Published for SISSA by Springer

Received: *August 15, 2014*
Revised: *November 27, 2014*
Accepted: *December 9, 2014*
Published: *January 7, 2015*

NLO evolution of 3-quark Wilson loop operator

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Abstract: It is well known that the high-energy scattering of a meson from some hadronic target can be described by the interaction of that target with a color dipole formed by two Wilson lines corresponding to the fast quark-antiquark pair. Moreover, the energy dependence of the scattering amplitude is governed by the evolution equation of this color dipole with respect to rapidity. Similarly, the energy dependence of scattering of a baryon can be described in terms of evolution of a three-Wilson-line operator with respect to the rapidity of the Wilson lines. We calculated the evolution of the 3-quark Wilson loop operator in the Next-to-Leading Order (NLO), and we presented a quasi-conformal evolution equation for a composite 3-Wilson-line operator. Futhermore, we obtained the linearized version of that evolution equation describing the amplitude of the odderon exchange at high energies.

Keywords: NLO Computations, Deep Inelastic Scattering (Phenomenology)

ArXiv ePrint: 1405.0443

doi:10.1007/JHEP01(2015)009

Contents

1 Introduction

The behavior of QCD amplitudes in high-energy Regge limit can be described in terms of the rapidity evolution of Wilson-line operators. A well-known general feature of high-energy scattering is that a fast particle moves along its straight-line classical trajectory and the only quantum effect is the eikonal phase factor acquired along this propagation path. In QCD, for the fast quark or gluon scattering off some target, this eikonal phase factor is a Wilson line - the infinite gauge link ordered along the straight line collinear to the particle's velocity. This observation serves as a starting point in the analysis of high-energy amplitudes by the operator expansion (OPE) in the Wilson lines developed in [1]. In a few sentences, the basic outline of the high-energy OPE is the following (for reviews, see refs. [2, 3]). First, we introduce a "rapidity divide" η between the rapidity of the projectile Y_P and the rapidity of the target Y_T and separate all Feynman loop integrals over longitudinal momentum (≡ rapidity) into two parts: the coefficient functions (called the impact factors) with $Y > \eta$ and the matrix elements of the Wilson-line operators with $Y < \eta$. As we mentioned above, interaction of the fast particles with the slow ones can be described in the eikonal approximation so the relevant operators are the Wilson lines. Second, we find the evolution equations for these Wilson-line operators with respect to our rapidity factorization scale η. Third, we solve these equations (analytically or numerically) and evolve the Wilson-line operators down to the energies of few GeV at which step we need to convolute the results with the initial conditions for the rapidity evolution. If the target can be described by perturbative QCD (like virtual photon or heavy-quark meson) these initial conditions can be calculated in pQCD. If the target is a proton or a nucleus, the initial conditions are usually taken in the form of Mueller-Glauber model. It should be mentioned that the alternative approach to the high-energy scattering in QCD based on the evolution of the target wave function is described by the Jalilian-Marian, Iancu, McLerran, Weigert, Leonidov and Kovner (JIMWLK) hamiltonian [4–10].

The high-energy OPE method is general but originally it was applied to the case of the deep inelastic scattering or meson scattering where the relevant Wilson-line operator is a color dipole. The leading order (LO) evolution of color dipoles was studied in the leading order in the paper [1] and independently by Yu. Kovchegov in refs. [11, 12] where it was applied to scattering from large nuclei. This equation is now known as the Balitsky-Kovchegov (BK) equation. In the linear 2-gluon approximation the C-even part of this equation goes into the Balitsky-Fadin-Kuraev-Lipatov (BFKL) equation [13–15] describing the C-even t-channel state known as pomeron. The running-coupling BK evolution of color dipole (rcBK) was obtained in refs. [16] and [17] the full NLO kernel was found in papers [18, 19]. The solutions of the rcBK equation are widely used now for pA and heavy-ion experiments at LHC and RHIC. However, recently it has been realized that many interesting processes are described by the evolution of the more complicated operators such as "color tripoles" (trace of three Wilson lines in the quark representation) and "color quadrupoles" (trace of four Wilson lines). To describe evolution of such operators the NLO BK was generalized to the full hierarchy of Wilson-line evolution in recent paper [20]. (see also the JIMWLK calculation in ref. [21]).

