$\mathcal{N} = 1$ Euler anomaly flow from dilaton effective action

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ABSTRACT: We consider $\mathcal{N} = 1$ supersymmetric gauge theories in the conformal window. The running of the gauge coupling is absorbed into the metric by applying a suitable matter superfield- and Weyl-transformation. The computation becomes equivalent to one of a free theory in a curved background carrying the information of the renormalisation group flow. We use the techniques of conformal anomaly matching and dilaton effective action, by Komargodski and Schwimmer, to rederive the difference of the Euler anomaly coefficient $\Delta a \equiv a_{\mathrm{UV}} - a_{\mathrm{IR}}$ for the $\mathcal{N} = 1$ theory. The structure of Δa is therefore in one-to-one correspondence with the Wess-Zumino dilaton action.

Contents

1   Introduction

An exact expression for the difference of the ultraviolet (UV) and infrared (IR) Euler anomaly $\Delta a \equiv a_{\mathrm{UV}} - a_{\mathrm{IR}}$ was derived for $\mathcal{N} = 1$ supersymmetric gauge theories by Anselmi, Freedman, Grisaru, and Johansen (AFGJ) [1]. Thereafter it has served as a fruitful laboratory for testing different techniques by rederiving the result. Examples include verification up to fourth loop order [2], the use of the local renormalisation group (RG) [3] and employing superspace techniques assuming a gradient flow equation [4]. In the latter case an expression valid outside the fixed point has been obtained [4] of a form conjectured earlier by a perturbative approach [5].

In this paper $\Delta a|_{\mathcal{N}=1}$ is derived by using the techniques of conformal anomaly matching and dilaton effective action. The latter were used by Komargodski and Schwimmer (KS) [6, 7] to derive the a-theorem $\Delta a \geq 0$ as conjectured in 1988 by Cardy [8]. A crucial ingredient is the introduction of an *external* field called the dilaton by coupling it to the renormalisation scale $\mu \to \mu e^{\tau(x)}$, thereby introducing a local scale interpretation analogous to the the local RG pioneered by Shore [9, 10]. The locality of the approach is crucial and served Jack and Osborn to derive a proof the a-theorem at weak coupling (i.e. perturbation theory) by using it as a source term in a field theory in a generic curved background. KS and later Komargodski [7] focused on the four point dilaton function and were able to prove the a-theorem based on analyticity assumptions. In essence the dilaton serves as a compensator field to the Weyl-rescaling

$$\mathrm{g}_{\mu\nu} \to e^{-2\alpha(x)} \mathrm{g}_{\mu\nu} \ . \tag{1.1}$$

The transformation (1.1) corresponds to changing distances locally and implies that coordinate and momenta invariants change as $x^2 \to e^{-2\alpha(x)}x^2$ and $p^2 \to e^{2\alpha(x)}p^2$. Variation of the logarithm of the partion function with respect to the Weyl-parameter results in the vacuum expectation value (VEV) of the trace of the energy momentum tensor (EMT). For a theory on a curved space, with no explicit scale symmetry breaking, the EMT is parametrised by [11, 12]

$$\langle \Theta \rangle = a\, E_4 + b\, W^2 + c\, H^2 + c'\Box H \,, \tag{1.2}$$

where the abbreviations

$$\Theta \equiv T^\rho_{\ \rho}\,, \quad H \equiv \frac{R}{d-1}\,, \tag{1.3}$$

are used throughout. The quantities $E_4 = R^2_{\mu\nu\alpha\beta} - 4R^2_{\mu\nu} + R^2$, $W^2 = R^2_{\mu\nu\alpha\beta} - \frac{4}{(d-2)}R^2_{\mu\nu} + \frac{2}{(d-1)(d-2)}R^2$ and R are the Euler density, the Weyl tensor squared and the Ricci-scalar; and $R_{\mu\nu\alpha\beta}$ and $R_{\mu\nu}$ denote the Riemann and Ricci tensors. The Euler density E_4 is a topological quantity and the Weyl tensor squared W^2 vanishes on a conformally flat space. The absence of c, and therefore the H^2-term, in a 4D conformal field theory (CFT) was shown in ref. [13, 14]. The $\Box R$ term can be removed by a finite H^2-counterterm in the action [11] and will therefore not be discussed any further throughout. The constants a, b, c and c' depend on the dynamics of the theory. Their free field values for various spins were computed in [11]. Note, the non-vanishing of a and b therefore establish the conformal or Weyl anomaly in 4D [11, 12].

This work is structured as follows. In section 2 the general framework is outlined by restating some of the results of [6] in a language appropriate for this work. The specific construction is presented and illustrated in sections 2.1 and 2.2 respectively. In section 3, in particular 3.1, the AFGJ Euler anomaly result is rederived within our framework using the Konishi anomaly. The paper ends with conclusions in section 4. Appendices A, B and C include a review of the derivation of the NSVZ beta function using the Konishi anomaly, the derivation of the the trace anomaly for a free theory for a dilaton background field without additional background curvature and renormalisation group equations for the generating functional.

