Published for SISSA by Springer

Received: *October 7, 2015*
Accepted: *December 27, 2015*
Published: *January 19, 2016*

Classical limit of irregular blocks and Mathieu functions

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Abstract: The Nekrasov-Shatashvili limit of the $\mathcal{N}=2$ SU(2) pure gauge (Ω-deformed) super Yang-Mills theory encodes the information about the spectrum of the Mathieu operator. On the other hand, the Mathieu equation emerges entirely within the frame of two-dimensional conformal field theory (2*d* CFT) as the classical limit of the null vector decoupling equation for some degenerate irregular block. Therefore, it seems to be possible to investigate the spectrum of the Mathieu operator employing the techniques of 2*d* CFT. To exploit this strategy, a full correspondence between the Mathieu equation and its realization within 2*d* CFT has to be established. In our previous paper [1], we have found that the expression of the Mathieu eigenvalue given in terms of the classical irregular block exactly coincides with the well known weak coupling expansion of this eigenvalue in the case in which the auxiliary parameter is the noninteger Floquet exponent. In the present work we verify that the formula for the corresponding eigenfunction obtained from the irregular block reproduces the so-called Mathieu exponent from which the noninteger order elliptic cosine and sine functions may be constructed. The derivation of the Mathieu equation within the formalism of 2*d* CFT is based on conjectures concerning the asymptotic behaviour of irregular blocks in the classical limit. A proof of these hypotheses is sketched. Finally, we speculate on how it could be possible to use the methods of 2*d* CFT in order to get from the irregular block the eigenvalues of the Mathieu operator in other regions of the coupling constant.

Keywords: Supersymmetric gauge theory, Field Theories in Lower Dimensions, Integrable Equations in Physics, Conformal and W Symmetry

ArXiv ePrint: 1509.08164

doi:10.1007/JHEP01(2016)115

Contents

1 Introduction

In a last few years much attention was paid to the study of the connections among two-dimensional conformal field theory (2d CFT), $\mathcal{N}=2$ supersymmetric gauge theories and integrable systems, cf. e.g. [2–22].[1] This kind of research was inspired by the discovery of certain dualities, in particular, the AGT [24] and Bethe/gauge [25–27] correspondences.[2]

The AGT correspondence states that the Liouville field theory (LFT) correlators on the Riemann surface $C_{g,n}$ with genus g and n punctures can be identified with the partition functions of a class $T_{g,n}$ of four-dimensional $\mathcal{N}=2$ supersymmetric SU(2) quiver gauge theories:

$$\left\langle \prod_{i=1}^{n} \mathsf{V}_{\Delta_i} \right\rangle^{\rm LFT}_{C_{g,n}} = Z^{(\sigma)}_{T_{g,n}}. \tag{1.1}$$

[1] See also the volume [23] edited by J. Teschner and refs. therein.
[2] See also [28–30].

Let us recall that for a given pant decomposition σ of the Riemann surface $C_{g,n}$, both sides of the equation above have an integral representation. Indeed, LFT correlators can be factorized according to the pattern given by the pant decomposition of $C_{g,n}$ and written as an integral over a continuous spectrum of the Liouville theory in which, for each pant decomposition σ, the integrand is built out of the holomorphic and the anti-holomorphic Virasoro conformal blocks $\mathcal{F}^{(\sigma)}_{c,\Delta_p}[\Delta_i](\mathsf{Z})$ and $\bar{\mathcal{F}}^{(\sigma)}_{c,\Delta_p}[\Delta_i](\bar{\mathsf{Z}})$ multiplied by the DOZZ 3-point functions [31, 32]. The Virasoro conformal block $\mathcal{F}^{(\sigma)}_{c,\Delta_p}[\Delta_i](\mathsf{Z})$ on $C_{g,n}$ depends on the following quantities: the cross ratios of the vertex operators locations denoted symbolically by Z, the external conformal weights $\{\Delta_i\}_{i=1,\ldots,n}$, the intermediate conformal weights $\{\Delta_p\}_{p=1,\ldots,3g-3+n}$ and the central charge c.

On the other hand, the partition function $Z^{(\sigma)}_{T_{g,n}}$ can be written as the integral over the holomorphic times the anti-holomorphic Nekrasov partition functions [33, 34]:

$$Z^{(\sigma)}_{T_{g,n}} = \int [da]\, \mathcal{Z}^{(\sigma)}_{\text{Nekrasov}}\, \bar{\mathcal{Z}}^{(\sigma)}_{\text{Nekrasov}},$$

where $[da]$ is some appropriate measure. The Nekrasov partition function can be written as a product of three factors $\mathcal{Z}_{\text{Nekrasov}} = \mathcal{Z}_{\text{class}}\mathcal{Z}_{1-\text{loop}}\mathcal{Z}_{\text{inst}}$. The first two factors $\mathcal{Z}_{\text{class}}\mathcal{Z}_{1-\text{loop}} =: \mathcal{Z}_{\text{pert}}$ describe the contribution coming from perturbative calculations. Supersymmetry implies that there are contributions to $\mathcal{Z}_{\text{pert}}$ only at the tree- ($\mathcal{Z}_{\text{class}}$) and 1-loop ($\mathcal{Z}_{1-\text{loop}}$) levels. $\mathcal{Z}_{\text{inst}}$ is the instanton contribution. The Nekrasov partition function $\mathcal{Z}_{\text{Nekrasov}}(\tilde{q}, \tilde{a}, \tilde{m}, \epsilon_1, \epsilon_2)$ depends on the set of parameters: $\tilde{q}$, $\tilde{a}$, $\tilde{m}$, ϵ_1, ϵ_2. The components of $\tilde{q} = \{\exp 2\pi\tau_1, \ldots, \exp 2\pi\tau_{3g-3+n}\}$ are the gluing parameters associated with the pant decomposition of $C_{g,n}$, where the $\tau_p = \frac{\theta_p}{2\pi} + \frac{4\pi i}{g_p^2}$ are the complexified gauge couplings. The multiplet $\tilde{m} = \{m_1, \ldots, m_n\}$ contains the mass parameters. Moreover, $\tilde{a} = \{a_1, \ldots, a_{3g-3+n}\}$, where the a's are the vacuum expectation values of the scalar fields in the vector multiplets. Finally, ϵ_1, ϵ_2 represent the complex Ω-background parameters.

Comparing the integral representations of both sides of eq. (1.1) it is possible, thanks to AGT hypothesis, to identify separately in the holomorphic and anti-holomorphic sectors the Virasoro conformal blocks $\mathcal{F}_{c,\Delta_p}[\Delta_i](\mathsf{Z})$ on $C_{g,n}$ and the instanton sectors $\mathcal{Z}_{\text{inst}}$ of the Nekrasov partition functions for the super Yang-Mills theories $T_{g,n}$.

Soon after its discovery, the AGT conjecture has been extended to the $2d$ conformal Toda/$4d$ SU(N) gauge theories correspondence [35, 36], and to the so-called 'nonconformal' cases [37–39] (see also [22, 40–42]), which will be of main interest in the present work.

The AGT correspondence works at the level of the quantum Liouville field theory. It is intriguing to ask, however, what happens if we proceed to the semiclassical limit of the Liouville correlation functions. This is the limit in which the central charge c, the external Δ_i and intermediate Δ_p conformal weights tend to infinity in such a way that their ratios are fixed $\Delta_p/c = \Delta_i/c = \text{const.}$, cf. [32]. For the standard parametrization of the central charge $c = 1 + 6\mathsf{Q}^2$, where $\mathsf{Q} = b + \frac{1}{b}$ and for *heavy* weights $(\Delta_p, \Delta_i) = \frac{1}{b^2}(\delta_p, \delta_i)$ with $\delta_p, \delta_i = \mathcal{O}(b^0)$, the classical limit corresponds to $b \to 0$. It is commonly believed that in the classical limit the conformal blocks behave exponentially with respect to Z:

$$\mathcal{F} \overset{b\to 0}{\sim} \mathrm{e}^{\frac{1}{b^2}f}.$$

The function f is known as the *classical conformal block*.

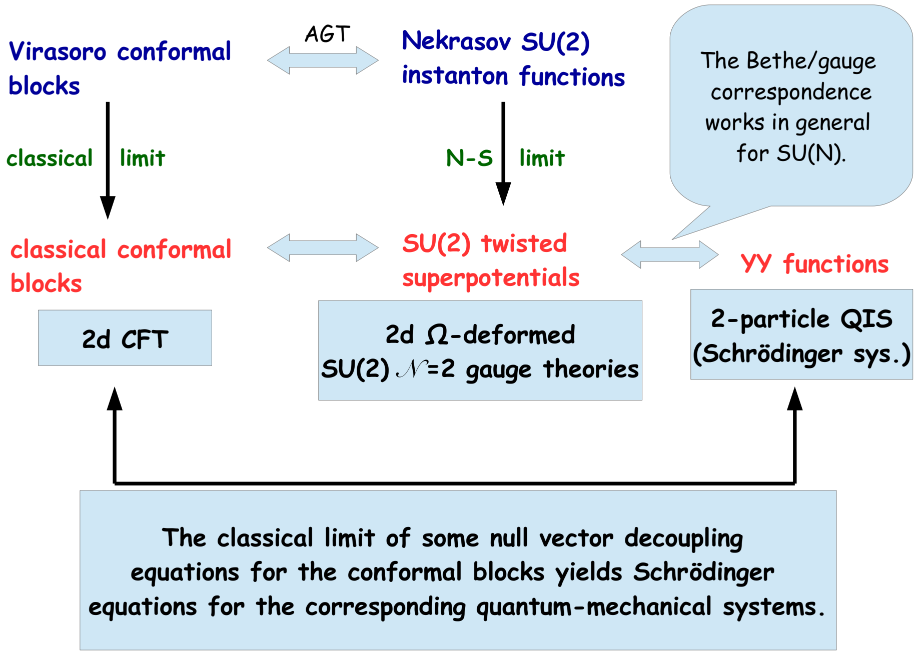

Figure 1. The triple correspondence in the case of the Virasoro classical conformal blocks links the latter to SU(2) instanton twisted superpotentials which describe the spectra of some quantum-mechanical systems. The Bethe/gauge correspondence on the r.h.s. connects the SU(N) $\mathcal{N}=2$ SYM theories with the N-particle quantum integrable systems. An extension of the above triple relation to the case N >2 needs to consider on the l.h.s. the classical limit of the $W_{\rm N}$ symmetry conformal blocks according to the known extension [35] of the AGT conjecture.

The AGT correspondence dictionary says that $b=\sqrt{\epsilon_2/\epsilon_1}$. Therefore, the semiclassical limit $b\to 0$ of the conformal blocks corresponds to the so-called *Nekrasov-Shatashvili limit* $\epsilon_2\to 0$ (ϵ_1 being kept finite) of the Nekrasov partition functions. In [25] it was observed that in the limit $\epsilon_2\to 0$ the Nekrasov partition functions have the following asymptotic behavior:

$$\mathcal{Z}_{\rm Nekrasov}(\,\cdot\,,\epsilon_1,\epsilon_2)\stackrel{\epsilon_2\to 0}{\sim}\exp\left\{\frac{1}{\epsilon_2}W(\,\cdot\,,\epsilon_1)\right\}, \tag{1.2}$$

where $W(\,\cdot\,,\epsilon_1)=W_{\rm pert}(\,\cdot\,,\epsilon_1)+W_{\rm inst}(\,\cdot\,,\epsilon_1)$ is the *effective twisted superpotential* of the corresponding two-dimensional gauge theories restricted to the two-dimensional Ω-background.

The twisted superpotentials play a pivotal role in the already mentioned *Bethe/gauge correspondence* [25–27] which maps supersymmetric vacua of the $\mathcal{N}=2$ theories to Bethe states of quantum integrable systems (QIS's). A result of that duality is that the twisted superpotentials are identified with the Yang-Yang (YY) functions [43] which describe the spectra of some QIS's. Therefore, combining both the classical/Nekrasov-Shatashvili limit of the AGT duality and the Bethe/gauge correspondence one thus gets a triple correspondence which connects the classical blocks with the twisted superpotentials and then with the Yang-Yang functions (cf. figure 1).

For example, the twisted superpotentials for the $\mathcal{N}=2$ SU(N) $N_f = 0$ (pure gauge) and the $\mathcal{N}=2^*$ SU(N) SYM theories determine respectively the spectra of the N-particle periodic Toda (pToda) and the elliptic Calogero-Moser (eCM) models [25]. In the case of the SU(2) gauge group these QIS's are simply quantum-mechanical systems whose dynamics is described by some Schrödinger equations. Concretely, for the 2-particle pToda and eCM models these Schrödinger equations correspond to the celebrated Mathieu and Lamé equations with energy eigenvalues expressed in terms of the twisted superpotentials. This correspondence can be used to investigate nonperturbative effects in the Mathieu and Lamé quantum-mechanical systems, cf. [44].[3] On the other hand, the Mathieu and Lamé equations emerge entirely within the framework of 2d CFT as the classical limit of the null vector decoupling (NVD) equations for the 3-point degenerate irregular block and for the 2-point block (projected 2-point function) on the torus with one degenerate *light* operator [1, 15, 46]. It turns out that the classical irregular block $f_{\rm irr}$ and the classical 1-point block on the torus $f_{\rm torus}$ determine the spectra of the Mathieu and Lamé operators in the same way as their gauge theory counterparts, i.e.: $W_{\rm inst}^{{\rm SU}(2),N_f=0}$ and $W_{\rm inst}^{{\rm SU}(2),\mathcal{N}=2^*}$.[4] Therefore, it seems that there is a way to study the spectrum of the Mathieu and Lamé operators using two-dimensional conformal field theory methods.[5] However, in order to exploit this possibility it is necessary to establish a full correspondence between the Mathieu and Lamé equations and their realizations within 2d CFT. The missing element is to understand how the solutions of the equations obtained in the classical limit from the NVD equations are connected to the eigenfunctions of the Mathieu and Lamé operators. It is also important to know what kind of solutions are possible to be obtained. An answer to these questions in the case of the Mathieu equation is our main goal in the present paper.

Our motivations for studying the Mathieu equation:

$$\psi'' + \left(\lambda - 2h^2 \cos 2x\right)\psi = 0$$

using the methods of 2d CFT are twofold. The first motivation is technical. In fact, the Mathieu equation is undoubtedly difficult to solve. Studying the literature on this topic one can find the following opinions:

> "Unfortunately, the analytic determination of Mathieu functions *presents great difficulties* (Whittaker [47], Frenkel and Portugal [48]), and they are difficult to

[3] See also [45].

[4] It is known that the Mathieu equation is a specific limit of the Lamé equation. It should be stressed that in 2d CFT these equations are obtained independently as classical limits of two different null vector decoupling (NVD) equations. As has been already mentioned, the Mathieu equation is a result of taking the classical limit of the NVD equation obeyed by the three-point degenerate irregular block and the Mathieu eigenvalue is expressed in terms of the classical irregular block (cf. sections 3 and 4). The Lamé equation emerges as the classical limit of the NVD equation fulfilled by the two-point degenerate block on the torus and the Lamé eigenvalue is determined by the classical torus block [46]. However, taking into account the relationship between the Mathieu and Lamé equations, the same should be true for their conformal field theory realizations. Moreover, a similar type of relationship should be for "quantum counterparts" of the Mathieu and Lamé equations, i.e. for the NVD equations. In the present work we leave these questions as an open problems to which we will return soon.

[5] For interesting questions which can be studied in this way, see the conclusions of the present work.

employ, *mainly because of the impossibility of analytically representing them in a simple and handy way* (Sips [49], Frenkel and Portugal [48])."[6]

Therefore, an elaboration of any "handy" methods to compute the solutions of the Mathieu equation, i.e. the Mathieu functions, are of great importance. In the present work we derive closed formulas for the Mathieu eigenvalue and eigenfunction in the case in which the auxiliary parameter is the non-integer Floquet exponent. These formulas contain the classical limit of irregular conformal blocks. The latter are certain special functions of $2d$ CFT that assume the form of a formal power series coefficients of which are calculable either algebraically or recursively, cf. [40]. Hence, it is quite easy to implement the calculation of the Mathieu eigenvalue and eigenfunction on computer using our expressions, cf. appendix A.

The second group of motivations for our approach is of conceptual character. Firstly, usual methods of solving the Mathieu equation are mostly variants of the perturbation theory. However, interesting effects which emerge in the Mathieu spectrum, as for example the *level splitting*, are non-perturbative in their nature. Indeed, let us recall that for the large coupling constant h^2 the Mathieu spectrum can be approximated by the energy levels of the harmonic oscillator. When the coupling constant tends to zero $h^2 \to 0$ the oscillatory levels split into energy bands. The dependence of the width of the energy band as a function of the coupling constant contains an exponential factor. This is non-perturbative effect which can be obtained by connecting solutions in the strongly and weakly-coupled regions via properly defined boundary conditions, cf. [50]. As has been already mentioned, this phenomenon has been thoroughly studied in [44] using methods of supersymmetric gauge theories. Because of the correspondence between $2d$ CFT and the $\mathcal{N}=2$ SYM theories (cf. figure 1) it should to be possible to investigate the non-perturbative effects in the Mathieu system using conformal field theory tools. In particular, it seems to be possible to find an analytic continuation of the Mathieu eigenvalue from the weakly-coupled to the other regions of the spectrum using duality relations for the four-point regular conformal blocks.[7] Secondly, the very fact that the Mathieu equation appears within the $2d$ CFT which is due to the Gaiotto's discovery of irregular states is appealing. The reason is that, it enables to test the conjectures that are usually assumed without a proof, like the existence of the classical limit of conformal blocks or factorization in the classical limit of conformal blocks that contain both the "heavy" and the "light" operators. Our earlier paper [1] as well as the present work pertains to exactly this point in the simplest accessible cases, namely, the irregular conformal blocks. Although it was not possible to perform the entire proof of the existence of the classical irregular conformal block (we were able to prove solely the existence of the first coefficient of the classical irregular block in the leading order in the inverse of the "Planck constant" $b^2 \sim \hbar$), the discovery that the spectrum of the Mathieu operator is expressed in terms of the classical irregular block, indirectly proves its existence. The same argument applies to the three-point irregular block with a certain degenerate operator which plays the role of the light field. In the present paper we observe

[6]Cf. http://mathworld.wolfram.com/MathieuFunction.html.

[7]Concrete questions which can be studied in this way are discussed in the conclusions of the present work.

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and prove up to the leading order approximation the factorization conjecture to work, which enables us to reproduce explicitly the Mathieu functions.

The organization of the paper is as follows. In section 2 the necessary tools of $2d$ CFT are introduced. In section 3 the simplest irregular blocks are defined and some of their properties are described. In particular, an exponentiation of the pure gauge irregular block within the classical limit is proved at the leading order. After that, the NVD equations for certain degenerate irregular blocks are derived. Section 4 is devoted to the derivation of the Mathieu equation within the formalism of $2d$ CFT. The calculation presented there provides formulas for the Mathieu eigenvalue and the related eigenfunction in terms of the classical limit of irregular blocks. It is shown that these formulas reproduce the well known noninteger order weak coupling expansion of the Mathieu eigenvalue and the corresponding Mathieu function. In subsection 4.2 a factorization property of the degenerate irregular block with the light operator and its representation in the classical limit as a product of light and heavy parts is proved at the leading order. This factorization property is crucial for deriving the Mathieu equation. Section 5 contains our conclusions. In particular, the problems that are still open and the possible extensions of the present work are discussed.

2 Conformal blocks in the operator formalism

2.1 Chiral vertex operators

Starting from the Belavin-Polyakov-Zamolodchikov axioms [51], Moore and Seiberg [52, 53] have constructed formalism of the so-called *rational conformal field theories* (RCFT's),[8] where

— the operator algebra of local fields contains purely holomorphic subalgebra $\mathcal{A}$ called *chiral* or *vertex algebra*;

— the Hilbert space of states of the theory is a direct sum of irreducible representations of the algebra $\mathcal{A}\oplus\mathcal{A}$:

$$\mathcal{H} \;=\; \bigoplus_{i=1}^{N}\mathcal{U}_i\otimes\mathcal{U}_i\;. \tag{2.1}$$

In RCFT's the sum in (2.1) is over a discrete finite set. However, one can generalize and successfully apply the Moore-Seiberg formalism to the case of two-dimensional conformal field theories with continuous spectrum, cf. e.g. [56, 57]. In such a case the direct sum in eq. (2.1) becomes a direct integral.

