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Beta-gamma system, pure spinors and Hilbert series of arc spaces

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Abstract: Algorithms are presented for calculating the partition function of constrained beta-gamma systems in terms of the generating functions of the individual fields of the theory, the latter obtained as the Hilbert series of the arc space of the algebraic variety defined by the constraint. Examples of a beta-gamma system on a complex surface with an A_1 singularity and pure spinors are worked out and compared with existing results.

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1 Introduction

A beta-gamma system is a two-dimensional conformal field theory modelled after the b-c ghost system with a set of possibly bosonic complex fields, denoted γ and their canonical conjugates, denoted β. It can be related to a certain large volume limit of the two-dimensional non-linear sigma-model with the fields γ identified as the complex coordinates of the target space [1–3]. A beta-gamma system is said to be free, field theoretically, or flat, geometrically, if the fields γ and β satisfy the commutation relations for free fields. In particular, a pair of γ's commute. The set of γ's then corresponds to the coordinates of the complex affine target space. A beta-gamma system is said to be curved, if the target space is curved. In this article we shall restrict to curved systems obtained as the coordinates of the target space satisfy one or more algebraic equations. This is obviously equivalent to imposing constraints on the γ's.

The action of a free beta-gamma system is linear in both the fields γ and β. The partition function of this field theory is obtained as the generating function of degeneracy of operators graded by quantum numbers associated to conserved charges of the classical action. The partition function of a constrained or curved system is then the generating function of degeneracy of operators satisfying the constraints.

The partition function can be computed by counting operators possessing equal conserved charges obtained through multiplication of β's, γ's and their derivatives with respect to the world-sheet coordinate. A direct construction of operators, however, becomes intractable in the presence of constraints, save for the simplest of instances, as the derivatives of the fields γ and β too are constrained by the derivatives of the constraints to all orders. Partition function of curved beta-gamma systems in several instances have been obtained by resorting to more indirect means [4–7]. A special case, which serves as the motivation for the majority of studies of the beta-gamma system in recent times, is the pure spinor constraint which is a quadratic one arising in an attempt to write a super-Poincarè invariant world-sheet string theory [8]. The partition function of pure spinors has been obtained as the character of representations of the SO(8) group [9–12]. In a variety of other examples the constraints are not quadratic. In the case where the target space can be realized as an orbifold, for example, $\mathbf{C}^2/\mathbf{Z}_N$ or $\mathbf{C}^3/\mathbf{Z}_M\times\mathbf{Z}_N$, with integral M and N, partition functions of beta-gamma systems have been obtained by lifting the geometric orbifold action to the partition function of the affine spaces $\mathbf{C}^2$ and $\mathbf{C}^3$, respectively [7]. This, however, relies upon the affine parametrization of the orbifolds.

In the present article we consider two examples of constraints. The first is a quadratic one among three γ's, the other being pure spinors, which also obeys a set of quadratic constraints. We use the constraints directly without solving them, thereby avoiding any reference to the affine parametrization. Regarding the constraints in γ's as describing an algebraic variety embedded in the affine space of the unconstrained ones the contribution of the various modes of γ's to the partition function is given by the Hilbert series of the arc space of the variety. However, in both the instances considered here the varieties possess an isolated singular point. This renders the definition of the conjugate fields non-unique. The total partition function is then obtained by resorting to some prescription. One efficient prescription is to implement the so-called field-antifield symmetry of the partition function in a multiplicative fashion [4]. We show that it can also be obtained from the combination of various modes of the fields which are invariant under a certain gauge transformation that keeps the action unchanged modulo the constraints, provided the β's are subjected to the same constraints. We exhibit the computations explicitly for two cases. We obtain the partition function of a beta-gamma system on the rational double point surface singularity in both the ways and compare with the result obtained earlier [7] by realizing the target space as an orbifold. We find that the latter prescription fares slightly better when compared with the orbifold results. This computation uses the known description of resolution of surface singularities in terms of arc spaces. For the pure spinors this description is not known. We obtain the partition function by implementing the field-antifield symmetry on the contribution of the pure spinors obtained as the Hilbert series of the arc space of the pure spinor constraint. This is different from implementing the field-antifield symmetry at every order of mass separately. Obtaining the Hilbert series entails a computation of Gröbner basis of the ideal generated by the pure spinor constraint by considering $10m$ equations in $16m$ variables for every mass level m. The algorithm for this computation is rather simple and has been implemented in `Macaulay2` [13]. The results match with the existing ones up to the first mass level.

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In section 2 we begin by recalling some features of the beta-gamma system and its partition function and lay out the two prescriptions used to evaluate the partition function. In section 3 we recall the notion of arc spaces and the blow up of surface singularities in these terms. We use both prescriptions to compute the partition function of the beta-gamma system on the rational double point surface singularity in the following section, comparing the results. In section 5 we obtain the partition function of the pure spinor system up to the first mass level by implementing the field-antifield symmetry. We conclude in section 6.

2 β-γ system on $\mathbf{C}^d$

2.1 Flat system

A beta-gamma system on the d-dimensional complex affine space $\mathbf{C}^d$ is a two-dimensional conformal field theory of a set of complex fields $\{\gamma^i\}$ of vanishing conformal dimension and their canonical conjugates $\{\beta_i\}$, $i = 1, 2, \cdots, d$. On the two-dimensional space, henceforth referred to as the world-sheet, the conjugate fields are one forms, namely, $\beta_i = \beta_{iz}\, dz + \beta_{i\bar{z}}\, d\bar{z}$, where z designates the coordinate of the world-sheet and a bar denotes its complex conjugate. For a flat beta-gamma system the fields γ are identified with the coordinates of the coordinate ring of the target space $\mathbf{C}^d = \mathbf{C}[x_1, x_2, \cdots, x_d]$ as $\gamma^i = x_i$. The coordinates commute pairwise as do the conjugates thereby having trivial operator products. The operator product between a β and a γ, on the other hand, is taken to be the free one, namely

$$\gamma^i(z)\beta_j(z') \sim \delta^i_j \frac{dz'}{z - z'}. \tag{2.1}$$

The action for a beta-gamma system is written as

$$S = \frac{1}{2\pi} \sum_{i=1}^{d} \int \beta_i \bar{\partial}\gamma^i \tag{2.2}$$

in the conformal gauge, where $\partial = \frac{\partial}{\partial z}$. The theory possesses two conserved currents, namely, the energy momentum tensor and a U(1) current corresponding to the scaling of the fields,

$$\gamma^i \longrightarrow \Lambda_i \gamma^i, \quad \beta_i \longrightarrow \Lambda_i^{-1}\beta_i. \tag{2.3}$$

The respective charges, namely, $L_0 = \oint dz z \beta_{iz}\partial\gamma^i$ and $J_0 = \oint dz \beta_i \gamma^i$, characterize the field theory. Introducing the modular parameter q and another one, t, corresponding to the scaling the partition function of the beta-gamma system is written as the character

$$\mathcal{Z} = \mathrm{Tr}\,(q^{L_0} t^{J_0}), \tag{2.4}$$

where Tr signifies a trace with respect to the states of the Hilbert space of the theory.