2 General framework

Consider a massless theory with fields ϕ and a coupling g. The path integral is given by

$$e^{W(g(\mu),\mu)} = \int [\mathcal{D}\phi]_\mu\; e^{-S_{\mathrm{W}}(g(\mu),\mu,\phi)} \,, \tag{2.1}$$

where the action S_{W} is to be understood in a Wilsonian sense and W is proportional to the negative free energy. For the purposes of this work S_{W} is interpreted to be on a renormalisation trajectory from the UV to an IR fixed point.

In massless theories correlation functions depend on ratios of q^2/μ^2 where q denotes an external momentum. Hence the renormalisation scale transforms as $\mu \to e^{-\alpha}\mu$ under the Weyl-rescaling. An *external* field, known as the dilaton τ, is introduced in the action

$$S_{\mathrm{W}}(g(\mu),\mu,\phi) \to S_\tau \equiv S_{\mathrm{W}}(g(\mu e^\tau),\mu e^\tau,\phi) \,. \tag{2.2}$$

transforming under Weyl-rescaling (1.1) as

$$\tau \to \tau + \alpha \,, \tag{2.3}$$

such that the product μe^{τ} is Weyl-invariant. The dilaton therefore serves as a spurion (or compensator) formally restoring scale invariance. In this work no dynamic nature is attributed to the dilaton field which is in line with [7] but not the first paper [6] on the a-theorem in 2011. The dilaton serves as a source term for the EMT and when made a local field the, yet to-be-defined, Wess-Zumino term carries the information on the Euler anomaly. Promoting the dilaton to a local field $\tau \to \tau(x)$ requires *local* Weyl invariance and demands changes similar to passing from global to local gauge invariance. The specific implementation will be discussed in the the explicit examples. The space-dependance of τ augments the couplings to local objects $g(\mu) \to g(\mu e^{\tau(x)})$. Note that the functional form μe^{τ} renders local Weyl-rescaling equivalent to a local RG transformation. The path integral becomes τ-dependent,

$$e^{W_{\tau}} = \int [\mathcal{D}\phi]_{\mu} e^{-S_{\tau}} = \int [\mathcal{D}\phi]_{\mu} e^{-S_{\mathrm{W}}(g(\mu e^{\tau}),\mu e^{\tau},\phi)} \,. \tag{2.4}$$

The quantity W_{τ} corresponds to the generating functional of the correlation functions (connected component) of the traces of the EMT. The Wess-Zumino action[1]

$$S_{\mathrm{WZ}} = \int d^4 x\, 2 \left(2\Box\tau(\partial\tau)^2 - (\partial\tau)^4 \right) + \mathcal{O}(R) \,, \tag{2.5}$$

was shown to be the source term [6] of the Euler anomaly. Above R stands for the non-dilaton curvature background. More precisely, using arguments of conformal anomaly matching it was shown that the difference of the UV and IR dilaton effective action, with with $g^*_{(\mathrm{UV,IR})} \equiv g(\infty, 0)$,

$$\Delta W_{\tau} \equiv \int_{g^*_{\mathrm{IR}}}^{g^*_{\mathrm{UV}}} dg\, \partial_g W_{\tau} = -\int_{-\infty}^{\infty} d\ln\mu\, \partial_{\ln\mu} W_{\tau} = W_{\tau}(g^*_{\mathrm{UV}}) - W_{\tau}(g^*_{\mathrm{IR}}) = -\Delta a\, S_{\mathrm{WZ}} + \dots \,, \tag{2.6}$$

contains a term proportional to S_{WZ} times the sought after quantity $\Delta a \equiv a(\mu_{\mathrm{UV}}) - a(\mu_{\mathrm{IR}})$ [6]. Hence determining Δa reduces to finding $\partial_{\ln\mu} W_{\tau}$. Note that the second equality in (2.6) follows from (C.1) in the limit $m \to 0$ and using $dg/\beta = d\ln\mu$.

2.1 Dilaton dependent conformal factor

In this work a theory is considered which can be reinterpreted as a free field theory in a conformally flat background

$$\tilde{\mathrm{g}}_{\rho\lambda} = e^{-2s(\tau)}\delta_{\rho\lambda} \,, \tag{2.7}$$

which carries the information on the RG flow parameters. Above $\delta_{\rho\lambda}$ denotes the flat Euclidian metric with positive signature. The adaption to Minkowski space is straightforward as it results in the appearance of various factors of i only. By using (C.5) from the

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[1]The Wess-Zumino action, above, is analogous to the Wess-Zumino term [15] of pions in connection with the the axial anomaly.

