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Next to subleading soft-graviton theorem in arbitrary dimensions

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Abstract: We study the soft graviton theorem recently proposed by Cachazo and Strominger. We employ the Cachazo, He and Yuan formalism to show that the next to subleading order soft factor for gravity is universal at tree level in arbitrary dimensions.

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Contents

1 Introduction

The study of the soft graviton amplitudes dates back to Weinberg [1, 2] where the leading soft behavior was obtained. In [3–5] a new soft graviton theorem, conjectured to be the Ward identities of a new symmetry of the quantum gravity S-matrix,[1] was proposed. Cachazo and Strominger [8] have recently shown that the new conjectured soft behavior, through subleading and next-to-subleading orders in the soft expansion, has a universal form in four spacetime dimensions at tree level.[2] An extension to gluons for the first subleading soft behavior at tree level was reported in [15] and in [16] it was demonstrated that the conformal invariance of tree level gauge theory amplitudes in four spacetime dimensions determines the form of the first subleading theorem.

Very recently it has been shown that the form of the subleading and next-to-subleading terms in the soft expansion in D dimensions — for both gauge theory and gravity — is greatly constrained by the requirements from Poincaré symmetry, gauge invariance, and a self consistency condition originating from the distributional nature of scattering amplitudes [17]. Moreover, the subleading and next-to-subleading terms were determined up to a numerical constant for each external leg.[3] Using Feynman diagram techniques the first subleading theorem for soft gluons and gravitons was also confirmed in [18]. The

[1]This new proposed symmetry is an extension of the Bondi, van der Burg, Metzner and Sachs (BMS) symmetry [6, 7].

[2]Early results for soft photons at subleading order were obtained in [9–12]. Gross and Jackiw, using dispersion relation methods, derived the subleading soft factor for graviton scattering off scalars in [13], and White revised the subject in [14] using path integral resummation techniques.

[3]However, the authors also point out that these numerical factors are completely fixed when specializing to four spacetime dimensions.

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authors of [19] have shown that on-shell gauge invariance determines the complete form of the first subleading soft theorem in gauge theory and the first two subleading soft theorems for gravity. Using the Cachazo, He, Yuan (CHY) formula [20], the tree level universality of the soft behavior to first subleading order has been shown to hold in D dimensions [21, 22]. The purpose of the present note is to use the CHY formula to prove the universal nature of the next-to-subleading soft graviton theorem at tree level in arbitrary dimensions.

Studies on loop corrections to subleading soft theorems have been presented in [23–25]. Progress in the context of string theory has been reported in [26, 27] and also in [28, 29] relevant for recent twistor constructions. More recent progress on soft theorems in the context of massless QED has appeared in [30, 31].

The conjecture of [8] states, for an on-shell tree level n-graviton amplitude M_n, that

$$M_n = \left(\frac{1}{\lambda}S^{(0)} + S^{(1)} + \lambda S^{(2)} + \mathcal{O}(\lambda^2)\right) M_{n-1}, \tag{1.1}$$

where n is taken to be the soft particle with momentum k_n and we scale the momentum $k_n \to \lambda k_n$ and take the limit when λ approaches zero. In the above,

$$S^{(0)} = \sum_{a=1}^{n-1} \frac{\epsilon_{\mu\nu} k_a^\mu k_a^\nu}{k_n \cdot k_a} \tag{1.2}$$

is Weinberg's soft theorem with $\epsilon_{\mu\nu}$ denoting the polarization tensor of the soft graviton and the gravitational constant has been set to 1. The conjectured forms of the subleading and next-to-subleading theorems are

$$S^{(1)} = -i \sum_{a=1}^{n-1} \frac{\epsilon_{\mu\nu} k_a^\mu k_{n\lambda} J_a^{\lambda\nu}}{k_n \cdot k_a}, \qquad S^{(2)} = -\frac{1}{2} \sum_{a=1}^{n-1} \frac{\epsilon_{\mu\nu} k_{n\rho} J_a^{\rho\mu} k_{n\lambda} J_a^{\lambda\nu}}{k_n \cdot k_a}. \tag{1.3}$$

In order to treat gluon and graviton polarizations on an equal footing one can choose to write the graviton polarization for the a^{th} particle as

$$\epsilon_{a\mu\nu} = \epsilon_{a\mu}\epsilon_{a\nu} \tag{1.4}$$

where $a = 1, \ldots, n-1$. Tracelessness and orthogonality to k_a translate into $\epsilon_a \cdot \epsilon_a = 0$ and $\epsilon_a \cdot k_a = 0$ respectively.[4]

The subleading contributions to the soft theorem depend on the total angular momentum operator, which is[5]

$$J_a^{\mu\nu} = i\left(k_a^{[\mu} \frac{\partial}{\partial k_{a\nu]}} + \epsilon_a^{[\mu} \frac{\partial}{\partial \epsilon_{a\nu]}}\right) \tag{1.5}$$

for the a^{th} particle. Note that in using this formula one should consider the polarization vectors ϵ_a^μ to be independent of the momenta k_a^μ.

This paper is organized as follows. In section 2 we review the CHY formalism [20] for tree level graviton amplitudes which is valid in arbitrary dimensions and, in this language,

[4] We do not use any other gauge condition in this work.

[5] We follow the convention $A_{(\mu}B_{\nu)} = A_\mu B_\nu + A_\nu B_\mu$ and $A_{[\mu}B_{\nu]} = A_\mu B_\nu - A_\nu B_\mu$.

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we set up the computation for the expansion of the amplitude up to next-to-subleading order in the soft parameter. We finish this section by stating the new soft theorem extended to D dimensions. In section 3 we explicitly evaluate the tree level n-graviton amplitude at next-to-subleading order in the soft expansion. In section 4 we compute the action of the conjectured $S^{(2)}$ operator (1.3) onto the $(n-1)$-graviton amplitude, as stated in (1.1), and show that it perfectly matches with the next-to-subleading amplitude $M^{(2)}$ of section 3, thus proving the theorem.

2 Review and setup of the problem

In this section we briefly review the CHY construction [20] for tree level graviton amplitudes. A key object is the *scattering equations*

$$\sum_{b\neq a}^{n} \frac{k_a{\cdot}k_b}{\sigma_{ab}} = 0, \qquad a,b = 1,\ldots,n. \tag{2.1}$$

with $\sigma_{ab} \equiv \sigma_a - \sigma_b$, where the σ_a are in general complex valued quantities. Due to the $\mathrm{SL}(2,\mathbb{C})$ symmetry of (2.1), these constitute a system of $n-3$ independent equations for the set $\{\sigma_a\}$ and one can arbitrarily fix three of the σ_a variables. We will call $\sigma_i, \sigma_j, \sigma_k$ the three fixed σs. The gauge fixed amplitude is

$$M_n = \int [d\sigma]_{n-4}\, d\sigma_n \prod_{a\neq i,j,k}^{n} \delta(f_a^n)\, E_n, \tag{2.2}$$

where we have employed the useful short notation

$$f_a^n \equiv \sum_{b\neq a}^{n} \frac{k_a{\cdot}k_b}{\sigma_{ab}}, \qquad [d\sigma]_{n-4} \equiv (\sigma_{pq}\sigma_{qr}\sigma_{rp})(\sigma_{ij}\sigma_{jk}\sigma_{ki}) \prod_{c\neq p,q,r}^{n-1} d\sigma_c. \tag{2.3}$$