Assuming that the fields possess mode expansions

$$\begin{aligned} \beta_i(z) &= z^{-1/2} \sum_{n\in\mathbf{Z}} z^{-n-1} \beta_{i(n+1)} \\ \gamma^i(z) &= z^{1/2} \sum_{n\in\mathbf{Z}} z^{-n-1} \gamma^i_{(n)} \end{aligned} \tag{2.5}$$

and the existence of a vacuum $|0\rangle$ to obey the highest weight conditions

$$\beta_{i(n+1)}|0\rangle = 0 \quad \gamma^i_{(n)}|0\rangle = 0, \quad n \geqslant 0, \tag{2.6}$$

the character of the beta-gamma system on $\mathbf{C}^d$ is obtained as [14, 15]

$$\mathcal{Z}_{\mathbf{C}^d} = (\mathcal{Z}_{\mathbf{C}})^d, \tag{2.7}$$

where the character of the affine complex plane is defined to be

$$\mathcal{Z}_{\mathbf{C}} = \frac{1}{1-t}\prod_{n=1}^{\infty}\frac{1}{(1-q^n t)(1-q^n/t)}. \tag{2.8}$$

This can be interpreted as the generating function of degeneracy of monomials of a given degree for q and t, where each γ contributes a t, each β contributes q/t to the partition function while each derivative ∂ contributes a q [7].

2.2 Curved system

One type of curved beta-gamma systems is obtained from a flat one by imposing constraints on the fields γ. The constraints we consider are non-linear but algebraic, which are well-defined as the γ's commute between themselves. This makes the target space into a non-affine algebraic variety. The various operators then correspond one-to-one with the regular functions formed from monomials on this variety. We present a means to evaluate the partition function as the Hilbert series of the arc space of the variety.

More specifically, we deal with quadratic constraints given by

$$\sum_{j=1}^{d}\Omega_{ij}\gamma^i\gamma^j = 0, \tag{2.9}$$

with one or more constant $d \times d$ matrices Ω. Computation of the partition function then entails enumeration of monomials in the fields β and γ as well as their derivatives with respect to the world-sheet coordinate modulo the constraint and its derivatives to all orders. The multiplication of fields in forming monomials are to be normal ordered as usual, but this does not affect their number.

Constructing monomials, alias operators, involving β's is ambiguous as they are not constrained *prima facie* unlike the γ's. They are to be constrained by prescribing extra conditions. Treating them as canonical momenta conjugate to the γ's, as is in fact necessitated by its connection with the sigma model [2] we end up with the usual problem of defining momenta on a singular space. Indirect means of constraining the β's are therefore to be devised. We consider two different ways to evaluate the partition function of this theory, starting from the separate contributions of the γ's and β's. In both the methods the contribution from the γ's alone , denoted Z_γ, is obtained first by counting monomials constructed solely from the γ's, wherein the constraint (2.9) is taken into account. The share from the β's is then derived from Z_γ using symmetries of the action relating the γ's and the β's.

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In the first method, implementing the so-called field-antifield symmetry Z_γ is split into two factors. The first is independent of q arising from the contribution of the zero modes. The other is a function of both q and t from the contribution of the massive modes. Thus,

$$Z_\gamma = Z_0(t) Z_m(q,t), \tag{2.10}$$

where the subscripts 0 and m refer to the zero and non-zero mass modes. The total partition function is obtained as [4]

$$Z(q,t) = Z_0(t) Z_m(q,t) Z_m(q,1/t). \tag{2.11}$$

This method has been used previously in a ghost-for-ghost scheme for pure spinors [11].

We propose an alternative using the gauge invariance of the action (2.2) under the transformation

$$\delta\gamma^i = 0, \qquad \delta\beta_i = \sum_{i,j=1}^{d} \Omega_{ij}\gamma^j. \tag{2.12}$$

The action (2.2) is invariant under this transformation modulo (2.9). This imposes a restriction on the possible combinations of suitably defined β's due to the constraints on γ's. Only β's appearing in gauge invariant combinations are then counted in the partition function. Two such combinations at first and second mass levels, for example, are the U(1) current and energy momentum tensor, respectively, which are composite operators. Although the number of gauge invariant operators at each mass level is finite, new operators emerge at each higher mass level, rendering the counting of such states intractable. As a consequence, the partition function generically contains negative terms in $1/t$. This is a hurdle in obtaining the partition function of β-γ systems in a closed form.

We use a new method to implement gauge invariance in the β-γ system directly at the level of the partition function, thereby, giving a rationale for the omission of negative terms in $1/t$. Assuming the existence of the conjugate fields we obtain their separate contribution to the partition function by subjecting them to the same constraint as the γ's, namely,

$$\sum_{j=1}^{d} \Omega_{ij}\beta_i\beta_j = 0. \tag{2.13}$$

Supposing we have a way of finding Z_γ, we can use the same method to count the totality of monomials of β's alone modulo this constraint. Let us denote it by Z_β. The full partition function of the theory is then obtained by combining Z_γ and Z_β in such a way that the condition of gauge invariance in (2.12) is respected.

The naïve product $Z_\gamma Z_\beta$ actually counts the set of all possible monomials constructed out of the fields β and γ and their derivative, satisfying, the constraints (2.13) and (2.9) and their derivatives. Out of this set, we need to subtract the gauge-noninvariant monomials, namely, the monomials which do not vanish modulo (2.9). We implement this as follows.