appendix[2,3]

$$\partial_{\ln\mu}W_\tau = \int d^4x\sqrt{\tilde{g}}\,\langle\Theta\rangle_\tau\,, \tag{2.8}$$

and (2.6) we may write a more explicit formula for the difference of the Euler anomaly Δa

$$\Delta a = \int_{-\infty}^{\infty} d\ln\mu \int d^4x\sqrt{\tilde{g}}\langle\Theta\rangle_\tau|_{S_{\rm WZ}}\,, \tag{2.9}$$

where where $|_{S_{\rm WZ}}$ denotes the following projection

$$\langle\Theta\rangle_\tau = \tilde{a}\tilde{E}_4 + \tilde{c}\tilde{H}^2 = \langle\Theta\rangle_\tau|_{S_{\rm WZ}} S_{\rm WZ} + \dots\,. \tag{2.10}$$

The coefficients $\tilde{a}$ and $\tilde{c}$ depend on the dynamics of the theory. The Wess-Zumino action $S_{\rm WZ}$ is defined in (2.5) and we follow the rule that the tilde denotes geometric quantities, e.g. $\tilde{E}_4$ and $\tilde{H}^2$, evaluated in the background metric $\tilde{g}_{\mu\nu}$.[4] The quantity Δa is determined once $\tilde{a}$ and $\tilde{c}$ are known. In the next section we will discuss a very simple toy model that illustrates these ideas and will serve as a stepping stone for the $\mathcal{N}=1$-computation.

2.2 Weyl anomaly of free scalar field in conformally flat background

We consider a scalar field theory on a flat space, focusing solely on the kinetic term

$$S_{\rm W}(\mu) = \int d^4x Z(\mu)\delta^{\rho\lambda}\partial_\rho\phi\partial_\lambda\phi\,, \tag{2.11}$$

thereby ignoring other contributions. The usefulness of which will, hopefully, become clear in section 3.1. Taking $Z(\mu) \to Z(\mu e^{\tau(x)})$ amounts to passing to S_τ (2.2). The factor $Z(\mu e^{\tau(x)})$ can be absorbed into the metric by a local Weyl-rescaling by choosing $\alpha = s$ in (1.1) with

$$s(\mu e^\tau) = -\frac{1}{2}\ln Z(\mu e^{\tau(x)}) \quad\Rightarrow\quad \tilde{g}_{\rho\lambda} = Z\delta_{\rho\lambda}\,. \tag{2.12}$$

The theory then becomes a field theory on a conformally flat space with metric $\tilde{g}_{\mu\nu}$ (2.12)

$$S_\tau(\mu) = \int d^4x\sqrt{\tilde{g}}\tilde{g}^{\rho\lambda}D_\rho^{(s)}\phi D_\lambda^{(s)}\phi\,. \tag{2.13}$$

Above $D_\rho^{(s)} = \partial_\rho - (\partial_\rho s)$ denotes the Weyl-covariant derivative[5] analogous to the covariant derivative in gauge theories. The coefficients of the trace anomaly (1.2) for a theory with metric $\tilde{g}_{\rho\lambda}$ and one free scalar field are given by (cf. [11, 16] or the explicit computation in appendix B)

$$\tilde{a} = a_{(0)}^{\rm free}\,, \quad \tilde{c} = 0\,, \quad \tilde{c}' = -2a_{(0)}^{\rm free}\,, \qquad a_{(0)}^{\rm free} = \frac{1}{360}\frac{1}{16\pi^2} = \frac{1}{5760\pi^2}\,, \tag{2.14}$$

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[2]Note that metric (2.7) is not a physical or geometric metric as it does not transform like (1.1) under Weyl-rescaling $s(\tau) \to s(\tau + \alpha)$ unless $s(x) = x$. The latter is the case in [6], $\hat{g}_{\rho\lambda} = e^{-2\tau(x)}\delta_{\rho\lambda}$, and constitutes one of the differences with respect to our approach.

[3]The subscript τ refers to the VEV of the trace of the EMT with respect to the partition function (2.4).

[4]Conformal flatness of $\tilde{g}$ implies that $\tilde{W}^2 = 0$.

[5]Adding the Weyl-covariant derivatives is equivalent to the replacement $\Box \to \Box - \frac{1}{6}\tilde{R}$ which is the usual conformally coupled scalar in a curved space of metric $\tilde{g}_{\rho\lambda}$.

or equivalently $\langle\Theta\rangle_\tau = a_{(0)}^{\text{free}}(\tilde{E}_4 - 2\tilde{\Box}\tilde{R})$. As stated earlier, the coefficient c' is of no importance for this work and is therefore discarded. The Euler density in terms of s is given by

$$\sqrt{\tilde{\text{g}}}\tilde{E}_4 = -8\left(\frac{1}{2}\Box(\partial s)^2 - \partial\cdot(\partial s(\Box s - (\partial s)^2))\right).\tag{2.15}$$