In the above, E_n is defined to be

$$E_n = 4\det(\Psi^{xy}_{xy})/\sigma^2_{xy}, \tag{2.4}$$

where $\Psi^{xyz\ldots}_{xyz'\ldots}$ is obtained from the $2n\times 2n$ antisymmetric matrix Ψ after removing rows $x, y, z, \ldots$ and columns $x, y, z', \ldots$ with $1 \leq x < y \leq n$. The explicit expression of Ψ is given by

$$\Psi = \begin{pmatrix} A & -C^{\mathrm{T}} \\ C & B \end{pmatrix} \tag{2.5}$$

with the $n\times n$ matrices A, B, C given by

$$A_{ab} = \frac{k_a{\cdot}k_b}{\sigma_{ab}}\delta_{a\neq b}, \quad B_{ab} = \frac{\epsilon_a{\cdot}\epsilon_b}{\sigma_{ab}}\delta_{a\neq b}, \quad C_{ab} = \frac{\epsilon_a{\cdot}k_b}{\sigma_{ab}}\delta_{a\neq b} - \delta_{ab}\sum_{c\neq a}^{n}\frac{\epsilon_a{\cdot}k_c}{\sigma_{ac}}, \tag{2.6}$$

where we use $\delta_{a\neq b} \equiv 1-\delta_{ab}$ in order to avoid cluttering our equations. In [20] it was shown that the quantity E_n is independent of the choice of x and y.

In order to expand the delta function appearing in (2.2) in powers of λ we separate it into two parts

$$\begin{aligned}\prod_{a\neq i,j,k}^{n}\delta(f_a^n) &= \frac{1}{\lambda}\delta\left(\sum_{b=1}^{n-1}\frac{k_n\cdot k_b}{\sigma_{nb}}\right)\prod_{a\neq i,j,k}^{n-1}\delta\left(\sum_{b\neq a}^{n-1}\frac{k_a\cdot k_b}{\sigma_{ab}}+\lambda\frac{k_a\cdot k_n}{\sigma_{an}}\right)\\ &= \delta(f_n^{n-1})\left(\frac{1}{\lambda}\delta^{(0)}+\delta^{(1)}+\lambda\delta^{(2)}\right)+\mathcal{O}(\lambda^2),\end{aligned}\tag{2.7}$$

where we define

$$\delta^{(0)}=\prod_{a\neq i,j,k}^{n-1}\delta(f_a^{n-1}),\quad \delta^{(1)}=\sum_{l\neq i,j,k}^{n-1}\frac{k_l\cdot k_n}{\sigma_{ln}}\delta'(f_l^{n-1})\left[\prod_{a\neq i,j,k,l}^{n-1}\delta(f_a^{n-1})\right],\tag{2.8}$$

$$\begin{aligned}\delta^{(2)}=&\frac{1}{2}\sum_{l\neq i,j,k}^{n-1}\frac{k_l\cdot k_n}{\sigma_{ln}}\delta'(f_l^{n-1})\sum_{m\neq i,j,k,l}\left[\frac{k_m\cdot k_n}{\sigma_{mn}}\delta'(f_m^{n-1})\prod_{b\neq i,j,k,l,m}^{n-1}\delta(f_b^{n-1})\right]\\ &+\frac{1}{2}\sum_{l\neq i,j,k}^{n-1}\left(\frac{k_l\cdot k_n}{\sigma_{ln}}\right)^2\delta''(f_l^{n-1})\prod_{b\neq i,j,k,l}^{n-1}\delta(f_b^{n-1}).\end{aligned}\tag{2.9}$$

We also need to expand E_n in (2.2) to second order in λ

$$E_n=E_n^{(0)}+\lambda E_n^{(1)}+\lambda^2E_n^{(2)}+\mathcal{O}(\lambda^3).\tag{2.10}$$

Plugging (2.7) and (2.10) into (2.2) we get

$$M_n=\frac{1}{\lambda}M_n^{(0)}+M_n^{(1)}+\lambda M_n^{(2)}+\mathcal{O}(\lambda^2),\tag{2.11}$$

where

$$\begin{aligned}M_n^{(0)}&=\int[d\sigma]_{n-4}\,d\sigma_n\delta(f_n^{n-1})\delta^{(0)}E_n^{(0)},\\ M_n^{(1)}&=\int[d\sigma]_{n-4}\,d\sigma_n\delta(f_n^{n-1})(\delta^{(1)}E_n^{(0)}+\delta^{(0)}E_n^{(1)}),\\ M_n^{(2)}&=\int[d\sigma]_{n-4}\,d\sigma_n\delta(f_n^{n-1})(\delta^{(2)}E_n^{(0)}+\delta^{(1)}E_n^{(1)}+\delta^{(0)}E_n^{(2)}).\end{aligned}\tag{2.12}$$

The soft theorem conjectures that the following equality should hold

$$M_n^{(i)}=S^{(i)}M_{n-1},\qquad i=0,1,2.\tag{2.13}$$

Weinberg's soft theorem, i.e., $M_n^{(0)}=S^{(0)}M_{n-1}$, can be derived as follows. To evaluate $M_n^{(0)}$ in (2.12) we also need $E_n^{(0)}$, the leading contribution to the determinant (2.4), which is $E_n^{(0)}=C_{nn}^2E_{n-1}$. In order to see that, we can set $\lambda=0$ in E_n. Then all the elements of the $(n-2)^{\text{th}}$ row vanish apart from the last one which equals $-C_{nn}$. Similarly all elements of the $(n-2)^{\text{th}}$ column are zero apart from the last one which is C_{nn}. Expansion of the determinant along the aforementioned row and column will yield another extra sign which completes the proof.

Separating all the dependence on σ_n in $M_n^{(0)}$, i.e.,

$$M_n^{(0)} = \int [d\sigma]_{n-4}\, \delta^{(0)} E_{n-1} \int d\sigma_n \delta(f_n^{n-1}) C_{nn}^2, \tag{2.14}$$

we can explicitly evaluate the integral over σ_n. Due to the absence of branch-cuts and the regularity of the integrand when $\sigma_n \to \infty$, we may treat the delta function as a pole and we can evaluate the integral by deforming the contour and using the residue theorem. Performing this one obtains

$$\int d\sigma_n \delta(f_n^{n-1}) C_{nn}^2 = \sum_{a=1}^{n-1} \frac{(\epsilon_n \cdot k_a)^2}{k_n \cdot k_a}. \tag{2.15}$$

Putting everything together into (2.14) yields

$$\begin{aligned} M_n^{(0)} &= \sum_{a=1}^{n-1} \frac{(\epsilon_n \cdot k_a)^2}{k_n \cdot k_a} \int [d\sigma]_{n-4} \prod_{l \neq i,j,k}^{n-1} \delta(f_l^{n-1}) E_{n-1} \\ &= \sum_{a=1}^{n-1} \frac{(\epsilon_n \cdot k_a)^2}{k_n \cdot k_a} M_{n-1}. \end{aligned} \tag{2.16}$$

From (1.2) one can easily see that $S^{(0)} M_{n-1}$ is precisely the last line of (2.16), thus proving Weinberg's leading soft-graviton theorem.

The computation of (2.13) for $i = 1$ in arbitrary dimensions was performed in [21, 22]. In the next section we start the computation of the next to subleading soft contribution ($i = 2$) by evaluating $M_n^{(2)}$ in (2.12). Then, in section 4, we will evaluate the action of $S^{(2)}$ on M_{n-1}. We will compare both sides of (2.13) by matching terms that contain the same support from the δ-distributions and we will find perfect matching, thus, proving the theorem.