We subtract $Z_\beta - 1$. Z_β is subtracted because the constraint (2.9) is quadratic and the gauge transformation (2.12) of β produces a single power of γ. Thus, one checks that monomials constructed solely from β's are not gauge invariant. The unity is subtracted so as to avoid over counting the constant monomial twice. The partition function is thus

$$\begin{aligned} Z'(q,t) &= Z_\gamma Z_\beta - Z_\beta + 1 - Z_\gamma + Z_\gamma \\ &= Z_\gamma + (Z_\gamma - 1)(Z_\beta - 1). \end{aligned} \tag{2.14}$$

There are more monomials to discard. Let us recall that Z_γ is the partition function of monomials in γ which *do not* vanish modulo (2.9). However, the constant monomial (the monomial $(\gamma_i)^0$ or unity) as well as ones with single powers of γ's, which are counted in Z_γ as the terms constant and linear in t, respectively, can not arise from a combination of β's and γ's by (2.12). These constitute non-invariant monomials. The totality of non-invariant monomials involving both types of fields is then

$$Z_\beta \left(Z_\gamma - 1 - t \left[\frac{dZ_\gamma(q,t)}{dt} \right]_{t=0} \right). \tag{2.15}$$

Moreover, the gauge transformation (2.12) on any monomial converts a β into a linear combination of γ's, thereby changing the grade of a monomial by a factor of q/t^2. Thus, the above expression (2.15) is to be subtracted from $Z'(q,t)$ after compensating for this change in grade. The resulting partition function is

$$Z(q,t) = Z_\gamma + (Z_\beta - 1)(Z_\gamma - 1) - \frac{q}{t^2}\, Z_\beta \left(Z_\gamma - 1 - t \left[\frac{dZ_\gamma(q,t)}{dt} \right]_{t=0} \right). \tag{2.16}$$

Let us recall that to evaluate the partition function in either way, we need to find Z_γ. For the second method we also need Z_β. These are obtained as Hilbert series of the arc spaces of the varieties described by (2.9) and (2.13), respectively, to which we turn next.

3 Arc spaces and Hilbert series

In this section we recall some features of the arc space of an algebraic variety [16] and define its Hilbert series. Relation between Hilbert series of arc spaces in a single variable and partitions has been noted earlier [17]. We restrict attention to complex numbers only but generalize the definition of Hilbert series to graded rings to incorporate the grades of q and t pertinent to beta-gamma systems. Let $\mathbf{C}[[\xi]]$ denote the formal power series (Puiseux series) ring of polynomials in a single variable ξ over the field of complex numbers $\mathbf{C}$. In the simplest case the arc space of an algebraic variety defined by a polynomial equation $f = 0$ in the coordinate ring $\mathbf{C}[x_1, x_2, \cdots, x_n]$ is the set of power series solutions to the equation $f(x(\xi)) = 0$, where $x(\xi) = (x_1(\xi), x_2(\xi), \cdots, x_n(\xi)) \in \mathbf{C}[[\xi]]^n$, with each component a power series in the formal variable ξ. This generalizes to more polynomials than one.

More formally, let $\mathcal{M} = \mathrm{Spec}\,\big(\mathbf{C}[x_1, x_2, \cdots, x_n]/(f_1, f_2, \cdots, f_m)\big)$ be an algebraic variety defined by m equations in the coordinate ring of $\mathbf{C}^n$. Let us write the coordinates x_i as formal power series in a formal variable ξ as

$$x_i = \sum_{j=1}^{r} x_i^{(j)} \xi^j. \tag{3.1}$$

Substituting these series in the polynomials f_k, $k = 1, 2, \cdots, m$ and truncating at the order ξ^r we obtain the set of polynomials $F_k^{(l)}$ as the coefficient of ξ^l in the expansion of $f_k(x_1, x_2, \cdots, x_n)$. The r-th *jet scheme* M_r of M is then defined as

$$\mathcal{M}_r = \mathrm{Spec}\left(\frac{\mathbf{C}[x_i^{(j)}; 1 \leqslant i \leqslant n, 0 \leqslant j \leqslant r]}{\{F_k^{(l)}; 1 \leqslant k \leqslant m, 0 \leqslant l \leqslant r\}}\right) \tag{3.2}$$

In particular, $\mathcal{M}_0 = \mathcal{M}$, the variety itself and $\mathcal{M}_1 = T\mathcal{M}$, the tangent space. The *arc space* of M is then defined as

$$\mathcal{M}_\infty = \mathrm{Spec}\left(\frac{\mathbf{C}[x_i^{(j)}; 1 \leqslant i \leqslant n, j \in \mathbf{N}]}{\{F_k^{(l)}; 1 \leqslant k \leqslant m, l \in \mathbf{N}\}}\right), \tag{3.3}$$

where $\mathbf{N}$ denotes the set of natural numbers, $\mathbf{N} = \{0, 1, 2, 3, \cdots\}$. In more mundane terms, the arc space on $\mathcal{M}$ is defined by the infinite set of equations obtained at each order of ξ by substituting an infinite series of the form (3.1) into the defining equations $f_k = 0$ of $\mathcal{M}$ for $k = 1, 2, \cdots m$.

A generating function for monomials in the variables $x_i^{(j)}$, modulo the relations $F_k^{(l)}$ is obtained by bestowing a grade to the variables $x_i^{(j)}$. This is defined to be the Hilbert series of the arc space $\mathcal{M}_\infty$, denoted $\mathcal{H}_{f_1, f_2, \cdots, f_m}$ or $\mathcal{H}_\mathcal{M}$. Evaluation of the Hilbert series requires computation of Gröbner basis from $F_k^{(l)}$, in general. However, for simple cases this complication may not exist. We shall associate a grade $q^j t$ to the variable $x_i^{(j)}$. The symbols are chosen to make the connection with the beta-gamma system conspicuous.

3.1 Arc space of the singular quadric in $\mathbf{C}^3$

Let us illustrate the computation of the Hilbert space with two simple examples. Further examples with a singly graded variable exist in literature [17]. The arc space of the affine space $\mathbf{C}[x]$ consists of all the powers of $x^{(j)}$ for all $j = 0, 1, \cdots, \infty$. The monomials are thereby obtained by arranging each of $x^{(j)}$ in a geometric series $1 + x^{(j)} + (x^{(j)})^2 + (x^{(j)})^2 \cdots = 1/(1 - x^{(j)})$ and multiplying them as $1/\prod_{j=1}^{\infty}(1 - x^{(j)})$. With the assignment of grades $q^j t$ to $x^{(j)}$ as above then yields the Hilbert series of the arc space of $\mathbf{C}[x]$ as

$$\mathcal{H}_\mathbf{C} = \frac{1}{1-t}\prod_{n=1}^{\infty}\frac{1}{(1 - q^n t)}. \tag{3.4}$$