In any 2d CFT there exist at least two chiral fields, i.e., the identity operator and its descendant — the holomorphic component of the energy-momentum tensor $T(z) = \sum_{n\in\mathbb{Z}} z^{-n-2}L_n$. Therefore, each chiral algebra $\mathcal{A}$ contains as a subalgebra the Virasoro algebra $\mathrm{Vir} = \bigoplus_{n\in\mathbb{Z}}\mathbb{C}\,L_n \bigoplus \mathbb{C}\,c$,

$$[L_n, L_m] \;=\; (n-m)L_{n+m} + \frac{c}{12}(n^3-n)\delta_{n+m,0}\;. \tag{2.2}$$

[8] A very similar formalism can be found in [54, 55].

In the Moore-Seiberg formalism the 'physical' fields of [51] are built out of more fundamental objects — the so-called *chiral vertex operators* (CVO's). These are intertwining operators acting between representations of the vertex algebra. In the present paper we confine ourselves to the simplest case when $\mathcal{A} = \mathrm{Vir}$ and define CVO's as operators acting between Verma modules.

Let $\mathcal{V}^n_{c,\Delta}$ be the free vector space generated by all vectors of the form

$$|\,\nu^n_{\Delta,I}\,\rangle = L_{-I}|\,\nu_\Delta\,\rangle = L_{-k_1}\dots L_{-k_{j-1}}L_{-k_j}|\,\nu_\Delta\,\rangle \tag{2.3}$$

where $I = (k_1, \dots, k_{j-1}, k_j)$ is an ordered $(k_1 \geq \dots \geq k_j \geq 1)$ sequence of positive integers of the length $|I| \equiv k_1 + \dots + k_j = n$, and $|\,\nu_\Delta\,\rangle$ is the highest weight vector:

$$L_0|\,\nu_\Delta\,\rangle = \Delta|\,\nu_\Delta\,\rangle, \qquad L_n|\,\nu_\Delta\,\rangle = 0 \quad \forall\, n > 0\,. \tag{2.4}$$

The $\mathbb{Z}$-graded representation of the Virasoro algebra determined on the space:

$$\mathcal{V}_{c,\Delta} \;=\; \bigoplus_{n=0}^{\infty}\mathcal{V}^n_{c,\Delta}$$

by the relations (2.2) and (2.4) is called the Verma module of the central charge c and the highest weight Δ. The dimension of the subspace $\mathcal{V}^n_{c,\Delta}$ of all homogeneous elements of degree n is given by the number $p(n)$ of partitions of n (with the convention $p(0) = 1$). It is an eigenspace of L_0 with the eigenvalue $\Delta + n$.

On $\mathcal{V}^n_{c,\Delta}$ there exists the symmetric bilinear form $\langle\,\cdot\,|\,\cdot\,\rangle$ uniquely defined by the relations

$$\langle\,\nu_\Delta\,|\,\nu_\Delta\,\rangle \;=\; 1 \quad\text{and}\quad (L_n)^\dagger \;=\; L_{-n}.$$

The Gram matrix $G_{c,\Delta}$ of the form $\langle\,\cdot\,|\,\cdot\,\rangle$ is block-diagonal in the basis $\{|\,\nu_{\Delta,I}\,\rangle\}$ with blocks

$$\Big[G^n_{c,\Delta}\Big]_{IJ} \;=\; \langle\,\nu^n_{\Delta,I}\,|\,\nu^n_{\Delta,J}\,\rangle \;=\; \langle\,\nu_\Delta\,|(L_{-I})^\dagger L_{-J}|\,\nu_\Delta\,\rangle.$$

In particular, one finds

— $n = 1$: $\{L_{-1}|\,\nu_\Delta\,\rangle\}$,

$$G^{n=1}_{c,\Delta} = \langle L_{-1}\nu_\Delta\,|\,L_{-1}\nu_\Delta\rangle = \langle\nu_\Delta\,|\,L_1L_{-1}\nu_\Delta\rangle = 2\Delta,$$

— $n = 2$: $\{L_{-2}|\,\nu_\Delta\,\rangle,\ L_{-1}L_{-1}|\,\nu_\Delta\,\rangle\}$,

$$G^{n=2}_{c,\Delta} = \begin{pmatrix}\langle L_{-2}\nu_\Delta\,|\,L_{-2}\nu_\Delta\rangle & \langle L^2_{-1}\nu_\Delta\,|\,L_{-2}\nu_\Delta\rangle \\ \langle L_{-2}\nu_\Delta\,|\,L^2_{-1}\nu_\Delta\rangle & \langle L^2_{-1}\nu_\Delta\,|\,L^2_{-1}\nu_\Delta\rangle\end{pmatrix} = \begin{pmatrix}\frac{c}{2} + 4\Delta & 6\Delta \\ 6\Delta & 4\Delta(2\Delta+1)\end{pmatrix},$$

— $n = 3$: $\{L_{-3}|\,\nu_\Delta\,\rangle,\ L_{-2}L_{-1}|\,\nu_\Delta\,\rangle,\ L_{-1}L_{-1}L_{-1}|\,\nu_\Delta\,\rangle\}$,

$$G^{n=3}_{c,\Delta} = \begin{pmatrix}2c + 6\Delta & 10\Delta & 24\Delta \\ 10\Delta & \Delta(c + 8\Delta + 8) & 12\Delta(3\Delta+1) \\ 24\Delta & 12\Delta(3\Delta+1) & 24\Delta(\Delta+1)(2\Delta+1)\end{pmatrix}.$$

The Verma module $\mathcal{V}_{c,\Delta}$ is irreducible if and only if the form $\langle\cdot|\cdot\rangle$ is non-degenerate. The criterion for irreducibility is vanishing of the determinant $\det G^n_{c,\Delta}$ of the Gram matrix, known as the Kac determinant, given by the formula [58–63]:

$$\det G^n_{c,\Delta} \;=\; C_n \prod_{\substack{r,s\in\mathbb{N},\\ s\le r\\ 1\le rs\le n}} \Phi_{rs}(c,\Delta)^{p(n-rs)}. \tag{2.5}$$

In the equation above C_n is a constant and

$$\Phi_{rs}(c,\Delta) \;=\; \left(\Delta+\tfrac{r^2-1}{24}(c-13)+\tfrac{rs-1}{2}\right)\left(\Delta+\tfrac{s^2-1}{24}(c-13)+\tfrac{rs-1}{2}\right)+\tfrac{(r^2-s^2)^2}{16}.$$

The Kac determinant vanishes for

$$\begin{aligned}\Delta_{rs}(c) &= \frac{(13-c)(r^2+s^2)+\sqrt{(c-25)(c-1)}(r^2-s^2)-24rs-2+2c}{48},\\ &r,s\in\mathbb{Z},\quad r\ge1,\ s\ge1,\quad 1\le rs\le n\end{aligned}$$

or

$$\begin{aligned}c_{rs}(\Delta) &= 13-6\left(T_{rs}(\Delta)+\frac{1}{T_{rs}(\Delta)}\right),\\ T_{rs}(\Delta) &= \frac{rs-1+2\,\Delta+\sqrt{(r-s)^2+4\,(r\,s-1)\,\Delta+4\,\Delta^2}}{r^2-1},\\ &r,s\in\mathbb{Z},\quad r\ge2,\ s\ge1,\quad 1\le rs\le n.\end{aligned}$$

For these values of Δ and c the representations $\mathcal{V}_{c,\Delta_{rs}(c)}$ or $\mathcal{V}_{c_{rs}(\Delta),\Delta}$ are reducible.

The set $\{\Delta_{rs}(c)\}$ of the degenerate conformal weights can be parametrized as follows

$$\Delta_{rs}(c) \;=\; \Delta_0+\frac{\beta_{rs}^2}{4}, \qquad\qquad \beta_{rs} \;=\; r\beta_+ + s\beta_-, \tag{2.6}$$

where

$$\beta_\pm(c)=\frac{\sqrt{1-c}\pm\sqrt{25-c}}{2\sqrt{6}}, \qquad\qquad \Delta_0=-\frac{1}{4}(\beta_++\beta_-)^2=\frac{c-1}{24}.$$

Sometimes, it is also convenient to use the alternative parametrization:[9]

$$\Delta_{rs}(c) \;=\; \frac{\mathsf{Q}^2}{4}-\frac{1}{4}\left(rb+\frac{s}{b}\right)^2 \tag{2.7}$$

for which the central charge is given by $c=1+6\mathsf{Q}^2$ with $\mathsf{Q}=b+b^{-1}$.

The non-zero element $|\,\chi_{rs}\,\rangle\in\mathcal{V}_{c,\Delta_{rs}(c)}$ of degree $n=rs$ is called a null vector if $L_0\,|\,\chi_{rs}\,\rangle=(\Delta_{rs}+rs)\,|\,\chi_{rs}\,\rangle$, and $L_k\,|\,\chi_{rs}\,\rangle=0$, $\forall\,k>0$. Hence, $|\,\chi_{rs}\,\rangle$ is the highest weight state which generates its own Verma module $\mathcal{V}_{c,\Delta_{rs}(c)+rs}$, which is a submodule of $\mathcal{V}_{c,\Delta_{rs}(c)}$. One can prove that each submodule of the Verma module $\mathcal{V}_{c,\Delta_{rs}(c)}$ is generated by a null

[9] Here $\beta_+\left(1+6\left(b+\frac{1}{b}\right)^2\right)=ib$ and $\beta_-\left(1+6\left(b+\frac{1}{b}\right)^2\right)\;=\;\frac{i}{b}$.

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vector. Then, the module $\mathcal{V}_{c,\Delta_{rs}(c)}$ is irreducible if and only if it does not contain null vectors with positive degree.

For non-degenerate values of Δ, i.e. for $\Delta \neq \Delta_{rs}(c)$, there exists in $\mathcal{V}^n_{c,\Delta}$ the 'dual' basis $\{|\,\nu^{t,n}_{\Delta,I}\,\rangle\}$ whose elements are defined by the relation $\langle\,\nu^{t,n}_{\Delta,I}\,|\,\nu^n_{\Delta,J}\,\rangle = \delta_{IJ}$ for all $|\,\nu^n_{\Delta,J}\,\rangle \in \{|\,\nu^n_{\Delta,J}\,\rangle\}$. The dual basis vectors $|\,\nu^{t,n}_{\Delta,I}\,\rangle$ have the following representation in the standard basis

$$|\,\nu^{t,n}_{\Delta,I}\,\rangle = \sum_{J,|J|=n} \left[G^n_{c,\Delta}\right]^{IJ} |\,\nu^n_{\Delta,J}\,\rangle,$$

where $\left[G^n_{c,\Delta}\right]^{IJ}$ is the inverse of the Gram matrix $\left[G^n_{c,\Delta}\right]_{IJ}$.

Let $\mathcal{V}_\Delta$ be the Verma module with the highest weight state $|\,\nu_\Delta\,\rangle$. The chiral vertex operator is the linear map

$$V^{\Delta_3\,\Delta_2\,\Delta_1}_{\infty\;\; z\;\; 0} : \mathcal{V}_{\Delta_2} \otimes \mathcal{V}_{\Delta_1} \to \mathcal{V}_{\Delta_3}$$

such that for all $|\,\xi_2\,\rangle \in \mathcal{V}_{\Delta_2}$ the operator

$$V(\xi_2|z) \equiv V^{\Delta_3\,\Delta_2\,\Delta_1}_{\infty\;\; z\;\; 0}(|\,\xi_2\,\rangle \otimes \cdot\,) : \mathcal{V}_{\Delta_1} \to \mathcal{V}_{\Delta_3}$$

satisfies the following conditions

$$[L_n, V(\nu_2|z)] = z^n \left(z\frac{\partial}{\partial z} + (n+1)\Delta_2\right) V(\nu_2|z)\;, \qquad n \in \mathbb{Z} \tag{2.8}$$

$$V(L_{-1}\xi_2|z) = \frac{\partial}{\partial z} V(\xi_2|z)\,, \tag{2.9}$$

$$V(L_n\xi_2|z) = \sum_{k=0}^{n+1} \binom{n+1}{k} (-z)^k \,[L_{n-k}, V(\xi_2|z)]\;, \qquad n > -1, \tag{2.10}$$

$$\begin{aligned} V(L_{-n}\xi_2|z) = {} & \sum_{k=0}^{\infty} \binom{n-2+k}{n-2} z^k\, L_{-n-k}\, V(\xi_2|z) \\ & + (-1)^n \sum_{k=0}^{\infty} \binom{n-2+k}{n-2} z^{-n+1-k}\; V(\xi_2|z)\; L_{k-1}, \qquad n > 1 \end{aligned} \tag{2.11}$$

and

$$\langle\,\nu_{\Delta_3}\,|V(\nu_{\Delta_2}|z)\,|\,\nu_{\Delta_1}\,\rangle \;=\; z^{\Delta_3-\Delta_2-\Delta_1}.$$

The commutation relation (2.8) defines the primary vertex operator corresponding to the highest weight state $|\,\nu_2\,\rangle \in \mathcal{V}_{\Delta_2}$. Eqs. (2.9)–(2.11) characterize the decendant CVO's.

2.2 The 3-point block

For a given triple $\Delta_1, \Delta_2, \Delta_3$ of conformal weights we define the trilinear map

$$\rho^{\Delta_3\,\Delta_2\,\Delta_1}_{\infty\;\; z\;\; 0} : \mathcal{V}_{\Delta_3} \otimes \mathcal{V}_{\Delta_2} \otimes \mathcal{V}_{\Delta_1} \to \mathbb{C}$$

induced by the matrix element of a single chiral vertex operator

$$\rho^{\Delta_3\,\Delta_2\,\Delta_1}_{\infty\;\; z\;\; 0}(\xi_3, \xi_2, \xi_1) \;=\; \langle\,\xi_3\,|V(\xi_2|z)\,|\,\xi_1\,\rangle\,, \qquad \forall\,|\,\xi_i\,\rangle \in \mathcal{V}_{\Delta_i}, \quad i = 1, 2, 3.$$

The form $\rho^{\Delta_3\,\Delta_2\,\Delta_1}_{\infty\;\; z\;\; 0}$ is uniquely determined by the conditions (2.8)–(2.11). In particular,

1. for L_0-eingenstates[10] $L_0|\,\xi_i\,\rangle = \Delta_i(\xi_i)|\,\xi_i\,\rangle$, $i = 1,2,3$ one gets

$$\rho_{\infty\;\,z\;\;0}^{\Delta_3\,\Delta_2\,\Delta_1}(\xi_3,\xi_2,\xi_1) = z^{\Delta_3(\xi_3)-\Delta_2(\xi_2)-\Delta_1(\xi_1)}\rho_{\infty\;1\;0}^{\Delta_3\,\Delta_2\,\Delta_1}(\xi_3,\xi_2,\xi_1)\,; \tag{2.12}$$

2. for basis vectors $\nu_{i,I} \equiv |\,\nu_{\Delta_i,I}\,\rangle \in \mathcal{V}_{\Delta_i}$, $i = 1,2,3$ one finds

$$\begin{aligned}
\rho_{\infty\;1\;0}^{\Delta_3\,\Delta_2\,\Delta_1}(\nu_{3,I},\nu_2,\nu_1) &= \gamma_{\Delta_3}\left[{}^{\Delta_2}_{\Delta_1}\right]_I \,,\\
\rho_{\infty\;1\;0}^{\Delta_3\,\Delta_2\,\Delta_1}(\nu_3,\nu_2,\nu_{1,I}) &= \gamma_{\Delta_1}\left[{}^{\Delta_2}_{\Delta_3}\right]_I \,,\\
\rho_{\infty\;1\;0}^{\Delta_3\,\Delta_2\,\Delta_1}(\nu_3,\nu_{2,I},\nu_1) &= (-1)^{|I|}\gamma_{\Delta_2}\left[{}^{\Delta_1}_{\Delta_3}\right]_I \,,
\end{aligned} \tag{2.13}$$

where for a given partition $I = (k_1,\ldots,k_{\ell(I)})$, $k_i \geq k_j \geq 1$, $i < j$,

$$\gamma_\Delta\left[{}^{\Delta_2}_{\Delta_1}\right]_I \;\equiv\; \prod_{i=1}^{\ell(I)}\left(\Delta + k_i\Delta_2 - \Delta_1 + \sum_{i<j}^{\ell(I)} k_j\right). \tag{2.14}$$

In terms of the trilinear form ρ (3-point block) one can spell out an important result known as the *null vector decoupling theorem* (Feigin-Fuchs [64]):[11]
Let $i,j,k \in \{1,2,3\}$ be chosen such that $j \neq i$, $k \neq i$, $j \neq k$. Let us assume that

(i) $\Delta_i = \Delta_{rs}(c) \equiv \frac{1}{24}(c-1) + \frac{1}{4}\beta_{rs}^2$, $r,s \in \mathbb{Z}_{>0}$ *(cf. parametrization (2.6)) and*

(ii) the vector $|\,\xi_i\,\rangle$ lies in the singular submodule generated by the null vector $|\,\chi_{rs}\,\rangle$, i.e.: $|\,\xi_i\,\rangle \in \mathcal{V}_{c,\Delta_{rs}(c)+rs} \subset \mathcal{V}_{c,\Delta_{rs}(c)}$.

Then, $\boxed{\rho_{z_3\;z_2\;z_1}^{\Delta_3\,\Delta_2\,\Delta_1}(\xi_3,\xi_2,\xi_1) = 0}$ *if and only if*

$$\Delta_j = \Delta_{\beta_j} \equiv \frac{1}{24}(c-1) + \frac{1}{4}\beta_j^2 \qquad \textit{and} \qquad \Delta_k = \Delta_{\beta_k} \equiv \frac{1}{24}(c-1) + \frac{1}{4}\beta_k^2$$

satisfy the fusion rules $\beta_j - \beta_k = \beta_{pq}$, where $p \in \{1-r, 3-r, \ldots, r-1\}$ and $q \in \{1-s, 3-s, \ldots, s-1\}$.

3 Quantum and classical zero flavor irregular blocks

3.1 Definition and basic properties

To begin with, let us consider the following (coherent) vector in the Verma module $\mathcal{V}_{c,\Delta}$ discovered by D. Gaiotto in [37] and constructed by A. Marshakov, A. Mironov and A. Morozov in [38]:[12]

$$|\,\Delta,\Lambda^2\,\rangle = \sum_I \Lambda^{2|I|}\left[G_{c,\Delta}^{|I|}\right]^{(1^{|I|})I} L_{-I}|\,\nu_\Delta\,\rangle = \sum_{n=0}^{\infty}\Lambda^{2n}\sum_{I,|I|=n}\left[G_{c,\Delta}^{n}\right]^{(1^n)I}|\,\nu_{\Delta,I}^n\,\rangle\,. \tag{3.1}$$

[10] Note that for the basis vectors $\{|\,\nu_{i,I}\,\rangle\}$ one has $\Delta_i(\nu_{i,I}) = \Delta_i + |I|$.

[11] Here we closely follow [56].

[12] With some abuse of nomenclature, we will call 'zero flavor' both the Gaiotto state and the irregular block. The reason for that is that the irregular block corresponds to the Nekrasov instanton function of the $\mathcal{N} = 2$ (Ω-deformed) pure gauge (zero flavor $N_f = 0$) super Yang-Mills theory, in accordance with the 'non-conformal' extension of the AGT conjecture, see below.

The sum in eq. (3.1) runs over all partitions or equivalently over their pictorial representations — Young diagrams. The symbol $(1^{|I|})$ in eq. (3.1) denotes a single-row Young diagram, where the total number of boxes $|I| = n$ equals the number of columns $\ell(I)$, i.e. $\ell(I) = |I| = n$.