3 Evaluation of $M_n^{(2)}$

We split the evaluation of $M_n^{(2)}$ into three parts

$$M_n^{(2)} = \int [d\sigma]_{n-4} (m_1 + m_2 + m_3), \qquad m_i = \int d\sigma_n\, \delta(f_n^{n-1}) \delta^{(3-i)} E_n^{(i-1)}. \tag{3.1}$$

3.1 Evaluation of m_1

Using (2.9), the first contribution, m_1, to $M_n^{(2)}$ is

$$\begin{aligned} m_1 = &\frac{1}{2} E_{n-1} \sum_{l \neq i,j,k}^{n-1} \delta'(f_l^{n-1}) \sum_{m \neq i,j,k,l}^{n-1} \delta'(f_m^{n-1}) \prod_{b \neq i,j,k,l,m}^{n-1} \delta(f_b^{n-1})\ I_1 \\ &+ \frac{1}{2} E_{n-1} \sum_{l \neq i,j,k}^{n-1} \delta''(f_l^{n-1}) \prod_{b \neq i,j,k,l}^{n-1} \delta(f_b^{n-1})\ I_2, \end{aligned} \tag{3.2}$$

where we have isolated the integration over σ_n to the following integral

$$I = k_l{\cdot}k_n\, k_m{\cdot}k_n \int d\sigma_n \delta(f_n^{n-1}) \frac{C_{nn}^2}{\sigma_{nl}\sigma_{nm}}. \tag{3.3}$$

Therefore, in (3.2), we have $I_1 = I|_{m\neq l}$ and $I_2 = I|_{m=l}$.

We now move on to compute the integral (3.3). We find

$$\begin{aligned} I = &\Bigg\{\left[\frac{k_m{\cdot}k_n\, \epsilon_n{\cdot}k_l}{\sigma_{ml}}\left(\frac{\epsilon_n{\cdot}k_l}{k_n{\cdot}k_l}\sum_{c\neq l}^{n-1}\frac{k_n{\cdot}k_c}{\sigma_{lc}} - 2\sum_{c\neq l}^{n-1}\frac{\epsilon_n{\cdot}k_c}{\sigma_{lc}}\right) - \frac{(\epsilon_n{\cdot}k_l)^2 k_m{\cdot}k_n}{\sigma_{ml}^2}\right] + (l\leftrightarrow m) \\ &\qquad + k_l{\cdot}k_n\, k_m{\cdot}k_n \sum_{c\neq l,m}^{n-1}\frac{(\epsilon_n{\cdot}k_c)^2}{\sigma_{lc}\sigma_{mc}k_n{\cdot}k_c}\Bigg\}\delta_{m\neq l} \\ &+ \Bigg\{(k_l{\cdot}k_n)^2\sum_{c\neq l}^{n-1}\frac{(\epsilon_n{\cdot}k_c)^2}{\sigma_{lc}^2\, k_n{\cdot}k_c} + (\epsilon_n{\cdot}k_l)\sum_{c\neq l}^{n-1}\frac{\epsilon_n{\cdot}k_l\, k_n{\cdot}k_c - 2\,\epsilon_n{\cdot}k_c\, k_n{\cdot}k_l}{\sigma_{lc}^2} \\ &\qquad + k_l{\cdot}k_n\left(\sum_{c\neq l}^{n-1}\frac{\epsilon_n{\cdot}k_c}{\sigma_{lc}} - \frac{\epsilon_n{\cdot}k_l}{k_n{\cdot}k_l}\sum_{c\neq l}^{n-1}\frac{k_n{\cdot}k_c}{\sigma_{lc}}\right)^2\Bigg\}\delta_{ml}\,. \end{aligned} \tag{3.4}$$

The first line in (3.4) is the contribution of a double pole at $\sigma_n = \sigma_l$ and a double pole at $\sigma_n = \sigma_m$, whereas the second line in (3.4) comes from the contribution of a single pole of the integrand at $\sigma_n = \sigma_c$, for all $c \neq l, m, n$. The first term in the third line comes from a single pole at $\sigma_n = \sigma_c$ for all $c \neq l, n$ and the remaining of (3.4) comes from a third order pole at $\sigma_n = \sigma_l$.

3.2 Evaluation of m_2

For the evaluation of m_2 we need to expand (2.4) to order λ. The derivative of the determinant of a $n \times n$ matrix with entries T_{ab} can be obtained from the formula

$$\frac{d}{d\lambda}\det(T) = \sum_{a=1}^{n}\sum_{b=1}^{n}(-1)^{a+b}\frac{dT_{ab}}{d\lambda}M_b^a, \tag{3.5}$$

where M_b^a denotes the determinant of the matrix obtained by removing the a^{th} row and the b^{th} column of T. Applying it onto E_n in equation (2.4) yields

$$\frac{dE_n}{d\lambda} = \sum_{a=1}^{n}\sum_{b=1}^{n}\left((-1)^{a+b}\frac{dA_{ab}}{d\lambda}\tilde{\psi}_b^a + 2(-1)^{a+b+n}\frac{dC_{ab}}{d\lambda}\tilde{\psi}_b^{n+a} + (-1)^{a+b}\frac{dB_{ab}}{d\lambda}\tilde{\psi}_{n+b}^{n+a}\right). \tag{3.6}$$

Here we have used the short notation

$$\tilde{\psi}_b^a \equiv \frac{4\,\det(\Psi_{12b}^{12a})}{\sigma_{12}^2}\delta_{a\neq\{1,2\}}\delta_{b\neq\{1,2\}}. \tag{3.7}$$

For convenience and without loss of generality we have chosen to remove the first two rows and the first two columns in (2.4). In (3.6) we have also used the identity $\tilde{\psi}_b^a = -\tilde{\psi}_a^b$. The derivatives of the different matrix elements are

$$\begin{aligned} \frac{dA_{ab}}{d\lambda} &= \frac{1}{\sigma_{ab}}\left(\delta_{an}k_n{\cdot}k_b + \delta_{bn}k_a{\cdot}k_n\right)\delta_{a\neq b}\,, \qquad \frac{dB_{ab}}{d\lambda} = 0\,, \\ \frac{dC_{ab}}{d\lambda} &= \frac{\epsilon_a{\cdot}k_n}{\sigma_{ab}}\delta_{bn}\delta_{a\neq b} - \delta_{ab}\frac{\epsilon_a{\cdot}k_n}{\sigma_{an}}\delta_{a\neq n}. \end{aligned} \tag{3.8}$$

Putting this into (3.6) yields

$$\frac{dE_n}{d\lambda} = 2\sum_{a=1}^{n-1}\frac{1}{\sigma_{na}}\left((-1)^{a+n}k_a{\cdot}k_n\,\tilde{\psi}^n_a + (-1)^a\epsilon_a{\cdot}k_n\,\tilde{\psi}^n_{n+a} + (-1)^{n-1}\epsilon_a{\cdot}k_n\,\tilde{\psi}^a_{n+a}\right). \tag{3.9}$$