Next let us work out the Hilbert series of the variety defined in $\mathbf{C}[x_1, x_2, x_3]$ by the quadratic polynomial $f = x_1 x_2 - x_3^2$, corresponding to the rational double point singular variety

$$x_1 x_2 - x_3^2 = 0. \tag{3.5}$$

This will provide part of the partition function Z_γ of the beta-gamma system discussed in the previous section. Substituting the power series (3.1) in $x_1 x_2 - x_3^2$ we obtain the polynomials

$$F^{(0)} = x_1^{(0)} x_2^{(0)} - (x_3^{(0)})^2, \tag{3.6}$$

$$F^{(1)} = x_1^{(0)} x_2^{(1)} + x_1^{(1)} x_2^{(0)} - 2x_3^{(0)} x_3^{(1)}, \tag{3.7}$$

$$F^{(2)} = x_1^{(2)} x_2^{(0)} + x_1^{(1)} x_2^{(1)} + x_1^{(0)} x_2^{(2)} - (x_3^{(1)})^2 - 2x_3^{(0)} x_3^{(1)}, \tag{3.8}$$

$$\vdots$$

at different orders of ξ. According to our assignment of grades to the variables every $F^{(l)}$, being quadratic, has t-grade t^2 and q-grade q^l. Now, considering only the three relations on the nine variables $x_1^{(0)}, \cdots, x_3^{(2)}$, the Hilbert series is [7, 18]

$$\frac{(1-t^2)(1-qt^2)(1-q^2t^2)}{(1-t)^3(1-qt)^3(1-q^2t)^3}. \tag{3.9}$$

Continuing *ad infinitum* for the countably infinite quadratic equations for the countable set of variables, we obtain the Hilbert series of the arc space of the variety $f = 0$ to be

$$H_{x_1 x_2 - x_3^2}(q,t) = \frac{(1-t^2)}{(1-t)^3} \prod_{n=1}^{\infty} \frac{(1-q^n t^2)}{(1-q^n t)^3} \tag{3.10}$$

$$\begin{aligned} &= \left(1 + 3t + 5t^2 + 7t^3 + 9t^4 + 11t^5 + 13t^6 + 15t^7 + 17t^8 + \cdots\right) \\ &\quad + q\left(3t + 8t^2 + 12t^3 + 16t^4 + 20t^5 + 24t^6 + 28t^7 + 32t^8 + \cdots\right) \\ &\quad + q^2\left(3t + 14t^2 + 27t^3 + 37t^4 + 47t^5 + 57t^6 + 67t^7 + 77t^8 + \cdots\right) \\ &\quad + q^3\left(3t + 17t^2 + 43t^3 + 68t^4 + 88t^5 + 108t^6 + 128t^7 + 148t^8 + \cdots\right) \\ &\quad + q^4\left(3t + 23t^2 + 66t^3 + 119t^4 + 166t^5 + 206t^6 + 246t^7 + 286t^8 + \cdots\right) \\ &\quad + q^5\left(3t + 26t^2 + 90t^3 + 180t^4 + 271t^5 + 352t^6 + 424t^7 + 49t^8 + \cdots\right) \\ &\quad + \mathcal{O}\left(q^6\right) \end{aligned} \tag{3.11}$$

Similarly, assuming that the β's obey the same constraint and noting that they do not possess zero modes, the Hilbert series for them is obtained as

$$H_{\beta_1 \beta_2 - \beta_3^2}(q,t) = \prod_{n=1}^{\infty} \frac{(1-q^{n+1}/t^2)}{(1-q^n/t)^3} \tag{3.12}$$

$$\begin{aligned} &= q\left(1 + \frac{3}{t} + \cdots\right) + q^2\left(\frac{5}{t^2} + \frac{3}{t} + \cdots\right) \\ &\quad + q^3\left(\frac{7}{t^3} + \frac{8}{t^2} + \frac{3}{t} + \cdots\right) + q^4\left(\frac{9}{t^4} + \frac{12}{t^3} + \frac{14}{t^2} + \frac{3}{t} + \cdots\right) \\ &\quad + q^5\left(\frac{11}{t^5} + \frac{16}{t^4} + \frac{27}{t^3} + \frac{17}{t^2} + \frac{3}{t} +\right) + \mathcal{O}\left(q^6\right) \end{aligned} \tag{3.13}$$

where the grade of $\beta_i^{(j)}$ is chosen to be q^i/t for $i = 1, 2, 3, \cdots$.

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3.1.1 Contribution from blow up

Resolution of rational surface singularities can be treated using arcs. The surface (3.5) with an A_1 singularity at the origin is blown up with a $\mathbf{P}^1$. The single exceptional divisor corresponds to a truncation of the Puiseux series (3.1) to [16, 19]

$$x_i = x_i^{(1)}\xi, \tag{3.14}$$

for $i = 1, 2, 3$. Putting this truncated series in the constraint (3.5) leads to the single equations

$$x_1^{(1)}x_2^{(1)} - (x_3^{(1)})^2 = 0. \tag{3.15}$$

The Hilbert series is

$$\mathcal{H}_{Bl_0(x_1x_2-x_3^2)}(q,t) = \frac{1-q^2t^2}{(1-qt)^3}. \tag{3.16}$$

4 Beta-Gamma system on surface with a rational double point

In this section we obtain the partition function of a beta-gamma system on the surface with an A_1 singularity using the Hilbert series obtained above in two different ways (2.11) and (2.16) as discussed before. Let us note that the coefficients $x_i^{(j)}$ are in one-to-one correspondence with the derivatives $\partial^j\gamma^i$ of the fields as well as with the modes in (2.5). The former identification is more better suited for our purposes here. This allows the identification of the Hilbert series as the relevant part of the partition function through counting monomials in the fields and their derivatives. In the case of the affine space, there is no constraint. Identifying the coefficients $x_i^{(j)}$ in (3.1) with $\partial^j\gamma^i$, $i = 1, 2, 3$, the Hilbert series for each $\mathbf{C}$ is given by (3.4). Each component of the arc space will have a conjugate with an inverse t-charge corresponding to the unconstrained β's as well. Thus the total partition function of a flat beta-gamma system is obtained by augmenting $\mathcal{H}_{\mathbf{C}}$ in (3.4) with the contribution from the conjugates, resulting into (2.8). This can also be thought as an instance of implementing the field-antifield symmetry according to (2.11).

