Gravitational memory, BMS supertranslations and soft theorems

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Abstract: The transit of a gravitating radiation pulse past arrays of detectors stationed near future null infinity in the vacuum is considered. It is shown that the relative positions and clock times of the detectors before and after the radiation transit differ by a BMS supertranslation. An explicit expression for the supertranslation in terms of moments of the radiation energy flux is given. The relative spatial displacement found for a pair of nearby detectors reproduces the well-known and potentially measurable gravitational memory effect. The displacement memory formula is shown to be equivalent to Weinberg's formula for soft graviton production.

Contents

1 Introduction

The passage of a finite pulse of radiation or other forms of energy through a region of spacetime produces a gravitational field which moves nearby detectors. The final positions of a pair of nearby detectors are generically displaced relative to the initial ones according to a simple and universal formula [1–11]. This effect is known as gravitational memory. Direct measurement of the gravitational memory effect may be possible in the coming decades, see e.g. the recent work [12, 13].

According to Bondi, Metzner, van der Burg and Sachs (BMS) [14, 15], the classical vacuum in general relativity is highly degenerate. The different vacua are related by the so-called 'supertranslations', which are spontaneously broken 'BMS symmetries'. In quantum language, these vacua differ by the addition of soft (*i.e* zero-energy) gravitons. In this paper we will show that the passage of radiation through a region induces a transition from one such vacuum to another. An explicit formula (involving moments of the radiation energy flux) is derived for the BMS supertranslation which relates the initial and final vacua. Moreover, relative positions and clock times of a family of detectors stationed in the vacuum are shown to be related by the same supertranslation. This observation provides a concrete operational meaning to BMS transformations.

The relative spatial displacement of nearby detectors following from the radiation-induced BMS transformation is precisely the standard gravitational memory. We find that

certain families of nearby detectors undergo, in addition to the standard spatial memory displacement, a relative time delay. It would be of interest to investigate potential experimental consequences.

Recently it has been shown [16] that the the observable consequences of BMS symmetry are embodied in the soft graviton scattering amplitudes which they universally determine. Herein we show that the Weinberg formula [17] for soft graviton production is essentially a rewriting of the formula for gravitational memory, establishing compatibility of [16] with the current work. However, while it is quite difficult to imagine a real experiment which directly measures soft gravitons, there is already a sizable literature on observation of gravitational memory. Hence the memory effect provides both a conceptually and observationally useful reformulation of BMS symmetry.

This paper is organized as follows. Section 2 establishes notation and briefly reviews BMS supertranslations. In section 3 we show that a finite duration radiation pulse crossing null infinity can be viewed as a domain wall mediating a transition between two inequivalent vacua of the gravitational field. Our central formula (3.7) is derived, involving the convolution of the radiative energy flux with a Green function, for the specific supertranslation which relates the initial and final vacua. Section 4 considers the effects of this transition on two types (inertial and fixed-angle) of detectors, and show that it can be understood as a supertranslation acting on the detector worldlines and clocks. This is shown to reproduce the spatial gravitational memory effect. It further elucidates a clock desynchronization effect with potentially observable consequences for (fixed-angle) detectors. In section 5 we show that the gravitational memory formula, in the form given by Braginsky and Thorne [3] is, after a change of variables and notation, identical to Weinberg's soft graviton formula. In section 6 we point out that black holes are not invariant under supertranslations and therefore, in tension with the standard lore, carry an infinite amount of hair which encodes memories of how they were formed. Appendix A contains details on subleading corrections to large-radius geodesics. Appendix B demonstrates compatibility of our results with the interesting recent analyses by Tolish et al. [10, 11] of a specific example of the memory effect.

The existence of a connection between gravitational memory and BMS symmetry is known and has been discussed periodically: see for example [18, 19]. We expect the relation between asymptotic symmetries and memory to extend to other systems such as gauge theories. In particular in gauge theories the passage of charge through $\mathcal{I}^+$ should be remembered by angle-dependent gauge transformations on charged detectors.

2 BMS review

The metric of an asymptotically flat spacetime in retarded Bondi coordinates takes the asymptotic form

$$
\begin{aligned}
ds^2 = &-du^2 - 2dudr + 2r^2\gamma_{z\bar{z}}dzd\bar{z} \\
&+2\frac{m_B}{r}du^2 + rC_{zz}dz^2 + rC_{\bar{z}\bar{z}}d\bar{z}^2 + D^zC_{zz}dudz + D^{\bar{z}}C_{\bar{z}\bar{z}}dud\bar{z} + \ldots
\end{aligned} \tag{2.1}
$$

where $\gamma_{z\bar{z}} = \frac{2}{(1+z\bar{z})^2}$ is the unit metric on S^2, D_z is the γ-covariant derivative and subleading terms are suppressed by powers of r.[1] The Bondi mass aspect m_B and C_{zz} are related by the constraint equation $G_{uu} = 8\pi G T_{uu}^M$ on $\mathcal{I}^+$

$$\partial_u m_B = \frac{1}{4}\left[D_z^2 N^{zz} + D_{\bar{z}}^2 N^{\bar{z}\bar{z}}\right] - T_{uu},$$