Note that all the dependence in λ is now contained in the $\tilde{\psi}$ determinants only, which also need to be evaluated at $\lambda = 0$ at the end. We further need to isolate any encounter of σ_n in (3.9), since we eventually want to integrate over that variable. We find

$$\begin{aligned} \tilde{\psi}^n_a &= C_{nn}\sum_{b=1}^{n-1}\left((-1)^{n+b}\,\frac{\epsilon_n{\cdot}k_b}{\sigma_{nb}}\psi^a_b - (-1)^b\,\frac{\epsilon_n{\cdot}\epsilon_b}{\sigma_{nb}}\psi^a_{n+b-1}\right), \\ \tilde{\psi}^n_{n+a} &= -C_{nn}\sum_{b=1}^{n-1}\left((-1)^{n+b}\,\frac{\epsilon_n{\cdot}k_b}{\sigma_{nb}}\psi^{n+a-1}_b - (-1)^b\,\frac{\epsilon_n{\cdot}\epsilon_b}{\sigma_{nb}}\psi^{n+a-1}_{n+b-1}\right), \\ \tilde{\psi}^a_{n+a} &= -C^2_{nn}\psi^a_{n+a-1}, \end{aligned} \tag{3.10}$$

where we have dropped the tilde sign to denote the further removal of the rows and columns that contain the variable σ_n, that is ψ^a_b denotes the determinant E_{n-1} after the removal of the a^{th} row and the b^{th} column. Then

$$\begin{aligned} E^{(1)}_n = 2C_{nn}\sum_{a=1}^{n-1}\sum_{b=1}^{n-1}\frac{(-1)^{a+b}}{\sigma_{na}\sigma_{nb}}&\Big(k_n{\cdot}k_a\,\epsilon_n{\cdot}k_b\,\psi^a_b - k_n{\cdot}\epsilon_b\,\epsilon_n{\cdot}\epsilon_a\,\psi^{n+a-1}_{n+b-1} \\ &+ (-1)^n\left(\epsilon_n{\cdot}k_a\,k_n{\cdot}\epsilon_b - k_n{\cdot}k_a\,\epsilon_n{\cdot}\epsilon_b\right)\psi^a_{n+b-1}\Big) \\ + 2C^2_{nn}\sum_{a=1}^{n-1}\frac{(-1)^n}{\sigma_{na}}\epsilon_a{\cdot}k_n\psi^a_{n+a-1}. \end{aligned} \tag{3.11}$$

We recall that m_2 takes the form

$$m_2 = \int d\sigma_n\delta(f^{n-1}_n)\sum_{l\neq i,j,k}^{n-1}\frac{k_n{\cdot}k_l}{\sigma_{ln}}\delta'(f^{n-1}_l)\prod_{m\neq i,j,k,l}^{n-1}\delta(f^{n-1}_m)E^{(1)}_n \tag{3.12}$$

thus, we will need the following integrals

$$I_3 \equiv -\,k_n{\cdot}k_l\int d\sigma_n\delta(f^{n-1}_n)\frac{C_{nn}}{\sigma_{nl}\sigma_{na}\sigma_{nb}}, \tag{3.13}$$

$$I_4 \equiv -\,k_n{\cdot}k_l\int d\sigma_n\delta(f^{n-1}_n)\frac{C^2_{nn}}{\sigma_{nl}\sigma_{na}}. \tag{3.14}$$

The integral I_4 is directly obtained from (3.4) since $I_4 = -(k_n{\cdot}k_a)^{-1} I|_{m=a}$. For I_3 we find

$$
\begin{aligned}
I_3 = k_n{\cdot}k_l &\left[\frac{\epsilon_n{\cdot}k_a}{k_n{\cdot}k_a}\frac{1}{\sigma_{al}\sigma_{ab}} + (a \leftrightarrow l) + (a \leftrightarrow b)\right]\delta_{l\neq a}\delta_{l\neq b}\delta_{a\neq b}\\
&+ k_n{\cdot}k_l\Bigg[\left(\frac{\epsilon_n{\cdot}k_a}{k_n{\cdot}k_a}\frac{1}{\sigma_{al}}\sum_{c\neq a}^{n-1}\frac{1}{\sigma_{ac}}\left(\frac{\epsilon_n{\cdot}k_c}{\epsilon_n{\cdot}k_a} - \frac{k_n{\cdot}k_c}{k_n{\cdot}k_a}\right) + \frac{\epsilon_n{\cdot}k_l}{k_n{\cdot}k_l}\frac{1}{\sigma_{al}^2} - \frac{\epsilon_n{\cdot}k_a}{k_n{\cdot}k_a}\frac{1}{\sigma_{al}^2}\right)\delta_{l\neq a}\delta_{ab}\\
&\qquad\qquad + (l \leftrightarrow a) + (\{a,l,b\} \to \{l,b,a\})\Bigg] \\
&+ \epsilon_n{\cdot}k_l\Bigg[-\frac{1}{\epsilon_n{\cdot}k_l}\sum_{c\neq l}^{n-1}\frac{\epsilon_n{\cdot}k_c}{\sigma_{lc}^2} - \frac{1}{k_n{\cdot}k_l\epsilon_n{\cdot}k_l}\sum_{c\neq l}^{n-1}\sum_{d\neq l}^{n-1}\frac{k_n{\cdot}k_c\epsilon_n{\cdot}k_d}{\sigma_{lc}\sigma_{ld}}\\
&\qquad\qquad + \frac{1}{k_n{\cdot}k_l}\sum_{c\neq l}^{n-1}\frac{k_n{\cdot}k_c}{\sigma_{lc}^2} + \frac{1}{(k_n{\cdot}k_l)^2}\left(\sum_{c\neq l}^{n-1}\frac{k_n{\cdot}k_c}{\sigma_{lc}}\right)^2\Bigg]\delta_{ab}\delta_{bl}.
\end{aligned}
\tag{3.15}
$$

As a check, note that from this expression the quantity $I_3/(k_n{\cdot}k_l)$ is symmetric under the exchange of any two pairs of (l, a, b) which is evident from the original definition in (3.13).

We now write m_2 making explicit the linear combination of the different types of minors we have, i.e.,

$$
m_2 = 2\sum_{l\neq i,j,k}^{n-1}\delta'(f_l^{n-1})\prod_{m\neq i,j,k,l}^{n-1}\delta(f_m^{n-1})D_l, \tag{3.16}
$$

where

$$
\begin{aligned}
D_l \equiv &\sum_{a\neq l}^{n-1}\sum_{b\neq l,a}^{n-1}\left(c_1\psi_b^a + c_2\psi_{n+b-1}^{n+a-1} + c_3\psi_{n+b-1}^a\right)I_{3\{l\neq a,\, l\neq b,\, a\neq b\}}\\
&+ \sum_{a\neq l}^{n-1}c_4\psi_{n+a-1}^a + \sum_{a\neq l}^{n-1}\left(c_5\psi_l^a + c_6\psi_{n+l-1}^{n+a-1} + c_7\psi_{n+l-1}^a + c_8\psi_{n+a-1}^l\right)I_{3\{l=b,\, l\neq a\}}\\
&+ c_9\psi_{l+n-1}^l.
\end{aligned}
\tag{3.17}
$$