Let us now discuss the case of the quadratic constraint (2.9) with

$$\Omega = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \tag{4.1}$$

The constraint is (3.5) with the identification $x_i = \gamma^i$. This singular variety can also be looked upon as the orbifold $\mathbf{C}^2/\mathbf{Z}_2$, where the $\mathbf{Z}_2$ acts on $(u, v) \in \mathbf{C}^2$ by changing signs of both. The partition function has been computed earlier and written in a closed form [7], by directly implementing the orbifolding on the partition function of the affine space (2.7) with $d = 2$. The partition function ([7], eq. (29)) expanded in a series with respect to q

and t is

$$
\begin{aligned}
Z_{\mathbf{C}^2/\mathbf{Z}_2}(q,t) = {} & (1+3t^2+5t^4+7t^6+9t^8+11t^{10}+\cdots) \\
& + q(4+12t^2+20t^4+28t^6+36t^8+44t^{10}+\cdots) \\
& + q^2\left(\frac{3}{t^2}+17+42t^2+70t^4+98t^6+126t^8+154t^{10}+\cdots\right) \\
& + q^3\left(\frac{12}{t^2}+52+120t^2+200t^4+280t^6+360t^8+440t^{10}+\cdots\right) \\
& + q^4\left(\frac{5}{t^4}+\frac{42}{t^2}+147+320t^2+525t^4+735t^6+945t^8+1155t^{10}+\cdots\right) \\
& + q^5\Bigg(\frac{20}{t^4}+\frac{120}{t^2}+372+776t^2+1260t^4 \\
& + 1764t^6+2268t^8+2772t^{10}+\cdots\Bigg)+\mathcal{O}\left(q^6\right). \qquad (4.2)
\end{aligned}
$$

The orbifold description and (2.9) are related by a quadratic identification of variables $(u,v)\in\mathbf{C}^2$ with the x's as $x_1=u^2, x_2=v^2, x_3=uv$. To compare results of x's to (u,v) variables, noting the t-charge assignment, a t is to be replaced with a t^2 in the formulas for partition function (2.16) as well as the Hilbert series. The comparison is, however, valid only in a local coordinate chart. Certain monomials which survive the orbifold action in terms of u,v variables are absent in the description in terms of x's. These correspond to missing states in the latter description. For example, there are four monomials $u\partial u, u\partial v, v\partial u, v\partial v$ with grade qt^2 which survive the orbifold projection. Only three of them appear in terms of x's as $\partial x_1, \partial x_2$ and $\partial x_3 = u\partial v + v\partial u$. The combination $u\partial v - v\partial u$ is absent. Inclusion of this combination calls for extending the set of regular functions on the variety $x_1x_2-x_3^2=0$ by rational functions of γ's that is, x's and their derivatives [5]. This is achieved by including the blow up modes in the Hilbert series with (3.16). While this mends the partition function at this level, states at higher grades still remain missing. This may be ascribed to the fact that a resolution of singularity by blowing up a point repairs the variety $\mathcal{M}$ up to its tangent space, $\mathcal{M}_1$ in general. Thus we do not expect the partition function obtained without resorting to the parametric representation to completely match (4.2).

4.1 Implementing field-antifield symmetry

Field-antifield symmetry possessed by the partition function takes the grade t to its inverse. The field-antifield symmetry can be imposed on the partition function according to (2.11). Taking $Z_\gamma = \mathcal{H}_{x_1x_2-x_3^2}$ the separation into massless and massive modes is obvious. The zero mode part $Z_0(t) = (1-t^2)/(1-t)^3$ transforms to

$$Z_0(t) = t^{-1}\, Z_0(1/t). \qquad (4.3)$$

Treating the β-γ system as the ghost system of an appropriate string theory, the index -1 (-2 if compared to (4.2)) of t corresponds to γ-charge anomaly [20]. It indicates that an appropriate number of antifields have to be introduced to define a consistent inner product on the full Hilbert space. At higher mass levels this is expected to be a symmetry, indicating that all the physical states appear in field-antifield pairs. By (2.10), (2.11) and (3.10) this yields the total partition function

$$\widehat{Z}(q,t) = \frac{(1-t^2)}{(1-t)^3}\prod_{n=1}^{\infty}\frac{(1-q^n t^2)}{(1-q^n t)^3}\frac{(1-q^n/t^2)}{(1-q^n/t)^3}. \tag{4.4}$$

Expanding in series in q and t this yields

$$\begin{aligned}\widehat{Z}(q,t^2) = &(1+3t^2+5t^4+7t^6+9t^8+11t^{10}+\cdots)\\ &+q(-1/t^4+4+11t^2+20t^4+28t^6+36t^8+44t^{10}+\cdots)\\ &+q^2(-3/t^6-4/t^4+14+38t^2+67t^4+98t^6+126t^8+154t^{10}+\cdots)\\ &+q^3(-5/t^8-11/t^6-14/t^4+40+106t^2+189t^4+275t^6+360t^8\\ &+440t^{10}+\cdots)\\ &+q^4(-7/t^{10}-20/t^8-38/t^6-40/t^4+105+275t^2+487t^4+715t^6\\ &+938t^8+1155t^{10}+\cdots)\\ &+q^5(-9/t^{12}-28/t^{10}-67/t^8-106/t^6-105/t^4+252+651t^2+1154t^4\\ &+1697t^6+2240t^8+2763t^{10}+\cdots)+\mathcal{O}\left(q^6\right),\end{aligned} \tag{4.5}$$

where we have used the grade t^2 to compare with (4.2). This exhibits missing states for terms with lower t-grades for every power of q. Also, the series contains negative terms in the partition function which are difficult to account for. Adding the blow up modes (3.16) exacerbates the mismatch, as noted earlier [2]. Further examples of partition function of beta-gamma systems, such as, a system with constraint $\gamma^2=0$ and also the conifold, may be evaluated in this way matching previous results [4].

4.2 Implementing gauge invariance

Let us now work out the partition function using the gauge invariance of the action, as explained in section 2.2. In this method one first counts all monomials arising from γ's and β's separately as the Hilbert of series of the respective arc spaces of the constraints (2.9) and (2.13), respectively. The partition function is then obtained by extracting the set of gauge invariant monomials from (2.16). Let us illustrate this with examples, which will also justify the formula (2.16). We shall use a generic symbol β and γ without the indices for this purpose.

Example 1. Let us compute the coefficient of q^2/t in the partition function. This grade is contributed by monomials of the generic form $\beta^2\gamma$. The number of such combinations is obtained from the coefficient of q^2/t^2 in (3.13) and that of q^0t in (3.11). The total number of such monomials respecting constraints (2.9) and (2.13) is thus $5\times 3=15$. Under the gauge transformation (2.12) a monomial of the form $\beta^2\gamma$ goes to one of the form $\beta\gamma^2$ changing the grade from q^2/t to qt, the change being a factor of q/t^2. In order to count the ones that vanish modulo (2.9) we note that the constraint is to be imposed on the portion of $\beta\gamma^2$ that is quadratic in the γ's. Now, Z_γ counts the monomials which *do not* vanish modulo the constraint. Thus the number of non-vanishing monomials of the form γ^2 is given by the coefficient of t^2 in (3.11), which is 5. Multiplying with the 3 β's this gives the number of monomials of the form $\beta\gamma^2$ that survive the constraint as 15. Thus,

$$\begin{aligned} \text{contribution of monomials of the form } \beta^2\gamma &= 15q^2/t \\ \text{contribution of non-vanishing monomials of the form } \beta\gamma^2 &= 15qt \end{aligned}$$

As indicated in (2.16), the subtraction of the second term to obtain the number of vanishing ones is effected in the partition function by multiplying the it with q/t^2 compensating for the change of grade due to the gauge transformation. We conclude that the number of gauge invariant monomials is thus zero. The partition function does not have a q^2/t term in its series expansion.