$$T_{uu} = \frac{1}{4}N_{zz}N^{zz} + 4\pi G \lim_{r\to\infty}[r^2 T_{uu}^M]. \tag{2.2}$$

where $N_{zz} = \partial_u C_{zz}$ is the Bondi news, T^M is the matter stress tensor and T_{uu} is the total energy flux through a given point on $\mathcal{I}^+$. The asymptotic form of the metric (2.1) is preserved by infinitesimal supertranslations [14, 15]

$$u \to u - f, \qquad\qquad r \to r - D^z D_z f,$$

$$z \to z + \frac{1}{r}D^z f, \qquad\qquad \bar{z} \to \bar{z} + \frac{1}{r}D^{\bar{z}} f, \qquad\qquad f = f(z, \bar{z}), \tag{2.3}$$

whose generating vector fields we denote

$$\zeta_f = f\partial_u + D^z D_z f\partial_r - \frac{1}{r}(D^{\bar{z}} f\partial_{\bar{z}} + D^z f\partial_z). \tag{2.4}$$

The Lie derivative action on the asymptotic data is

$$\mathcal{L}_f m_B = f\partial_u m_B,$$

$$\mathcal{L}_f C_{zz} = fN_{zz} - 2D_z^2 f. \tag{2.5}$$

According to BMS [14, 15] two spacetimes related by supertranslations should be regarded as physically inequivalent in the sense that they correspond to different points in the phase space.

3 BMS vacuum transitions

Consider spacetimes which, prior to some retarded time u_i on $\mathcal{I}^+$, are asymptotically well-approximated by Schwarzschild with

$$m_B = M_i = \text{constant}, \qquad C_{zz} = 0, \tag{3.1}$$

while for $u > u_f$ they are also nearly asymptotically Schwarzschild[2]

$$m_B = M_f = \text{constant}, \qquad C_{zz} \neq 0. \tag{3.2}$$

During the intermediate interval $u_i < u < u_f$ the Bondi news and/or total radiation flux T_{uu} is nonzero on $\mathcal{I}^+$.[3] Christodoulou and Klainerman [20] considered spacetimes of this type with $M_f = 0$, where u_i and u_f must be taken early and late enough to capture most of the long time tails. For nonzero M_f the late time geometry could for example be a stable star or black hole.

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[1]In particular we have corrections $\frac{1}{4r^2}C_{zz}C^{zz}dudr + \gamma_{z\bar{z}}C_{zz}C^{zz}dzd\bar{z}$ which contribute to the Einstein equations at the same order.

[2]We exclude for simplicity cases with nonzero initial or final ADM momentum.

[3]Generic spacetimes may have long time radiation tails outside this interval, but for our purposes making the radiation flux outside the interval arbitrarily small is good enough.
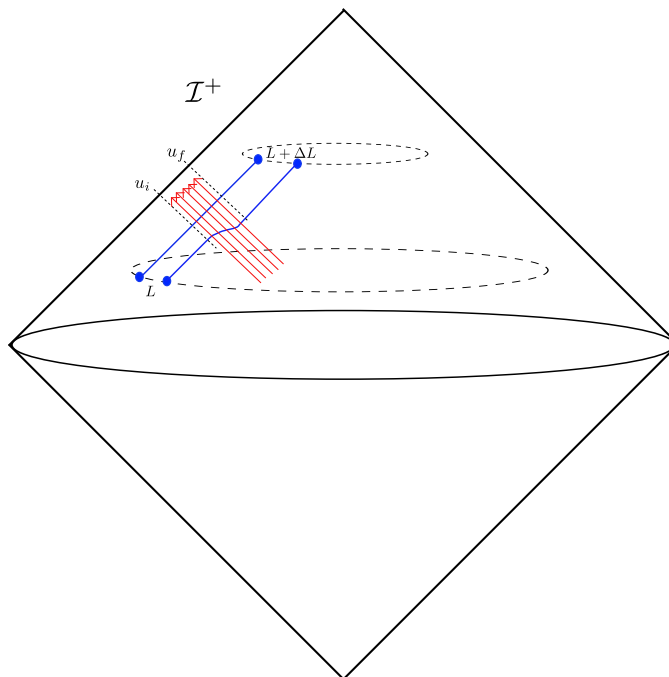
Figure 1. Remembrance of things passed. We consider a transit of radiation through a set of detectors in the vicinity of the future null infinity $\mathcal{I}^+$. Detectors are located at large r_0 and inserted at different points on the sphere S^2 separated by distance L. Change in the vacuum state is detected by the net displacement ΔL. The new vacuum is related to the old one by the supertranslation $C(z, \bar{z})$.