In [38] it was shown that the vector (3.1) obeys the Gaiotto defining conditions:

$$L_0|\Delta,\Lambda^2\rangle = \left(\Delta + \frac{\Lambda}{2}\frac{\partial}{\partial\Lambda}\right)|\Delta,\Lambda^2\rangle, \quad L_1|\Delta,\Lambda^2\rangle = \Lambda^2|\Delta,\Lambda^2\rangle, \quad L_n|\Delta,\Lambda^2\rangle = 0 \;\; \forall\, n \geq 2. \tag{3.2}$$

The zero flavor $N_f = 0$ qunatum irregular block is defined as the inner product of the Gaiotto state [37, 38]:

$$\mathcal{F}_{c,\Delta}(\Lambda) = \langle\, \Delta, \Lambda^2 \,|\, \Delta, \Lambda^2 \,\rangle = \sum_{n=0}^{\infty} \Lambda^{4n} \left[G^n_{c,\Delta}\right]^{(1^n)(1^n)} \tag{3.3}$$

$$= 1 + \Lambda^4 \,\frac{1}{2\Delta} + \Lambda^8 \,\frac{c + 8\Delta}{4\Delta(2c\Delta + c + 2\Delta(8\Delta - 5))} \tag{3.4}$$

$$+\Lambda^{12}\,\frac{(11c - 26)\Delta + c(c+8) + 24\Delta^2}{24\Delta\left((c-7)\Delta + c + 3\Delta^2 + 2\right)\left(2(c-5)\Delta + c + 16\Delta^2\right)} + \ldots\,. \tag{3.5}$$

In fact, there are much more Gaiotto's states and therefore irregular blocks.[13] In the present paper we confine ourselves to study irregular blocks which are built out of (3.1). Possible extensions of the present work taking into account the existence of the other Gaiotto states will be discussed soon in a forthcoming publication.[14]

Let $C_{g,n}$ denotes a Riemann surface with genus g and n punctures. Let x be the modular parameter of the 4-punctured Riemann sphere $C_{0,4}$. Then, the s-channel conformal block on $C_{0,4}$ is defined as the following formal x-expansion:

$$\mathcal{F}_{c,\Delta}\left[\begin{smallmatrix}\Delta_2 & \Delta_3\\ \Delta_1 & \Delta_4\end{smallmatrix}\right](x) = x^{\Delta-\Delta_3-\Delta_4}\left(1 + \sum_{n=1}^{\infty} x^n\, \mathcal{F}^n_{c,\Delta}\left[\begin{smallmatrix}\Delta_2 & \Delta_3\\ \Delta_1 & \Delta_4\end{smallmatrix}\right]\right), \tag{3.6}$$

where

$$\begin{aligned}\mathcal{F}^n_{c,\Delta}\left[\begin{smallmatrix}\Delta_2 & \Delta_3\\ \Delta_1 & \Delta_4\end{smallmatrix}\right] &= \sum_{|I|=|J|=n} \rho^{\Delta_1\,\Delta_2\,\Delta}_{\infty\;\,1\;\,0}(\nu_{\Delta_1}, \nu_{\Delta_2}, \nu_{\Delta,I}) \left[G^n_{c,\Delta}\right]^{IJ} \rho^{\Delta\,\Delta_3\,\Delta_4}_{\infty\;1\;\,0}(\nu_{\Delta,J}, \nu_{\Delta_3}, \nu_{\Delta_4})\\ &= \sum_{|I|=|J|=n} \gamma_\Delta\left[\begin{smallmatrix}\Delta_2\\ \Delta_1\end{smallmatrix}\right]_I \left[G^n_{c,\Delta}\right]^{IJ} \gamma_\Delta\left[\begin{smallmatrix}\Delta_3\\ \Delta_4\end{smallmatrix}\right]_J\,.\end{aligned} \tag{3.7}$$

Let $q = \mathrm{e}^{2\pi i\tau}$ be the elliptic variable on the torus with modular parameter τ, then the conformal block on $C_{1,1}$ is given by the following formal q-series:

$$\mathcal{F}^{\tilde{\Delta}}_{c,\Delta}(q) = q^{\Delta - \frac{c}{24}}\left(1 + \sum_{n=1}^{\infty} \mathcal{F}^{\tilde{\Delta},n}_{c,\Delta}\, q^n\right),$$

[13] See for instance [65] and refs. therein.
[14] Cf. conclusions.

where

$$\mathcal{F}_{c,\Delta}^{\tilde{\Delta},n} = \sum_{|I|=|J|=n} \rho_{\infty\ 1\ 0}^{\Delta\ \tilde{\Delta}\ \Delta}(\nu_{\Delta,I}, \nu_{\tilde{\Delta}}, \nu_{\Delta,J}) \left[G_{c,\Delta}^{n}\right]^{IJ}.$$

The irregular block (3.3) can be recovered from the conformal blocks on the torus and on the sphere in a properly defined decoupling limit of the external conformal weights [38, 39]. Indeed, employing the AGT inspired parametrization of the external weights $\tilde{\Delta}$, Δ_i and the central charge c, i.e.:

$$\tilde{\Delta} = \frac{M\,(\epsilon - M)}{\epsilon_1\epsilon_2}, \qquad \Delta_i = \frac{\alpha_i(\epsilon - \alpha_i)}{\epsilon_1\epsilon_2}, \qquad c = 1 + 6\frac{\epsilon^2}{\epsilon_1\epsilon_2}, \qquad \epsilon = \epsilon_1 + \epsilon_2\ ,$$
$$\alpha_1 = \frac{1}{2}\,(\epsilon + \mu_1 - \mu_2)\,, \quad \alpha_2 = \frac{1}{2}\,(\mu_1 + \mu_2)\,, \quad \alpha_3 = \frac{1}{2}\,(\mu_3 + \mu_4)\,, \quad \alpha_4 = \frac{1}{2}\,(\epsilon + \mu_3 - \mu_4)\,,$$

and introducing the dimensionless expansion parameter $\Lambda = \hat{\Lambda}/(-\epsilon_1\epsilon_2)^{\frac{1}{2}}$ it is possible to prove the following limits [38, 39]:

$$q^{\frac{c}{24}-\Delta}\,\mathcal{F}_{c,\Delta}^{\tilde{\Delta}}(q) \quad \xrightarrow[qM^4=\hat{\Lambda}^4]{M\to\infty} \quad \mathcal{F}_{c,\Delta}(\Lambda),$$
$$x^{\Delta_3+\Delta_4-\Delta}\,\mathcal{F}_{c,\Delta}\left[\begin{smallmatrix}\Delta_2 & \Delta_3\\ \Delta_1 & \Delta_4\end{smallmatrix}\right](x) \;\xrightarrow[x\mu_1\mu_2\mu_3\mu_4=\hat{\Lambda}^4]{\mu_1,\mu_2,\mu_3,\mu_4\to\infty}\; \mathcal{F}_{c,\Delta}(\Lambda). \tag{3.8}$$

Due to the 'non-conformal' AGT relation, the $N_f = 0$ irregular block can be expressed through the SU(2) pure gauge Nekrasov instanton partition function [22, 37, 40, 42]:

$$\mathcal{F}_{c,\Delta}(\Lambda) = \mathcal{Z}_{\text{inst}}^{\text{SU(2)},N_f=0}(\hat{\Lambda}, a, \epsilon_1, \epsilon_2). \tag{3.9}$$

The identity (3.9), which in particular is understood as term by term equality between the coefficients of the expansions of both sides, holds for

$$\Lambda = \frac{\hat{\Lambda}}{\sqrt{-\epsilon_1\epsilon_2}}, \qquad \Delta = \frac{\epsilon^2 - 4a^2}{4\epsilon_1\epsilon_2}, \qquad c = 1 + 6\frac{\epsilon^2}{\epsilon_1\epsilon_2} \equiv 1 + 6\mathsf{Q}^2 \tag{3.10}$$

where

$$\mathsf{Q} \;=\; b + \frac{1}{b} \;\equiv\; \sqrt{\frac{\epsilon_2}{\epsilon_1}} + \sqrt{\frac{\epsilon_1}{\epsilon_2}} \quad \Leftrightarrow \quad b = \sqrt{\frac{\epsilon_2}{\epsilon_1}}. \tag{3.11}$$

In [25] it was observed that in the limit $\epsilon_2 \to 0$ the Nekrasov partition functions $\mathcal{Z}_{\text{Nekrasov}} = \mathcal{Z}_{\text{pert}}\mathcal{Z}_{\text{inst}}$ behave exponentially. In particular, for the instantonic sector we have

$$\mathcal{Z}_{\text{inst}}(\,\cdot\,, \epsilon_1, \epsilon_2) \;\overset{\epsilon_2\to 0}{\sim}\; \exp\left\{\frac{1}{\epsilon_2}\,W_{\text{inst}}(\,\cdot\,, \epsilon_1)\right\}. \tag{3.12}$$

Therefore, taking into account the AGT relation (3.9), the fact that $b = (\frac{\epsilon_2}{\epsilon_1})^{\frac{1}{2}}$ and the Nekrasov-Shatashvili limit (3.12) of the instanton function, one can expect that the irregular block has the following exponential behavior in the limit $b \to 0$:

$$\mathcal{F}_{1+6Q^2,\Delta}(\Lambda) \;\overset{b\to 0}{\sim}\; \exp\left\{\frac{1}{b^2}\,f_{\delta}^{\mathbf{0}}\left(\hat{\Lambda}/\epsilon_1\right)\right\} \quad , \tag{3.13}$$

where $\Delta = \frac{1}{b^2}\delta$, $\delta = \mathcal{O}(b^0)$. The semiclassical asymptotic behavior (3.13) is a very nontrivial statement concerning the quantum $N_f = 0$ irregular block.[15] First, the existence of the *classical zero flavor irregular block* $f_\delta^{\mathbf{0}}\left(\hat{\Lambda}/\epsilon_1\right)$ can be checked by direct calculation. Indeed, from the power expansion of the quantum irregular block (3.3) and eq. (3.13) one finds

$$f_\delta^{\mathbf{0}}\left(\hat{\Lambda}/\epsilon_1\right) \;=\; \lim_{b\to 0} b^2 \log \mathcal{F}_{1+6Q^2,\frac{1}{b^2}\delta}\left(\hat{\Lambda}/(\epsilon_1 b)\right) \;=\; \sum_{n=1}^{\infty} \left(\hat{\Lambda}/\epsilon_1\right)^{4n} f_\delta^{\mathbf{0},n}, \tag{3.14}$$

where the coefficients $f_\delta^{\mathbf{0},n}$ up to $n = 6$ take the form:

$$\begin{aligned}
f_\delta^{\mathbf{0},1} &= \frac{1}{2\delta},\; f_\delta^{\mathbf{0},2} = \frac{5\delta - 3}{16\delta^3(4\delta+3)},\; f_\delta^{\mathbf{0},3} = \frac{9\delta^2 - 19\delta + 6}{48\delta^5\,(4\delta^2 + 11\delta + 6)},\\
f_\delta^{\mathbf{0},4} &= \frac{5876\,\delta^5 - 16489\,\delta^4 - 22272\,\delta^3 + 17955\,\delta^2 + 9045\,\delta - 4050}{512\,\delta^7(\delta+2)(4\delta+3)^3(4\delta+15)},\\
f_\delta^{\mathbf{0},5} &= \frac{17884\,\delta^6 - 96187\,\delta^5 - 156432\,\delta^4 + 388737\,\delta^3 - 7317\,\delta^2 - 138348\,\delta + 34020}{1280\,\delta^9(\delta+2)(\delta+6)(4\delta+3)^3(4\delta+15)},\\
f_\delta^{\mathbf{0},6} &= \big[7756224\,\delta^{11} - 19228160\,\delta^{10} - 456215812\,\delta^9\\
&\quad - 971240994\,\delta^8 + 1505016987\,\delta^7 + 5076827496\,\delta^6\\
&\quad + 930371157\,\delta^5 - 4398704919\,\delta^4 - 1494083556\,\delta^3\\
&\quad + 1212636096\,\delta^2 + 293932800\,\delta - 128595600\big]\\
&\quad \times \big[6144\,\delta^{11}(\delta+2)^3(\delta+6)(4\delta+3)^5(4\delta+15)(4\delta+35)\big]^{-1}.
\end{aligned} \tag{3.15}$$

As a further consistency check of our approach, let us observe that combining (3.9)–(3.11) and (3.13) it is possible to identify the classical irregular block with the SU(2) $N_f = 0$ effective twisted superpotential:

$$f_\delta^{\mathbf{0}}\left(\hat{\Lambda}/\epsilon_1\right) = \frac{1}{\epsilon_1}\, W_{\rm inst}^{{\rm SU}(2),\,N_f=0}\left(\hat{\Lambda}, a, \epsilon_1\right), \tag{3.16}$$

where $\delta = \frac{1}{4} - \frac{a^2}{\epsilon_1^2}$. We stress that the *classical conformal weight* δ in eq. (3.16) above is expressed in terms of the gauge theory parameters a, ϵ_1. Indeed, it is easy to see that

$$\delta = \lim_{b\to 0} b^2 \Delta = \lim_{\epsilon_2\to 0} \frac{\epsilon_2}{\epsilon_1}\Delta = \frac{1}{4} - \frac{a^2}{\epsilon_1^2}.$$

By comparison of the expansion (3.14)–(3.15) with that of the twisted superpotential obtained independently from the instanton partition function, the identity (3.16) may be confirmed up to desired order.

3.2 Towards a proof of the classical limit

The equations (3.15) constitute a direct premise for the existence of the classical irregular conformal block. The rigorous proof of this statement, however, has not yet been performed,

[15] Cf. considerations in subsection 3.2 and conclusions of the present paper.

although there are many convincing arguments in favor of its validity.[16] In what follows we discuss the leading order of the coefficients of the quantum irregular block and extend the discussion beyond the leading order to provide yet more arguments for the existence of the classical irregular block.

Classical irregular block at the leading order. In order to find the leading contribution to the classical irregular block we examine the coefficients of the expansion of the quantum irregular block. Since these are functions of the matrix elements of the Virasoro algebra, we analyze their dependence on c and Δ to find out how they scale with respect to b within the classical limit.

The quantum irregular block can be rewritten explicitly as

$$\mathcal{F}_{c,\Delta}(\Lambda) = \sum_{n\geq 0} \left(\frac{\hat{\Lambda}}{\epsilon_1 b}\right)^{4n} \left(G^{(n)}_{c,\Delta}\right)^{(1^n)\,(1^n)} = \sum_{n\geq 0} \left(\frac{\hat{\Lambda}}{\epsilon_1 b}\right)^{4n} \frac{M_{p(n),p(n)}(\Delta, c)}{\det G^{(n)}_{c,\Delta}}, \tag{3.17}$$

where $M_{p(n),p(n)}(\Delta, c)$ is the greatest principal minor of the Kac determinant at the level n $(|\Delta\rangle \equiv |\nu_\Delta\rangle)$[17]

$$\det G^{(n)}_{c,\Delta} = \det\left(\langle\Delta|L_{I_u} L_{-I_v}|\Delta\rangle\right), \qquad I_u, I_v \vdash n, \quad u, v \in \{1, \ldots, p(n)\}. \tag{3.18}$$

As it was mentioned earlier the matrix elements $\langle\Delta|L_{I_u} L_{-I_v}|\Delta\rangle$ are polynomials in Δ, and c. Although in the classical limit both parameters are important we restrict our attention to Δ dependence. The reason is that, due to Virasoro algebra (2.2), Δ appears as a factor in a matrix element $\langle\Delta|L_{I_u} L_{-I_v}|\Delta\rangle$ either additively accompanied by c or alone, which takes place when I_u and I_v have part one in common with nonzero multiplicity. In order to find the greatest power of Δ in general matrix element at level n we take advantage of the argument used by Kac and Raina in ref. [63]. Let us first consider the diagonal matrix element. Making use of the following notation: $\ell_u := \ell(I_u)$,

$$I_u = (k_1(I_u), \ldots, k_{\ell_u}(I_u), 0, \ldots) = (1^{m_1(I_u)}\, 2^{m_2(I_u)} \ldots), \;\; |I_u| = \sum_{s\geq 1} k_s(I_u) = \sum_{i\geq 1} i\, m_i(I_u) = n,$$

as well as Virasoro algebra (2.2) we obtain for the arbitrary diagonal matrix element

$$\begin{aligned}\langle\Delta|L_{I_u} L_{-I_u}|\Delta\rangle &= \langle\Delta|L_{i_l}^{m_{i_l}(I_u)} \ldots L_{i_2}^{m_{i_2}(I_u)} L_{i_1}^{m_{i_1}(I_u)} L_{-i_1}^{m_{i_1}(I_u)} L_{-i_2}^{m_{i_2}(I_u)} \ldots L_{-i_l}^{m_{i_l}(I_u)}|\Delta\rangle \\ &\overset{\deg}{\sim} \prod_{s=1}^{l} \langle\Delta|L_{i_s}^{m_{i_s}(I_u)} L_{-i_s}^{m_{i_s}(I_u)}|\Delta\rangle = \prod_{i=1}^{l} \langle\Delta|L_{i}^{m_{i}(I_u)} L_{-i}^{m_{i}(I_u)}|\Delta\rangle,\end{aligned} \tag{3.19}$$

which, in accord with the formula

$$\begin{aligned}\underbrace{[L_i, \ldots, [L_i}_{m}, L_{-i}^m] \ldots] &= \left(2L_0 + \frac{c}{12}(i^2-1); i\right)_m, \\ (a;k)_n := \prod_{i=0}^{n-1}(a+ki), &\qquad (a;1)_n = (a)_n := \Gamma(a+n)/\Gamma(a),\end{aligned} \tag{3.20}$$

[16] See the discussion in the conclusions on this point.

[17] We use conventions as in eq. (2.3). $I \vdash n$ means that I is a partition of n.

amounts to

$$\langle\Delta|L_{I_u}L_{-I_u}|\Delta\rangle \overset{\text{deg}}{\sim} \prod_{i\geq 1} i^{m_i(I_u)} m_i(I_u)! \left(2\Delta + \frac{c}{12}(i^2-1); i\right)_{m_i(I_u)}. \tag{3.21}$$

The symbol '$\overset{\text{deg}}{\sim}$' indicates that a polynomial on the left hand side and the one on the right hand side are equal up to the term with the greatest power in variable Δ which, by definition, determines its degree. Let us now consider arbitrary off-diagonal term $\langle\Delta|L_{I_u}L_{-I_v}|\Delta\rangle$. Let us assume that both partitions have, say, N parts $\{i_s\}_{s=1}^N$ in common with nonzero multiplicities $m_{i_s}(I_u)$ and $m_{i_s}(I_v)$. Then, repeating the above reasoning, one finds that the general off-diagonal element takes the form

$$\langle\Delta|L_{I_u}L_{-I_v}|\Delta\rangle = \langle\Delta|L_{i_p}^{m_{i_p}(I_u)}\cdots L_{i_N}^{m_{i_N}(I_u)}\cdots L_{i_1}^{m_{i_1}(I_u)} L_{-i_1}^{m_{i_1}(I_v)}\cdots L_{-i_N}^{m_{i_N}(I_v)}\cdots L_{-i_q}^{m_{i_q}(I_v)}|\Delta\rangle.$$

Using the generalized formula in eq. (3.20) for $m \leq n$

$$\begin{aligned}\underbrace{[L_i,\ldots,[L_i}_{m},L_{-i}^n]\ldots] &= L_{-i}^{n-m} i^m \frac{n!}{(n-m)!}\prod_{s=n-m}^{n-1}\left(2L_0+\frac{c}{12}(i^2-1)+is\right)\\ &= L_{-i}^{n-m} i^m \frac{n!}{(n-m)!}\frac{\left(2L_0+\frac{c}{12}(i^2-1);i\right)_n}{\left(2L_0+\frac{c}{12}(i^2-1);i\right)_{n-m}},\end{aligned} \tag{3.22}$$

we get

$$\begin{aligned}&\langle\Delta|L_{I_u}L_{-I_v}|\Delta\rangle\\ &\quad\overset{\text{deg}}{\sim} \prod_{i\geq 1} i^{\min\{m_i(I_u),m_i(I_v)\}} \frac{m_i(I_v)!}{\vartheta\big(m_i(I_v)-m_i(I_u)\big)!}\frac{\left(2\Delta+\frac{c}{12}(i^2-1);i\right)_{m_i(I_v)}}{\left(2\Delta+\frac{c}{12}(i^2-1);i\right)_{\vartheta(m_i(I_v)-m_i(I_u))}},\end{aligned} \tag{3.23}$$

where $\vartheta(x) = x\theta(x)$ and $\theta(x)$ is Heaviside's theta function. These general results in eqs. (3.21) and (3.23) allows us to draw the following conclusions for the matrix elements as a polynomials in Δ and c. For any $I_u, I_v \vdash n$

1. $\deg_\Delta\langle\Delta|L_{I_u}L_{-I_v}|\Delta\rangle \leq \min\{\ell_u,\ell_v\}$,
2. $\deg_\Delta\langle\Delta|L_{I_u}L_{-I_u}|\Delta\rangle = \ell_u$,
3. for $u \neq v$ and $\ell_u = \ell_v$, $\deg_\Delta\langle\Delta|L_{I_u}L_{-I_v}|\Delta\rangle < \ell_u$.