Example 2. As the second example let us consider the coefficient of the q^2t term in the partition function. These arise from three types of monomials, viz.$m_1 = \beta^2\gamma^3$, $m_2 = \beta\gamma\partial\gamma$ and $m_3 = \gamma^2\partial\beta$. Number of β^2 is counted as 5 from the coefficient of q^2/t^2 in (3.13), while that of γ^3 is 7 from the coefficient of the q^0t^3 term of (3.11), leading to $[m_1] = 5 \times 7 = 35$ monomials of type m_1. Under (2.12) these go over to monomials of the type $\beta\gamma^4$. The number of non-vanishing monomials of this form, corresponding to the gauge non-invariant combinations, is counted as $3\times 9 = 27$ from the coefficient of q/t in (3.13) and the coefficient of t^4 in (3.11). Similarly, the number of monomials of type m_2 is obtained as $[m_2] = 3 \times 8 = 24$ from the coefficient of q/t in (3.13) and that of qt^2 in (3.11). These give rise to monomials of the form $\gamma^2\partial\gamma$. Non-vanishing combinations of this form, the gauge non-invariant ones, are counted as the coefficient of qt^3 in (3.11) to be 12. Finally, the number of monomials of type m_3 is $3 \times 5 = 15$, obtained from the coefficient of q^2/t in (3.12) and that of t^2 in (3.11). Under (2.12) these go over to monomials of the form $\gamma^2\partial\gamma$, which have been considered above. Considering all these, the number of gauge invariant monomials are

$$35 + 24 + 15 - 27 - 12 = 35.$$

Adding the three $\partial^2\gamma$'s, namely, $\partial^2 x_1$, $\partial^2 x_2$ and $\partial^2 x_3$, which contribute to this order as also seen from the coefficient of q^2t in (3.11), the coefficient of q^2t in the partition function is 38.

Example 3. For certain grades we end up with an over-determined system, however, yielding negative coefficients. For example, monomials with grade q^3/t^2 arise from the $7 \times 3 = 21$ monomials of the form $\beta^3\gamma$, as seen from the coefficients of q^3/t^3 in (3.13) and that of q^0t in (3.11). Under the gauge transformation (2.12) these produce terms of the form $\beta^2\gamma^2$. The number of such monomials that survive modulo the constraint is 5×5 as seen from the coefficients of q^2/t^2 in (3.13) and that of q^0t^2 in (3.11). The difference $21 - 25 = -4$ signifies that there is no gauge invariant combination of the form $\beta^3\gamma$.

This method thus provides a rationale for ignoring terms with negative coefficients in the partition function.

Generalizing these examples, we obtain (2.16). A monomial of the form $\partial^a\beta^b\partial^c\gamma^d$ with grade $q^{a+b+c}t^{d-b}$ is counted from $Z_\gamma Z_\beta$ by multiplying the coefficients of q^{a+b}/t^b from (3.13) and that of q^ct^d from (3.11). Under the transformation (2.12) such a monomial reduces to $\partial^a\beta^{b-1}\partial^c\gamma^{d+1}$, with grade $q^{a+b+c-1}t^{d-b+2}$. The corresponding partition function is then given by $Z_\beta Z_\gamma q/t^2$. But the latter counts non-vanishing monomials modulo the constraints. The gauge invariant ones are the vanishing ones. We thus need to subtract them from $Z_\gamma Z_\beta$. However, monomials formed from β's alone can not be gauge invariant as each β produces a single γ under the gauge transformation (2.12), while the constraint (2.9) is quadratic in γ's. Moreover, a gauge transformation of a monomial of the form $\partial^a\beta^b\partial^c\gamma^d$ can not produce terms that are independent of γ's or linear in them. This explains the last two terms of (2.16).

Let us now present the complete formula. Using the expression (3.10) for Z_γ and (3.12) for Z_β in (2.16) we obtain the partition function as

$$\begin{aligned}\tilde{Z}(q,t^2) = &(1+3t^2+5t^4+7t^6+9t^8+11t^{10}+\cdots)\\ &+q(4+11t^2+20t^4+28t^6+36t^8+44t^{10}+\cdots)\\ &+q^2(14+38t^2+67t^4+98t^6+126t^8+154t^{10}+\cdots)\\ &+q^3\left(-\frac{4}{t^4}+40+106t^2+189t^4+275t^6+360t^8+440t^{10}+\cdots\right)\\ &+q^4\left(-\frac{8}{t^6}-\frac{27}{t^4}+105+275t^2+487t^4+715t^6+938t^8+1155t^{10}+\cdots\right)\\ &+q^5\bigg(-\frac{12}{t^8}-\frac{49}{t^6}-\frac{86}{t^4}+252+651t^2+1154t^4+1697t^6\\ &+2240t^8+2763t^{10}+\cdots\bigg)+\mathcal{O}(q^6),\end{aligned} \tag{4.6}$$

which matches with (4.5) in the positive terms. Let us note that the expression is written for $\tilde{Z}(q,t^2)$ rather than $\tilde{Z}(q,t)$ to compare with (4.5), which correspond to a different assignment of charges. While this formula too is plagued by the presence of negative terms, as discussed in Example 3 above, the negative terms may be ignored.

Incorporating certain rational functions by blowing up the singularity, discussed before, ameliorates the results to a certain extent. Instead of (3.10) if we add the contribution of the exceptional divisor of the blow up to the Hilbert series (3.16) to set

$$Z_\gamma = \mathcal{H}_{x_1x_2-x_3^2} + qt\ \mathcal{H}_{Bl_0(x_1x_2-x_3^2)}, \tag{4.7}$$

where the factor qt^2 accounts for the unit codimension of the exceptional divisors, then using this expressions with (3.12) and (2.16) the corrected partition function reads

$$\begin{aligned}Z(q,t^2) = &(1+3t^2+5t^4+7t^6+9t^8+11t^{10}+\cdots)\\ &+q(4+12t^2+20t^4+28t^6+36t^8+44t^{10}+\cdots)\\ &+q^2(17+38t^2+70t^4+98t^6+126t^8+154t^{10}+\cdots)\\ &+q^3\left(-\frac{4}{t^4}+\frac{5}{t^2}+40+115t^2+189t^4+280t^6+360t^8+440t^{10}+\cdots\right)\\ &+q^4\left(-\frac{8}{t^6}-\frac{20}{t^4}-\frac{1}{t^2}+123+279t^2+502t^4+715t^6+945t^8+1155t^{10}+\cdots\right)\end{aligned}$$

$$+ q^5\Big(- \frac{12}{t^8} - \frac{40}{t^6} - \frac{89}{t^4} + \frac{26}{t^2} + 264 + 685t^2 + 1162t^4 + 1718t^6 + 2240t^8$$
$$+ 2772t^{10} + \cdots \Big) + \mathcal{O}(q^6). \tag{4.8}$$