The coefficients c_i are

$$
\begin{aligned}
&c_1 = (-1)^{a+b}k_n{\cdot}k_a\,\epsilon_n{\cdot}k_b\,;\;\; c_2 = -(-1)^{a+b}\epsilon_b{\cdot}k_n\,\epsilon_n{\cdot}\epsilon_a\,;\;\; c_3 = (-1)^n\left(\epsilon_n{\cdot}k_a\,\epsilon_b{\cdot}k_n - k_n{\cdot}k_a\,\epsilon_n{\cdot}\epsilon_b\right);\\
&c_4 = (-1)^n\left(\epsilon_n{\cdot}k_a\,k_n{\cdot}\epsilon_a - k_n{\cdot}k_a\,\epsilon_n{\cdot}\epsilon_a\right)I_{3\{a=b,\, l\neq a\}} + (-1)^n\epsilon_a{\cdot}k_n\,I_{4\{l\neq a\}};\\
&c_5 = (-1)^{a+l}\left(k_n{\cdot}k_a\,\epsilon_n{\cdot}k_l - k_n{\cdot}k_l\,\epsilon_n{\cdot}k_a\right);\;\; c_6 = (-1)^{a+l}\left(\epsilon_a{\cdot}k_n\,\epsilon_n{\cdot}\epsilon_l - \epsilon_l{\cdot}k_n\,\epsilon_n{\cdot}\epsilon_a\right);\\
&c_7 = (-1)^{a+l+n}\left(\epsilon_n{\cdot}k_a\,\epsilon_l{\cdot}k_n - k_n{\cdot}k_a\,\epsilon_n{\cdot}\epsilon_l\right);\;\; c_8 = (-1)^{a+l+n}\left(\epsilon_n{\cdot}k_l\,\epsilon_a{\cdot}k_n - k_n{\cdot}k_l\,\epsilon_n{\cdot}\epsilon_a\right);\\
&c_9 = (-1)^n\left(\epsilon_n{\cdot}k_l\,k_n{\cdot}\epsilon_l - k_n{\cdot}k_l\,\epsilon_n{\cdot}\epsilon_l\right)I_{3\{l=a=b\}} + (-1)^n\epsilon_l{\cdot}k_n\,I_{4\{l=a\}}.
\end{aligned}
\tag{3.18}
$$

In the above we have used the identity $\psi_b^a = -\psi_a^b$.

3.3 Evaluation of m_3

We define $\tilde{\psi}_{cd}^{ab}$ and ψ_{cd}^{ab} to be respectively the determinants E_n and E_{n-1} after the removal of the rows a, b and the columns c, d.

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For the evaluation of m_3 we need to take the second derivative of (2.4) with respect to λ. From (3.9) we have

$$\frac{d^2E_n}{d\lambda^2} = 2\sum_{a=1}^{n-1}\frac{1}{\sigma_{na}}\left((-1)^{a+n}k_a{\cdot}k_n\,\frac{d\tilde{\psi}_a^n}{d\lambda} + (-1)^a\epsilon_a{\cdot}k_n\,\frac{d\tilde{\psi}_{n+a}^n}{d\lambda} + (-1)^{n-1}\epsilon_a{\cdot}k_n\,\frac{d\tilde{\psi}_{n+a}^a}{d\lambda}\right). \tag{3.19}$$

With the definition θ_{ij} to be 0 when $i > j$ and -1 when $i < j$ we find

$$\begin{aligned}
\frac{d\tilde{\psi}_a^n}{d\lambda} &= \sum_{b=1}^{n-1}\frac{1}{\sigma_{bn}}\left((-1)^{n+b-1}k_b{\cdot}k_n\tilde{\psi}_{an}^{bn} + (-1)^{n-1}\epsilon_b{\cdot}k_n\tilde{\psi}_{a,n+b}^{bn} + (-1)^b\epsilon_b{\cdot}k_n\tilde{\psi}_{an}^{n,n+b}\right)\\
&\quad + \sum_{b\neq a}^{n-1}(-1)^{n+\theta_{ab}}\frac{\epsilon_b{\cdot}k_n}{\sigma_{bn}}\tilde{\psi}_{ab}^{n,n+b},\\
\frac{d\tilde{\psi}_{n+a}^n}{d\lambda} &= \sum_{b=1}^{n-1}\frac{1}{\sigma_{bn}}\left((-1)^{n+b}k_b{\cdot}k_n\tilde{\psi}_{n,n+a}^{bn} + (-1)^{b-1}\epsilon_b{\cdot}k_n\tilde{\psi}_{n,n+a}^{n,n+b} + (-1)^n\epsilon_b{\cdot}k_n\tilde{\psi}_{b,n+a}^{n,n+b}\right)\\
&\quad + \sum_{b\neq a}^{n-1}(-1)^{n+\theta_{ab}}\frac{\epsilon_b{\cdot}k_n}{\sigma_{bn}}\tilde{\psi}_{n+a,n+b}^{bn},\\
\frac{d\tilde{\psi}_{n+a}^a}{d\lambda} &= \sum_{b=1}^{n-1}\frac{1}{\sigma_{bn}}\left((-1)^{n+b}k_b{\cdot}k_n\tilde{\psi}_{b,n+a}^{an} + (-1)^n\epsilon_b{\cdot}k_n\tilde{\psi}_{b,n+a}^{a,n+b} + (-1)^{b-1}\epsilon_b{\cdot}k_n\tilde{\psi}_{n,n+a}^{a,n+b}\right)\\
&\quad + \sum_{b\neq a}^{n-1}\frac{(-1)^{\theta_{ab}}}{\sigma_{bn}}\Big((-1)^{n+b}k_b{\cdot}k_n\tilde{\psi}_{n,n+a}^{ab}\\
&\qquad + (-1)^{n+\theta_{ab}}\epsilon_b{\cdot}k_n\tilde{\psi}_{n+a,n+b}^{ab} + (-1)^b\epsilon_b{\cdot}k_n\tilde{\psi}_{n+a,n+b}^{an}\Big).
\end{aligned} \tag{3.20}$$

We can further expand the n and $2n$ rows and columns of the minors appearing in (3.20). With the help of the identity $\psi_{cd}^{ab} = \psi_{ab}^{cd}$ we arrive at the following result

$$E_n^{(2)} = C_{nn}^2A_1 + C_{nn}A_2 + A_3, \tag{3.21}$$

where

$$A_1 = \sum_{a=1}^{n-1}\sum_{b=1}^{n-1}\frac{\epsilon_a{\cdot}k_n\epsilon_b{\cdot}k_n}{\sigma_{na}\sigma_{nb}}\psi_{b,n+a-1}^{a,n+b-1} + \sum_{a=1}^{n-1}\sum_{b\neq a}^{n-1}\frac{\epsilon_a{\cdot}k_n\epsilon_b{\cdot}k_n}{\sigma_{na}\sigma_{nb}}\psi_{n+a-1,n+b-1}^{ab}, \tag{3.22}$$