Since the degree of a matrix element as a polynomial in Δ depends on the length of partition those of greatest degree yield the leading contribution to both $M_{p(n),p(n)}$ and $\det G_{\Delta,c}^{(n)}$ within the classical limit. In the following discussion the explicit form of particular matrix elements prove useful

$$\begin{aligned}\langle\Delta|L_1^nL_{-1}^n|\Delta\rangle &= n!(2\Delta)_n,\\ \langle\Delta|L_{I_u}L_{-1}^n|\Delta\rangle &= n!\prod_{i\geq 1}\Big((i+1)\Delta+\sum_{s>i} s\, m_s(I_u);i\Big)_{m_i(I_u)}.\end{aligned} \tag{3.24}$$

The above results enable to conclude that ($L_{-1}^n = L_{-(1^n)} = L_{-I_{p(n)}}$)

$$\deg_\Delta \langle \Delta | L_{I_u} L_{-I_u} | \Delta \rangle = \deg_\Delta \langle \Delta | L_{I_u} L_{-I_{p(n)}} | \Delta \rangle. \tag{3.25}$$

Moreover, let us note that the minor produced by crossing out the i^{th} diagonal element when treated as a polynomial in Δ and c has the same coefficient in the highest degree term as the product of diagonal elements of the Gram matrix, namely

$$M_{u,u}(\Delta, c) \overset{\deg}{\sim} \frac{\prod_{j=1}^{p(n)} \langle \Delta, c | L_{I_j} L_{-I_j} | \Delta, c \rangle}{\langle \Delta, c | L_{I_u} L_{-I_u} | \Delta, c \rangle}, \tag{3.26}$$

With these results at hand we can proceed to estimate the contribution to the classical conformal block at the leading order within the limit $b \to 0$. In order to do this we expand the Kac determinant along the $p(n)^{\text{th}}$ row

$$\frac{\det G_{c,\Delta}^{(n)}}{M_{p(n),p(n)}(\Delta, c)} = \langle \Delta | L_1^n L_{-1}^n | \Delta \rangle + \sum_{u=1}^{p(n)-1} (-1)^{p(n)+u} \frac{M_{u,p(n)}(\Delta, c)}{M_{p(n),p(n)}(\Delta, c)} \langle \Delta | L_{I_u} L_{-1}^n | \Delta \rangle. \tag{3.27}$$

Let us observe that in view of our analysis concerning matrix elements as polynomials in Δ the leading contribution of the latter to $M_{u,p(n)}$ can be found as follows. By means of the elementary operations on columns we obtain

$$\begin{aligned} M_{u,p(n)} &= \det(c_1, \ldots, c_{u-1}, c_{u+1}, \ldots, c_{p(n)-1}, c_{p(n)}) \\ &= (-1)^{p(n)-u} \det(c_1, \ldots, c_{u-1}, c_{p(n)}, c_{u+1}, \ldots, c_{p(n)-1}) \\ &= (-1)^{p(n)-u} \tilde{M}_{u,p(n)}, \end{aligned} \tag{3.28}$$

where $c_u \equiv \{\langle \Delta | L_{I_j} L_{-I_u} | \Delta \rangle\}_{j=1}^{p(n)-1}$ denotes i^{th} column of $G^{(n)}$ with the last entry removed. Using eq. (3.26) we find that

$$\tilde{M}_{u,p(n)} \overset{\deg}{\sim} \frac{\langle \Delta | L_{I_u} L_{-1}^n | \Delta \rangle}{\langle \Delta | L_{I_u} L_{-I_u} | \Delta \rangle} M_{p(n),p(n)}, \tag{3.29}$$

which, when placed in eq. (3.28), yields the leading contribution to $M_{u,p(n)}$. Combining eqs. (3.28), (3.29) and (3.27) we obtain

$$\frac{\det G_{c,\Delta}^{(n)}}{M_{p(n),p(n)}(\Delta, c)} \overset{\deg}{\sim} \langle \Delta | L_1^n L_{-1}^n | \Delta \rangle + \sum_{u=1}^{p(n)-1} \frac{\langle \Delta | L_{I_u} L_{-1}^n | \Delta \rangle}{\langle \Delta | L_{I_u} L_{-I_u} | \Delta \rangle} \langle \Delta | L_{I_u} L_{-1}^n | \Delta \rangle. \tag{3.30}$$

Within the classical limit $\Delta \to \infty$, $c \to \infty$, $c/\Delta = \text{const.}$ for $b \to 0$ the conformal weight and the central charge scale as $\Delta \sim \delta/b^2$ and $c \sim 6/b^2$. From eqs. (3.24) and (3.19) we infer that

$$\langle \Delta | L_1^n L_{-1}^n | \Delta \rangle \overset{b \to 0}{\sim} b^{-2n} n! (2\delta)^n, \qquad \langle \Delta | L_{I_u} L_{-1}^n | \Delta \rangle \overset{b \to 0}{\sim} b^{-2\ell_u} n! \prod_{i \geq 1} \big((i+1)\delta\big)^{m_i(I_u)},$$

$$\langle \Delta | L_{I_u} L_{-I_u} | \Delta \rangle \overset{b \to 0}{\sim} b^{-2\ell_u} 2^{-\ell_u} \prod_{i \geq 1} m_i(I_u)! \, i^{m_i(I_u)} \big(4\delta + i^2 - 1\big)^{m_i(I_u)},$$

and

$$\frac{\langle\Delta|L_{I_u}L_{-1}^n|\Delta\rangle}{\langle\Delta|L_{I_u}L_{-I_u}|\Delta\rangle} \overset{b\to 0}{\sim} \frac{n!}{\prod\limits_{i\geq 1} m_i(I_u)!}\prod_{i\geq 1}\left(\frac{2(i+1)\delta}{i\,(4\delta+i^2-1)}\right)^{m_i(I_u)}. \tag{3.31}$$

Hence, the formula in eq. (3.30) within the classical limit amounts to

$$\frac{\det G_{c,\Delta}^{(n)}}{M_{p(n),p(n)}(\Delta,c)} \overset{b\to 0}{\sim} n!(2\delta)^n b^{-2n}+n!\sum_{u=1}^{p(n)-1}(2\delta)^{\ell_u}b^{-2\ell_u}\frac{n!}{\prod\limits_{i\geq 1} m_i(I_u)!}\prod_{i\geq 1}\left(\frac{2(i+1)^2\delta}{i\,(4\delta+i^2-1)}\right)^{m_i(I_u)}.$$

Since $n=\ell_{\max}$ then it is seen that the first term dominates over the rest for $b\to 0$. Therefore, the coefficient of the irregular block in eq. (3.17) within this limit reads

$$\left(G_{c,\Delta}^{(n)}\right)^{(1^n)\,(1^n)} = \frac{M_{p(n),p(n)}(\Delta,c)}{\det G_{c,\Delta}^{(n)}} \overset{b\to 0}{\sim} \frac{b^{2n}}{n!(2\delta)^n}, \tag{3.32}$$

This, in accord with eq. (3.14), yields the first coefficient $f_\delta^{\mathbf{0},1}$ of the classical irregular block expansion given in eq. (3.15), namely

$$\mathcal{F}_{c,\Delta}(\Lambda)=\sum_{n\geq 0}\left(\frac{\hat\Lambda}{\epsilon_1 b}\right)^{4n}\left(G_{c,\Delta}^{(n)}\right)^{(1^n)\,(1^n)} \overset{b\to 0}{\sim} \sum_{n\geq 0}\left(\frac{\hat\Lambda}{\epsilon_1 b}\right)^{4n}\frac{b^{2n}}{n!(2\delta)^n}=\exp\left\{\frac{1}{b^2}\left(\frac{\hat\Lambda}{\epsilon_1}\right)^4\frac{1}{2\delta}\right\}.$$

Classical irregular block beyond the leading order. The above computations show that in the estimations of the quantum irregular block coefficients based on the leading order contribution all but the first term in the classical irregular block expansion are neglected. Therefore a more accurate analysis is required. In general the sought expression takes the form

$$f_\delta^{\mathbf{0}}\left(\hat\Lambda/\epsilon_1\right)=\lim_{b\to 0} b^2\log\left[1+\sum_{n\geq 1}\left(\frac{\hat\Lambda}{\epsilon_1 b}\right)^{4n}\mathcal{F}^{(n)}(\Delta,c)\right]. \tag{3.33}$$

where for the sake of brevity we have introduced notation $\mathcal{F}^{(n)}\equiv\left(G^{(n)}\right)^{(1^n)\,(1^n)}$. The logarithm in eq. (3.33) has the following expansion

$$\begin{aligned}
&\log\left[1+\sum_{n\geq 1}\left(\frac{\hat\Lambda}{\epsilon_1 b}\right)^{4n}\mathcal{F}^{(n)}(\Delta,c)\right]\\
&\quad=\sum_{n\geq 1}\left(\frac{\hat\Lambda}{\epsilon_1}\right)^{4n}\sum_{\substack{\{m_i\}\geq 0\\ \sum i m_i=n}}(-1)^{\sum m_i+1}\left(\sum m_i-1\right)!\prod_{i\geq 1}\frac{\left[b^{-4i}\mathcal{F}^{(i)}(\delta/b^2,6/b^2)\right]^{m_i}}{m_i!}.
\end{aligned} \tag{3.34}$$

In order to find the limit of the above coefficient of $\hat\Lambda/\epsilon_1$ the knowledge of $\mathcal{F}^{(i)}$ is necessary. Unfortunately the exact form of $M_{p(n),p(n)}$ as a polynomial in Δ and c is not known and it is necessary to compute it term by term which is the major obstacle in finding the limit. In order for the limit in eq. (3.33) to exist each coefficient should be proportional to b^2. Thus the complete rigorous proof of the mentioned limit is still an open problem which in order to be solved must be attacked in fact by another methods.[18]

[18] Cf. conclusions.

3.3 The null vector decoupling equations

In this subsection we shall derive the partial differential equations obeyed by the $N_f = 0$ *degenerate irregular blocks*, cf. [15]. We define the latters as matrix elements of the degenerate chiral vertex operators $V_{\pm}(z) = V(|\nu_{\Delta_\pm}\rangle|z)$ between the states (3.1):

$$\begin{aligned}\Psi^{\mathbf{0}}_{\pm}(\Lambda, z) &= \langle\, \Delta', \Lambda^2\, | V_{\pm}(z) |\, \tilde{\Delta}, \Lambda^2\, \rangle \\ &= \rho^{\Delta' \Delta_\pm \tilde{\Delta}}_{\infty\ z\ \ 0}\left(|\Delta', \Lambda^2\rangle\, ,\, |\nu_{\Delta_\pm}\rangle\, ,\, |\tilde{\Delta}, \Lambda^2\rangle \right).\end{aligned} \tag{3.35}$$

In the above equation:

$$\Delta_+ \equiv \Delta_{21} = -\frac{3}{4}b^2 - \frac{1}{2}, \qquad \Delta_- \equiv \Delta_{12} = -\frac{3}{4b^2} - \frac{1}{2}.$$

Moreover, in order to apply the null vector decoupling theorem we will assume that the weights $\Delta_1 \equiv \tilde{\Delta}$ and $\Delta_3 \equiv \Delta'$ of the *in* and *out* states are related by the fusion rule:[19]

$$\Delta_1 \equiv \tilde{\Delta} = \Delta\!\left(\sigma - \frac{b}{4}\right), \qquad \Delta_3 \equiv \Delta' = \Delta\!\left(\sigma + \frac{b}{4}\right), \quad \text{where} \quad \Delta(\sigma) \equiv \frac{\mathsf{Q}^2}{4} - \sigma^2. \tag{3.36}$$

Let us consider the descendant chiral vertex operator

$$\chi_+(z) = \left(\widehat{L}_{-2}(z) - \frac{3}{2(2\Delta_+ + 1)}\,\widehat{L}^2_{-1}(z)\right) V_+(z) \equiv V\!\left(\left(L_{-2} + \frac{1}{b^2}L^2_{-1}\right)|\,\nu_{\Delta_+}\,\rangle\,|\,z\right) \tag{3.37}$$

which corresponds to the null vector

$$|\,\chi_+\,\rangle = \chi_+(0)|\,0\,\rangle = \left(L_{-2} + \frac{1}{b^2}L^2_{-1}\right)|\,\nu_{\Delta_+}\,\rangle$$

appearing at the second level of the Verma module $\mathcal{V}_{\Delta_+}$. Then, by the NVD theorem, we have that

$$\langle\, \Delta', \Lambda^2\, |\, \chi_+(z)\, |\, \tilde{\Delta}, \Lambda^2\, \rangle \;=\; \rho^{\Delta' \Delta_+ \tilde{\Delta}}_{\infty\ z\ \ 0}\left(|\Delta', \Lambda^2\rangle\, ,\, |\,\chi_+\,\rangle\, ,\, |\tilde{\Delta}, \Lambda^2\rangle \right) \;=\; 0. \tag{3.38}$$

In order to convert eq. (3.38) to the PDE obeyed by the degenerate irregular block $\Psi^{\mathbf{0}}_{+}(\Lambda, z)$, one needs to employ the following Ward identity:

$$\begin{aligned}\langle\, \Delta', \Lambda^2\, |\, T(w) V_+(z)\, |\, \tilde{\Delta}, \Lambda^2\, \rangle = \Bigg[&\frac{z}{w(w-z)}\frac{\partial}{\partial z} + \frac{\Delta_+}{(w-z)^2} + \left(\frac{\Lambda^2}{w} + \frac{\Lambda^2}{w^3}\right) \\ &+ \frac{1}{2w^2}\left(\frac{\Lambda}{2}\frac{\partial}{\partial \Lambda} + \tilde{\Delta} + \Delta' - \Delta_+ - z\frac{\partial}{\partial z}\right)\Bigg]\,\Psi^{\mathbf{0}}_{+}(\Lambda, z)\,.\end{aligned} \tag{3.39}$$

Using the formula [51]:

$$\widehat{L}_{-k}(z) \;=\; \frac{1}{2\pi i}\oint\limits_{C_z} dw (w-z)^{1-k}\, T(w)$$

[19] In the parameterization $\Delta_{\beta_i} = \frac{1}{24}(c-1) + \frac{1}{4}\beta_i$ (see also (2.6)) used in the NVD theorem the fusion rule (3.36) reads as follows: $\Delta_{\beta_3} = \Delta_{\beta_1 - \beta_+}$. In another commonly used parametrization, in which $\Delta(\alpha_i) = \alpha_i(\mathsf{Q} - \alpha_i)$, we have $\Delta(\alpha_3) = \Delta(\alpha_1 + \frac{b}{2})$.

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it is now possible to compute the matrix element $\langle\, \Delta', \Lambda^2 \,|\, \widehat{L}_{-2}(z) V_+(z) \,|\, \tilde{\Delta}, \Lambda^2 \,\rangle$ with the help of eq. (3.39). Applying Cauchy's integral formula one finds that

$$\begin{aligned}\langle\, \Delta', \Lambda^2 \,|\, \widehat{L}_{-2}(z) V_+(z) \,|\, \tilde{\Delta}, \Lambda^2 \,\rangle = \Bigg[& -\frac{1}{z}\frac{\partial}{\partial z} + \left(\frac{\Lambda^2}{z} + \frac{\Lambda^2}{z^3} \right) \\ & + \frac{1}{2z^2}\left(\frac{\Lambda}{2}\frac{\partial}{\partial \Lambda} + \tilde{\Delta} + \Delta' - \Delta_+ - z\frac{\partial}{\partial z} \right) \Bigg] \Psi_+^{\mathbf{0}}(\Lambda, z)\,. \end{aligned} \tag{3.40}$$

Finally, taking into account that the matrix element of the descendant operator $\widehat{L}_{-1}^2(z)V_+(z)$ yields $\partial_z^2 \Psi_+^{\mathbf{0}}(\Lambda, z)$, we get from (3.37), (3.38) and (3.40) the desired partial differential equation, determining $\Psi_+^{\mathbf{0}}(\Lambda, z)$:

$$\left[\frac{1}{b^2}\, z^2 \frac{\partial^2}{\partial z^2} - \frac{3z}{2}\frac{\partial}{\partial z} + \Lambda^2 \left(z + \frac{1}{z} \right) + \frac{\Lambda}{4}\frac{\partial}{\partial \Lambda} + \frac{\tilde{\Delta} + \Delta' - \Delta_+}{2} \right] \Psi_+^{\mathbf{0}}(\Lambda, z) \;=\; 0\,. \tag{3.41}$$

Replacing Δ_+ with Δ_- and repeating all the steps described above one can get an analogous equation for $\Psi_-^{\mathbf{0}}(\Lambda, z)$. In the next section we will consider the limit $b \to 0$ of eq. (3.41). A part of this analysis has been already done in our previous work [1]. The new result here is the derivation from the degenerate zero flavor irregular block of the formula for the eigenfunction of the Mathieu operator.

4 The classical irregular block and the spectrum of the Mathieu operator

4.1 The classical limit of the null vector decoupling equation

Let us turn for a while to the zero flavor degenerate irregular block introduced in eq. (3.35). From (3.1) and (2.12) we have that

$$\begin{aligned}\Psi_+^{\mathbf{0}}(\Lambda, z) \;&=\; \langle\, \Delta', \Lambda^2 \,|V_+(z)|\, \tilde{\Delta}, \Lambda^2 \,\rangle \;=\; z^{\Delta' - \Delta_+ - \tilde{\Delta}} \sum_{r,s \geq 0} \Lambda^{2(r+s)} z^{r-s} \\ & \quad \times \sum_{|I|=r} \sum_{|J|=s} \left[G_{c,\Delta'}^r \right]^{(1^r) I} \rho_{\infty\; 1\; 0}^{\Delta' \Delta_+ \tilde{\Delta}} \left(\nu_{\Delta',I}, \nu_{\Delta_+}, \nu_{\tilde{\Delta},J} \right) \left[G_{c,\tilde{\Delta}}^s \right]^{J (1^s)} \qquad (4.1\text{a}) \\ & \equiv\; z^{\kappa}\, \Phi_+^{\mathbf{0}}(\Lambda, z), \qquad (4.1\text{b}) \end{aligned}$$

where $\kappa \equiv \Delta' - \Delta_+ - \tilde{\Delta}$. Let us observe that $\Phi_+^{\mathbf{0}}(\Lambda, z)$ can be split into two parts, i.e. when $r = s$ and $r \neq s$: $\Phi_+^{\mathbf{0}}(\Lambda, z) = \Phi_{r=s}^{\mathbf{0}}(\Lambda) + \Phi_{r \neq s}^{\mathbf{0}}(\Lambda, z)$, where

(*i*) for $r = s$,

$$\Phi_{r=s}^{\mathbf{0}}(\Lambda) = \sum_{r \geq 0} \Lambda^{4r} \sum_{|I|=|J|=r} \left[G_{c,\Delta'}^r \right]^{(1^r)\, I} \rho_{\infty\; 1\; 0}^{\Delta' \Delta_+ \tilde{\Delta}} \left(\nu_{\Delta',I}, \nu_{\Delta_+}, \nu_{\tilde{\Delta},J} \right) \left[G_{c,\tilde{\Delta}}^r \right]^{J\, (1^r)}, \tag{4.2a}$$

(ii) for $r \neq s$,

$$\Phi^{\mathbf{0}}_{r\neq s}(\Lambda, z) = \sum_{\substack{r\neq s\\ r,s\geq 0}} \Lambda^{2(r+s)} z^{r-s} \sum_{\substack{|I|=r\\ |J|=s}} \left[G^{r}_{c,\Delta'}\right]^{(1^r)\,I} \rho^{\Delta'\,\Delta_+\,\tilde{\Delta}}_{\infty\;\;1\;\;0}\left(\nu_{\Delta',I}, \nu_{\Delta_+}, \nu_{\tilde{\Delta},J}\right)\left[G^{s}_{c,\tilde{\Delta}}\right]^{J\,(1^s)}. \tag{4.2b}$$

Then, one can write

$$\begin{aligned}\Psi^{\mathbf{0}}_{+}(\Lambda, z) &= z^{\kappa}\;\exp\left\{\log\left(\Phi^{\mathbf{0}}_{r=s}(\Lambda) + \Phi^{\mathbf{0}}_{r\neq s}(\Lambda, z)\right)\right\}\\ &= z^{\kappa}\;\exp\left\{\log \Phi^{\mathbf{0}}_{r=s}(\Lambda) + \log\left(1 + \frac{\Phi^{\mathbf{0}}_{r\neq s}(\Lambda, z)}{\Phi^{\mathbf{0}}_{r=s}(\Lambda)}\right)\right\}\\ &= z^{\kappa}\,\mathrm{e}^{\mathcal{Y}^{\mathbf{0}}(\Lambda)}\,\mathrm{e}^{\mathcal{X}^{\mathbf{0}}(\Lambda,z)},\end{aligned} \tag{4.3}$$

where the following notation has been introduced

$$\mathcal{Y}^{\mathbf{0}}(\Lambda) \;=\; \log \Phi^{\mathbf{0}}_{r=s}(\Lambda), \qquad \mathcal{X}^{\mathbf{0}}(\Lambda, z) \;=\; \log\left(1 + \frac{\Phi^{\mathbf{0}}_{r\neq s}(\Lambda, z)}{\Phi^{\mathbf{0}}_{r=s}(\Lambda)}\right). \tag{4.4}$$

Note that the 'diagonal' part $\Phi^{\mathbf{0}}_{r=s}$ of the degenerate irregular block and thus $\mathcal{Y}^{\mathbf{0}}$ do not depend on z.

