquad 2\theta_3^2(2\tau) = \theta_3^2 + \theta_4^2. \end{aligned} \tag{A.4}$$

The series representation of the Eisenstein series E_2, E_4 and E_6 are given by

$$\begin{aligned} E_2(q) &= 1 - 24\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}, \\ E_4(q) &= 1 + 240\sum_{n=1}^{\infty}\frac{n^3q^n}{1-q^n}, \\ E_6(q) &= 1 - 504\sum_{n=1}^{\infty}\frac{n^5q^n}{1-q^n}. \end{aligned} \tag{A.5}$$

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The functions E_4 and E_6 can be written in terms of theta functions using the following expressions

$$\begin{aligned} E_4 &= \frac{1}{2}\left(\theta_3^8+\theta_4^8+\theta_2^8\right), \\ E_6 &= \frac{1}{2}\left(-\theta_2^6\left(\theta_3^4+\theta_4^4\right)\theta_2^2+\theta_3^6\left(\theta_4^4-\theta_2^4\right)\theta_3^2+\theta_4^6\left(\theta_3^4+\theta_2^4\right)\theta_4^2\right). \end{aligned} \tag{A.6}$$

Eisenstein series with the U(1) chemical potential are defined by

$$\begin{aligned} E_{4,1}(z) &= \frac{1}{2}\left(\theta_3^6\theta_3^2(z)+\theta_4^6\theta_4^2(z)+\theta_2^6\theta_2^2(z)\right), \\ E_{6,1}(z) &= \frac{1}{2}\left(-\theta_2^6\left(\theta_3^4+\theta_4^4\right)\theta_2^2(z)+\theta_3^6\left(\theta_4^4-\theta_2^4\right)\theta_3^2(z)+\theta_4^6\left(\theta_3^4+\theta_2^4\right)\theta_4^2(z)\right). \end{aligned} \tag{A.7}$$

The decomposition of these series in terms of even and odd parts are defined by

$$\begin{aligned} E_{4,1} &= E_{4,1}^{\text{even}}\theta_{\text{even}}+E_{4,1}^{\text{odd}}(z)\theta_{\text{odd}}(z), \\ E_{6,1} &= E_{6,1}^{\text{even}}\theta_{\text{even}}+E_{6,1}^{\text{odd}}(z)\theta_{\text{odd}}(z). \end{aligned} \tag{A.8}$$

where

$$\theta_{\text{even}}(z)=\theta_3(2\tau,2z) \qquad \theta_{\text{odd}}(z)=\theta_2(2\tau,2z). \tag{A.9}$$

Any Jacobi form of index 1, $f_{s,1}(\tau,z)$ such as $E_{4,1},\ E_{6,1}$) can be decomposed as:

$$f_{s,1}(\tau,z)=f_{s,1}^{\text{even}}(\tau)\theta_{\text{even}}(\tau,z)+f_{s,1}^{\text{odd}}(\tau)\theta_{\text{odd}}(\tau,z). \tag{A.10}$$

Then the definition of $\Gamma_{3,2}^{(r,s)}\otimes f_{s,1}$ is iven by

$$\Gamma_{3,2}^{r,s}\otimes f_{s,1}=\Gamma_{3,2}^{r,s}(\text{even})f_{s,1}^{\text{even}}+\Gamma_{3,2}^{r,s}(\text{odd})f_{s,1}^{\text{odd}}, \tag{A.11}$$

where

$$\begin{aligned} \Gamma_{3,2}^{(r,s)}(\text{even}) &= \sum_{\substack{m_1,m_2,n_2\in\mathbb{Z},\\ n_1=\mathbb{Z}+\frac{r}{N},b\in2\mathbb{Z}}} q^{\frac{p_L^2}{2}}\bar{q}^{\frac{p_R^2}{2}}e^{2\pi i m_1 s/N} \\ \Gamma_{3,2}^{(r,s)}(\text{odd}) &= \sum_{\substack{m_1,m_2,n_2,\in\mathbb{Z},\\ n_1=\mathbb{Z}+\frac{r}{N},b\in2\mathbb{Z}+1}} q^{\frac{p_L^2}{2}}\bar{q}^{\frac{p_R^2}{2}}e^{2\pi i m_1 s/N}. \end{aligned} \tag{A.12}$$

where p_L,p_R are given in (2.21) and N is the order of the g' action.