$$\begin{aligned}
A_2 &= 2\sum_{a=1}^{n-1}\sum_{b=1}^{n-1}\sum_{c=1}^{n-1}\frac{(-1)^{b+c}}{\sigma_{na}\sigma_{nb}\sigma_{nc}}\epsilon_a{\cdot}k_n(\epsilon_b{\cdot}k_n\epsilon_n{\cdot}k_c - k_c{\cdot}k_n\epsilon_b\epsilon_n)\psi_{c,n+a-1}^{a,n+b-1}\\
&\quad + 2\sum_{a=1}^{n-1}\sum_{c=1}^{n-1}\sum_{b\neq a}^{n-1}\frac{(-1)^{b+c+n+\theta_{ab}}}{\sigma_{na}\sigma_{nb}\sigma_{nc}}\epsilon_a{\cdot}k_n\\
&\qquad \left[(k_b{\cdot}k_n\epsilon_n{\cdot}k_c - k_ck_n\epsilon_n{\cdot}k_b)\psi_{c,n+a-1}^{ab} + (\epsilon_b{\cdot}k_n\epsilon_c{\cdot}\epsilon_n - \epsilon_c{\cdot}k_n\epsilon_b{\cdot}\epsilon_n)\psi_{n+a-1,n+b-1}^{a,n+c-1}\right]\\
&\quad + 2\sum_{a=1}^{n-1}\sum_{b\neq a}^{n-1}\sum_{c\neq a}^{n-1}\frac{(-1)^{b+c+\theta_{ab}+\theta_{ac}}}{\sigma_{na}\sigma_{nb}\sigma_{nc}}\epsilon_a{\cdot}k_n(\epsilon_c{\cdot}k_n\epsilon_n{\cdot}k_b - k_b{\cdot}k_n\epsilon_c{\cdot}\epsilon_n)\psi_{n+a-1,n+c-1}^{ab},
\end{aligned} \tag{3.23}$$

$$A_3 = \sum_{a=1}^{n-1}\sum_{b=1}^{n-1}\sum_{c=1}^{n-1}\sum_{d=1}^{n-1} \frac{(-1)^{a+b+c+d}}{\sigma_{na}\sigma_{nb}\sigma_{nc}\sigma_{nd}} \Big(\epsilon_n{\cdot}k_a\epsilon_n{\cdot}k_b\epsilon_c{\cdot}k_n\epsilon_d{\cdot}k_n + k_a{\cdot}k_nk_b{\cdot}k_n\epsilon_c{\cdot}\epsilon_n\epsilon_d{\cdot}\epsilon_n$$

$$- 2k_a{\cdot}k_n\epsilon_n{\cdot}k_b\epsilon_c{\cdot}k_n\epsilon_d{\cdot}\epsilon_n\Big)\psi^{b,n+c-1}_{a,n+d-1}$$

$$+ 2\sum_{a=1}^{n-1}\sum_{b=1}^{n-1}\sum_{c=1}^{n-1}\sum_{d\neq a}^{n-1} \frac{(-1)^{a+b+c+d+n+\theta_{ad}}}{\sigma_{na}\sigma_{nb}\sigma_{nc}\sigma_{nd}} \Big(k_a{\cdot}k_n\epsilon_n{\cdot}k_d(k_b{\cdot}k_n\epsilon_c{\cdot}\epsilon_n - \epsilon_n{\cdot}k_b\epsilon_c{\cdot}k_n)\psi^{ad}_{b,n+c-1}$$

$$+ \epsilon_a{\cdot}k_n\epsilon_d{\cdot}\epsilon_n(k_b{\cdot}k_n\epsilon_c\epsilon_n - \epsilon_n{\cdot}k_b\epsilon_c{\cdot}k_n)\psi^{b,n+c-1}_{n+a-1,n+d-1}\Big)$$

$$+ \sum_{a=1}^{n-1}\sum_{c=1}^{n-1}\sum_{b\neq c}^{n-1}\sum_{d\neq a}^{n-1} \frac{(-1)^{a+b+c+d+\theta_{ad}+\theta_{cb}}}{\sigma_{na}\sigma_{nb}\sigma_{nc}\sigma_{nd}} \Big(2k_a{\cdot}k_n\epsilon_b{\cdot}\epsilon_n\epsilon_c{\cdot}k_n\epsilon_n{\cdot}k_d\psi^{ad}_{n+b-1,n+c-1}$$

$$+ k_a{\cdot}k_n\epsilon_n{\cdot}k_bk_c{\cdot}k_n\epsilon_n{\cdot}k_d\psi^{bc}_{ad} + \epsilon_a{\cdot}k_n\epsilon_b{\cdot}\epsilon_n\epsilon_c{\cdot}k_n\epsilon_d{\cdot}\epsilon_n\psi^{n+b-1,n+c-1}_{n+a-1,n+d-1}\Big). \tag{3.24}$$

In order to finish the calculation of m_3 the only new integral we need to evaluate is

$$I_5 \equiv \int d\sigma_n \delta(f_n^{n-1}) \frac{1}{\sigma_{na}\sigma_{nb}\sigma_{nc}\sigma_{nd}} \tag{3.25}$$

for which we obtain

$$I_5 = \frac{1}{(k_n{\cdot}k_a)^2}\left[\sum_{l\neq a}^{n-1}\frac{k_n{\cdot}k_l}{\sigma_{al}^2} + \frac{1}{k_n{\cdot}k_a}\left(\sum_{l\neq a}^{n-1}\frac{k_n{\cdot}k_l}{\sigma_{al}}\right)^2\right]\delta_{ab}\delta_{bc}\delta_{cd}$$

$$+ \left\{\frac{1}{k_n{\cdot}k_b}\frac{1}{\sigma_{ab}}\left[\frac{1}{k_n{\cdot}k_b}\sum_{l\neq b}^{n-1}\frac{k_n{\cdot}k_l}{\sigma_{bl}} - \frac{1}{\sigma_{ab}}\right]\delta_{a\neq b}\delta_{bc}\delta_{cd} + \text{cyclic}\,\{a,b,c,d\}\right\}$$

$$+ \left\{\left(\frac{1}{k_n{\cdot}k_a} + \frac{1}{k_n{\cdot}k_c}\right)\frac{1}{\sigma_{ac}^2}\delta_{ab}\delta_{cd}\delta_{a\neq c} + \text{cyclic}\{b,c,d\}\right\} \tag{3.26}$$

$$+ \left\{\frac{1}{\sigma_{ac}\sigma_{ad}}\frac{1}{k_n{\cdot}k_a}\delta_{ab}\delta_{c\neq d}\delta_{a\neq c,d} + \text{cyclic}\{b,c,d\}\right\}$$

$$+ \left\{\frac{1}{\sigma_{ba}\sigma_{bd}}\frac{1}{k_n{\cdot}k_b}\delta_{bc}\delta_{a\neq d}\delta_{b\neq a,d} + \text{cyclic}\{b,c,d\}\right\}.$$

4 Action of $S^{(2)}$ on the amplitude

From (1.3), the complete expression for $S^{(2)}$ including the spin contribution can be written as

$$S^{(2)} = S^{(2)}_{\text{orb}} + S^{(2)}_{\text{so}} + S^{(2)}_{\text{spin}}, \tag{4.1}$$

where the orbital, spin-orbit and spin parts are respectively given by

$$S^{(2)}_{\text{orb}} = \frac{1}{2}\sum_{a=1}^{n-1}K^{\text{orb}}_{a\mu\nu}\frac{\partial^2}{\partial k_{a\mu}\partial k_{a\nu}}, \quad S^{(2)}_{\text{so}} = \sum_{a=1}^{n-1}K^{\text{so}}_{a\mu\nu}\frac{\partial^2}{\partial k_{a\mu}\partial \epsilon_{a\nu}}, \quad S^{(2)}_{\text{spin}} = \frac{1}{2}\sum_{a=1}^{n-1}K^{\text{spin}}_{a\mu\nu}\frac{\partial^2}{\partial \epsilon_{a\mu}\partial \epsilon_{a\nu}}, \tag{4.2}$$