The substitution of (4.1b) into eq. (3.41) yields

$$\begin{aligned}\Bigg[\frac{1}{b^2}\,z^2\frac{\partial^2}{\partial z^2} + \left(\frac{2\kappa}{b^2} - \frac{3}{2}\right) z\frac{\partial}{\partial z} + \frac{\Lambda}{4}\frac{\partial}{\partial\Lambda} + \frac{\kappa(\kappa-1)}{b^2} - \frac{3\kappa}{2}&\\ + \Lambda^2\left(z + \frac{1}{z}\right) + \frac{\tilde{\Delta} + \Delta' - \Delta_+}{2}\Bigg]\,\Phi^{\mathbf{0}}_{+}(\Lambda, z) \;=\; 0.&\end{aligned} \tag{4.5}$$

Our aim now is to find the limit $b \to 0$ of eq. (4.5). To this purpose it is convenient to replace the parameter σ in $\tilde{\Delta}$ and Δ' (cf. (3.36)) with $\xi = b\sigma$ and Λ with the new parameter $\hat{\Lambda} = \Lambda\epsilon_1 b$. After this rescaling, we have

$$\Delta', \tilde{\Delta} \overset{b\to 0}{\sim} \frac{1}{b^2}\,\delta, \qquad \text{where} \qquad \delta \;=\; \lim_{b\to 0} b^2\Delta' \;=\; \lim_{b\to 0} b^2\tilde{\Delta} \;=\; \frac{1}{4} - \xi^2, \tag{4.6a}$$

$$\tilde{\Delta} + \Delta' - \Delta_+ \overset{b\to 0}{\sim} \frac{1}{b^2}\,2\left(\frac{1}{4} - \xi^2\right) \;=\; \frac{1}{b^2}\,2\delta, \tag{4.6b}$$

$$\kappa \overset{b\to 0}{\longrightarrow} \frac{1}{2} - \xi, \qquad \kappa\,(\kappa - 1) \overset{b\to 0}{\longrightarrow} -\left(\frac{1}{4} - \xi^2\right) \;=\; -\delta. \tag{4.6c}$$

Note that $\Delta_+ \overset{b\to 0}{\sim} \mathcal{O}(b^0)$.

The next step needed to complete our task is to determine the behavior of the normalized degenerate irregular block $\Phi^{\mathbf{0}}_{+} = z^{-\kappa}\Psi^{\mathbf{0}}_{+}$ when $b \to 0$. For $\Lambda = \hat{\Lambda}/(\epsilon_1 b)$ and $\Delta', \tilde{\Delta} \overset{b\to 0}{\sim} \frac{1}{b^2}\,\delta$, it is reasonable to expect that

$$\Phi^{\mathbf{0}}_{+}(\Lambda, z) \;=\; z^{-\kappa}\,\langle\,\Delta', \Lambda^2\,|\,V_+(z)\,|\,\tilde{\Delta}, \Lambda^2\,\rangle \overset{b\to 0}{\sim} \varphi^{\mathbf{0}}\left(\hat{\Lambda}/\epsilon_1, z\right)\,\exp\left\{\frac{1}{b^2} f^{\mathbf{0}}_{\delta}\left(\hat{\Lambda}/\epsilon_1\right)\right\}\;. \tag{4.7}$$

Moreover, comparing the r.h.s. of eq. (4.7) with eqs. (4.3)–(4.4) one arrives at the following results:

$$\varphi^{\mathbf{0}}\left(\hat{\Lambda}/\epsilon_1, z\right) \;=\; \lim_{b\to 0}\exp\left\{\mathcal{X}^{\mathbf{0}}(\Lambda, z)\right\} = \lim_{b\to 0}\left(1 + \frac{\Phi^{\mathbf{0}}_{r\neq s}(\hat{\Lambda}/(\epsilon_1 b), z)}{\Phi^{\mathbf{0}}_{r=s}(\hat{\Lambda}/(\epsilon_1 b))}\right), \tag{4.8}$$

$$f^{\mathbf{0}}_{\delta}\left(\hat{\Lambda}/\epsilon_1\right) \;=\; \lim_{b\to 0} b^2\mathcal{Y}^{\mathbf{0}}(\Lambda) = \lim_{b\to 0} b^2\log\Phi^{\mathbf{0}}_{r=s}\left(\hat{\Lambda}/(\epsilon_1 b)\right). \tag{4.9}$$

The meaning of eq. (4.7) is that the light ($\Delta_+ \overset{b\to 0}{\sim} \mathcal{O}(b^0)$) degenerate chiral vertex operator does not contribute to the classical limit. In other words, its presence in the matrix element does not affect the 'classical dynamics' (i.e. the 'classical action'). Let us note that eq. (4.7) is a 'chiral version' of Zamolodchikovs' conjecture [32] (see also [69]) concerning the semi-classical behavior of the Liouville correlators with heavy and light vertices on the sphere. Let us stress that there are only a few explicitly known tests verifying Zamolodchikovs' hypothesis. For instance, the derivation of the large intermediate conformal weight limit $\Delta \to \infty$ of the 4-point block on the sphere as well as its expansion in powers of the so-called elliptic variable is based on that assumption in the case of the semiclassical behavior of the 5-point function with the light degenerate vertex operator [66, 70]. The calculation performed in this section is a new test of the semiclassical behavior of the form (4.7) (see also [71]). Regardless of the attempts to prove (cf. subsection 4.2), eq. (4.7) can be well confirmed, first, by direct calculations, secondly, by its consequences. Indeed, one can check order by order that the limits (4.8) and (4.9) exist. Moreover, the latter limit reproduces the classical zero flavor irregular block.

Therefore, from (4.5) and (4.7) for $b \to 0$ one gets

$$\left[z^2\frac{\partial^2}{\partial z^2} + 2(\frac{1}{2}-\xi)z\frac{\partial}{\partial z} + \frac{\hat{\Lambda}^2}{\epsilon_1^2}\left(z+\frac{1}{z}\right) + \frac{\hat{\Lambda}}{4}\frac{\partial}{\partial\hat{\Lambda}}f^{\mathbf{0}}_{\delta}\left(\hat{\Lambda}/\epsilon_1\right)\right]\varphi^{\mathbf{0}}\left(\hat{\Lambda}/\epsilon_1, z\right) \;=\; 0\,. \tag{4.10}$$

The nontrivial point here is the observation that $\lim_{b\to 0} b^2\hat{\Lambda}\partial_{\hat{\Lambda}}\varphi^{\mathbf{0}} = 0$. This result has been checked up to high orders of the expansion of (4.8).

At this point, we define the new function $\psi^{\mathbf{0}}(\hat{\Lambda}/\epsilon_1, z)$ related to the old one by

$$\varphi^{\mathbf{0}}\left(\hat{\Lambda}/\epsilon_1, z\right) \;=\; z^{\xi}\,\psi^{\mathbf{0}}\left(\hat{\Lambda}/\epsilon_1, z\right). \tag{4.11}$$

The analogue of eq. (4.10) in the case of $\psi^{\mathbf{0}}(\hat{\Lambda}/\epsilon_1, z)$ is

$$\left[z^2\frac{\partial^2}{\partial z^2} + z\frac{\partial}{\partial z} + \frac{\hat{\Lambda}^2}{\epsilon_1^2}\left(z+\frac{1}{z}\right) + \frac{\hat{\Lambda}}{4}\frac{\partial}{\partial\hat{\Lambda}}f^{\mathbf{0}}_{\delta}\left(\hat{\Lambda}/\epsilon_1\right) - \xi^2\right]\psi^{\mathbf{0}}\left(\hat{\Lambda}/\epsilon_1, z\right) \;=\; 0\,. \tag{4.12}$$

Since for $z = \mathrm{e}^w$ the derivatives transform as $\left(z^2\partial_z^2 + z\partial_z\right)\psi^{\mathbf{0}}(z) = \partial_w^2\psi^{\mathbf{0}}(\mathrm{e}^w)$, it turns out that eq. (4.12) becomes

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}w^2} + 2\frac{\hat{\Lambda}^2}{\epsilon_1^2}\cosh(w) + \frac{\hat{\Lambda}}{4}\frac{\partial}{\partial\hat{\Lambda}}f^{\mathbf{0}}_{\delta}\left(\hat{\Lambda}/\epsilon_1\right) - \xi^2\right]\psi^{\mathbf{0}}\left(\hat{\Lambda}/\epsilon_1, \mathrm{e}^w\right) \;=\; 0. \tag{4.13}$$

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Finally, the substitution $w = -2ix$, $x \in \mathbb{R}$ in (4.13) yields

$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + 8\frac{\hat{\Lambda}^2}{\epsilon_1^2}\cos 2x + \hat{\Lambda}\,\frac{\partial}{\partial\hat{\Lambda}} f_\delta^{\mathbf{0}}\left(\hat{\Lambda}/\epsilon_1\right) - 4\xi^2\right]\psi^{\mathbf{0}}\left(\hat{\Lambda}/\epsilon_1, \mathrm{e}^{-2ix}\right) \;=\; 0. \tag{4.14}$$

In conclusion, what we have obtained is the following claim:

1. *For the coupling constant* $h = 2\hat{\Lambda}/\epsilon_1$ *and the Floquet exponent* $\nu = 2\xi$ *the eigenvalue* λ *of the Mathieu operator:*

$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + 2h^2\cos 2x\right]\psi^{\mathbf{0}} \;=\; \lambda\,\psi^{\mathbf{0}} \tag{4.15}$$

   is given by the following formula

$$\lambda \;=\; 4\xi^2 - \hat{\Lambda}\,\frac{\partial}{\partial\hat{\Lambda}} f_\delta^{\mathbf{0}}\left(\hat{\Lambda}/\epsilon_1\right) \quad , \tag{4.16}$$

   where $\delta = \frac{1}{4} - \xi^2$*.*

2. *The corresponding eigenfunction is of the form (cf. (4.8) and (4.11))*

$$\psi^{\mathbf{0}} \;\equiv\; \psi^{\mathbf{0}}\left(\hat{\Lambda}/\epsilon_1, \mathrm{e}^{-2ix}\right) \;=\; \mathrm{e}^{2ix\xi}\lim_{b\to 0}\left(1 + \frac{\Phi^{\mathbf{0}}_{r\neq s}(\hat{\Lambda}/(\epsilon_1 b), \mathrm{e}^{-2ix})}{\Phi^{\mathbf{0}}_{r=s}(\hat{\Lambda}/(\epsilon_1 b))}\right) \quad . \tag{4.17}$$

Indeed, using formulae (3.15) for the coefficients of the classical irregular block $f_\delta^{\mathbf{0}}(\hat{\Lambda}/\epsilon_1)$ with $\delta = \frac{1}{4} - \xi^2$, after postulating the relation $\xi = \nu/2$ and taking into account that $h^2 = 4\hat{\Lambda}^2/\epsilon_1^2$, one finds that

$$\begin{aligned}
\lambda &= 4\xi^2 - \hat{\Lambda}\,\partial_{\hat{\Lambda}}\left[\sum_{n=1}^{\infty}\left(\hat{\Lambda}/\epsilon_1\right)^{4n} f_\delta^{\mathbf{0},n}\right] \\
&= 4\left(\frac{\nu^2}{4}\right) - \frac{4h^4}{16}\,f^{\mathbf{0},1}_{\frac{1}{4}-\frac{\nu^2}{4}} - \frac{8h^8}{256}\,f^{\mathbf{0},2}_{\frac{1}{4}-\frac{\nu^2}{4}} - \frac{12h^{12}}{4096}\,f^{\mathbf{0},3}_{\frac{1}{4}-\frac{\nu^2}{4}} - \ldots \\
&= \nu^2 + \frac{h^4}{2\left(\nu^2-1\right)} + \frac{\left(5\nu^2+7\right)h^8}{32\left(\nu^2-4\right)\left(\nu^2-1\right)^3} + \frac{\left(9\nu^4+58\nu^2+29\right)h^{12}}{64\left(\nu^2-9\right)\left(\nu^2-4\right)\left(\nu^2-1\right)^5} + \ldots\,.
\end{aligned} \tag{4.18}$$

Hence, the formula (4.16) reproduces the well known weak coupling (small h^2) expansion of λ for the noninteger Floquet exponent $\nu \notin \mathbb{Z}$, cf. [50].

4.2 The classical asymptotic of the degenerate irregular block with the light insertion

In order to understand the factorization phenomenon into 'heavy' and 'light' factors of the degenerate irregular block within the classical limit it suffices to examine the behavior of the two factors in eq. (4.3) as $b \to 0$. According to eq. (4.1b) the main ingredients of the degenerate irregular block expansion are the inverse of the Gram matrix and the three form rho. Their dependence on Δ is crucial for study of the classical limit. As we will see in what follows it is enough to confine oneself to the leading order in b. In section 3.2 we found the leading behavior of the $p(n) \times p(n)$ component of the inverse Gram matrix in the classical limit. However, from that analysis one can infer also the leading behavior for all the elements of $p(n)^{\text{th}}$ column of the inverse Gram matrix. Indeed, from eq. (3.29) we find that

$$\left(G_{c,\Delta}^{(n)}\right)^{(1^n)\,I_u} \overset{\deg}{\sim} (-1)^{p(n)-u} \frac{\langle\Delta|L_{I_u} L_{-1}^n|\Delta\rangle}{\langle\Delta|L_{I_u} L_{-I_u}|\Delta\rangle} \left(G_{c,\Delta}^{(n)}\right)^{(1^n)\,(1^n)},$$

and by virtue of eqs. (3.31) and (3.32) we obtain the leading behavior within the classical limit of the arbitrary matrix element of $p(n)^{\text{th}}$ column of the inverse Gram matrix

$$\left(G_{c,\Delta}^{(n)}\right)^{(1^n)\,I_u} \overset{b\to 0}{\sim} \frac{b^{2n}}{(2\delta)^n} \frac{(-1)^{p(n)-u}}{\prod\limits_{i\geq 1} m_i(I_u)!} \prod_{i\geq 1} \left(\frac{2(i+1)\delta}{i\,(4\delta+i^2-1)}\right)^{m_i(I_u)}. \tag{4.19}$$

As for the rho form its analysis is much more involved. By definition, for any two partitions $I \vdash r,\ J \vdash s$ it takes the form

$$\rho_{\infty\ \ 1\ \ 0}^{\Delta'\,\Delta_+\,\tilde{\Delta}}\left(\nu_{\Delta',I}, \nu_{\Delta_+}, \nu_{\tilde{\Delta},J}\right) = \langle\Delta'|L_I V_+(z) L_{-J}|\tilde{\Delta}\rangle\big|_{z=1}. \tag{4.20}$$

Making use of the Virasoro algebra (2.2) this can be developed into the form

$$\begin{aligned}\langle\Delta'|L_I V_+(z) L_{-J}|\tilde{\Delta}\rangle =& \langle\Delta'|V_{\Delta_+}(z)\mathrm{ad}_I(L_{-J})\,|\tilde{\Delta}\rangle + \langle\Delta'|\mathrm{ad}_{\ddot{I}^{(1)}}(V_+(z))\,\mathrm{ad}_{\dot{I}^{(1)}}(L_{-J})\,|\tilde{\Delta}\rangle \\ &+ \langle\Delta'|\mathrm{ad}_{\ddot{I}^{(2)}}(V_+(z))\,\mathrm{ad}_{\dot{I}^{(2)}}(L_{-J})\,|\tilde{\Delta}\rangle + \ldots + \langle\Delta'|\mathrm{ad}_I(V_+(z))\,L_{-J}|\tilde{\Delta}\rangle,\end{aligned} \tag{4.21}$$

where for the sake of brevity we have used the following notation

$$\mathrm{ad}_I(L_{-J}) := [L_{k_{\ell(I)}(I)}, \ldots, [L_{k_1(I)}, L_{-J}]\ldots],$$

and

$$\underset{m\in\{1,\ldots,\ell(I)\}}{\forall} \quad I = \dot{I}^{(m)} \cup \ddot{I}^{(m)},\ \ \dot{I}^{(m)} := \left(k_1(I), \ldots, k_{\ell(I)-m}(I)\right).$$

Let us examine a matrix element that contributes to the sum in eq. (4.21). It takes the form

$$\begin{aligned}&\langle\Delta'|\mathrm{ad}_{\ddot{I}^{(m)}}(V_+(z))\,\mathrm{ad}_{\dot{I}^{(m)}}(L_{-J})\,|\tilde{\Delta}\rangle \\ &\quad= \langle\Delta'|[L_{k_{\ell(I)}(I)}, \ldots, [L_{k_{\ell(I)-m+1}(I)}, V_+(z)]\ldots][L_{k_{\ell(I)-m}(I)}, \ldots, [L_{k_1(I)}, L_{-J}]\ldots]|\tilde{\Delta}\rangle.\end{aligned} \tag{4.22}$$

It is nonzero provided $t \equiv |\dot{I}^{(m)}| = \sum k_i(\dot{I}^{(m)}) \leq |J| = s$. For definiteness let us assume that $t < s$. From the commutator formula between the Virasoro generator and vertex operator (2.8) we obtain

$$\mathrm{ad}_{\ddot{I}^{(m)}}(V_+(z)) = \prod_{i=\ell(I)-m+1}^{\ell(I)} z^{k_i(I)}\Big(z\partial_z + \big(k_i(I)+1\big)\Delta_+\Big)V_+(z).$$

Hence, the matrix element (4.22) assumes the form

$$\begin{aligned}&\langle\Delta'|\mathrm{ad}_{\ddot{I}^{(m)}}(V_+(z))\,\mathrm{ad}_{\dot{I}^{(m)}}(L_{-J})\,|\tilde{\Delta}\rangle\\&\qquad=\prod_{i=\ell(I)-m+1}^{\ell(I)} z^{k_i(I)}\Big(z\partial_z+\big(k_i(I)+1\big)\Delta_+\Big)\langle\Delta'|V_+(z)\mathrm{ad}_{\dot{I}^{(m)}}(L_{-J})\,|\tilde{\Delta}\rangle.\end{aligned}\tag{4.23}$$

The nested commutator encoded in $\mathrm{ad}_{\dot{I}^{(m)}}(L_{-J})$ on the right hand side of the above formula provides possible factors containing $\tilde{\Delta}$ and c. These factors typically appear if one or more parts of $\dot{I}^{(m)}$ coincides with those in the partition J. Let us rewrite the matrix element on the right hand side of eq. (4.23) in terms of multiplicities

$$\begin{aligned}&\langle\Delta'|V_+(z)\mathrm{ad}_{\dot{I}^{(m)}}(L_{-J})\,|\tilde{\Delta}\rangle=\langle\Delta'|V_+(z)[L_{k_{\ell(I)-m}(I)},\ldots,[L_{k_1(I)},L_{-J}]\ldots]|\tilde{\Delta}\rangle\\&\quad=\langle\Delta'|V_+(z)\,\underbrace{[L_{i_l},\ldots,[L_{i_l}}_{m_{i_l}(I)},\ldots,\underbrace{[L_{i_1},\ldots,[L_{i_1}}_{m_{i_1}(I)},L_{-j_1}^{m_{j_1}(J)}\cdots L_{-j_n}^{m_{j_n}(J)}]\ldots]|\tilde{\Delta}\rangle,\end{aligned}\tag{4.24a}$$

where

$$i_1:=k_1(I),\,i_2:=k_{m_{i_1}(I)+1}(I),\ldots,\,i_l:=k_{1+\sum\limits_{u=1}^{l-1}m_{i_u}(I)}(I)=k_{\ell(I)-m}(I),\tag{4.24b}$$

and similarly for parts j_u. Let us assume for definiteness that $j_u=i_u$ for $u\in\{1,\ldots,N\}$, $N\leq l$. Then, according to the formula in eq. (3.22) an overall factor that appears in front of the resulting matrix element, up to leading term in $\tilde{\Delta}$, reads

$$\langle\Delta'|V_+(z)\mathrm{ad}_{\dot{I}^{(m)}}(L_{-J})\,|\tilde{\Delta}\rangle\overset{\deg}{\sim}\mathsf{Poly}_{\dot{I}^{(m)},\dot{J}}(\tilde{\Delta},c)\langle\Delta'|V_+(z)L_{-\ddot{J}}|\tilde{\Delta}\rangle,\tag{4.25a}$$

where $\dot{I}^{(m)}$, $\dot{J}\vdash t$ and $\ddot{J}\vdash s-t$.