Using the explicit form $s = -\frac{1}{2}\ln Z(\mu e^\tau)$ the Euler term becomes

$$\sqrt{\tilde{\text{g}}}\tilde{E}_4 = -\Big[\gamma^2\Box(\partial\tau)^2 + (2\gamma\dot{\gamma} - 2\gamma^2)\partial^\lambda(\partial_\lambda\tau\Box\tau) - \gamma^3\partial^\lambda(\partial_\lambda\tau(\partial\tau)^2)$$
$$-6\gamma\dot{\gamma}(\partial\tau)^2\Box\tau - 3\gamma^2\dot{\gamma}(\partial\tau)^4\Big],\tag{2.16}$$

where here and below we use the abbreviation $\dot{\gamma}\equiv\frac{d}{d\log\mu}\gamma$ and the following expressions

$$\partial_\rho\gamma = \dot{\gamma}\,\partial_\rho\tau\,,\quad \partial_\rho s = -\frac{1}{2}\frac{\partial\ln Z(\mu e^\tau)}{\partial(\mu e^\tau)}\partial_\rho(\mu e^\tau) = -\frac{1}{2}\gamma\,\partial_\rho\tau\,,\quad \gamma = \frac{\partial\ln Z(\mu)}{\partial\ln\mu}\,,\tag{2.17}$$

have been used. The quantity Δa is obtained by integrating over $d\ln\mu$ and projecting on S_{WZ}. In doing so γ and $\dot{\gamma}$ can be treated as being space-independent, since expanding $\gamma(\mu e^\tau) = \gamma(\mu) + O(\tau(x))$ leads to terms which are not contained in S_{WZ}. Furthermore it is then clear that the first line in (2.16) can be discarded since it is a total derivative and therefore inequivalent to the S_{WZ} (2.5) bulk-term. In order to project the second line of (2.16) on S_{WZ} (2.5) it is convenient (following [6, 7])[6]

$$\Box\tau = (\partial\tau)^2\,,\tag{2.18}$$

under which all four-derivative invariants vanish, except for

$$S_{\text{WZ}}|_{(2.18)} = \int d^4x\, 2(\Box\tau)^2\,.\tag{2.19}$$

Using (2.9) and performing the integral over $d\ln\mu$ we get

$$\Delta a = \frac{1}{2}a_{(0)}^{\text{free}}\big[3A_1 + A_2\big]\,,\tag{2.20}$$

where

$$A_1 = \int_{-\infty}^{\infty} d\ln\mu\, 2\gamma\dot{\gamma} = \int_{\gamma_{\text{IR}}}^{\gamma_{\text{UV}}} d\gamma\, 2\gamma = (\gamma_{\text{UV}}^2 - \gamma_{\text{IR}}^2)\,,$$
$$A_2 = \int_{-\infty}^{\infty} d\ln\mu\, 3\gamma^2\dot{\gamma} = \int_{\gamma_{\text{IR}}}^{\gamma_{\text{UV}}} d\gamma\, 3\gamma^2 = (\gamma_{\text{UV}}^3 - \gamma_{\text{IR}}^3)\,,\tag{2.21}$$

and $\gamma_{\text{IR,UV}}\equiv\gamma(g_{\text{IR,UV}}^*)$ are the values of the anomalous dimensions at the respective fixed points. For further reference the final result (2.20) is stated with explicit coefficients A_1 and A_2

$$\Delta a = \frac{1}{2}\left((\gamma_{\text{UV}}^3 - \gamma_{\text{IR}}^3) + 3(\gamma_{\text{UV}}^2 - \gamma_{\text{IR}}^2)\right)a_{(0)}^{\text{free}}\,.\tag{2.22}$$

This result constitutes an important intermediate result for the derivation of $\Delta a|_{\mathcal{N}=1}$.

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[6]We note in passing that eq. (2.18) is the lowest order equation of motion [6] when a dynamic nature is attributed the dilaton.

3 $\mathcal{N} = 1$ supersymmetric gauge theory

The theory considered in this section is a $\mathcal{N} = 1$ supersymmetric gauge theory with flavour symmetry $\mathrm{SU}(N_f) \times \mathrm{SU}(N_f)$ and gauge group $\mathrm{SU}(N_c)$. The action can be written in terms of the usual vector superfield V and matter superfields $(\Phi_f, \tilde{\Phi}_f)$ as, e.g. [17],

$$S_{\mathrm{W}}(\mu) = \int d^6 z \frac{1}{g^2(\mu)} \mathrm{tr} W^2 + \mathrm{h.c.} + \frac{1}{8} Z(\mu) \sum_f \left[\int d^8 z \Phi_f^\dagger e^{-2V} \Phi_f + \int d^8 z \tilde{\Phi}_f^\dagger e^{-2V} \tilde{\Phi}_f \right] ,$$

(3.1)

where W^2 is the supersymmetric gauge field kinetic term, g is referred to as the holomorphic coupling constant parametrisation and $d^6 z$ and $d^8 z$ include integration over the fermionic superspace variables.

The main tool in deriving $\Delta a|_{\mathcal{N}=1}$ is the use of the Konishi anomaly [18–20]. The latter is illustrated in appendix A as a method to derive the NSVZ beta function. In section 3.1 the Konishi anomaly is used to write the Wilsonian action such that the RG flow can be absorbed into the metric. This procedure makes it amenable to the free field theory computation in the dilaton background discussed in section 2.2.