The initial and final regions of $\mathcal{I}^+$ before u_i and after u_f are in the vacuum in the sense that $N_{zz} = 0$: the radiative modes are unexcited. According to BMS, the vacuum is not unique. It is characterized by any u-independent C_{zz} obeying

$$D_{\bar{z}}^2 C_{zz} - D_z^2 C_{\bar{z}\bar{z}} = 0. \tag{3.3}$$

The general solution to this equation is

$$C_{zz} = -2D_z^2 C(z, \bar{z}). \tag{3.4}$$

Comparison with (2.5) implies that the different vacua are related by supertranslations under which $C \to C + f$. The supertranslation which relates the initial and final vacua can be determined by integrating the constraint (2.2) over the transition interval $u_i < u < u_f$. Defining

$$\Delta C_{zz} = C_{zz}(u_f) - C_{zz}(u_i), \quad \Delta m_B = M_f - M_i, \tag{3.5}$$

and using (3.3) one finds

$$D_z^2 \Delta C^{zz} = 2 \int_{u_i}^{u_f} du \, T_{uu} + 2\Delta m_B. \tag{3.6}$$

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Note that the second term just subtracts the constant zero mode of the first.[4] The supertranslation ΔC which produces such a ΔC_{zz} is obtained by inverting $D_z^2 D_{\bar z}^2$:

$$\Delta C(z, \bar z) = 2 \int d^2 z' \gamma_{z' \bar z'} G(z, \bar z; z', \bar z') \left(\int_{u_i}^{u_f} du\, T_{uu}(z', \bar z') + \Delta m_B \right) \tag{3.7}$$

where

$$G(z, \bar z; z', \bar z') = -\frac{1}{\pi} \sin^2 \frac{\Theta}{2} \log \sin^2 \frac{\Theta}{2}, \quad \sin^2 \frac{\Theta(z, z')}{2} \equiv \frac{|z - z'|^2}{(1 + z' \bar z')(1 + z \bar z)},$$

$$D_z^2 D_{\bar z}^2 G(z, \bar z; z', \bar z') = -\gamma_{z \bar z} \delta^2(z - z') + \cdots. \tag{3.8}$$

If we plug (3.8) into (3.7) and act with $D_z^2 D_{\bar z}^2$ using $\partial_z \partial_{\bar z} \log |z - z_i|^2 = 2\pi\, \delta^{(2)}(z - z_i)$ the delta function piece produces the r.h.s. of (3.6) while the remaining terms integrate to zero due to the energy-momentum conservation. $C(z, \bar z)$ is unique up to the 4 global spacetime translations

$$f_{global} = c_0 + \frac{c_1(1 - z \bar z) + c_2 z + c_3 \bar z + \bar c_3 z}{1 + z \bar z} \tag{3.9}$$

which do not affect C_{zz}.

This discussion could be generalized to allow for initial and final momentum, or multiple vacuum transitions induced by multiple well-separated radiation intervals.

To summarize, the passage of radiation through $\mathcal{I}^+$ changes the vacuum by a BMS transformation. The BMS transformation relating the initial and final vacuum is given in (3.7) by an integral of the total radiation flux over the transition interval.

4   Gravitational memory

In this section we will relate the BMS transformation of the vacuum to the gravitational memory effect. Towards this end we introduce two families of observers or detectors at large r. The first, which we refer to as BMS (or fixed-angle) detectors, travel along worldlines at fixed radius and angle:

$$X_{\rm BMS}^\mu(s) = (s, r_0, z_0, \bar z_0), \tag{4.1}$$

where r_0 is large. The assertion that BMS diffeomorphisms are physically nontrivial is equivalent to the statement that it is meaningful to discuss observations at a fixed value of z near $\mathcal{I}^+$. Such observations are convenient as they behave simply under the action of BMS. The second family of detectors are inertial ones moving along geodesics

$$\partial_s^2 X_{\rm geo}^\mu(s) + \Gamma_{\nu\lambda}^\mu \partial_s X_{\rm geo}^\nu(s) \partial_s X_{\rm geo}^\lambda(s) = 0. \tag{4.2}$$

At large r_0 the BMS detectors are nearly inertial. One may readily check (see appendix A) that

$$X_{\rm BMS}^{u,r}(s) = X_{\rm geo}^{u,r}(s) + \mathcal{O}\left(\frac{1}{r_0}\right), \qquad X_{\rm BMS}^z(s) = X_{\rm geo}^z(s) + \mathcal{O}\left(\frac{1}{r_0^2}\right). \tag{4.3}$$

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[4]Had we allowed for net momentum loss as well as energy loss the right hand side would also contain a term subtracting the angular momentum $\ell = 1$ mode.

The truly inertial detectors however do not remain at fixed r or z, so over a long period of time $u > r_0$ the radius can become small. Hence we must consider only retarded time lapses which are parametrically less than r_0.

The relevant type of detector — BMS or inertial — depends on the application in question. For example the eLisa detectors move on geodesic orbits and so are perhaps best modeled by inertial detectors. On the other hand the LIGO detector are at fixed separations on the earth and are not geodesic. It would be interesting to understand what type of detector array is well-approximated by BMS detectors.