We now list the set of identities relating $\mathcal{E}_2$ and Eisenstein series as well as theta function which have been used to obtain the results in this paper. First we have the identity

$$\mathcal{E}_2(\tau)^2=\frac{1}{4}\left(2\theta_3^8+2\theta_4^8-\theta_2^8\right), \tag{A.13}$$

and we define $\mathcal{E}_2^2$ in the presence of the U(1) chemical potential using the relation

$$\mathcal{E}_{2,1}(\tau,z)^2 = \frac{1}{4}\left(2\theta_3^6\theta_3(z)^2 + 2\theta_4^6\theta_4(z)^2 - \theta_2^6\theta_2(z)^2\right). \tag{A.14}$$

We have then the identity

$$\mathcal{E}_{2,1}(\tau,z)^2(E_6 + 2\mathcal{E}_2(\tau)E_4) = \mathcal{E}_2(\tau)^2(E_{6,1} + 2\mathcal{E}_2(\tau)E_{4,1}). \tag{A.15}$$

These are the following identities between $\mathcal{E}_2$ and Eisenstein series at 2τ.

$$\begin{aligned} E_6(2\tau) &= \frac{1}{8}\mathcal{E}_2(\tau)\left(11\mathcal{E}_2^2(\tau) - 3E_4\right), \\ E_4(2\tau) &= \frac{1}{4}\left(5\mathcal{E}_2^2(\tau) - E_4\right). \end{aligned} \tag{A.16}$$

We note that $\mathcal{E}_2^3$ can be rewritten in terms of Eisenstein series and a single power of $\mathcal{E}_2$ using the relation

$$\mathcal{E}_2^3(\tau) = \frac{1}{4}(E_6 + 3E_4\mathcal{E}_2(\tau)). \tag{A.17}$$

Their modular transformed versions can be simplified as:

$$\begin{aligned} E_6(\tau/2) &= \mathcal{E}_2(\tau/2)(-11\mathcal{E}_2^2(\tau/2) + 12E_4), \\ E_4(\tau/2) &= (5\mathcal{E}_2^2(\tau/2) - 4E_4), \\ \mathcal{E}_2^3(\tau/2) &= (-2E_6 + 3E_4\mathcal{E}_2(\tau/2)). \end{aligned} \tag{A.18}$$

Finally we also quote the identities obtained in in [12] relating $\mathcal{E}_2$ and theta functions.

$$\begin{aligned} -\left(\theta_3^8\theta_4^4 + \theta_4^8\theta_3^4\right) &= -\frac{2}{3}\left(E_6 + 2\mathcal{E}_2(\tau)E_4\right), \\ \theta_3^8\theta_2^4 + \theta_2^8\theta_3^4 &= -\frac{2}{3}\left(E_6 - \mathcal{E}_2\left(\frac{\tau}{2}\right)E_4\right), \\ \theta_2^8\theta_4^4 - \theta_2^8\theta_4^4 &= -\frac{2}{3}\left(E_6 - \mathcal{E}_2\left(\frac{\tau+1}{2}\right)E_4\right). \end{aligned} \tag{A.19}$$

For simplifications in the section 3.2 dealing with the $2B$ orbifold we need to relate theta functions and $\mathcal{E}_4$. This is given by

$$\theta_4^4(2\tau) = -(\mathcal{E}_2 - 2\mathcal{E}_4). \tag{A.20}$$

Finally we have the interesting identity relating the $(0,2)$ sector of the new supersymmetric index for the $2B$ model given in (3.60) to Eisenstein series

$$\Phi_{R^+}^{(0,2)}\theta_2^6 + \Phi_{NS^+}^{(0,2)}\theta_3^6 - \Phi_{NS^-}^{(0,2)}\theta_4^6 = \frac{1}{3}E_6 - \frac{4}{3}\mathcal{E}_2(\tau)E_4. \tag{A.21}$$