3.1 $\Delta a|_{\mathcal{N}=1}$ from dilaton effective action and Konishi anomaly

We consider the $\mathcal{N} = 1$ supersymmetric gauge theory with Wilsonian effective action given in (A.1). Choosing a rescaling factor, with $\gamma_* = -b_0/N_f$ (A.6),

$$(\Phi_f, \tilde{\Phi}_f) \to \left(\frac{\mu'}{\mu} \right)^{\gamma_*/2} (\Phi_f, \tilde{\Phi}_f)$$

(3.2)

on the matter fields the Konishi anomaly turns the action into the following form

$$S_{\mathrm{W}}(\mu) = \int d^6 z \frac{1}{g(\mu')^2} \mathrm{tr} W^2 + \mathrm{h.c.} +$$
$$+ \frac{1}{8} \sum_f \left[\int d^8 z \hat{Z}(\mu) \Phi_f^\dagger e^{-2V} \Phi_f + \int d^8 z \hat{Z}(\mu) \tilde{\Phi}_f^\dagger e^{-2V} \tilde{\Phi}_f \right] ,$$

(3.3)

with

$$\hat{Z}(\mu) \equiv Z(\mu) \left(\frac{\mu'}{\mu} \right)^{\gamma_*} ,$$

(3.4)

where $\mu' > \mu$ is an arbitrary scale which can be thought off as a UV cut-off Λ_{UV}. Crucially, the RG flow is absorbed into the precoefficient $\hat{Z}(\mu)$ in front of the matter term. Eq. (3.3) is the analogue of the action (2.11) for the scalar field to the degree that the running of the theory is parametrised by a coefficient in front of the matter kinetic term.

Again following the procedure in (2.2) a dilaton is introduced through

$$\hat{Z}(\mu) \to \hat{Z}_\tau(\mu) = \hat{Z}(\mu e^{\tau(x)}) = Z(\mu e^{\tau(x)}) \left(\frac{\mu'}{\mu e^{\tau(x)}} \right)^{\gamma_*} .$$

(3.5)

To keep the procedure manifestly supersymmetric, following [21] the dilaton is promoted to a (chiral) superfield T such that

$$T| = \tau + i\omega , \quad \hat{Z}(\mu e^T)| = \hat{Z}(\mu e^\tau) .$$

(3.6)

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Above ω is the axion and the bar stands for projection on to the lowest component of the multiplet. It will be seen that $\hat{Z}$ in (3.3) can be absorbed into the background geometry by a local Weyl-rescaling. To preserve local SUSY invariance the Weyl transformations are promoted to super-Weyl transformations. Under the latter, the $\mathrm{tr}W^2$-term is invariant whereas the matter term transforms as follows (c.f [22])

$$\int d^8 z \Phi^\dagger e^{-2V} \Phi \to \int d^8 z e^{-A} e^{-A^\dagger} \Phi^\dagger e^{-2V} \Phi \ , \tag{3.7}$$

with the superfield $A = \alpha + i\beta + \dots$ being the super-Weyl parameter corresponding to α in (1.1). Note that such a formalism is automatically local Weyl-invariant and that there is no need to introduce the Weyl-covariant derivatives as in (2.12). Furthermore, the transformation (3.7) amounts to a Weyl-rescaling of the vielbein[7]

$$e_\rho^a \to e^{-\frac{A}{2}} e^{-\frac{A^\dagger}{2}} |\ e_\rho^a = e^{-\alpha} e_\rho^a \ . \tag{3.8}$$

Upon identifying $\alpha = s$ in eq. (2.12)

$$e_\rho^a \to \tilde{e}_\rho^a = \sqrt{\hat{Z}_\tau(\mu)} e_\rho^a \ . \tag{3.9}$$

The action $S_W(\mu e^T)$ (3.3) can then be written in a manifestly locally supersymmetric form; cf. section 6.3 in [23]. Eq. (3.9) results in

$$\tilde{g}_{\rho\lambda} = \tilde{e}_\rho^a \tilde{e}_\lambda^a = e^{-2s(\mu e^\tau)} \delta_{\rho\lambda} \ , \quad s(\mu e^\tau) = -\frac{1}{2} \ln \hat{Z}(\mu e^{\tau(x)}) \ . \tag{3.10}$$

Notice that the UV scale μ' is arbitrary and that therefore a physical quantity like $\langle\Theta\rangle_\tau$ should not depend on it. Since the geometric terms $\tilde{E}_4$ and $\tilde{H}^2$ are independent of μ',[8] the form of (2.10) implies that $\tilde{a}$ and $\tilde{c}$ are μ'-independent and therefore constants. This means that $\tilde{a}$ and $\tilde{c}$ assumed the values at the (free) UV fixed point and the geometric quantities are to be evaluated in the background metric $\tilde{g}_{\rho\lambda}$ carrying the dynamic information. This allows us to recycle, in large parts, the computation in section 2.2 as outlined below.