4.1 BMS detector memory

Let us now consider what happens to the BMS detector worldlines in the setup of the previous section when they encounter a pulse of radiation passing to $\mathcal{I}^+$. Let us denote the initial positions of a pair of nearby detectors, detector 1 and detector 2, by z_1 and z_2. They are initially separated by a finite distance

$$L = \frac{2r_0|\delta z|}{1 + z_1 \bar{z}_1}, \quad \delta z \equiv z_1 - z_2 \tag{4.4}$$

where we take δz to be order $\frac{1}{r_0}$ and subleading corrections to L are suppressed. As $z_{1,2}$ are fixed in (4.1), but the metric undergoes a transition described by (3.5), the radiation induces a change in the proper distance between the detectors. Computing the new distance between z_1 and z_2 using the metric (2.1) gives

$$\Delta L = \frac{r_0}{2L} \Delta C_{zz}(z_1, \bar{z}_1)\delta z^2 + \text{c.c.} = \frac{(1 + z_1\bar{z}_1)^2}{8} \frac{L}{r_0} \left(\Delta C_{zz}(z_1, \bar{z}_1)\frac{\delta z}{\delta \bar{z}} + \text{c.c.} \right), \tag{4.5}$$

where $\Delta C_{zz}(z_1, \bar{z}_1)$ is given according to (3.7) in terms of the energy flux as

$$\Delta C_{zz}(z, \bar{z}) = \frac{4}{\pi} \int d^2z' \gamma_{z'\bar{z}'} \frac{\bar{z} - \bar{z}'}{z - z'} \frac{(1 + z'\bar{z})^2}{(1 + z'\bar{z}')(1 + z\bar{z})^3} \left(\int_{u_i}^{u_f} du\, T_{uu}(z', \bar{z}') + \Delta m_B \right) \tag{4.6}$$

This is precisely the standard formula for gravitational memory [1, 4, 5, 8].

Not only will the distances between BMS detectors be shifted, but if they are equipped with initially synchronized clocks they will no longer be synchronized after passage of the radiation. This can be checked by sending a light ray from detector 1 to detector 2, stamping it with the time at detector 2 and then returning it to detector 1. If the clocks remain synchronized, the time stamp from detector 2 will be exactly midway between the light emission and reception times at detector 1. A light ray emitted from z_1 will travel to z_2 in a retarded time interval $\delta_{12}u$ obeying

$$2r_0^2 \gamma_{z\bar{z}} \delta z \delta \bar{z} + r_0 \Delta C_{zz} \delta z \delta z + D^z \Delta C_{zz} \delta_{12}u \delta z - (\delta_{12}u)^2 + \text{c.c.} = 0. \tag{4.7}$$

On the other hand, on the return trip, the change in z has the opposite sign so the retarded time interval $\delta_{21}u$ obeys

$$2r_0^2 \gamma_{z\bar{z}} \delta z \delta \bar{z} + r_0 \Delta C_{zz} \delta z \delta z - D^z \Delta C_{zz} \delta_{21}u \delta z - (\delta_{21}u)^2 + \text{c.c.} = 0. \tag{4.8}$$

The difference is[5]

$$\delta_{12}u - \delta_{21}u = D^z \Delta C_{zz} \delta z + \text{c.c.}. \tag{4.9}$$

Since this is nonzero the clocks are not synchronized.

An alternate way of computing the memory and clock desynchronization is as follows. The proper distance and time delay observed in the above mentioned experiments are invariant under all diffeomorphisms, including BMS transformations. We may therefore eliminate all ΔC_{zz} terms in the late time metric by the inverse of the BMS transformation (3.7) which by construction obeys

$$2D_z^2 f = \Delta C_{zz}, \tag{4.10}$$

so that $f = -\Delta C$ This will have the effect of resetting all the clocks and relabeling the positions of the family of BMS observers by (2.3). To see that this agrees with the previous analysis let $\zeta = \delta z \partial_z + \delta \bar{z} \partial_{\bar{z}}$ denote the initial separation vector between the detectors. Using (2.4) the action of the supertranslation (2.3) on this separation is

$$\mathcal{L}_f \zeta = -\frac{1}{2} \left(D_z f \delta z + D_{\bar{z}} f \delta \bar{z} \right) \partial_u - \frac{1}{2} \left(D_z^2 D^z f \delta z + D_{\bar{z}}^2 D^{\bar{z}} f \delta \bar{z} \right) \partial_r$$
$$+ \left(\frac{\gamma^{z\bar{z}}}{r} D_{\bar{z}}^2 f \delta \bar{z} + \frac{1 + z\bar{z}}{2r} [2\bar{z} D_{\bar{z}} f + (1 + z\bar{z}) D_z D_{\bar{z}} f] \delta z \right) \partial_z + \text{c.c.}. \tag{4.11}$$

To compare to the original coordinate system we evaluate the norm of the vector at $(z + \frac{1}{r} D^z f, \bar{z} + \frac{1}{r} D^{\bar{z}} f)$. The proper distance changes by

$$\Delta L = \frac{r_0}{2L} \Delta C_{zz}(z_1, \bar{z}_1) \delta z^2 + \text{c.c.} \tag{4.12}$$

which agrees, as it must, with (4.5). The extra terms that appear in (4.11) cancel against the change of the metric of the flat space evaluated at the shifted point.

To compare the time delay of the two detectors two effects must be taken into account. First the transformation of $u \to u - f$ resets the clocks by a relative amount $D_z f \delta z + \text{c.c.}$. A second effect arises because the relative radius changes by

$$\delta r = -D_z^2 D^z f \delta z + \text{c.c.} \tag{4.13}$$

Due to the presence of the term $2dudr$ in the metric, this implies a difference proportional to δr in the time lapses for light rays traveling from detector 1 to detector 2 and the reverse. Adding these two effects, and using

$$[D_z, D_{\bar{z}}] D^{\bar{z}} f = -D_z f, \tag{4.14}$$

one finds

$$\delta_{12}u - \delta_{21}u = D^z \Delta C_{zz} \delta z + \text{c.c.}, \tag{4.15}$$

as expected.