with

$$
\begin{aligned}
K^{\rm orb}_{a\mu\nu} &\equiv k_n{\cdot}k_a\, \epsilon_{n\mu}\epsilon_{n\nu} - \epsilon_n{\cdot}k_a\, \epsilon_{n(\mu}k_{n\nu)} + \frac{(\epsilon_n{\cdot}k_a)^2}{k_n{\cdot}k_a} k_{n\mu}k_{n\nu}\,,\\
K^{\rm so}_{a\mu\nu} &\equiv \epsilon_a{\cdot}k_n\, \epsilon_{n\mu}\epsilon_{n\nu} - \epsilon_n{\cdot}\epsilon_a\, \epsilon_{n\mu}k_{n\nu} - \frac{\epsilon_n{\cdot}k_a\, \epsilon_a{\cdot}k_n}{k_n{\cdot}k_a}\epsilon_{n\nu}k_{n\mu} + \frac{\epsilon_n{\cdot}k_a\, \epsilon_n{\cdot}\epsilon_a}{k_n{\cdot}k_a} k_{n\mu}k_{n\nu}\,,\\
K^{\rm spin}_{a\mu\nu} &\equiv \frac{(\epsilon_a{\cdot}k_n)^2}{k_n{\cdot}k_a}\epsilon_{n\mu}\epsilon_{n\nu} - \frac{\epsilon_a{\cdot}k_n\, \epsilon_n{\cdot}\epsilon_a}{k_n{\cdot}k_a}\epsilon_{n(\mu}k_{n\nu)} + \frac{(\epsilon_n{\cdot}\epsilon_a)^2}{k_n{\cdot}k_a} k_{n\mu}k_{n\nu}\,.
\end{aligned}
\tag{4.3}
$$

Then the action of $S^{(2)}$ on the amplitude is

$$
\begin{aligned}
S^{(2)}M_{n-1} &= S^{(2)}\int [d\sigma]_{n-4} \prod_{l\neq i,j,k}^{n-1} \delta(f_l^{n-1}) E_{n-1}\\
&= \int [d\sigma]_{n-4}\,(s_1+s_2+s_3+s_4)\,,
\end{aligned}
\tag{4.4}
$$

where we have separated the calculation into the following four parts

$$
\begin{aligned}
s_1 &= E_{n-1}\, S^{(2)}_{\rm orb} \prod_{l\neq i,j,k}^{n-1}\delta(f_l^{n-1}), & s_2 &= \sum_{a=1}^{n-1} K^{\rm orb}_{a\mu\nu}\frac{\partial E_{n-1}}{\partial k_{a\mu}}\frac{\partial}{\partial k_{a\nu}}\prod_{l\neq i,j,k}^{n-1}\delta(f_l^{n-1}),\\
s_3 &= \sum_{a=1}^{n-1} K^{\rm so}_{a\mu\nu}\frac{\partial E_{n-1}}{\partial \epsilon_{a\nu}}\frac{\partial}{\partial k_{a\mu}}\prod_{l\neq i,j,k}^{n-1}\delta(f_l^{n-1}), & s_4 &= \prod_{l\neq i,j,k}^{n-1}\delta(f_l^{n-1})\, S^{(2)}E_{n-1}.
\end{aligned}
\tag{4.5}
$$

In the subsequent computations we will make use of the identities

$$
\frac{\partial f_l^n}{\partial k_{a\mu}} = \frac{k_l^\mu}{\sigma_{la}}\delta_{l\neq a} + \delta_{la}\sum_{d\neq l}^{n}\frac{k_d^\mu}{\sigma_{ld}}\,, \qquad \frac{\partial^2 f_l^n}{\partial k_{a\mu}\partial k_{a\nu}} = 0,
\tag{4.6}
$$

and also

$$
\begin{aligned}
\frac{\partial E_{n-1}}{\partial k_{a\mu}} &= 2\sum_{b\neq a}^{n-1}\frac{1}{\sigma_{ab}}\Big((-1)^{a+b}{k_b}^\mu \psi^a_b + (-1)^{a+b+n+1}{\epsilon_b}^\mu\psi^a_{n+b-1} + (-1)^n {\epsilon_b}^\mu \psi^b_{n+b-1}\Big)\,,\\
\frac{\partial E_{n-1}}{\partial \epsilon_{a\mu}} &= 2\sum_{b\neq a}^{n-1}\frac{1}{\sigma_{ab}}\Big((-1)^{a+b+n}{k_b}^\mu \psi^b_{n+a-1} + (-1)^{n+1}{k_b}^\mu\psi^a_{n+a-1} + (-1)^{a+b}{\epsilon_b}^\mu \psi^{n+a-1}_{n+b-1}\Big)\,.
\end{aligned}
\tag{4.7}
$$

In the following we omit the upper index of the scattering equations f_l^{n-1} and we simply write them as f_l.

4.1 Evaluation of s_1

We find

$$
\begin{aligned}
s_1 = \frac{1}{2}\,E_{n-1}&\sum_{l\neq i,j,k}^{n-1}\delta'(f_l)\sum_{m\neq i,j,k,l}^{n-1}\delta'(f_m)\prod_{b\neq i,j,k,l,m}^{n-1}\delta(f_b)\sum_{a=1}^{n-1}K^{\rm orb}_{a\mu\nu}\frac{\partial f_m}{\partial k_{a\mu}}\frac{\partial f_l}{\partial k_{a\nu}}\\
&+\frac{1}{2}\,E_{n-1}\sum_{l\neq i,j,k}^{n-1}\delta''(f_l)\prod_{b\neq i,j,k,l}^{n-1}\delta(f_b)\sum_{a=1}^{n-1}K^{\rm orb}_{a\mu\nu}\frac{\partial f_l}{\partial k_{a\mu}}\frac{\partial f_l}{\partial k_{a\nu}}.
\end{aligned}
\tag{4.8}
$$

After some straightforward algebra and using (4.3) and (4.6) we obtain

$$\sum_{a=1}^{n-1} K^{\rm orb}_{a\mu\nu}\frac{\partial f_m}{\partial k_{a\mu}}\frac{\partial f_l}{\partial k_{a\nu}} = \delta_{ml}\, I_2 + \delta_{m\neq l}\, I_1. \tag{4.9}$$

thus, comparing with (3.2), we obtain the desired result $s_1 = m_1$.

4.2 Evaluation of s_2 and s_3

The combination $s_2 + s_3$ has the same delta function support as m_2, thus, we will compare these two expressions. For s_2 we obtain

$$s_2 = \sum_{l\neq i,j,k}^{n-1} \delta'(f_l) \prod_{m\neq i,k,j,l}^{n-1} \delta(f_m) \sum_{a=1}^{n-1} K^{\rm orb}_{a\mu\nu}\frac{\partial E_{n-1}}{\partial k_{a\mu}}\frac{\partial f_l}{\partial k_{a\nu}} \tag{4.10}$$

and for s_3 we get

$$s_3 = \sum_{l\neq i,j,k}^{n-1} \delta'(f_l) \prod_{m\neq i,k,j,l}^{n-1} \delta(f_m) \sum_{a=1}^{n-1} K^{\rm so}_{a\mu\nu}\frac{\partial E_{n-1}}{\partial \epsilon_{a\nu}}\frac{\partial f_l}{\partial k_{a\mu}}. \tag{4.11}$$

After some tedious but straightforward algebra and using (4.3), (4.6) and (4.7) we can expand $s_2 + s_3$ in the same form of m_2 as shown in (3.16) and (3.17). We have explicitly computed each of the coefficients of the corresponding expansion for $s_2 + s_3$ and see that they all precisely match those of (3.18), thus, arriving at $s_2 + s_3 = m_2$ as expected.