A free theory in a curved background is in particular conformal and therefore free of the $\tilde{R}^2$-term (i.e. $\tilde{c} = 0$). Since the dilaton couples to the matter part only, the trace anomaly is exhausted by the free field theory computation of the matter-fields in the curved background with metric (3.10). Equivalence to the example in the previous section is achieved through the formal replacement $Z \to \hat{Z}$ (following from (3.10)) which implies $\gamma \to \delta\gamma \equiv \gamma - \gamma_*$ *and* the change in the number of degrees of freedom ν. More precisely the matter superfield consists of a complex scalar and a Weyl fermion which contribute [11]

$$\nu \equiv 2\Big|_{\mathbb{C}\text{-scalar}} + \frac{11}{2}\Big|_{\text{Weyl-fermion}} = \frac{15}{2} \tag{3.11}$$

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[7]Under a super-Weyl transformation, the supersymmetric generalization of vielbein transforms as $E_\mu^a \to e^{-\frac{A}{2}} e^{-\frac{A^\dagger}{2}} E_\mu^a$, which corresponds to the standard Weyl transformation $e_\mu^a \to e^{-\alpha} e_\mu^a$ after projecting on the lowest component of E_μ^a. In the interest of clarity we would like to add that $e_\mu^a = \delta_\mu^a$ on flat space.

[8]To see this notice that these terms depend on derivatives of s only (c.f (2.15)). The latter are related to the anomalous dimension $\gamma(\mu e^\tau)$ through the relation (2.17) which is independent of μ'.

in units of a real scalar field. This number has to be multiplied by the number of flavours $2N_f$ (two matter-field per flavour) and colours N_c (the $SU(N_c)$ Casimir of the adjoint representation). Hence Δa is given by $2N_f N_c \nu \Delta a|^{(2.22)}_{\gamma_{\text{UV,IR}} \to \delta\gamma_{\text{UV,IR}}}$. Now, $(\gamma_{\text{UV}}, \gamma_{\text{IR}}) = (0, \gamma_*)$ implies $(\delta\gamma_{\text{UV}}, \delta\gamma_{\text{IR}}) = (-\gamma_*, 0)$ and therefore

$$\Delta a|_{\mathcal{N}=1} = \frac{15}{2} N_c N_f (-\gamma_*^3 + 3\gamma_*^2) a_{(0)}^{\text{free}} . \tag{3.12}$$

We note that (3.12) is indeed the same as the non-perturbative result quoted in (eq. 4.18) in [1] when taking into account the explicit form of γ_* (A.6). The formula above is valid in the conformal window $3/2N_c < N_f < 3N_c$ where the UV theory is asymptotically free and the IR theory acquires a non-trivial fixed point. Within these boundaries the anomalous dimension γ_* takes on the values -1 to 0 and the quantity Δa is therefore manifestly positive in accordance with the a-theorem. The latter has been proven for $\mathcal{N}=1$ supersymmetric theories by using R-symmetries and is known as a-maximization [24]. The adaption to gauge groups other than $SU(N_c)$, provided they are asymptotically free, amounts to replacing the $SU(N_c)$-Casimir N_c by the corresponding Casimir of the group.

4 Discussion and conclusion

In this short paper we rederived the difference of the Euler term in $\mathcal{N}=1$ supersymmetric gauge theories (3.12) valid in the conformal window. By an appropriate rescaling of the matter superfield *and* choosing the Weyl-parameter α (1.1) to equal the logarithm of the matter prefactor (3.10), the computation was shown to be equivalent to one of the (free) UV theory in a curved background carrying the information on the flow. This allowed for $\Delta a|_{\mathcal{N}=1}$ to be computed from the free field theory example, in section 2.2, with a simple formal replacement for γ_{IR} and γ_{UV}. It is noted that the structure of $\Delta a|_{\mathcal{N}=1}$ is completely given by the Wess-Zumino term of the dilaton effective action. The aspect of matching the computation with a free theory bears some resemblance to the original AFGJ-derivation [1] in that independence on an RG-scale is exploited in evaluating certain quantities in the UV where they correspond to free field theory computations. An extension to the non-supersymmetric case is not straightforward because it relies on the one-loop exactness of the rescaling anomaly in supersymmetric gauge theories. From sections A and 3.1 it is seen that an exact expression of Δa in non-supersymmetric theories is related to finding an exact beta function. The Konishi anomaly is a rescaling anomaly which in $\mathcal{N}=1$ supersymmetric theories is, by holomorphicity, bound to the axial anomaly. The latter is generally one-loop exact by topological protection of the axial charge. In non-supersymmetric theories there is no holomorphicity and the Konishi anomaly is an unknown function which could be determined order by order in perturbation theory. An extension of some of the ideas in this paper to non-supersymmetric theories, in direction other than the Konishi anomaly, is presented elsewhere [25].

We end the paper with remarks of the speculative and qualitative kind. Reformulating a gauge theory as a free theory in a curved background is reminiscent of the anti-de Sitter space/conformal field theory duality which has given rise to a lot of work and inspiration

over the past two decades. The extension to theories with more than one relevant coupling is not immediate. One might wonder whether bi-gravity, whose renormalisation group flow has been studied in [26], might be a possible avenue for a theory with two relevant couplings. A practical requirement is that the UV theory is to asymptotically free in order to retain computability.