In conclusion the effects of a radiation pulse passing through $\mathcal{I}^+$ on a family of BMS observers is characterized by the induced supertranslation (3.7). They may be equivalently described as leaving the worldlines unchanged and supertranslating the metric, or leaving the metric unchanged and supertranslating the observers. In either case they imply the familiar gravitational memory effect as well as clock desynchronization.

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[5]The total elapsed time is, to leading order in r_0, $\delta_{12}u + \delta_{21}u = 2L$.

4.2 Inertial detector memory

Most discussions of gravitational memory involve inertial (rather than BMS) detectors moving on geodesics (4.2) that are nearly, but not exactly, worldlines of constant (r, z) and varying u. According to (4.3), the difference between the two worldlines is suppressed by powers of r. It immediately follows that the spatial gravitational memory formula (4.5) applies equally at large r to either BMS or inertial detectors.

The situation is more subtle for the relative time delay. In that case, we found above that there are two contributions which cancel at leading order, and the final result (4.15) is the sum of the subleading terms for each contribution. These subleading terms are in fact sensitive to the difference between the BMS and inertial worldlines. Direct computation reveals that, for inertial observers, the relative time delay actually vanishes at the order (4.15), as we show in appendix A. In Bondi coordinates, this cancelation looks miraculous. However it is in fact a consequence of the equivalence principle, which implies the existence of Fermi normal coordinates in which the connection vanishes everywhere along the worldlines of two neighboring geodesics. It follows there can be no discrepancy of order L in the proper times and (4.15) hence must be cancelled by subleading geodesic corrections.

5 Memory and soft theorems

Recently it has been shown [16, 21] that Weinberg's soft graviton theorem [17] is equivalent to — or more precisely is the Ward identity of — BMS invariance of the quantum gravity $\mathcal{S}$-matrix. In the preceding we have seen that the gravitational memory effect captures the consequences of BMS symmetry. In this section we show how to directly understand the relation between the memory effect and the soft theorem without an interpolating discussion of BMS symmetry.

Weinberg's soft graviton theorem [17] is a universal relation between $(n \to m + 1)$-particle with one final soft graviton and $(n \to m)$-particle quantum field theory scattering amplitudes given by

$$\lim_{\omega \to 0} \mathcal{A}_{m+n+1}\big(p_1, \ldots p_n; p'_1, \ldots p'_m, (\omega k, \epsilon_{\mu\nu})\big)$$
$$= \sqrt{8\pi G} S_{\mu\nu} \epsilon^{\mu\nu} \mathcal{A}_{m+n}\big(p_1, \ldots p_n; p'_1, \ldots p'_m\big) + \mathcal{O}(\omega^0), \quad (5.1)$$

where

$$S_{\mu\nu} = \left(\sum_{j=1}^{m} \frac{p_{j\mu} p_{j\nu}}{k \cdot p_j} - \sum_{j=1}^{n} \frac{p'_{j\mu} p'_{j\nu}}{k \cdot p'_j} \right)^{TT}. \quad (5.2)$$

In this expression $k = (\omega, \omega \vec{k})$ with $\vec{k}^2 = 1$ is the four-momentum and $\epsilon_{\mu\nu}$ the transverse-traceless polarization tensor of the graviton. The superscript TT denotes the transverse-traceless projection (as detailed in [22]) and μ, ν indices refer to asymptotically Minkowskian coordinates with flat metric $\eta_{\mu\nu}$.

Here we explicate the relation between memory and soft theorems in the general context considered by Braginsky and Thorne [3]. They analyzed the possible detection of "burst

memory waves" produced by the collision and scattering of large massive objects such as stars or black holes. They found that such collisions resulted in a net difference in the transverse traceless part of the asymptotic metric at $\mathcal{I}^+$ given by[6]

$$\Delta h_{\mu\nu}^{TT}(\vec{k}) = \frac{1}{r_0}\sqrt{\frac{G}{2\pi}}\left(\sum_{j=1}^{n}\frac{p'_{j\mu}p'_{j\nu}}{k \cdot p'_j} - \sum_{j=1}^{m}\frac{p_{j\mu}p_{j\nu}}{k \cdot p_j}\right)^{TT}. \tag{5.3}$$

Here we have n (m) incoming (outgoing) objects with asymptotic momenta $p_{j\mu}$ ($p'_{j\mu}$). $k = (1, \vec{k})$ is the null vector pointing from the collision region to null infinity, and serves as a coordinate on the S^2 at $\mathcal{I}^+$. Equation (5.3) was derived by solving the linearized Einstein equation with a retarded propagator. The gravitational memory of the collision is then simply constructed from (5.3) via (4.5).