$$\begin{aligned}\mathsf{Poly}_{\dot{I}^{(m)},\dot{J}}(\tilde{\Delta},c):=\prod_{u=1}^{N}&\,i_u^{\min\{m_{i_u}(I),m_{i_u}(J)\}}\frac{m_{i_u}(J)!}{\vartheta\big(m_{i_u}(J)-m_{i_u}(I)\big)!}\\&\times\frac{\Big(2\tilde{\Delta}+\frac{c}{12}(i_u^2-1);i_u\Big)_{m_{i_u}(J)}}{\Big(2\tilde{\Delta}+\frac{c}{12}(i_u^2-1);i_u\Big)_{\vartheta(m_{i_u}(J)-m_{i_u}(I))}},\end{aligned}\tag{4.25b}$$

where $\vartheta(x)=x\theta(x)$ and $\theta(x)$ is Heaviside's theta function. $\mathsf{Poly}_{\dot{I}^{(m)},\dot{J}}(\tilde{\Delta},c)$ is a polynomial in $\tilde{\Delta}$ and c and '$\overset{\deg}{\sim}$' has the same meaning as in section 3.2. Its degree in $\tilde{\Delta}$, in general, varies between

$$0\leq\deg_{\tilde{\Delta}}\mathsf{Poly}_{I_i,I_j}(\tilde{\Delta},c)\leq t,\qquad I_i,I_j\vdash t,\quad i,j=\{1,\ldots,p(t)\}.$$

Since $\Big(2\tilde{\Delta}+\frac{c}{12}(i^2-1);i\Big)_n\overset{\deg}{\sim}2^n\tilde{\Delta}^n$ as a polynomial in $\tilde{\Delta}$ then from eq. (4.25b) we get

$$\deg_{\tilde{\Delta}}\mathsf{Poly}_{\dot{I}^{(m)},\dot{J}}(\tilde{\Delta},c)=\sum_{u=1}^{N}\Big(m_{i_u}(J)-\vartheta\big(m_{i_u}(J)-m_{i_u}(I)\big)\Big)\leq\sum_{u=1}^{l}m_{i_u}(I)=\ell(\dot{I}^{(m)}).\tag{4.26}$$

Therefore, the degree of the polynomial is *maximal* if $\ell(\dot{I}^{(m)}) = \sum m_{i_u}(I) = t = |\dot{I}^{(m)}| = \sum i_u\, m_{i_u}(I)$ which due to the fact that $i_u > i_v$ for $u < v$ following from eq. (4.24b) entails that

$$\sum_{u=1}^{N}(i_u - 1)m_{i_u}(I) = 0 \quad \Rightarrow \quad i_1 := k_1 = 1 \wedge m_1(I) = t,$$

and $m_{i_u}(I) = 0$ for $u > 1$, i.e., $\dot{I}^{(m)} = (1^t)$. Moreover, $\dot{I}^{(m)}$ by definition consists of first m parts of I. Hence, if I is to be a partition *it must assume the form* (1^r).

The matrix element (4.23) can now be rewritten as

$$\begin{aligned}&\langle\Delta'|\mathrm{ad}_{\ddot{I}^{(m)}}(V_+(z))\,\mathrm{ad}_{\dot{I}^{(m)}}(L_{-J})\,|\tilde{\Delta}\rangle\\ &\quad\overset{\deg}{\sim} \mathsf{Poly}_{\dot{I}^{(m)},j}(\tilde{\Delta},c) \prod_{i=\ell(I)-m+1}^{\ell(I)} z^{k_i(I)}\Big(z\partial_z + \big(k_i(I)+1\big)\Delta_+\Big)\langle\Delta'|V_+(z)L_{-\ddot{J}}|\tilde{\Delta}\rangle. \end{aligned}\tag{4.27}$$

Note that the matrix element in the last line of the above equation is nothing but the gamma vector given in eq. (2.13) with $I \to \ddot{J}$. Performing necessary computations we find the typical form of the contribution to the sum (4.21), namely

$$\begin{aligned}&\langle\Delta'|\mathrm{ad}_{\ddot{I}^{(m)}}(V_+(z))\,\mathrm{ad}_{\dot{I}^{(m)}}(L_{-J})\,|\tilde{\Delta}\rangle\big|_{z=1}\\ &\qquad\overset{\deg}{\sim} \mathsf{Poly}_{\dot{I}^{(m)},j}(\tilde{\Delta},c)(-1)^{\ell(\ddot{J})}\prod_{i=1}^{\ell(\ddot{J})}\left(\Delta' - k_i(\ddot{J})\Delta_+ - \tilde{\Delta} - \sum_{s>i}^{\ell(\ddot{J})} k_s(\ddot{J})\right)\\ &\qquad\qquad\times \prod_{j=\ell(I)-m+1}^{\ell(I)}\left(\Delta' + k_j(I)\Delta_+ - \tilde{\Delta} + \sum_{s>j}^{\ell(I)} k_s(I) - s + t\right). \end{aligned}\tag{4.28}$$

The above formula enables one to estimate the contribution of the corresponding term in the sum (4.21) within the classical limit. Since for $b \to 0$ conformal weights scale as $\Delta', \tilde{\Delta} \sim \delta/b^2$ and $\Delta_+ \sim -1/2$ we conclude that the two products in the second and third line of eq. (4.28) reduce to the ξ dependent numerical factors and the only factor that determines the classical behavior of the matrix element is the polynomial $\mathsf{Poly}_{\dot{I}^{(m)},j}(\tilde{\Delta},c)$. The degree of the latter depends on the number of factors. Therefore, the term with the greatest number of factors will dominate entire sum within the classical limit. The term in question is the first one in eq. (4.21). This analysis allows us to conclude that

$$\begin{aligned}\langle\Delta'|L_I V_+(1)L_{-J}|\tilde{\Delta}\rangle &= \langle\Delta'|V_+(z)\mathrm{ad}_I(L_{-J})\,|\tilde{\Delta}\rangle\big|_{z=1} + \dots\\ &= (-1)^{\ell(\ddot{J})}\mathsf{Poly}_{\dot{I}^{(m)},j}(\tilde{\Delta},c)\prod_{i=1}^{\ell(\ddot{J})}\left(\Delta' - k_i(\ddot{J})\Delta_+ - \tilde{\Delta} - \sum_{s>i}^{\ell(\ddot{J})} k_s(\ddot{J})\right) + \dots\end{aligned}$$

and in the classical limit and within the leading order approximation the rho form reads

$$\rho_{\infty\ 1\ 0}^{\Delta'\,\Delta_+\,\tilde{\Delta}}\left(\nu_{\Delta',I}, \nu_{\Delta_+}, \nu_{\tilde{\Delta},J}\right) \overset{b\to 0}{\sim} \mathsf{Poly}_{\dot{I}^{(m)},j}(\delta/b^2, 6/b^2)C_{\ddot{J}}^{(-)}(\xi),$$

where we introduced the label for the ξ dependent numerical factor

$$C^{(-)}_{\ddot{J}}(\xi) := \prod_{i=1}^{\ell(\ddot{J})} \left(\sum_{s>i}^{\ell(\ddot{J})} k_s(\ddot{J}) - \frac{1}{2} k_i(\ddot{J}) + \xi \right). \tag{4.29}$$

The argument concerning the maximal degree of the polynomial entails that the dominant contribution to the sum over all partitions with fixed $|I| = r$, $|J| = s$ comes from the term with the maximal possible multiplicity, i.e., if $r < s$ then $I = (1^r)$ and $J = \ddot{J} \cup (1^r)$. In this case

$$\begin{aligned} &\rho_{\infty\ 1\ \ 0}^{\Delta'\,\Delta_+\,\tilde{\Delta}} \left(\nu_{\Delta',(1^r)}, \nu_{\Delta_+}, \nu_{\tilde{\Delta},J} \right) \\ &\qquad\qquad = r!(2\tilde{\Delta})_r (-1)^{\ell(\ddot{J})} \prod_{i=1}^{\ell(\ddot{J})} \left(\Delta' - k_i(\ddot{J})\Delta_+ - \tilde{\Delta} - \sum_{s>i}^{\ell(\ddot{J})} k_s(\ddot{J}) \right) + \dots \\ &\qquad\qquad \overset{b\to 0}{\sim} r!(2\delta)^r b^{-2r} C^{(-)}_{\ddot{J}}(\xi) = \left[G^r_{c,\delta} \right]_{(1^r)\,(1^r)} C^{(-)}_{\ddot{J}}(\xi). \end{aligned} \tag{4.30}$$

We are at the point where we have all necessary ingredients to prove the factorization phenomenon for the three point irregular conformal block stated in eq. (4.7). Let us consider the case where $r = s$. Then from eqs. (3.32) and (4.30) we get that $|\ddot{J}| = 0$ as well as $C^{(-)}_{\ddot{J}}(\xi) = 1$ which, when applied to eq. (4.2a), yields

$$\begin{aligned} \Phi^{\mathbf{0}}_{r=s}(\Lambda) &= \sum_{r\geq 0} \Lambda^{4r} \sum_{|I|=|J|=r} \left[G^r_{c,\Delta'} \right]^{(1^r)\,I} \rho_{\infty\ 1\ \ 0}^{\Delta'\,\Delta_+\,\tilde{\Delta}} \left(\nu_{\Delta',I}, \nu_{\Delta_+}, \nu_{\tilde{\Delta},J} \right) \left[G^r_{c,\tilde{\Delta}} \right]^{J\,(1^r)} \\ &\overset{b\to 0}{\sim} \sum_{r\geq 0} \left(\frac{\hat{\Lambda}}{\epsilon_1} \right)^{4r} b^{-4r} \left[G^r_{c,\delta} \right]^{(1^r)\,(1^r)} \left[G^r_{c,\delta} \right]_{(1^r)\,(1^r)} \left[G^r_{c,\delta} \right]^{(1^r)\,(1^r)} \\ &= \exp\left\{ \frac{1}{b^2} \left(\frac{\hat{\Lambda}}{\epsilon_1} \right)^4 \frac{1}{2\delta} \right\}. \end{aligned} \tag{4.31}$$

Let us now consider the case when $r \neq s$. According to eq. (4.2b) we have

$$\begin{aligned} \Phi^{\mathbf{0}}_{r\neq s}(\Lambda, z) &= \sum_{\substack{r\neq s \\ r,s\geq 0}} \Lambda^{2(r+s)} z^{r-s} F^{(r,s)}_c \left(\Delta', \Delta_+, \tilde{\Delta} \right) \\ &= \sum_{s>r\geq 0} \Lambda^{2(r+s)} \left(F^{(s,r)}_c \left(\Delta', \Delta_+, \tilde{\Delta} \right) z^{s-r} + F^{(r,s)}_c \left(\Delta', \Delta_+, \tilde{\Delta} \right) z^{-(s-r)} \right), \end{aligned} \tag{4.32}$$

where

$$F^{(r,s)}_c \left(\Delta', \Delta_+, \tilde{\Delta} \right) := \sum_{\substack{|I|=r \\ |J|=s}} \left[G^r_{c,\Delta'} \right]^{(1^r)\,I} \rho_{\infty\ 1\ \ 0}^{\Delta'\,\Delta_+\,\tilde{\Delta}} \left(\nu_{\Delta',I}, \nu_{\Delta_+}, \nu_{\tilde{\Delta},J} \right) \left[G^s_{c,\tilde{\Delta}} \right]^{J\,(1^s)}. \tag{4.33}$$

Thus the function $\Phi^{\mathbf{0}}_{r\neq s}(\Lambda, z)$ splits into two parts with positive and negative power of variable z

$$\Phi^{\mathbf{0}}_{r\neq s}(\Lambda, z) = \phi^{r\neq s}_1(\Lambda, z) + \phi^{r\neq s}_2(\Lambda, z), \tag{4.34a}$$

where

$$\begin{aligned}\phi_1^{r\neq s}(\Lambda,z) &= \sum_{s>r\geq 0}\Lambda^{2(r+s)}F_c^{(s,r)}\left(\Delta',\Delta_+,\tilde{\Delta}\right)z^{s-r},\\ \phi_2^{r\neq s}(\Lambda,z) &= \sum_{s>r\geq 0}\Lambda^{2(r+s)}F_c^{(r,s)}\left(\Delta',\Delta_+,\tilde{\Delta}\right)z^{-(s-r)}.\end{aligned} \tag{4.34b}$$

Let us consider the second one and compute the classical limit of $F^{(r,s)}$. From eqs. (4.19) and (4.30) and recalling the notation $J = \ddot{J}\cup(1^r)$ we obtain

$$\begin{aligned}F_c^{(r,s)}\left(\Delta',\Delta_+,\tilde{\Delta}\right) &\overset{b\to 0}{\sim} \sum_{|\ddot{J}|=n}\left[G^r_{c,\delta}\right]^{(1^r)\,(1^r)}\left[G^r_{c,\delta}\right]_{(1^r)\,(1^r)}C_{\ddot{J}}^{(-)}(\xi)\left[G^r_{c,\delta}\right]^{(1^s)\,,\ddot{J}\cup(1^r)}\\ &= b^{2(r+s)}\frac{1}{b^{2r}(2\delta)^r r!}\frac{1}{(2\delta)^{s-r}}\sum_{|\ddot{J}|=s-r}C_{\ddot{J}}^{(-)}(\xi)\prod_{i\geq 2}\frac{1}{m_i(\ddot{J})!}\left(\frac{2(i+1)\delta}{i\,(4\delta+i^2-1)}\right)^{m_i(\ddot{J})}.\end{aligned}$$

Let us denote

$$\zeta_n^{(-)}(\xi) := \frac{1}{(2\delta)^n}\sum_{|\ddot{J}|=n}C_{\ddot{J}}^{(-)}(\xi)\prod_{i\geq 2}\frac{1}{m_i(\ddot{J})!}\left(\frac{2(i+1)\delta}{i\,(4\delta+i^2-1)}\right)^{m_i(\ddot{J})}. \tag{4.35}$$

Inserting the result for the classical limit of $F^{(r,s)}$ to $\phi_2^{r\neq s}$ amounts to

$$\begin{aligned}\phi_2^{r\neq s}(\Lambda,z) &\overset{b\to 0}{\sim} \sum_{s>r\geq 0}\left(\frac{\hat{\Lambda}}{\epsilon_1}\right)^{2(r+s)}z^{-(s-r)}\frac{1}{b^{2r}(2\delta)^r r!}\zeta_{s-r}^{(-)}(\xi)\\ &= \sum_{s\geq 1}\sum_{r=0}^{s-1}\left(\frac{\hat{\Lambda}}{\epsilon_1}\right)^{4r}\frac{1}{b^{2r}(2\delta)^r r!}\left(\frac{\hat{\Lambda}}{\epsilon_1}\right)^{2(s-r)}z^{-(s-r)}\zeta_{s-r}^{(-)}(\xi)\\ &= \sum_{r\geq 0}\sum_{s\geq r+1}\left(\frac{\hat{\Lambda}}{\epsilon_1}\right)^{4r}\frac{1}{b^{2r}(2\delta)^r r!}\left(\frac{\hat{\Lambda}}{\epsilon_1}\right)^{2(s-r)}z^{-(s-r)}\zeta_{s-r}^{(-)}(\xi)\\ &\overset{s-r=n}{=} \sum_{r\geq 0}\frac{1}{r!}\left(\frac{1}{b^2}\left(\frac{\hat{\Lambda}}{\epsilon_1}\right)^4\frac{1}{2\delta}\right)^r\sum_{n\geq 1}\left(\frac{\hat{\Lambda}}{\epsilon_1}\right)^{2n}\zeta_n^{(-)}(\xi)z^{-n}.\end{aligned}$$

The last line of the above formula is noting but the exponent of the first coefficient of the irregular classical block as in eq. (4.31) and the second one is the leading order approximation to the Mathieu function. The same argument applies to $\phi_1^{r\neq s}$, such that we can conclude with the following formula

$$\Phi^{\mathbf{0}}_{r\neq s}(\Lambda,z) \overset{b\to 0}{\sim} \exp\left\{\frac{1}{b^2}\left(\frac{\hat{\Lambda}}{\epsilon_1}\right)^4\frac{1}{2\delta}\right\}\sum_{n\geq 1}\left(\frac{\hat{\Lambda}}{\epsilon_1}\right)^{2n}\left(\zeta_n^{(-)}(\xi)z^{-n}+\zeta_n^{(+)}(\xi)z^n\right). \tag{4.36}$$

Having found the classical limit of $\Phi^{\mathbf{0}}_{r=s}(\Lambda)$ in eq. (4.31) and $\Phi^{\mathbf{0}}_{r\neq s}(\Lambda,z)$ in eq. (4.36) we can combine them to find the limit that defines the factor in eq. (4.8) deriving from the

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light field. As a result we find

$$\varphi^{\mathbf{0}}\Big(\hat{\Lambda}/\epsilon_1, z\Big) = \lim_{b\to 0}\left(1+\frac{\Phi^{\mathbf{0}}_{r\neq s}(\hat{\Lambda}/(\epsilon_1 b), z)}{\Phi^{\mathbf{0}}_{r=s}(\hat{\Lambda}/(\epsilon_1 b))}\right) = \sum_{n\geq 0}\left(\frac{\hat{\Lambda}}{\epsilon_1}\right)^{2n}\Big(\zeta_n^{(-)}(\xi)z^{-n}+\zeta_n^{(+)}(\xi)z^n\Big)\,. \tag{4.37}$$

The above formula does not depend on b and is a finite expression which shows in the leading order approximation that the factorization of the light operator insertion in the three point irregular conformal block indeed takes place.