Acknowledgments

A $\mathcal{N} = 1$ effective action and the Konishi anomaly

Using arguments of holomorphicity it can be argued that the running of the coupling g, of the Wilsonian effective action of the supersymmetric gauge theory (3.1), is one-loop exact [17, 27] and reads

$$S_{\rm W}(\mu) = \left(\frac{1}{g^2(\mu')} - \frac{b_0}{8\pi^2} \ln \frac{\mu'}{\mu} \right) \int d^6 z {\rm tr} W^2 + {\rm h.c.}$$
$$+ \frac{1}{8} Z(\mu) \sum_f \left[\int d^8 z \Phi_f^\dagger e^{-2V} \Phi_f + \int d^8 z \tilde{\Phi}_f^\dagger e^{-2V} \tilde{\Phi}_f \right] , \quad (A.1)$$

where $b_0 \equiv 3N_c - N_f$ and $\mu' > \mu$ is an arbitrary scale which can be identified with the UV cut-off $\Lambda_{\rm UV}$. Rescaling the matter fields by

$$(\Phi_f, \tilde{\Phi}_f) \to Z^{-1/2} (\Phi_f, \tilde{\Phi}_f) , \quad (A.2)$$

is accompanied by the Konishi anomaly [18–20], and leads to the effective action [27]

$$S_{\rm W}(\mu) = \int d^6 z \frac{1}{g^2(\mu)} {\rm tr} W^2 + {\rm h.c.} + \sum_f \left[\int d^8 z \Phi_f^\dagger e^{-2V} \Phi_f + \int d^8 z \tilde{\Phi}_f^\dagger e^{-2V} \tilde{\Phi}_f \right] , \quad (A.3)$$

where the running has been removed from the matter term and all the running is absorbed in front of the gauge field term which in this case defines the running gauge coupling to be

$$\frac{1}{g^2(\mu)} = \frac{1}{g^2(\mu')} - \frac{1}{8\pi^2} \left(b_0 \ln \frac{\mu'}{\mu} - N_f \ln Z \right) . \quad (A.4)$$

The μ'-independence of $g(\mu)$ implies an RGE which solves to the holomorphic NSVZ beta function [28–30]

$$\beta \equiv \frac{\partial}{\partial \ln \mu} g = -g^3 \frac{N_f}{16\pi^2} (\gamma - \gamma_*) . \quad (A.5)$$

Above we have used the following notation

$$\gamma \equiv \frac{\partial \ln Z(\mu)}{\partial \ln \mu} , \quad \gamma_* \equiv -b_0/N_f = 1 - 3N_c/N_f . \quad (A.6)$$

In the range $3/2N_c < N_f < 3N_c$ the theory is in the so-called conformal window. Theories of the latter kind are asymptotically free in the UV and flow to a non-trivial ($\gamma_{IR} = \gamma_* \neq 0$) IR fixed point. The lower bound $3/2N_c$ follows from the unitarity bound on the scaling dimension of the composite squark field $\Delta_{\tilde{q}\tilde{q}} = 2 + \gamma_* \geq 1$ and the upper bound of $3N_c$ is derived from the requirement of asymptotic freedom.

B Free theory trace anomaly in dilaton background field

In this appendix the trace anomaly of a free scalar field theory (2.13) is evaluated on a conformally flat background $\tilde{g}_{\rho\lambda} = e^{-2s(x)}\delta_{\rho\lambda}$. The path integral is Gaussian and evaluates to

$$e^W = \int \mathcal{D}\phi\, e^{-S^{(0)}} = \sqrt{\det \Delta^{(0)}} = \exp \frac{1}{2}\mathrm{Tr}\ln \Delta^{(0)} \,, \tag{B.1}$$

where $\Delta^{(0)} = e^{-2s}(-\square + (\square s - (\partial s)^2)$ is the kinetic operator obtained from (2.13) by integration by parts. The contribution can be evaluated using Schwinger's formula

$$W = \frac{1}{2}\mathrm{Tr}\ln \Delta^{(0)} = \frac{1}{2}\int_0^\infty \frac{dt}{t}\mathrm{Tr}(e^{-t\Delta^{(0)}}) \,. \tag{B.2}$$

This expression requires regularisation since it is UV-divergent as $t \to 0$. Noting that the mass dimension of the t-variable is two, a UV cutoff Λ_{UV} is introduced as follows

$$W_{\mathrm{reg}} = \frac{1}{2}\int_{\Lambda_{UV}^{-2}}^\infty \frac{dt}{t}\mathrm{Tr}(e^{-t\Delta^{(0)}}) \,. \tag{B.3}$$