Evidently there are strong similarities between (5.3) and (5.2). To make it more manifest we note the Fourier transform of $h_{\mu\nu}^{TT}(\omega, \vec{k})$ on $\mathcal{I}^+$ can be written, using the stationary phase approximation at large r [21, 23]

$$h_{\mu\nu}^{TT}(\omega, \vec{k}) = 4\pi i \lim_{r \to \infty} r \int du \, e^{i\omega u} h_{\mu\nu}^{TT}(u, r\vec{k}), \tag{5.4}$$

Assuming that $h_{\mu\nu}^{TT}(u, r\vec{k})$ approaches finite but different values at $u \to \pm\infty$ and large $r = r_0$ it then follows[7] that (5.3) is proportional to the coefficient of the pole in ω

$$\Delta h_{\mu\nu}^{TT}(\vec{k}) = \frac{1}{4\pi i r_0} \lim_{\omega \to 0}\left(-i\omega h_{\mu\nu}^{TT}(\omega, \vec{k})\right). \tag{5.5}$$

Next we note that to linear order, the expectation value of the asymptotic metric fluctuation produced in the process of $n \to m$ scattering obeys

$$\begin{aligned}
\lim_{\omega \to 0}\omega h_{\mu\nu}^{TT}(\omega, k)\epsilon^{\mu\nu} &= \lim_{\omega \to 0}\frac{\omega \mathcal{A}_{m+n+1}\big(p_1, \ldots p_n; p'_1, \ldots p'_m, (\omega k, \epsilon_{\mu\nu})\big)}{\mathcal{A}_{m+n}\big(p_1, \ldots p_n; p'_1, \ldots p'_m\big)} \\
&= \sqrt{8\pi G}\epsilon^{\mu\nu}\lim_{\omega \to 0}\omega S_{\mu\nu}(\omega k) \\
&= \sqrt{8\pi G}\epsilon^{\mu\nu}\left(\sum_{j=1}^{m}\frac{p_{j\mu}p_{j\nu}}{k \cdot p_j} - \sum_{j=1}^{n}\frac{p'_{j\mu}p'_{j\nu}}{k \cdot p'_j}\right)^{TT}.
\end{aligned} \tag{5.6}$$

Inserting this into (5.5) we then see that this is equivalent as claimed to the Braginsky-Thorne result (5.3).[8]

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[6]This is equation (1) of [3] written with the normalization $h_{\mu\nu} \equiv \frac{1}{\sqrt{32\pi G}}(g_{\mu\nu} - \eta_{\mu\nu})$, in the mostly plus $(-+\ldots+)$ signature and in covariant gauge.

[7]In the formulas above we assume that $\omega r \gg 1$ when taking the limits.

[8]Notice that while the usual scattering amplitude computations involve infrared divergences the memory formula (4.6) is formulated in terms of physical and thus manifestly finite quantities. In particular if one imagines the evolution of state $|\Psi\rangle$ in the theory of quantum gravity formula (4.6) should be understood as the relation between the expectation values $\langle\Psi|\Delta C_{zz}(z, \bar{z})|\Psi\rangle$ and $\langle\Psi|T_{uu}(z, \bar{z})|\Psi\rangle$. These expectation values when formulated in terms of scattering amplitudes are cross section-like observables, namely the
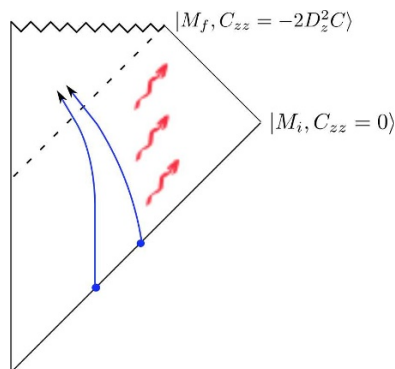
Figure 2. Supertranslation hair of black holes. We consider formation of a black hole by infalling matter. During the process radiation is necessarily created and leaks to future null infinity, where it mediates the transition to a new vacuum state. When the black hole settles down the state at $\mathcal{I}^+$ is characterized by the supertranslation hair $C(z, \bar{z})$.

6 Measuring black hole hair

One often hears that black holes have no hair. This statement does not take into account the subtleties associated with asymptotic structure at $\mathcal{I}$. In particular, as we discussed in section 3, a supertranslation maps the Schwarzschild solution to a physically inequivalent configuration. Hence black holes have a lush infinite head of supertranslation hair. This may bear on the information puzzle.

The present discussion clarifies the nature of supertranslation hair and how it can be measured classically. Let us consider, as a special case of the Braginsky-Thorne construction, N incoming stars which collide and collapse into a black hole. We station an array of evenly-spaced detectors near future null infinity. The relative positions of these detectors will shift due to the memory effect as given in (5.3) and (4.5). Hence the detector positions can record an infinite amount of data about how the black hole was formed. Black hole formed by different initial star configurations will carry different supertranslation hair.

Acknowledgments

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observables that involve the square of the amplitude $|\mathcal{A}|^2$ integrated over the phase space with the proper weights. As usual in quantum mechanics to compute the expectation values one has to repeat the experiment many times. In the consideration of this section we considered the classical version of this experiment. In this case we imagine that we consider one realization of the process and both the initial and final states are fixed. For this situation the expectation values above become $\langle\Psi|n^*\rangle\langle n^*|\mathcal{O}|\Psi\rangle$ and by canceling the overall factor $\langle\Psi|n^*\rangle$ we get the relation between scattering amplitudes. In the case of the memory formula this relation becomes nothing but the soft theorem.