4.3 Mathieu functions

Our next point is to demonstrate that the formula (4.17) fits the *noninteger order* Mathieu function which corresponds to the eigenvalue given by (4.16), (4.18). As a starting point let us recall that $\Phi^{\mathbf{0}}_{r=s}$ and $\Phi^{\mathbf{0}}_{r\neq s}$ in eq. (4.17) are two parts of the normalized $N_f = 0$ degenerate irreagular block $\Phi^{\mathbf{0}}_{+}(\Lambda, z) = \Phi^{\mathbf{0}}_{r=s}(\Lambda) + \Phi^{\mathbf{0}}_{r\neq s}(\Lambda, z)$ (cf. eqs. (4.1a)–(4.5)). Explicitely,

$$\Phi^{\mathbf{0}}_{r=s}(\Lambda) = \sum_{r\geq 0}\Lambda^{4r}\,F_c^{(r,r)}\Big(\Delta', \Delta_+, \tilde{\Delta}\Big)\,, \tag{4.38}$$

$$\Phi^{\mathbf{0}}_{r\neq s}(\Lambda, z) = \sum_{s\geq 1}\sum_{r=0}^{s-1}\Lambda^{2(r+s)}\left(F_c^{(s,r)}\Big(\Delta', \Delta_+, \tilde{\Delta}\Big)\; z^{s-r} + F_c^{(r,s)}\Big(\Delta', \Delta_+, \tilde{\Delta}\Big)\; z^{-(s-r)}\right),$$

where (cf. eqs. (4.2a) and (4.2b))

$$F_c^{(r,s)}\Big(\Delta', \Delta_+, \tilde{\Delta}\Big) \;\equiv\; \sum_{\substack{|I|=r\\|J|=s}}\Big[G^r_{c,\Delta'}\Big]^{(1^r)\,I}\rho^{\Delta'\,\Delta_+\,\tilde{\Delta}}_{\infty\;\;1\;\;0}\Big(\nu_{\Delta',I}, \nu_{\Delta_+}, \nu_{\tilde{\Delta},J}\Big)\Big[G^s_{c,\tilde{\Delta}}\Big]^{J\,(1^s)}. \tag{4.39}$$

As has been already noted, $\Phi^{\mathbf{0}}_{r\neq s}(\Lambda, z)$ has two linearly independent components to which it can be split, namely

$$\Phi^{\mathbf{0}}_{r\neq s}(\Lambda, z) = \phi_1^{r\neq s}(\Lambda, z) + \phi_2^{r\neq s}(\Lambda, z), \tag{4.40}$$

where

$$\phi_1^{r\neq s}(\Lambda, z) = \sum_{s\geq 1}\sum_{r=0}^{s-1}\Lambda^{2(r+s)}F_c^{(s,r)}\Big(\Delta', \Delta_+, \tilde{\Delta}\Big)\; z^{s-r}\,, \tag{4.41}$$

$$\phi_2^{r\neq s}(\Lambda, z) = \sum_{s\geq 1}\sum_{r=0}^{s-1}\Lambda^{2(r+s)}F_c^{(r,s)}\Big(\Delta', \Delta_+, \tilde{\Delta}\Big)\; z^{-(s-r)}\,. \tag{4.42}$$

Let us consider the ratio $\Phi^{\mathbf{0}}_{r\neq s}/\Phi^{\mathbf{0}}_{r=s}$ in eq. (4.17). From (4.38), (4.40), (4.41), (4.42) we get

$$\frac{\Phi^{\mathbf{0}}_{r\neq s}(\Lambda, z)}{\Phi^{\mathbf{0}}_{r=s}(\Lambda)} = \frac{\phi_1^{r\neq s}(\Lambda, z)}{\Phi^{\mathbf{0}}_{r=s}(\Lambda)} + \frac{\phi_2^{r\neq s}(\Lambda, z)}{\Phi^{\mathbf{0}}_{r=s}(\Lambda)}\,, \tag{4.43}$$

where

$$\frac{1}{\Phi^{\mathbf{0}}_{r=s}(\Lambda)}\,\phi_1^{r\neq s}(\Lambda,z) = \frac{1}{1+\sum\limits_{s\geq 1}\Lambda^{4s}\,F^{(s,s)}}\,\sum_{s\geq 1}\sum_{r=0}^{s-1}\Lambda^{2(r+s)}F^{(s,r)}\,z^{s-r}\,,$$

$$\frac{1}{\Phi^{\mathbf{0}}_{r=s}(\Lambda)}\,\phi_2^{r\neq s}(\Lambda,z) = \frac{1}{1+\sum\limits_{s\geq 1}\Lambda^{4s}\,F^{(s,s)}}\,\sum_{s\geq 1}\sum_{r=0}^{s-1}\Lambda^{2(r+s)}F^{(r,s)}\,z^{-(s-r)}.$$

Let us observe that in both equations written above one can expand the factor $(1+\sum_{s\geq 1}\Lambda^{4s}F^{(s,s)})^{-1}$ according to the formula for the sum of the geometric series. Then, collecting the resulting expressions up to 12 order in Λ one can obtain

— for the first term in eq. (4.43):

$$\begin{aligned}\frac{\phi_1^{r\neq s}(\Lambda,z)}{\Phi^{\mathbf{0}}_{r=s}(\Lambda)} &= \Lambda^2\,z\,F^{(1,0)} + \Lambda^4\,z^2\,F^{(2,0)} \\ &+\Lambda^6\left[z^3\,F^{(3,0)} + z\left(F^{(2,1)} - F^{(1,0)}F^{(1,1)}\right)\right] \\ &+\Lambda^8\left[z^4\,F^{(4,0)} + z^2\left(F^{(3,1)} - F^{(1,1)}F^{(2,0)}\right)\right] \\ &+\Lambda^{10}\left[z^5\,F^{(5,0)} + z^3\left(F^{(4,1)} - F^{(1,1)}F^{(3,0)}\right)\right. \\ &\qquad\left. + \,z\left(F^{(1,0)}(F^{(1,1)})^2 - F^{(1,1)}F^{(2,1)} - F^{(1,0)}F^{(2,2)} + F^{(3,2)}\right)\right] \\ &+\Lambda^{12}\left[z^6\,F^{(6,0)} + z^4\left(F^{(5,1)} - F^{(1,1)}F^{(4,0)}\right)\right. \\ &\qquad\left. + \,z^2\left((F^{(1,1)})^2F^{(2,0)} - F^{(2,0)}F^{(2,2)} - F^{(1,1)}F^{(3,1)} + F^{(4,2)}\right)\right] + \ldots\,,\end{aligned}$$

— for the second term in eq. (4.43):

$$\begin{aligned}\frac{\phi_2^{r\neq s}(\Lambda,z)}{\Phi^{\mathbf{0}}_{r=s}(\Lambda)} &= \Lambda^2\,z^{-1}\,F^{(0,1)} + \Lambda^4\,z^{-2}\,F^{(0,2)} \\ &+\Lambda^6\left[z^{-3}\,F^{(0,3)} + z^{-1}\left(F^{(1,2)} - F^{(0,1)}F^{(1,1)}\right)\right] \\ &+\Lambda^8\left[z^{-4}\,F^{(0,4)} + z^{-2}\left(F^{(1,3)} - F^{(1,1)}F^{(0,2)}\right)\right] \\ &+\Lambda^{10}\left[z^{-5}\,F^{(0,5)} + z^{-3}\left(F^{(1,4)} - F^{(1,1)}F^{(0,3)}\right)\right. \\ &\qquad\left. + \,z^{-1}\left(F^{(0,1)}(F^{(1,1)})^2 - F^{(1,1)}F^{(1,2)} - F^{(0,1)}F^{(2,2)} + F^{(2,3)}\right)\right] \\ &+\Lambda^{12}\left[z^{-6}\,F^{(0,6)} + z^{-4}\left(F^{(1,5)} - F^{(1,1)}F^{(0,4)}\right)\right. \\ &\qquad\left. + \,z^{-2}\left((F^{(1,1)})^2F^{(0,2)} - F^{(0,2)}F^{(2,2)} - F^{(1,1)}F^{(1,3)} + F^{(2,4)}\right)\right] + \ldots\,.\end{aligned}$$

Thus, eventually we get

$$\frac{\Phi^{\mathbf{0}}_{r\neq s}(\Lambda,z)}{\Phi^{\mathbf{0}}_{r=s}(\Lambda)} = \Lambda^2\,\mathcal{K}_2 + \Lambda^4\,\mathcal{K}_4 + \Lambda^6\,\mathcal{K}_6 + \Lambda^8\,\mathcal{K}_8 + \Lambda^{10}\,\mathcal{K}_{10} + \Lambda^{12}\,\mathcal{K}_{12} + \ldots\,, \tag{4.44}$$

where

$$
\begin{aligned}
\mathcal{K}_2 &= z^{-1}\, F^{(0,1)} + z\, F^{(1,0)}, \\
\mathcal{K}_4 &= z^2\, F^{(2,0)} + z^{-2}\, F^{(0,2)}, \\
\mathcal{K}_6 &= z^3\, F^{(3,0)} + z^{-3}\, F^{(0,3)} + z\, \Big(F^{(2,1)} - F^{(1,0)}F^{(1,1)}\Big) + z^{-1}\Big(F^{(1,2)} - F^{(0,1)}F^{(1,1)}\Big), \\
\mathcal{K}_8 &= z^4\, F^{(4,0)} + z^{-4}\, F^{(0,4)} + z^2\, \Big(F^{(3,1)} - F^{(1,1)}F^{(2,0)}\Big) + z^{-2}\Big(F^{(1,3)} - F^{(1,1)}F^{(0,2)}\Big), \\
\mathcal{K}_{10} &= z^5\, F^{(5,0)} + z^{-5}\, F^{(0,5)} + z^3\, \Big(F^{(4,1)} - F^{(1,1)}F^{(3,0)}\Big) + z^{-3}\Big(F^{(1,4)} - F^{(1,1)}F^{(0,3)}\Big) \\
&\quad + z\,\Big(F^{(1,0)}(F^{(1,1)})^2 - F^{(1,1)}F^{(2,1)} - F^{(1,0)}F^{(2,2)} + F^{(3,2)}\Big) \\
&\quad + z^{-1}\Big(F^{(0,1)}(F^{(1,1)})^2 - F^{(1,1)}F^{(1,2)} - F^{(0,1)}F^{(2,2)} + F^{(2,3)}\Big),
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{K}_{12} &= z^6\, F^{(6,0)} + z^{-6}\, F^{(0,6)} + z^4\, \Big(F^{(5,1)} - F^{(1,1)}F^{(4,0)}\Big) + z^{-4}\Big(F^{(1,5)} - F^{(1,1)}F^{(0,4)}\Big) \\
&\quad + z^2\Big((F^{(1,1)})^2F^{(2,0)} - F^{(2,0)}F^{(2,2)} - F^{(1,1)}F^{(3,1)} + F^{(4,2)}\Big) \\
&\quad + z^{-2}\Big((F^{(1,1)})^2F^{(0,2)} - F^{(0,2)}F^{(2,2)} - F^{(1,1)}F^{(1,3)} + F^{(2,4)}\Big)\ .
\end{aligned}
$$

Now, having computed the coefficients[20] $F_c^{(r,s)}\Big(\Delta', \Delta_+, \tilde{\Delta}\Big)$ for[21]

$$
\tilde{\Delta} = \Delta\bigg(\xi/b - \frac{b}{4}\bigg), \qquad \Delta' = \Delta\bigg(\xi/b + \frac{b}{4}\bigg), \qquad \Delta_+ = -\frac{3}{4}b^2 - \frac{1}{2}, \qquad c = 1 + 6\left(b + \frac{1}{b}\right)^2,
$$

setting $z = \mathrm{e}^{-2ix}$ and taking into account that $\Lambda = \hat{\Lambda}/(\epsilon_1 b)$ one can find the limit[22]

$$
\lim_{b\to 0}\left(1 + \frac{\Phi^{\mathbf{0}}_{r\neq s}(\hat{\Lambda}/(\epsilon_1 b), \mathrm{e}^{-2ix})}{\Phi^{\mathbf{0}}_{r=s}(\hat{\Lambda}/(\epsilon_1 b))}\right)
$$

order by order, namely

— order Λ^2:

$$
\Lambda^2\,\mathcal{K}_2 \xrightarrow{b\to 0} \left(\frac{\hat{\Lambda}}{\epsilon_1}\right)^2\left(\frac{\mathrm{e}^{-2ix}}{2\xi - 1} - \frac{\mathrm{e}^{2ix}}{2\xi + 1}\right) \equiv \left(\frac{\hat{\Lambda}}{\epsilon_1}\right)^2 \mathcal{P}_2(\xi, x)\ ,
$$

— order Λ^4:

$$
\Lambda^4\,\mathcal{K}_4 \xrightarrow{b\to 0} \left(\frac{\hat{\Lambda}}{\epsilon_1}\right)^4\left(\frac{\mathrm{e}^{4ix}}{4(\xi+1)(2\xi + 1)} + \frac{\mathrm{e}^{-4ix}}{4(\xi-1)(2\xi - 1)}\right) \equiv \left(\frac{\hat{\Lambda}}{\epsilon_1}\right)^4 \mathcal{P}_4(\xi, x)\ ,
$$

[20] Cf. eq. (4.39) and appendix A.
[21] Here, $\Delta(\sigma) \equiv \mathsf{Q}^2/4 - \sigma^2$, cf. eqs. (3.36) and (4.6a).
[22] Cf. eq. (4.17).

— order Λ^6:

$$\Lambda^6\,\mathcal{K}_6 \overset{b\to 0}{\longrightarrow} \left(\frac{\hat{\Lambda}}{\epsilon_1}\right)^6 \left(\frac{\big(4\xi^2-8\xi+7\big)\,\mathrm{e}^{-2ix}}{4(\xi-1)(2\xi-1)^3(2\xi+1)} - \frac{\big(4\xi^2+8\xi+7\big)\,\mathrm{e}^{2ix}}{4(\xi+1)(2\xi-1)(2\xi+1)^3} + \frac{\mathrm{e}^{-6ix}}{12(\xi-1)(2\xi-3)(2\xi-1)} - \frac{\mathrm{e}^{6ix}}{12(\xi+1)(2\xi+1)(2\xi+3)}\right) \equiv \left(\frac{\hat{\Lambda}}{\epsilon_1}\right)^6 \mathcal{P}_6(\xi,x)\ ,$$

— order Λ^8:

$$\Lambda^8\,\mathcal{K}_8 \overset{b\to 0}{\longrightarrow} \left(\frac{\hat{\Lambda}}{\epsilon_1}\right)^8 \left(\frac{\big(2\xi^2-5\xi+5\big)\,\mathrm{e}^{-4ix}}{3(\xi-1)(2\xi-3)(2\xi-1)^3(2\xi+1)} + \frac{\big(2\xi^2+5\xi+5\big)\,\mathrm{e}^{4ix}}{3(\xi+1)(2\xi-1)(2\xi+1)^3(2\xi+3)} + \frac{\mathrm{e}^{-8ix}}{96(\xi-2)(\xi-1)(2\xi-3)(2\xi-1)} + \frac{\mathrm{e}^{8ix}}{96(\xi+1)(\xi+2)(2\xi+1)(2\xi+3)}\right) \equiv \left(\frac{\hat{\Lambda}}{\epsilon_1}\right)^8 \mathcal{P}_8(\xi,x)\ ,$$

— order Λ^{10}:

$$\begin{aligned}\Lambda^{10}\,\mathcal{K}_{10} \overset{b\to 0}{\longrightarrow} \left(\frac{\hat{\Lambda}}{\epsilon_1}\right)^{10} \Bigg(&\frac{\big(4\xi^2-12\xi+13\big)\,\mathrm{e}^{-6ix}}{32(\xi-2)(\xi-1)(2\xi-3)(2\xi-1)^3(2\xi+1)} \\ &- \frac{\big(4\xi^2+12\xi+13\big)\,\mathrm{e}^{6ix}}{32(\xi+1)(\xi+2)(2\xi-1)(2\xi+1)^3(2\xi+3)} \\ &+ \frac{\big(-32\xi^6-80\xi^5-64\xi^4-32\xi^3-94\xi^2+13\xi-116\big)\,\mathrm{e}^{2ix}}{6(\xi-1)(\xi+1)(2\xi-1)^3(2\xi+1)^5(2\xi+3)} \\ &+ \frac{\big(32\xi^6-80\xi^5+64\xi^4-32\xi^3+94\xi^2+13\xi+116\big)\,\mathrm{e}^{-2ix}}{6(\xi-1)(\xi+1)(2\xi-3)(2\xi-1)^5(2\xi+1)^3} \\ &+ \frac{\mathrm{e}^{-10ix}}{480(\xi-2)(\xi-1)(2\xi-5)(2\xi-3)(2\xi-1)} \\ &- \frac{\mathrm{e}^{10ix}}{480(\xi+1)(\xi+2)(2\xi+1)(2\xi+3)(2\xi+5)}\Bigg) \\ \equiv \left(\frac{\hat{\Lambda}}{\epsilon_1}\right)^{10} &\mathcal{P}_{10}(\xi,x)\ ,\end{aligned} \tag{4.45}$$

— order Λ^{12}:

$$\Lambda^{12}\,\mathcal{K}_{12}\xrightarrow{b\to0}\left(\frac{\hat\Lambda}{\epsilon_1}\right)^{12}\Bigg(\frac{\left(2\xi^2-7\xi+8\right)\mathrm{e}^{-8ix}}{60(\xi-2)(\xi-1)(2\xi-5)(2\xi-3)(2\xi-1)^3(2\xi+1)}$$
$$+\frac{\left(2\xi^2+7\xi+8\right)\mathrm{e}^{8ix}}{60(\xi+1)(\xi+2)(2\xi-1)(2\xi+1)^3(2\xi+3)(2\xi+5)}$$
$$+\frac{\left(704\xi^8-3904\xi^7+8528\xi^6-9440\xi^5+7780\xi^4-5876\xi^3+4739\xi^2-7672\xi+4817\right)\mathrm{e}^{-4ix}}{384(\xi-2)(\xi-1)^3(\xi+1)(2\xi-3)(2\xi-1)^5(2\xi+1)^3}$$
$$+\frac{\left(704\xi^8+3904\xi^7+8528\xi^6+9440\xi^5+7780\xi^4+5876\xi^3+4739\xi^2+7672\xi+4817\right)\mathrm{e}^{4ix}}{384(\xi-1)(\xi+1)^3(\xi+2)(2\xi-1)^3(2\xi+1)^5(2\xi+3)}$$
$$+\frac{\mathrm{e}^{12ix}}{5760(\xi-3)(\xi-2)(\xi-1)(2\xi-5)(2\xi-3)(2\xi-1)}$$
$$+\frac{\mathrm{e}^{-12ix}}{5760(\xi+1)(\xi+2)(\xi+3)(2\xi+1)(2\xi+3)(2\xi+5)}\Bigg)\equiv\left(\frac{\hat\Lambda}{\epsilon_1}\right)^{12}\mathcal{P}_{12}(\xi,x). \tag{4.46}$$

The above analysis yields the expansion:

$$\lim_{b\to0}\left(1+\frac{\Phi^{\mathbf{0}}_{r\neq s}(\hat\Lambda/(\epsilon_1 b),\mathrm{e}^{-2ix})}{\Phi^{\mathbf{0}}_{r=s}(\hat\Lambda/(\epsilon_1 b))}\right)=1+\left(\frac{\hat\Lambda}{\epsilon_1}\right)^2\mathcal{P}_2(\xi,x)+\left(\frac{\hat\Lambda}{\epsilon_1}\right)^4\mathcal{P}_4(\xi,x)$$
$$+\left(\frac{\hat\Lambda}{\epsilon_1}\right)^6\mathcal{P}_6(\xi,x)+\ldots+\left(\frac{\hat\Lambda}{\epsilon_1}\right)^{12}\mathcal{P}_{12}(\xi,x)+\ldots$$

which for $\xi=\nu/2$, $h=2\hat\Lambda/\epsilon_1$ and after multiplication by $\mathrm{e}^{i\nu x}$ (cf. eq. (4.17)) gives the sought eigenfunction:

$$\psi^{\mathbf{0}}_\nu(x)=\mathrm{e}^{i\nu x}+\frac{h^2}{4}\mathcal{R}_2(\nu,x)+\frac{h^4}{32}\mathcal{R}_4(\nu,x)+\frac{h^6}{128}\mathcal{R}_6(\nu,x)+\frac{h^8}{768}\mathcal{R}_8(\nu,x)+\ldots. \tag{4.47}$$

For instance, the coefficients $\mathcal{R}_n(\nu,x)$, $n=2,4,6,8$ explicitly read as follows[23]

$$\mathcal{R}_2=\frac{\mathrm{e}^{i(\nu-2)x}}{\nu-1}-\frac{\mathrm{e}^{i(\nu+2)x}}{\nu+1}\,,$$
$$\mathcal{R}_4=\frac{\mathrm{e}^{i(\nu+4)x}}{(\nu+1)(\nu+2)}+\frac{\mathrm{e}^{i(\nu-4)x}}{(\nu-2)(\nu-1)}\,,$$
$$\mathcal{R}_6=\frac{\left(\nu^2-4\nu+7\right)\mathrm{e}^{i(\nu-2)x}}{(\nu-2)(\nu-1)^3(\nu+1)}-\frac{\left(\nu^2+4\nu+7\right)\mathrm{e}^{i(\nu+2)x}}{(\nu-1)(\nu+1)^3(\nu+2)}$$
$$+\frac{\mathrm{e}^{i(\nu-6)x}}{3(\nu-3)(\nu-2)(\nu-1)}-\frac{\mathrm{e}^{i(\nu+6)x}}{3(\nu+1)(\nu+2)(\nu+3)}\,,$$
$$\mathcal{R}_8=\frac{\left(\nu^2-5\nu+10\right)\mathrm{e}^{i(\nu-4)x}}{(\nu-3)(\nu-2)(\nu-1)^3(\nu+1)}+\frac{\left(\nu^2+5\nu+10\right)\mathrm{e}^{i(\nu+4)x}}{(\nu-1)(\nu+1)^3(\nu+2)(\nu+3)}$$
$$+\frac{\mathrm{e}^{i(\nu-8)x}}{8(\nu-4)(\nu-3)(\nu-2)(\nu-1)}+\frac{\mathrm{e}^{i(\nu+8)x}}{8(\nu+1)(\nu+2)(\nu+3)(\nu+4)}\,.$$

[23]Coefficients $\mathcal{R}_n$, $n=10,12$ are given by (4.45), (4.46) for $\xi=\nu/2$, $h=2\hat\Lambda/\epsilon_1$.