Using $W = W(\mu/\Lambda_{UV})$ and (C.5) one gets[9]

$$\int d^4x \sqrt{\tilde{g}}\langle\Theta\rangle = -\lim_{\Lambda_{UV}\to\infty}\frac{\partial}{\partial \ln \Lambda_{UV}}W_{\mathrm{reg}} = -\lim_{\Lambda_{UV}\to\infty}\mathrm{Tr}(e^{-\frac{\Delta^{(0)}}{\Lambda_{UV}^2}}) = -b_4 \,, \tag{B.4}$$

where b_4 is a coefficient of the asymptotic Heat Kernel expansion

$$\mathrm{Tr}(e^{-t\Delta^{(0)}}) = \sum_{n\geq 0} b_n t^{\frac{n-d}{2}} \,. \tag{B.5}$$

Using the plane-wave basis to evaluate the trace we obtain

$$\int d^4x \sqrt{\tilde{g}}\langle\Theta\rangle = -\frac{1}{16\pi^2}\frac{1}{90}\int d^4x \left[3\square^2 s - 2\square(\partial s)^2 - 4\partial\cdot(\partial s((\partial s)^2 - \square s))\right] \,, \tag{B.6}$$

which decomposes into the the following invariants

$$\int d^4x \sqrt{\tilde{g}}\langle\Theta\rangle = \frac{1}{16\pi^2}\frac{1}{90}\int d^4x \sqrt{\tilde{g}}\left(-\frac{1}{2}\tilde{\square}\tilde{R} + \frac{1}{4}\tilde{E}_4\right) \,, \tag{B.7}$$

where the geometric quantities $\tilde{R}$ and $\tilde{E}_4$ are defined with respect to the metric $\tilde{g}_{\rho\lambda}(s)$ given above. The result quoted in (2.14) follows by comparing the equation above to (1.2).

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[9]In general there are also quadratic and quartic divergences which need to be subtracted by suitable counterterms. In a supersymmetric theory those divergences cancel to zero.

C Renormalisation group equations for W

In this appendix we summarise the RG equation obeyed by W and how they relate to the trace anomaly. For future reference and completeness an explicit scale symmetry breaking term in form of a matter mass term is added. The quantum vacuum transition amplitude W obeys an RG equation

$$\left(\frac{\partial}{\partial \ln \mu} + \beta \frac{\partial}{\partial g} - \gamma_m \frac{\partial}{\partial \ln m}\right) W = 0 , \quad \gamma_m \equiv -\frac{\partial \ln m}{\partial \ln \mu} , \tag{C.1}$$

which follows from $\frac{dW}{d\mu} = 0$.[10] Assuming a space-dependent metric, dimensional analysis gives an equation of the form

$$\left(\frac{\partial}{\partial \ln \mu} + \frac{\partial}{\partial \ln m} + 2 \int d^4x \; \mathrm{g}^{\mu\nu}(x) \frac{\delta}{\delta \mathrm{g}^{\mu\nu}(x)}\right) W = 0 . \tag{C.2}$$

Equations (C.1), (C.2) can be combined into an RG equation with no explicit μ-derivative

$$\left(\beta \frac{\partial}{\partial g} - (1 + \gamma_m)\frac{\partial}{\partial \ln m} - 2 \int d^4x \; \mathrm{g}^{\mu\nu}(x)\frac{\delta}{\delta \mathrm{g}^{\mu\nu}(x)}\right) W = 0 . \tag{C.3}$$

The adaption of these equations to W_τ involves replacing $\mu \to \mu e^\tau$ everywhere. Note, if τ is made space-dependent then $g(\mu e^\tau)$ and $m(\mu e^\tau)$ and the partial derivatives are to be replaced by functional derivatives $\frac{\partial}{\partial g} \to \int d^4x \frac{\delta}{\delta g(x)}$ and $\frac{\partial}{\partial m} \to \int d^4x \frac{\delta}{\delta m(x)}$ respectively.

A definition of the trace of the EMT is given by

$$\langle \Theta \rangle = -2 \, \frac{\mathrm{g}^{\mu\nu}(x)}{\sqrt{\mathrm{g}(x)}} \frac{\delta}{\delta \mathrm{g}^{\mu\nu}(x)} W \tag{C.4}$$

where $\mathrm{g}(x)$ denotes the determinant of the metric. Combining (C.4) with (C.2) the following equations are obtained

$$\int d^4x \sqrt{\mathrm{g}} \; \langle \Theta \rangle_{\mathrm{anom}} = \frac{\partial}{\partial \ln \mu} W , \quad \int d^4x \sqrt{\mathrm{g}} \; \langle \Theta \rangle_{\mathrm{expl}} = \frac{\partial}{\partial \ln m} W , \tag{C.5}$$

where the subscripts "anom" and "expl" refer to anomalous and explicit scale breaking respectively.

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[10]Throughout this paper $\gamma = -\gamma_m$ is the anomalous dimension of the squark composite operator whereas other authors [4, 5] use the anomalous dimension of the superfield Φ as $\gamma = \gamma_\Phi$. The relation between the two is $-2\gamma_\Phi = \gamma_m$; i.e. $\gamma|_{\text{this work}} = 2\gamma|_{[4, 5]}$.

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