A Corrections to geodesics and Fermi coordinates

Let us consider an asymptotic time-like observer with the velocity v^μ in the Bondi retarded coordinates. Its four-velocity is given by

$$v^\mu = \left(1, \frac{m_B(u, z, \bar{z})}{r}, -\frac{1}{8}\frac{(1+z\bar{z})^4}{r^2}D_z C_{\bar{z}\bar{z}}(u, z, \bar{z}), -\frac{1}{8}\frac{(1+z\bar{z})^4}{r^2}D_{\bar{z}} C_{zz}(u, z, \bar{z})\right). \quad \text{(A.1)}$$

Notice that for the first two components we work to the $\frac{1}{r}$ order and for the last two to the $\frac{1}{r^2}$. The reason for this will become clear below. This four-vector describes a geodesic $v^\mu\nabla_\mu v^{u,r} = O(\frac{1}{r^2})$, $v^\mu\nabla_\mu v^{z,\bar{z}} = O(\frac{1}{r^3})$. For the norm we have $v^\mu v_\mu = -1 + O(\frac{1}{r^2})$.

Equally well (by choosing different initial conditions) we could have chosen the four-velocity to be

$$\begin{aligned}
v^\mu = \Bigg(&1 + \frac{m_B(u_0, z, \bar{z})}{r}, \frac{m_B(u, z, \bar{z}) - m_B(u_0, z, \bar{z})}{r}, \\
&-\frac{1}{8}\frac{(1+z\bar{z})^4}{r^2}D_z[C_{\bar{z}\bar{z}}(u, z, \bar{z}) - C_{\bar{z}\bar{z}}(u_0, z, \bar{z})], \\
&-\frac{1}{8}\frac{(1+z\bar{z})^4}{r^2}[D_{\bar{z}}C_{zz}(u, z, \bar{z}) - D_{\bar{z}}C_{zz}(u_0, z, \bar{z})]\Bigg).
\end{aligned} \quad \text{(A.2)}$$

In (A.1) we set u-independent initial values to zero. We do the same thing below since we are interested in the vacuum-to-vacuum transitions described in the bulk of the paper.

If we integrate over u for a long enough time the corrections are not small since the correspondent integrals diverge. Below we always assume r to be large enough (and the measurement time to be small enough) so that the corrections are small.

We also consider the orthonormal spatial basis

$$\begin{aligned}
n^\mu &= \left(-1, 1 - \frac{m_B}{r}, \frac{1}{8}\frac{(1+z\bar{z})^4}{r^2}D_z C_{\bar{z}\bar{z}}, \frac{1}{8}\frac{(1+z\bar{z})^4}{r^2}D_{\bar{z}}C_{zz}\right), \\
m^\mu &= \left(0, 0, \frac{1}{r}\frac{1+z\bar{z}}{2}, -\frac{1}{r}\frac{1+z\bar{z}}{2}\frac{(1+z\bar{z})^2 C_{zz}}{4r}\right), \\
\bar{m}^\mu &= \left(0, 0, -\frac{1}{r}\frac{1+z\bar{z}}{2}\frac{(1+z\bar{z})^2 C_{\bar{z}\bar{z}}}{4r}, \frac{1}{r}\frac{1+z\bar{z}}{2}\right).
\end{aligned} \quad \text{(A.3)}$$

All of these are parallel transported along v^μ to leading order in $\frac{1}{r}$ so that we have $v^\mu\nabla_\mu e_i^\nu = O(\frac{1}{r^2})$ and are orthogonal to v^μ, namely $v_\mu e_i^\mu = O(\frac{1}{r^2})$. They are also normalized in the usual way $n.n = 1 + O(\frac{1}{r^2})$, $m.\bar{m} = \frac{1}{2} + O(\frac{1}{r^2})$, $n.m = n.\bar{m} = m.m = \bar{m}.\bar{m} = O(\frac{1}{r^2})$. Physically, these vectors describe a set of gyroscopes that is carried by an observer.

Based on this we can introduce Fermi normal coordinates which are the coordinates that describe physics that the observer experiences in the vicinity of his location. Namely introducing $e_0^\mu = v^\mu$, $e_3^\mu = n^\mu$, $e^\mu = m^\mu$, $\bar{e}^\mu = \bar{m}^\mu$ we introduce corresponding coordinates x_i^μ such that the metric takes the form

$$ds^2 = -dx_0^2 + dx_3^2 + dx d\bar{x} + O(x^2) \quad \text{(A.4)}$$

and the leading corrections are related to $R_{abcd} = R_{\mu\nu\rho\sigma}e_a^\mu e_b^\nu e_c^\rho e_d^\sigma$ [24, 25] where the Riemann tensor is evaluated along the geodesic $\gamma(\tau)$. The equations for geodesics take the form

$$\ddot{x}^i = R^i{}_{00j}x^j. \tag{A.5}$$

For the case in hand the only contribution at the $\frac{1}{r}$ order comes from $R_{x00x} = \frac{(1+z\bar{z})^2}{8r}\partial_u^2 C_{zz}$ and $R_{\bar{x}00\bar{x}} = \frac{(1+z\bar{z})^2}{8r}\partial_u^2 C_{\bar{z}\bar{z}}$ which describe the ordinary gravitational memory. The same analysis was done in [4].