B Threshold integrals

In this appendix we detail the steps in performing the integral in (4.18). First we write the integrand in a from so that we can identity integrals which has already been performed. Adding and subtracting terms in the integrand we obtain

$$\begin{aligned}\Delta_G(T,U,V)-\Delta_{G'}(T,U,V) &= \int_{\mathcal{F}}\frac{d^2\tau}{\tau_2}\{\mathcal{B}_G-\mathcal{B}_{G'}\}, \qquad (B.1)\\ &= \int_{\mathcal{F}}\frac{d^2\tau}{\tau_2}\Big\{\Gamma^{(0,0)}\otimes 2(n_2-n_1)A(z)\\ &\quad -\Gamma^{0,1}\otimes\left[24A(z)(\frac{n_1-12}{18})-12B(z)\mathcal{E}_2(\tau)\left(\frac{2}{3}-\frac{\hat b}{2}\right)\right]\\ &\quad -\Gamma^{(1,0)}\otimes\left[24A(z)\left(\frac{n_1-12}{18}\right)+6B(z)\mathcal{E}_2\left(\frac{\tau}{2}\right)\left(\frac{2}{3}-\frac{\hat b}{2}\right)\right]\\ &\quad -\Gamma^{(1,1)}\otimes\left[24A(z)\left(\frac{n_1-12}{18}\right)+6B(z)\mathcal{E}_2\left(\frac{\tau+1}{2}\right)\left(\frac{2}{3}-\frac{\hat b}{2}\right)\right]\Big\},\\ &= -24\left(\left(\frac{1}{2}-\frac{3\hat b}{8}\right)\mathcal{I}_1+\left(\frac{n_1}{72}-\frac{1}{3}+\frac{\hat b}{8}\right)(\mathcal{I}_2+\mathcal{I}_3)\right), \qquad (B.2)\end{aligned}$$

where

$$\begin{aligned}\mathcal{I}_1 &= \int_{\mathcal{F}}\frac{\mathrm{d}^2\tau}{\tau_2}\Big\{\Gamma^{(0,0)}_{3,2}\otimes 4A(z)+\Gamma^{(0,1)}_{3,2}\otimes\left(\frac{4}{3}A-\frac{2}{3}B\mathcal{E}_2(\tau)\right)+\Gamma^{(1,0)}_{3,2}\otimes\left(\frac{4}{3}A+\frac{1}{3}B\mathcal{E}_2\left(\frac{\tau}{2}\right)\right)\\ &\quad+\Gamma^{(1,1)}_{3,2}\otimes\left(\frac{4}{3}A+\frac{1}{3}B\mathcal{E}_2\left(\frac{\tau+1}{2}\right)\right)\Big\},\\ \mathcal{I}_2 &= \int_{\mathcal{F}}\frac{\mathrm{d}^2\tau}{\tau_2}\Gamma^{(0,0)}_{3,2}\otimes 8A,\\ \mathcal{I}_3 &= \int\frac{\mathrm{d}^2\tau}{\tau_2}[\Gamma^{(0,0)}+\Gamma^{(0,1)}+\Gamma^{(1,0)}+\Gamma^{(1,1)}]\otimes 4A. \qquad (B.3)\end{aligned}$$

Using the results of the integrals in (B.5) and (B.17) in (B.1) we obtain

$$\begin{aligned}\Delta_G(T,U,V)-\Delta_{G'}(T,U,V) &= 48\left(\left(\frac{1}{2}-\frac{3\hat b}{8}\right)\log(\det(\mathrm{Im}(\Omega))^6\,|\Phi_6(\mathrm{U,T,V})|^2)\right. \qquad (B.4)\\ &\quad+\left(\frac{n_1}{72}-\frac{1}{3}+\frac{\hat b}{8}\right)\log(\det\left(\mathrm{Im}(\Omega)\right)^{10}|\Phi_{10}(\mathrm{U,T,V})|^2\right)\\ &\quad\left.+\left(\frac{n_1}{72}-\frac{1}{3}+\frac{\hat b}{8}\right)\log(\det\left(\mathrm{Im}(\Omega)\right)^{10}|\Phi_{10}(\mathrm{2U,T/2,V})|^2\right)\right).\end{aligned}$$