4.3 Evaluation of s_4

Having matched all the previous terms on both sides, our last task is to show that $s_4 = m_3$.

From (4.5), (4.1) and (4.2) we have

$$s_4 = \prod_{l\neq i,j,k}^{n-1} \delta(f_l^{n-1}) \sum_{a=1}^{n-1}\left[\frac{1}{2}K^{\rm orb}_{a\mu\nu}\frac{\partial^2 E_{n-1}}{\partial k_{a\nu}\partial k_{a\mu}} + K^{\rm so}_{a\mu\nu}\frac{\partial^2 E_{n-1}}{\partial \epsilon_{a\nu}\partial k_{a\mu}} + \frac{1}{2}K^{\rm spin}_{a\mu\nu}\frac{\partial^2 E_{n-1}}{\partial \epsilon_{a\nu}\partial \epsilon_{a\mu}}\right]. \tag{4.12}$$

Appropriately differentiating (4.7) we find

$$\begin{aligned}
\frac{\partial^2 E_{n-1}}{\partial k_{a\nu}\partial k_{a\mu}} &= 2\sum_{b\neq a}^{n-1}\sum_{c\neq a}^{n-1}\frac{1}{\sigma_{ab}\sigma_{ca}}\Big((-1)^{b+c+\theta_{ba}+\theta_{ac}} {k_b}^\mu {k_c}^\nu \psi^{ac}_{ab} \\
&+(-1)^{a+b+n+\theta_{ac}}({k_b}^\mu{\epsilon_c}^\nu + {\epsilon_c}^\mu {k_b}^\nu)\psi^{ac}_{b,n+c-1} + (-1)^{b+c+n+\theta_{ba}}({k_b}^\mu{\epsilon_c}^\nu + {\epsilon_c}^\mu {k_b}^\nu)\psi^{ab}_{a,n+c-1} \\
&+(-1)^{a+b}({\epsilon_b}^\mu{\epsilon_c}^\nu + {\epsilon_c}^\mu{\epsilon_b}^\nu)\psi^{a,n+c-1}_{c,n+b-1} - {\epsilon_b}^\mu{\epsilon_c}^\nu(\psi^{b,n+c-1}_{c,n+b-1} + (-1)^{b+c}\psi^{a,n+c-1}_{a,n+b-1})\Big) \\
&+ 2\sum_{b\neq a}^{n-1}\sum_{c\neq a,b}^{n-1}\frac{1}{\sigma_{ab}\sigma_{ca}}\Big((-1)^{a+b+n+\theta_{cb}}({k_b}^\mu{\epsilon_c}^\nu + {\epsilon_c}^\mu{k_b}^\nu)\psi^{bc}_{a,n+c-1} \\
&+(-1)^{a+b+\theta_{bc}+\theta_{ac}}({\epsilon_b}^\mu{\epsilon_c}^\nu + {\epsilon_c}^\mu{\epsilon_b}^\nu)\psi^{ac}_{n+b-1,n+c-1} - {\epsilon_b}^\mu{\epsilon_c}^\nu\psi^{bc}_{n+b-1,n+c-1}\Big),
\end{aligned} \tag{4.13}$$

$$\frac{\partial^2 E_{n-1}}{\partial \epsilon_{a\nu}\partial k_{a\mu}} = 2\sum_{b\neq a}^{n-1}\sum_{c\neq a}^{n-1}\frac{1}{\sigma_{ab}\sigma_{ac}}\Big[(-1)^{b+n}{k_b}^\mu{k_c}^\nu((-1)^{c+\theta_{ca}}\psi^{ac}_{b,n+a-1} + (-1)^{a+\theta_{ab}}\psi^{ab}_{a,n+a-1})$$

$$+(-1)^{b+c}(k_b{}^\mu \epsilon_c{}^\nu - \epsilon_c{}^\mu k_b{}^\nu)\psi^{a,n+a-1}_{b,n+c-1} - (-1)^{b+c}k_b{}^\mu \epsilon_c{}^\nu \psi^{a,n+c-1}_{b,n+a-1}$$

$$+\epsilon_b{}^\mu k_c{}^\nu((-1)^{b+c+\theta_{ab}+\theta_{ac}}\psi^{ac}_{n+a-1,n+b-1} - \psi^{ab}_{n+a-1,n+b-1})$$

$$+\epsilon_b{}^\mu k_c{}^\nu((-1)^{a+b}\psi^{a,n+a-1}_{a,n+b-1} + (-1)^{a+c}\psi^{b,n+a-1}_{c,n+b-1} - \psi^{b,n+a-1}_{a,n+b-1})$$

$$+\epsilon_b{}^\mu \epsilon_c{}^\nu(-1)^{c+n+\theta_{ab}}((-1)^b\psi^{a,n+c-1}_{n+a-1,n+b-1} - (-1)^a\psi^{b,n+c-1}_{n+a-1,n+b-1})\Big]$$

$$+\,2\sum_{b\neq a}^{n-1}\sum_{c\neq a,b}^{n-1}\frac{1}{\sigma_{ab}\sigma_{ac}}\Big[(-1)^{b+c+n+\theta_{bc}}k_b{}^\mu k_c{}^\nu \psi^{bc}_{a,n+a-1}$$

$$+(-1)^{a+c+n+\theta_{cb}}\epsilon_b{}^\mu \epsilon_c{}^\nu \psi^{b,n+a-1}_{n+b-1,n+c-1} + (-1)^{b+c+n+\theta_{bc}}\epsilon_b{}^\mu \epsilon_c{}^\nu \psi^{a,n+a-1}_{n+b-1,n+c-1}$$

$$+(-1)^{a+c+\theta_{ba}+\theta_{bc}}\epsilon_b{}^\mu k_c{}^\nu \psi^{bc}_{n+a-1,n+b-1}\Big], \tag{4.14}$$

$$\frac{\partial^2 E_{n-1}}{\partial\epsilon_{a\nu}\partial\epsilon_{a\mu}} = 2\sum_{b\neq a}^{n-1}\sum_{c\neq a}^{n-1}\frac{1}{\sigma_{ab}\sigma_{ac}}\Big[k_b{}^\mu k_c{}^\nu\left((-1)^{b+c}\psi^{b,n+a-1}_{c,n+a-1} + \psi^{a,n+a-1}_{a,n+a-1}\right)$$

$$+(-1)^{b+c+\theta_{ba}+\theta_{ca}}\epsilon_b{}^\mu \epsilon_c{}^\nu \psi^{n+a-1,n+b-1}_{n+a-1,n+c-1} - (-1)^{a+b}(k_b{}^\mu k_c{}^\nu + k_c{}^\mu k_b{}^\nu)\psi^{a,n+a-1}_{b,n+a-1}$$

$$+(-1)^{c+n+\theta_{ca}}(k_b{}^\mu \epsilon_c{}^\nu + \epsilon_c{}^\mu k_b{}^\nu)\left((-1)^b\psi^{b,n+a-1}_{n+a-1,n+c-1} - (-1)^a\psi^{a,n+a-1}_{n+a-1,n+c-1}\right)\Big]. \tag{4.15}$$

We now have all the ingredients to perform the comparison of s_4 with m_3. The algebra is tedious but straightforward since it only involves changes of the summation order and renaming dummy indices. We have performed the analysis and found agreement of the two expressions which completes the proof of the soft-graviton theorem.

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