It is clear in Fermi coordinates that there is no relative time shift linear in x unless the original observer is accelerated. This is the case for BMS observers and it is the source of time desynchronization linear order in x.

We can also analyze the nearby geodesic directly in Bondi coordinates as explained in the main body of the paper. The result of course does not depend on the coordinate system we used.

B Massive particle decay

It is instructive to compare our results to those of Tolish et. el. [10, 11] . The starting point of their work is the geodesic deviation equation for a small perturbation around the flat space

$$\frac{d^2 D_\mu}{dt^2} = -R_{t\mu t\nu}D^\nu,$$
$$\Delta D_\mu = M_{\mu\nu}D^\nu. \tag{B.1}$$

For the cases considered in [11], symmetries imply that

$$R_{t\mu t\nu} = W(\theta_\mu\theta_\nu - \phi_\mu\phi_\nu) \tag{B.2}$$

where θ^μ and ϕ^μ are the unit vector fields on the sphere.

In the Bondi coordinates we have for the tensor $\hat{M}_{\mu\nu} = \theta_\mu\theta_\nu - \phi_\mu\phi_\nu$

$$\hat{M}_{zz} = \frac{2}{(1+z\bar{z})^2}\frac{\bar{z}}{z}, \quad \hat{M}_{\bar{z}\bar{z}} = \frac{2}{(1+z\bar{z})^2}\frac{z}{\bar{z}}. \tag{B.3}$$

Consider now the following situation. A particle at rest of mass M decays into a massless particle with energy E moving in the $\hat{z}$-direction and the particle of mass M' moving in the $-\hat{z}$-direction. On the sphere it corresponds to $z = \bar{z} = 0$ for $\hat{z}$ and $z = \bar{z} = \infty$ for $-\hat{z}$. The contribution of the massless particle is [11]

$$M_{\mu\nu} = \frac{E}{r}(1 + \cos\theta)\hat{M}_{\mu\nu} = \frac{E}{r}\frac{2}{1+z\bar{z}}\hat{M}_{\mu\nu} \tag{B.4}$$

whereas for the massive one [11]

$$M_{\mu\nu} = \frac{E^2}{Mr}\frac{\sin\theta^2}{1 - \frac{E}{M}(1 - \cos\theta)}\hat{M}_{\mu\nu} = \frac{1}{r}\frac{|\vec{p}|^2\sin\theta^2}{p^0 + |\vec{p}|\cos\theta}\hat{M}_{\mu\nu} \tag{B.5}$$

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where $p^0 = M - E$, $|\vec{p}| = E$ by energy-momentum conservation. In the formulae above we set $G = 1$.

We now would like to reproduce the same formulas using the soft theorem which states that the memory is given by the soft factor[9]

$$M_{\mu\nu} = \frac{1}{2}\sqrt{32\pi G}\, h_{\mu\nu}^{TT} = \frac{2G}{r}\left(\sum_i \frac{p_\mu^i p_\nu^i}{(p^i.n)} - \sum_f \frac{p_\mu^f p_\nu^f}{(p^f.n)}\right)^{TT}, \quad\quad (B.7)$$

where we adopted field theoretical normalization $h_{\mu\nu} = \frac{1}{\sqrt{32\pi G}}\left(g_{\mu\nu} - \eta_{\mu\nu}\right)$ and $n = (1, \vec{n})$ is the unit four-vector in the direction of observation. Plugging the momenta in the formula above reproduces the result of [11]. For massless particles in the $(z, \bar{z})$ coordinates it becomes

$$M_{zz} = \frac{4G}{r}\sum_i E_i \frac{\bar{z}_i - \bar{z}}{z_i - z}\frac{(1 + z_i\bar{z})^2}{(1 + z_i\bar{z}_i)(1 + z\bar{z})^3}. \quad\quad (B.8)$$

In the example above we have $z_i = \bar{z}_i = 0$. Notice also that the $(z, \bar{z})$-dependent kernel that appeared in (B.8) is identical to the one that appeared in (4.6) . Indeed, as pointed out in [7] the Christodoulou memory effect can be thought as a generalization of the usual soft factor where instead of a finite set of particles approaching infinity we imagine arbitrary energy flux of gravitational radiation. Any fixed energy scattering can produce at most finite number of massive particles. It means that a generic final state can be thought as the finite number of massive particles plus arbitrary complicated profile of radiation. The memory due to the radiation is captured by (4.6), whereas for massive particles the contribution to the memory is simply given by (B.7).

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[9]The prescription for taking transverse-traceless part is thoroughly reviewed in [22]

$$h_{\mu\nu}^{TT} = h_{\mu\nu} - n_\mu h_{\nu\lambda}n^\lambda - n_\nu h_{\mu\lambda}n^\lambda + n_\mu n_\nu(h_{\lambda d}n^\lambda n^d) - \frac{1}{2}(\delta_{\mu\nu} - n_\mu n_\nu)h_{\lambda d}\left(\delta^{\lambda d} - n^\lambda n^d\right). \quad (B.6)$$

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