Let us first recall the results of one loop integration or the theta lifts which are known from earlier work

$$\begin{aligned}\mathcal{I}_1 &= -2\log\left(\det(\mathrm{Im}(\Omega))^{10}\,|\Phi_{10}(\mathrm{U,T,V})|^2\right), \qquad (B.5)\\ \mathcal{I}_2 &= -2\log\left(\det(\mathrm{Im}(\Omega))^{6}\,|\Phi_{6}(\mathrm{U,T,V})|^2\right).\end{aligned}$$

The first equation is the result for the theta lift of the elliptic genus of $K3$ and the second equation is the result for the theta lift of the elliptic genus of the $2A$ orbifold of $K3$. The new integral which we need to obtain the difference of one loop gauge thresholds for the non-standard embeddings is the following

$$\mathcal{I}_3 = \int \frac{\mathrm{d}^2\tau}{\tau_2} \left[\Gamma^{(0,0)} + \Gamma^{(0,1)} + \Gamma^{(1,0)} + \Gamma^{(1,1)}\right] \otimes 4A. \tag{B.6}$$

To evaluate this integral we can use the general result in [22] for integrals of this form which we will now state. Given the integral of the form

$$\begin{aligned}
\tilde{I}(U,T,V) &= \sum_{r,s=0}^{N-1} \sum_{b=0}^{1} \tilde{I}_{r,s,b} \ , && \text{(B.7)}\\
\tilde{I}_{r,s,b} &= \int_{\mathcal{F}} \frac{\mathrm{d}^2\tau}{\tau_2} \sum_{\substack{m_1,m_2,n_2\in\mathcal{Z}\\ n_1\in\mathcal{Z}+\frac{r}{N}\\ j\in 2\mathcal{Z}+b}} q^{p_L^2/2}\bar{q}^{p_R^2/2} e^{2\pi i s m_1/N} h_b^{r,s}, && \text{(B.8)}\\
h_b^{r,s}(\tau) &= \sum_{n\in Z - b^2/4} c_b^{r,s}(4n) q^n,\\
F^{r,s}(\tau,z) &= h_0^{r,s}(\tau)\theta_3(2\tau,2z) + h_1^{r,s}(\tau)\theta_2(2\tau,2z)\\
&= \sum_{b=0,1} \sum_{n\in\mathbb{Z}/N, j\in 2\mathbb{Z}+b} c_b^{r,s}(4n-j^2) q^n z^j \ ,
\end{aligned}$$

with the condition

$$c_0^{(r,s)}(u) = 0 \quad \text{for } u<0, \qquad c_1^{(r,s)}(u) = 0 \quad \text{for } u<-1, \tag{B.9}$$

the result for the integral is given by

$$\tilde{I}(U,T,V) = -2\log\left[\det \mathrm{Im}\Omega^{\mathrm{k}}\right] - 2\log\left[\det \tilde{\Phi}(\mathrm{U,T,V})\right] - 2\log\left[\det \bar{\tilde{\Phi}}(\mathrm{U,T,V})\right], \tag{B.10}$$

where

$$\tilde{\Phi}(U,T,V) = e^{2\pi i(\tilde{\alpha}U + \tilde{\beta}T + V)} \tag{B.11}$$
$$\prod_{b=0,1}\prod_{r=0}^{N-1} \prod_{\substack{k'\in\mathcal{Z}+\frac{r}{N}, l\in\mathbb{Z},\\ j\in 2\mathbb{Z}+b\\ k',l\geq 0,\, j<0 k'=l=0}} \left(1 - e^{2\pi i(k'T + lU + jV)}\right)^{\sum_{s=0}^{N-1} e^{2\pi i s l/N} c_b^{r,s}(4k'l - j^2)},$$

and

$$\begin{aligned}
\tilde{\beta} &= \frac{1}{24N} Q_{0,0}, && \text{(B.12)}\\
\tilde{\alpha} &= \frac{1}{24N}\chi(M) - \frac{1}{2N}\sum_{s=0}^{N-1} Q_{0,s} \frac{e^{-2\pi i s/N}}{(1-e^{2\pi i s/N})^2},\\
Q_{r,s} &= N\left(c_0^{r,s}(0) + 2c_1^{r,s}(-1)\right),\\
Q_{0,0} &= \chi(M) = 24.
\end{aligned}$$