The partition function matches (4.2) up to the first mass level completely, which is expected since the first mass level corresponds to the tangent space of the variety, which is repaired by a blow up. As discussed above, the negative terms correspond to a over-determined system and need be ignored yielding the partition function for the blown up rational double point

$$\begin{aligned} Z(q,t^2) = &(1 + 3t^2 + 5t^4 + 7t^6 + 9t^8 + 11t^{10} + \cdots) \\ &+ q(4 + 12t^2 + 20t^4 + 28t^6 + 36t^8 + 44t^{10} + \cdots) \\ &+ q^2(17 + 38t^2 + 70t^4 + 98t^6 + 126t^8 + 154t^{10} + \cdots) \\ &+ q^3\left(\frac{5}{t^2} + 40 + 115t^2 + 189t^4 + 280t^6 + 360t^8 + 440t^{10} + \cdots\right) \\ &+ q^4(123 + 279t^2 + 502t^4 + 715t^6 + 945t^8 + 1155t^{10} + \cdots) \\ &+ q^5\Big(\frac{26}{t^2} + 264 + 685t^2 + 1162t^4 + 1718t^6 + 2240t^8 \\ &+ 2772t^{10} + \cdots\Big) + \mathcal{O}(q^6). \end{aligned} \tag{4.9}$$

Terms with sufficiently high order of t^2 match as well for all powers of q, indicating that the number of missing states are finite at each mass level. As discussed in the examples above, the series can be verified at low orders in q and t^2 by explicitly constructing all possible combinations of β's and x's with arbitrary coefficients and discarding the gauge non-invariant monomials.

5 Partition function of Pure spinors

In this section we present the results for the case of pure spinors. The action for the pure spinor system is

$$S = \frac{1}{2\pi}\int \omega^T \bar{\partial}\lambda, \tag{5.1}$$

where $\lambda = \big(x_1, x_2, \cdots, x_{16}\big)^T$ is a sixteen dimensional complex vector subject to the constraint

$$\lambda\gamma^\mu\lambda = 0 \tag{5.2}$$

and ω denotes its conjugate. Here T denotes matrix transpose and γ^μ denotes the ten-dimensional gamma matrices with $\mu = 0, 1, \cdots, 9$. The action possesses the classical gauge symmetry

$$\delta_\epsilon\omega = \epsilon^\mu\gamma_\mu\lambda. \tag{5.3}$$

We write down the partition function of the pure spinor system using the field-antifield symmetry (2.11). To obtain Z_γ we need to evaluate the Hilbert series of the arc space of (5.2). The arc space is obtained by substituting the expansion (3.1) for the sixteen

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complex coordinates in (5.2) leading to constraints among the variables $x_i^{(j)}$. Since there are ten gamma matrices, there are ten equations for every power of ξ, giving rise to the arc-space of (5.2). These are not algebraically independent, however. The Hilbert series of the arc space counts the monomials in the variables $x_i^{(j)}$ modulo the constraints defining the arc space. Since there are more than one equations at every mass level, the computation of Hilbert series is more complicated than the previous case requiring the Gröbner basis of the ideal generated by the constraints at each level, that is for each power of q. We resort to `Macaulay2` to compute the Hilbert series. The first jet scheme of the variety (5.2) is obtained by writing

$$\lambda = \lambda^{(0)} + \lambda^{(1)}\xi := \left(x_1^{(0)} + x_1^{(1)}\xi, x_2^{(0)} + x_2^{(1)}\xi, \cdots, x_{16}^{(0)} + x_{16}^{(1)}\xi\right)^T \tag{5.4}$$

and substituting in (5.2). The twenty resulting equations, namely,

$$\lambda^{(0)}\gamma^\mu\lambda^{(0)} = 0 \tag{5.5}$$

$$\lambda^{(0)}\gamma^\mu\lambda^{(1)} + \lambda^{(1)}\gamma^\mu\lambda^{(0)} = 0 \tag{5.6}$$

define the first jet scheme. The Hilbert series is obtained from the set of equations with grades t for $\lambda^{(0)}$ and qt for $\lambda^{(1)}$. Computing the Hilbert series in `Macaulay2` yields

$$\begin{aligned}\Big(1 &+ 5t - 10qt^2 + 5t^2 - 34qt^3 + t^3 + q^3t^4 + 45q^2t^4 - 16qt^4 \\ &- 11q^3t^5 + 65q^2t^5 - 65q^3t^6 + 11q^2t^6 + 16q^4t^7 - 45q^3t^7 - q^2t^7 - q^5t^8 \\ &+ 34q^4t^8 - 5q^5t^9 + 10q^4t^9 - 5q^5t^{10} - q^5t^{11}\Big)/(1-t)^{11}(1-qt)^{16}.\end{aligned} \tag{5.7}$$

This result is correct to order q only since we truncated the series (5.4) at $\lambda^{(1)}$. Expanded in powers of q and retaining term up to order q we obtain

$$\mathcal{H}_{\lambda\gamma^\mu\lambda} = \frac{t^3 + 5t^2 + 5t + 1}{(1-t)^{11}} + q\frac{2\left(23t^3 + 35t^2 + 8t\right)}{(1-t)^{11}} + O\left(q^2\right), \tag{5.8}$$

In order to implement the field-antifield symmetry according to (2.11) the Hilbert series is taken to be the contribution of γ's to the partition function. We write it as

$$Z_\gamma(q,t) = \mathcal{H}_{\lambda\gamma^\mu\lambda} = \frac{1 + 5t + 5t^2 + t^3}{(1-t)^{11}}\tilde{Z}(q,t) \tag{5.9}$$

by pulling out the factor $Z_0(t) = (1 + 5t + 5t^2 + t^3)/(1-t)^{11}$. Then the partition function of the beta-gamma system (5.1) with the pure spinor constraint (5.2) is given by (2.11) as

$$Z_{PS}(q,t) = \frac{1 + 5t + 5t^2 + t^3}{(1-t)^{11}}\tilde{Z}(q,t)\tilde{Z}(q,1/t). \tag{5.10}$$

Expanded in powers of q this yields

$$Z_{PS}(q,t) = \frac{t^3 + 5t^2 + 5t + 1}{(1-t)^{11}} + q\frac{2(t+1)\left(23t^2 + 20t + 23\right)}{(1-t)^{11}} + \mathcal{O}(q^2). \tag{5.11}$$

This matches with the expression obtained earlier [11] up to the first mass level.