Finally, let us observe that the combination $\frac{1}{2}(\psi^{\mathbf{0}}_\nu(x)+\psi^{\mathbf{0}}_\nu(-x))$ yields the *noninteger order* ($\nu\notin\mathbb{Z}$) *Mathieu cosine function:*[24]

$$\begin{aligned}\frac{1}{2}\left(\psi^{\mathbf{0}}_\nu(x)+\psi^{\mathbf{0}}_\nu(-x)\right) &= \cos(\nu x)-\frac{h^2}{4}\left[\frac{\cos((\nu+2)x)}{\nu+1}-\frac{\cos((\nu-2)x)}{\nu-1}\right]\\ &\quad+\frac{h^4}{32}\left[\frac{\cos((\nu-4)x)}{(\nu-2)(\nu-1)}+\frac{\cos((\nu+4)x)}{(\nu+1)(\nu+2)}\right]\\ &\quad+\frac{h^6}{128}\left[\frac{\left(\nu^2-4\nu+7\right)\cos((\nu-2)x)}{(\nu-2)(\nu-1)^3(\nu+1)}-\frac{\left(\nu^2+4\nu+7\right)\cos((\nu+2)x)}{(\nu-1)(\nu+1)^3(\nu+2)}\right.\\ &\qquad\left.+\frac{\cos((\nu-6)x)}{3(\nu-3)(\nu-2)(\nu-1)}-\frac{\cos((\nu+6)x)}{3(\nu+1)(\nu+2)(\nu+3)}\right]+\ldots\\ &\equiv \mathrm{ce}_\nu(x,h^2).\end{aligned}$$

Therefore, the eigenfunction $\psi^{\mathbf{0}}_\nu(x)$ given by our formulae (4.17), (4.47) is nothing but the Floquet solution $\mathrm{me}_\nu(x,h^2)$, $\nu\notin\mathbb{Z}$ known as the *Mathieu exponent* (cf. example 17.1 in [50]).[25] The second solution is of the form $\psi^{\mathbf{0}}_\nu(-x)\equiv\mathrm{me}_\nu(-x,h^2)$ and it is the Floquet solution for $-\nu$ and the same eigenvalue (4.18). It is known that $\mathrm{me}_\nu(x,h^2)$ and $\mathrm{me}_\nu(-x,h^2)$ obey (http://dlmf.nist.gov/28.12.iii):

$$\begin{aligned}\mathrm{ce}_\nu(x,h^2) &= \frac{1}{2}\left(\mathrm{me}_\nu(x,h^2)+\mathrm{me}_\nu(-x,h^2)\right),\\ \mathrm{se}_\nu(x,h^2) &= \frac{1}{2}i\left(\mathrm{me}_\nu(x,h^2)-\mathrm{me}_\nu(-x,h^2)\right).\end{aligned}\tag{4.49}$$

Functions ce_ν and se_ν constitute another fundamental system of solutions.

So far only the solutions of the noninteger order ($\nu\notin\mathbb{Z}$) have been discussed. Hence, the question arises at this point of how to get from the classical limit of the irregular block the Mathieu eigenvalues and eigenfunctions corresponding to the integer values of the Floquet exponent. Recall, such solutions are periodic.[26] In particular, one can construct the solutions of periods π or 2π ($q=h^2$):

— the cosine-elliptic $\mathrm{ce}_m(x;q)$, $m=0,1,2,\ldots$, that corresponds at $q=0$ with $\cos mx$, for instance:

$$\begin{aligned}\mathrm{ce}_1(x;q) &= \cos x-\frac{1}{8}q\cos 3x+\frac{1}{64}q^2\left(\frac{1}{3}\cos 5x-\cos 3x\right)\\ &\quad-\frac{1}{512}q^3\left(\frac{1}{3}\cos 3x-\frac{4}{9}\cos 5x+\frac{1}{18}\cos 7x\right)+\ldots;\end{aligned}$$

[24] Cf. subsection 2.16 in a book by McLachlan [72], see also example 17.1 in [50].

[25] The coefficient $\mathcal{R}_4$ in our formula for $\psi^{\mathbf{0}}_\nu(x)$ differs from that presented in http://dlmf.nist.gov/28.15, where

$$\begin{aligned}\mathrm{me}_\nu(x,h^2) &= \mathrm{e}^{i\nu x}-\frac{h^2}{4}\left(\frac{1}{\nu+1}\mathrm{e}^{i(\nu+2)x}-\frac{1}{\nu-1}\mathrm{e}^{i(\nu-2)x}\right)\\ &\quad+\frac{h^4}{32}\left(\frac{1}{(\nu+1)(\nu+2)}\mathrm{e}^{i(\nu+4)x}+\frac{1}{(\nu-1)(\nu-2)}\mathrm{e}^{i(\nu-4)x}-\frac{2(\nu^2+1)}{(\nu^2-1)^2}\mathrm{e}^{i\nu x}\right)+\ldots.\end{aligned}\tag{4.48}$$

However, the expression (4.48) does not satisfy (4.49).

[26] Cf. appendix B.

— the sine-elliptic $\mathrm{se}_m(x;q)$, $m=1,2,\ldots$, that corresponds at $q=0$ with $\sin mx$, for example:

$$\begin{aligned}\mathrm{se}_1(x;q) &= \sin x-\frac{1}{8}q\sin 3x+\frac{1}{64}q^2\left(\sin 3x+\frac{1}{3}\sin 5x\right)\\&\quad-\frac{1}{512}q^3\left(\frac{1}{3}\sin 3x-\frac{4}{9}\sin 5x+\frac{1}{18}\sin 7x\right)+\ldots.\end{aligned}$$

The functions ce_m and se_m have period π if m is even and period 2π if m is odd. The corresponding eigenvalues λ denoted by $a_m(q)$ for ce_m and $b_m(q)$ for se_m are called *characteristic numbers*, e.g.:

$$\begin{aligned}a_1(q) &= 1+q-\frac{1}{8}q^2-\frac{1}{64}q^3-\frac{1}{1536}q^4+\frac{11}{36864}q^5+\ldots,\\ b_1(q) &= 1-q-\frac{1}{8}q^2+\frac{1}{64}q^3-\frac{1}{1536}q^4-\frac{11}{36864}q^5+\ldots.\end{aligned}$$

For any $q>0$ characteristic numbers form the band/gap structure: $a_0<b_1<a_1<b_2<a_2\ldots$. For large m the leading terms of the a_m and b_m are (http://dlmf.nist.gov/28.6.E14):

$$\left.\begin{matrix}a_m(q)\\ b_m(q)\end{matrix}\right\}=m^2+\frac{q^2}{2\left(m^2-1\right)}+\frac{\left(5m^2+7\right)q^4}{32\left(m^2-4\right)\left(m^2-1\right)^3}+\frac{\left(9m^4+58m^2+29\right)q^6}{64\left(m^2-9\right)\left(m^2-4\right)\left(m^2-1\right)^5}+\ldots.$$

Notice that for $m=\nu$ the above expression matches (B.5) and therefore can be recovered from the classical irregular block with $\delta=\frac{1}{4}(1-m^2)$ (cf. (4.16), (4.18)) or from the gauge theory counterpart of (4.16), i.e.:

$$\frac{1}{\epsilon_1}\,\hat{\Lambda}\partial_{\hat{\Lambda}}W^{\mathrm{SU}(2),N_f=0}\left(\hat{\Lambda},a,\epsilon_1\right),$$

where $a=\frac{1}{2}m\epsilon_1$, cf. [1, 44]. To conclude, regardless of the coincidence described above, work is in progress in order to find a mechanism which allows to derive from the conformal blocks the eigenvalues in the case of the finite integer values of m and the corresponding integer order solutions.

5 Concluding remarks and open problems

In the present paper we have shown that the $N_f=0$ classical irregular block solves the eigenvalue problem for the Mathieu operator. The statement that the Mathieu eigenvalue for small $h^2=4\hat{\Lambda}^2/\epsilon_1^2$ (weakly-coupled region) and noninteger characteristic exponent $\nu\notin\mathbb{Z}$ is determined by the classical zero flavor irregular block has been already stated in our previous work [1]. The new result of the present work is the derivation of an expression of the corresponding eigenfunction. Moreover, it has been shown that the formula (4.17) reproduces the known solution of the Mathieu equation with the eigenvalue (4.18). Therefore, we have established a link between the Mathieu equation and its realization within two-dimensional CFT. This result paves the way for a new interesting line of research. Concretely, one can try to study other regions of the spectrum of the Mathieu operator by

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means of 2d CFT tools. Indeed, two interesting questions arise at this point: (i) How is it possible to derive from the irregular block the solutions with integer values of the Floquet parameter? (ii) How within 2d CFT one can get the solutions in the other regions of the spectrum? The answer to the first question needs more studies. It seems that also the second question is reasonable. Let us remember that the quantum irregular block can be obtained from the four-point block on the sphere in the so-called decoupling limit of the external conformal weights (cf. eq. (3.8)). However, to our knowledge, such limit has been discussed only for the s-channel four-point block [38]. Therefore, it arises at this point the question of what happens if we take the decoupling limit from the four-point blocks in the other channels, i.e. t or u. The latter case appears to be especially interesting since in the u-channel four-point block the invariant ratio is $\frac{1}{x}$. Hence, the question is whether we can obtain the irregular block with the expansion parameter $\frac{1}{\Lambda} = (\frac{\hat{\Lambda}}{\epsilon_1 b})^{-1}$. Secondly, if this is possible, do we get in the classical limit the classical irregular block determining the strongly-coupled region of the Mathieu spectrum? In addition, let us note that the s and u-channel four-point blocks are related by the *braiding relation* which, in general, has the form of an integral transform with a complicated kernel — the so-called *braiding matrix* (cf. e.g. [74]). However, when one of the four external conformal weights becomes degenerate, then the integral transform reduces to the known formula for the analytic continuation of the hypergeometric function from the vicinity of a point x to that of the point $\frac{1}{x}$. It seems to be technically possible to take combinations of the decoupling and classical limits on both sides of the braiding relation (at least) in the degenerate case and to obtain as a result a 'duality relation' for the classical irregular blocks. As one can expect, in this way it becomes feasible to establish a formula that continues the Mathieu eigenvalue from the weakly-coupled region to that of strong coupling. Work is in progress in order to verify this hypothesis.

The derivation of the Mathieu equation within the formalism of two-dimensional conformal field theory is based on conjectures concerning the asymptotic behavior of irregular blocks in the classical limit, cf. eqs. (3.13) and (4.7). We recall that the coefficient of the irregular block expansion is a ratio of polynomials in Δ and c. Our idea of proving the existence of the classical irregular block was to estimate the degree of the irregular block coefficient as a polynomial in the parameter b. To accomplish this task we used methods which had previously been used to prove the Kac determinant formula, cf. [63]. However, these techniques have proven to be too weak to give a complete answer. As a result, we have obtained the classical irregular block at the leading order only. It is therefore necessary to use other methods to try to prove eq. (3.13). It seems that there are two available approaches to solve this problem. The first way is to use a representation of the Gaiotto states in terms of Jack polynomials. As S. Yanagida has shown in ref. [73], it is possible to represent the Gaiotto state for the pure gauge theory using the Jack polynomials. The coefficients relating the Jack polynomials to the Gaiotto state are found explicitly in [73]. The inner product, by means of which one computes the norm of the pure gauge Gaiotto state, induces by the bosonization map and the parameter dependent isomorphism, the inner product in the space spanned by the Jack polynomials. However, these Jack polynomials are not orthogonal within this inner product and one has to expand them in a new basis of

Jack polynomials depending on a different parameter and orthogonal with respect to the induced inner product. The coefficients relating the two bases of the Jack polynomials are not known explicitly. The lack of the explicit form of the coefficients hampers the effort to find the classical limit of the pure gauge conformal block.

The second and in our opinion more promising way to get eq. (3.13) in particular and more in general analogous results for the regular blocks, is the application of the Fock space free field realization of the conformal blocks. In the case of discrete spectrum this approach is even mathematically rigorous [75–78]. As a result one gets the Dotsenko-Fateev(-like) integral representations of the conformal blocks and hence one can try to use the methods of matrix models (beta-ensembles) in the computation of the classical limit of these blocks, cf. [80]. However, it should be stressed that the link between the integral and power series representations of conformal blocks is not completely understood cf. e.g. [81]. In our opinion, also the operator realization of conformal blocks requires further work. For instance, the Fock space representation of the chiral vertex operator with the three independent general conformal weights, whose compositions in the matrix element lead to integral formulas, needs to be developed, cf. [56, 57].

The main claim of this work — formulas (4.16) and (4.17) — possess fascinating generalizations. The simplest extension is to consider irregular blocks with $N_f = 1, 2$ flavors.[27] As a preview of the results which will be reported in our next papers let us only mention that also in the cases $N_f = 1, 2$ the classical limit of the irregular blocks exists and yields a consistent definition of the classical blocks. In these cases we have also found explicit formulas for the eigenvalues and the eigenfunctions of operators emergent in the classical limit of the null vector decoupling equations. For $N_f = 2$ one gets a Schrödinger operator containing a generalization of the Mathieu potential. These further developments will confirm the validities of $2d$ CFT technics in the investigation of different regions of the spectra of the investigated operators.

A Coefficients $F^{(r,s)}$

The first few coefficient $F_c^{(r,s)}(\Delta_1, \Delta_2, \Delta_3)$, e.g.:

$$
\begin{aligned}
F^{(1,0)} &= \frac{\Delta_1 + \Delta_2 - \Delta_3}{2\Delta_1}, \\
F^{(0,1)} &= \frac{-\Delta_1 + \Delta_2 + \Delta_3}{2\Delta_3}, \\
F^{(2,0)} &= \frac{\frac{1}{2}(\Delta_1 + \Delta_2 - \Delta_3)(\Delta_1 + \Delta_2 - \Delta_3 + 1)(c + 8\Delta_1) - 6\Delta_1(\Delta_1 + 2\Delta_2 - \Delta_3)}{2\Delta_1(2\Delta_1(c + 8\Delta_1 - 5) + c)}, \\
F^{(0,2)} &= \frac{\frac{1}{2}(\Delta_1 - \Delta_2 - \Delta_3 - 1)(\Delta_1 - \Delta_2 - \Delta_3)(c + 8\Delta_3) - 6\Delta_3(-\Delta_1 + 2\Delta_2 + \Delta_3)}{2\Delta_3(2\Delta_3(c + 8\Delta_3 - 5) + c)},
\end{aligned}
$$

and next up to the order Λ^8 (cf. (4.44)) can be easily and quickly computed using computer. The time of computation dramatically increases for the coefficients appearing in higher orders of the expansion (4.44) and results become very complicated. However, it seems to

[27] By the time this research was in progress the paper [79] appeared which mentioned this problem.

be possible to find recurrence relations for $F_c^{(r,s)}(\Delta_1, \Delta_2, \Delta_3)$ which really would improve speed of calculations and as a result it would gave an efficient method of calculation of the Mathieu function.

B Mathieu equation

The standard form of the Mathieu equation with parameters (a, q) (DLMF eq. (28.2)) or equivalently [50] (λ, h^2) reads as follows

$$\psi'' + (a - 2q\cos 2z)\,\psi \;=\; 0 \quad \leftrightarrow \quad \psi'' + \left(\lambda - 2h^2\cos 2z\right)\psi \;=\; 0. \tag{B.1}$$

A solution ψ with given initial constant values of ψ and ψ' at some point z_0 is an entire function of the three variables: z, a, q $(\Leftrightarrow z, \lambda, h^2)$.

The Floquet theorem states that the Mathieu eq. (B.1) has a nontrivial solution $\psi(z)$ such that

$$\psi(z+\pi) \;=\; \sigma\,\psi(z) \tag{B.2}$$

with σ being a root of the eq.:

$$\begin{vmatrix} \psi_1(\pi) - \sigma & \psi_2(\pi) \\ \psi_1'(\pi) & \psi_2'(\pi) - \sigma \end{vmatrix} \;=\; 0\;,$$

where $\psi_1(z)$ and $\psi_2(z)$ are even and odd, respectively, normalized $(\psi_1\psi_2' - \psi_1'\psi_2)\Big|_{z=0} = 1$ linearly independent solutions. Equivalently, the coefficients c_1 and c_2 in the solution $\psi = c_1\psi_1 + c_2\psi_2$, which obeys the condition (B.2), are given as an eigenvector of eq.:

$$\begin{pmatrix} \psi_1(\pi) & \psi_2(\pi) \\ \psi_1'(\pi) & \psi_2'(\pi) \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \;=\; \sigma \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}\;.$$

A solution of the Mathieu eq. in the form given by the Floquet theorem is called a Floquet solution. In order to gain more understanding of the Floquet solution let us define the quantities ν and y such that

$$\sigma \;=\; \mathrm{e}^{i\pi\nu}, \qquad y(z) \;=\; \mathrm{e}^{-i\nu z}\,\psi(z),$$

where $\psi(z)$ fulfills (B.2). The definition has the effect that

$$y(z+\pi) = \mathrm{e}^{-i\nu(z+\pi)}\,\psi(z+\pi) = \mathrm{e}^{-i\nu z}\psi(z) = y(z)$$

which shows that $y(z)$ is a periodic function of z with period π. Moreover,

$$\psi(z) \;=\; \mathrm{e}^{i\nu z}\,y(z) \tag{B.3}$$

showing that a Floquet solution $\psi(z)$ consists of a periodic function of z multiplied by a complex exponential in z. The quantity ν which controls the exponential behavior is known as the characteristic or Floquet exponent of ψ.

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The Floquet exponent ν is determined by the eq.:

$$\cos\pi\nu \;=\; \psi_1(\pi;a,q). \tag{B.4}$$

Eq. (B.4) allows to express the eigenvalue a (or λ) in terms of q (or h^2) and Floquet parameter ν. However, usefulness of eq. (B.4) is restricted by an ability to calculate the normalized Mathieu function ψ_1. The eigenfunction ψ_1 can be found as an expansion in terms of other functions, in particular, in terms of trigonometric functions. Indeed, the meaning of eq. (B.4) is that the Floquet exponent is determined by the value at $z=\pi$ of the solution which is even around $z=0$. To the lowest order, i.e. for $q=h^2=0$, the even solution around $z=0$ is $\psi_1^{(0)}(z)=\cos\sqrt{\lambda}z$. Therefore, from (B.4) for $q=h^2=0$ one gets $\nu=\sqrt{\lambda}$, and more in general $\nu^2=\lambda+O\big(h^2\big)$. One can derive various terms of this expansion perturbatively. Indeed, for small $q=h^2$ the eigenvalue λ as a function of ν and h^2 explicitly reads as follows (DLMF eq. 28.15 and cf. example 17.1 in [50])

$$\lambda_\nu(h^2) \;=\; \nu^2+\frac{h^4}{2\left(\nu^2-1\right)}+\frac{\left(5\nu^2+7\right)h^8}{32\left(\nu^2-4\right)\left(\nu^2-1\right)^3}+\frac{\left(9\nu^4+58\nu^2+29\right)h^{12}}{64\left(\nu^2-9\right)\left(\nu^2-4\right)\left(\nu^2-1\right)^5}+\dots\,. \tag{B.5}$$

The expansion (B.5) holds for noninteger values of $\nu\notin\mathbb{Z}$. The corresponding eigenfunction for small $q=h^2$ and $\nu\notin\mathbb{Z}$ is of the form

$$\mathrm{me}_\nu(z,h^2)=\mathrm{e}^{i\nu z}-\frac{h^2}{4}\left(\frac{1}{\nu+1}\mathrm{e}^{i(\nu+2)z}-\frac{1}{\nu-1}\mathrm{e}^{i(\nu-2)z}\right)+\dots\,.$$

The Mathieu equation admits periodic solutions. Indeed, the Floquet solution will be periodic for special values of σ. A necessary condition for periodicity is that $|\sigma|=1$. Since the Floquet solution (B.3) contains the factor y that is periodic with period π, ψ will be periodic with period

a) π if $\nu=0,2,\dots\Leftrightarrow\sigma=1$,
b) 2π if $\nu=1,3,\dots\Leftrightarrow\sigma=-1$,
c) $s\pi$ if $\nu=2r/s$, where $r,s>2$ are integers with no common divisors.

For physical reasons the solutions of most importance are those with periods π or 2π, and these are the ce_m and se_m introduced in the main text.

Acknowledgments

The authors are grateful to Franco Ferrari for useful discussions, very valuable advices and careful reading of the manuscript. M.P. is also grateful to Franco Ferrari for his kind hospitality during stays in Szczecin.

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