Now examining the integral we have in (B.6), it can be seen that we can use the above result to perform the integral. Comparing the form in (B.7) and (B.6) we see that we have $N = 2$, therefore $r, s \in \{0, 1\}$ and all the coefficients

$$c_b^{r,s}(u) = \frac{1}{2}c_b(u). \tag{B.13}$$

where $c_b(u)$ are the coefficients in the expansion of the elliptic genus of $K3$ which is given by

$$8A(\tau, z) = \sum_{b=0,1}\sum_{n\in\mathbb{Z}, j\in 2\mathbb{Z}+b} c_b\left(4n - j^2\right) q^n z^j \,. \tag{B.14}$$

Thus we have

$$Q_{r,s} = 24, \qquad \tilde{\alpha} = 2, \qquad \tilde{\beta} = \frac{1}{2}. \tag{B.15}$$

We can further simplify the expression in (B.11) as follows

$$\begin{aligned}
\tilde{\Phi}(U,T,V) &= e^{2\pi i(2U+T/2+V)} \prod_{b=0,1}\prod_{r=0}^{1} \prod_{\substack{k'\in\mathbb{Z}+\frac{r}{2}, l\in\mathbb{Z}, \\ j\in 2\mathbb{Z}+b \\ k',l\geq 0, j<0 k'=l=0}} \left(1 - e^{2\pi i(k'T+2lU+jV)}\right)^{c_b^{r,s}(4k'l-j^2)} \\
&= e^{2\pi i(2U+T/2+V)} \prod_{b=0,1}\left[\prod_{\substack{k'\in\mathbb{Z}, l\in\mathbb{Z}, \\ j\in 2\mathcal{Z}+b \\ k',l\geq 0, j<0 k'=l=0}} \left(1 - e^{2\pi i(2k'T/2+l(2U)+jV)}\right)^{c_b(8k'l-j^2)}\right. \\
&\qquad \left. \times \prod_{\substack{k'\in\mathbb{Z}, l\in\mathbb{Z}, \\ j\in 2\mathbb{Z}+b \\ k',l\geq 0,\, j<0 \\ k'=l=0}} \left(1 - e^{2\pi i((2k'+1)T/2+l(2U)+jV)}\right)^{c_b(4(2k'+1)l-j^2)}\right], \\
&= e^{2\pi i(2U+T/2+V)} \prod_{b=0,1}\prod_{r=0}^{1} \prod_{\substack{k'\in\mathbb{Z},\ l\in\mathbb{Z},\ j\in 2\mathbb{Z}+b \\ k',l\geq 0,\ j<0\ k'=l=0}} \left(1 - e^{2\pi i(k'T/2+l(2U)+jV)}\right)^{c_b(4k'l-j^2)} \\
&= \Phi_{10}(2U, T/2, V).
\end{aligned} \tag{B.16}$$

In the last line we have used the definition of Φ_{10} which is the theta lift of the elliptic genus of $K3$. Thus the result of the integral in (B.6) is given by

$$\begin{aligned}
\mathcal{I}_3 &= \int \frac{\mathrm{d}^2\tau}{\tau_2} \left[\Gamma^{(0,0)} + \Gamma^{(0,1)} + \Gamma^{(1,0)} + \Gamma^{(1,1)}\right] \otimes 4A, \\
&= -2\log(\det\left(\mathrm{Im}(\Omega)\right)^{10} |\Phi_{10}(2\mathrm{U}, \mathrm{T}/2, \mathrm{V})|^2\right).
\end{aligned} \tag{B.17}$$

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C Mathematica files

There are 2 Mathematica files included in the supplementary attachments. Both the Mathematica files begin with definitions of the generalized theta functions, Dedekind eta function, Jacobi forms of index 1 and Eisenstein series.

1. **z4wilson.nb** (online resource 1): the partition function of the shifted $E_8 \times E_8$ lattice together with the left moving bosonic partition function on $K3$ is written in terms of generalized theta functions and compared with the the $(0,1)$ sector of (4.11).

2. **relations.nb** (online resource 2): different relations given in the appendix A and used in the main text are checked by q expansions. The formula for $N_h - N_v$ as a function of $\hat{b}$ given in (4.8) is checked against the general expression (3.43). $N_h - N_v$ is also evaluated for the $2B$ model.

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