6 Conclusion

We obtain the partition function of beta-gamma systems with algebraic constraints on the fields γ. We showed that the partition function of a beta-gamma system can be evaluated by identifying the contribution from the γ's as the Hilbert series of arc spaces of the algebraic variety given by the constraint. Two examples are worked out explicitly. In the first we consider the A_1 surface singularity given by a quadratic constraint in three γ's. The partition function evaluated using the constraint without solving it with a parametric representation is expected to be different from that obtained using its description as an orbifold, dealt with in an earlier publication [7]. We demonstrate two different ways of computing the partition function in this case. The first one implements the so-called field-antifield symmetry in a multiplicative fashion. This, however, gives rise to terms with negative coefficients in the partition function, which can not be accounted for as the partition is the generating function of degeneracy of operators. We show that the partition function can be obtained alternatively as the generating function of monomials invariant under the classical gauge symmetry of the action modulo the constraint. This is implemented by subtracting the number of monomials not vanishing under the gauge transformation from the totality of monomials. Hence the terms with negative coefficients signify an over-determined system and may thus be omitted. The positive terms of both the expressions, on the other hand, match. Moreover, using the description of the blow up of the codimension two singularity in terms of arc spaces it is shown that the partition function matches with the orbifold partition function up to the first mass level, as the blow up repairs the tangent space of the orbifold. An advantage of the algorithm presented here lies in the fact that it can be straightforwardly extended to the case where the constraints are not reducible, such as pure spinor system. Moreover, this gives the partition function a geometric significance. A computation using Poisson brackets confirms (4.9) at the lowest grades.

We also obtain the partition function of the pure spinor system using the Hilbert series of the arc space of the pure spinor constraint looked upon as a variety embedded in the sixteen-dimensional complex affine space. This requires the computation of Gröbner bases in the polynomial rings involved. We used `Macaulay2` to obtain the Hilbert series which, though straightforward as an algorithm, is extremely memory-intensive and we are restricted here to the first mass level. The code is appended below. However, we show that the computation up to this level implementing the field-antifield symmetry in a product formula matches with previously known results [11]. The computation of gauge invariant monomials is more complicated since the resolution in terms of arc spaces is not known in addition to the variety of gauge invariants appearing at higher mass levels.

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A The naïve Macaulay2 code used for purespinors

```
baseRing = ZZ;

makeVars = (N,K) -> flatten toList apply(0..N, I -> flatten toList
apply(1..K, a -> x_[I,a]));

-- X_Ia:    I := mass level
--          a := index of fields, 1 ... 16


lstDeg = (N,K) -> flatten toList apply(0..N, I -> toList apply(1..K, a ->
{I,1}));

-- degrees corresponding to mass level change
-- all fields (in the a index) are equal degree

-- R = baseRing[makeVars(1,16),Degrees => lstDeg(1,16)];
                                   -- CHANGE N of (N,K) for each mass level




unit = matrix{{1,0},{0,1}}
tau1 = matrix{{0,1},{1,0}}
ep = matrix{{0,1},{-1,0}}    -- this is (i tau2)
tau3 = matrix{{1,0},{0,-1}}

g1 =  ep ** ep ** ep
g2 =  unit ** tau1 ** ep
g3 =  unit ** tau3 ** ep
g4 =  tau1 ** ep ** unit
g5 =  tau3 ** ep ** unit
g6 =  ep ** unit ** tau1
g7 =  ep ** unit ** tau3
g8 =  unit ** unit ** unit

zer = id_(R^8) * 0

G1 = matrix{{zer,g1},{transpose(g1),zer}}
G2 = matrix{{zer,g2},{transpose(g2),zer}}
G3 = matrix{{zer,g3},{transpose(g3),zer}}
G4 = matrix{{zer,g4},{transpose(g4),zer}}
G5 = matrix{{zer,g5},{transpose(g5),zer}}
G6 = matrix{{zer,g6},{transpose(g6),zer}}
G7 = matrix{{zer,g7},{transpose(g7),zer}}
G8 = matrix{{zer,g8},{transpose(g7),zer}}
G9 = id_(R^16)
G0 = matrix{{id_(R^8),zer},{zer,-id_(R^8)}}

-- MASS LEVEL 0

m0 = matrix{toList apply(1..16, a -> x_[0,a])}
m0t = transpose(m0)

m0Ideal1 = m0*G1*m0t
m0Ideal2 = m0*G2*m0t
m0Ideal3 = m0*G3*m0t
m0Ideal4 = m0*G4*m0t
m0Ideal5 = m0*G5*m0t
m0Ideal6 = m0*G6*m0t
m0Ideal7 = m0*G7*m0t
m0Ideal8 = m0*G8*m0t
m0Ideal9 = m0*G9*m0t
m0Ideal0 = m0*G0*m0t

-- MASS LEVEL 1

m1 = matrix{toList apply(1..16, a -> x_[1,a])}
m1t = transpose(m1)

m1Ideal1 = m1*G1*m0t + m0*G1*m1t
m1Ideal2 = m1*G2*m0t + m0*G2*m1t
m1Ideal3 = m1*G3*m0t + m0*G3*m1t
m1Ideal4 = m1*G4*m0t + m0*G4*m1t
m1Ideal5 = m1*G5*m0t + m0*G5*m1t
m1Ideal6 = m1*G6*m0t + m0*G6*m1t
m1Ideal7 = m1*G7*m0t + m0*G7*m1t
m1Ideal8 = m1*G8*m0t + m0*G8*m1t
m1Ideal9 = m1*G9*m0t + m0*G9*m1t
m1Ideal0 = m1*G0*m0t + m0*G0*m1t
```


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```
arc1 = ideal(
m0Ideal1,
m0Ideal2,
m0Ideal3,
m0Ideal4,
m0Ideal5,
m0Ideal6,
m0Ideal7,
m0Ideal8,
m0Ideal9,
m0Ideal0,
m1Ideal1,
m1Ideal2,
m1Ideal3,
m1Ideal4,
m1Ideal5,
m1Ideal6,
m1Ideal7,
m1Ideal8,
m1Ideal9,
m1Ideal0
);

hf = hilbertSeries arc1
reduceHilbert hf
```


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