As we briefly mentioned, in the high-energy OPE framework the amplitude in the Regge limit can be written as a convolution of the impact factors and the matrix elements of the Wilson line operators. The impact factors consist of the wavefunctions of the incoming and outgoing particles, which describe their splitting into the quarks and gluons propagating through the shockwave formed by the target particle. The propagation of the fast particle is described by a Wilson line, i.e. an infinite gauge link ordered along the classical trajectory of the particle. As we discussed above, for the high-energy scattering of a virtual photon or a meson the relevant operator is a color dipole whereas in the case of a proton scattering such an operator is the baryon or the 3-Quark Wilson Loop (3QWL) operator $\varepsilon^{i'j'h'}\varepsilon_{ijh}U^i_{1i'}U^j_{2j'}U^h_{3h'}$. The rapidity evolution of such a "color tripole" has been frequently discussed in recent years. In the linearized LO approximation it was studied in the C-odd case within the JIMWLK formalism, and it was proved to be equivalent to the C-odd Bartels-Kwiecinski-Praszalowicz (BKP) [22]–[23] equation in [24]. The Green function obeying the BKP equation describes the C-odd t-channel state known as odderon. The authors of [24] showed that the the LO linearized evolution equation for the 3QWL operator can be reduced to the BKP equation after the transformation from the coordinate to the momentum space. The full non-linear LO evolution equation for the 3QWL operator was derived within Wilson line approach [1] in ref. [25] and the connected contribution to the NLO kernel of the equation was calculated in [26]. In the momentum representation the evolution of the 3QWL operator was first studied in [27] and the nonlinear equation was worked out in [28]. In the C-odd case the linear NLO evolution equation for the odderon Green function was obtained in [29].

Here we present the full non-linear NLO evolution equation for this 3QWL operator. We calculate the evolution of the 3QWL operator with "rigid cutoff" in the rapidity of the Wilson lines, construct the composite 3QWL operator obeying the quasi-conformal evolution equation similarly to the case of the color dipole discussed in [18], and present its linearized kernel in the 3-gluon approximation. In addition, we linearize the BK equation in the same approximation and show that it contains the non-dipole 3QWL operators. It is worth noting that all the results are written in the $\overline{MS}$ scheme.

After completion of this paper we were informed about the JIMWLK calculation of the NLO evolution of the 3-Wilson-line operator [30]. Both evolution kernels reproduce NLO BK in the dipole limit $\vec{r}_1 \to \vec{r}_2$ and survive other checks but, as it is written, the result of [30] differs from our result since ref. [30] has a much larger basis of operators in the evolution equation (see the discussion in section 9 of this paper).

The paper is organized as follows. In section 2 we remind the general logic of high energy OPE and in section 3 we list all the building blocks necessary for construction of the NLO 3QWL kernel. Sections 4 and 5 present the derivation of the NLO kernel for the 3QWL operator and section 6 describes the calculation of the quasi-conformal kernel for the composite 3QWL operator. The linearized kernel is given in section 7. The main results of the paper are listed in section 8. Conclusions are in section 9. Appendices comprise the necessary technical details.

2 Rapidity factorization and evolution of Wilson lines

Let us consider the proton scattering off a hadron target like another proton or a nucleus. First, we assume that due to saturation the characteristic transverse momenta of the exchanged and the produced gluons are relatively high ($Q_s \sim 2-3\,\mathrm{GeV}$ for pA scattering at LHC) so the application of perturbation theory is justified. Alternatively, one may think about high-energy scattering of a charmed baryon.

If pQCD is applicable, in accordance with general logic of the high-energy OPE we factorize all amplitudes in rapidity. First, we integrate over the gluons with the rapidity Y close to that of the projectile proton Y_p. To this end we introduce the rapidity divide $\eta \leq Y_p$ which separates the "fast" gluons from the "slow" ones.

It is convenient to use the background field formalism: we integrate over the gluons with $\alpha > \sigma = e^\eta$ and leave the gluons with $\alpha < \sigma$ as a background field to be integrated over later. Since the rapidities of the background gluons are very different from the rapidities

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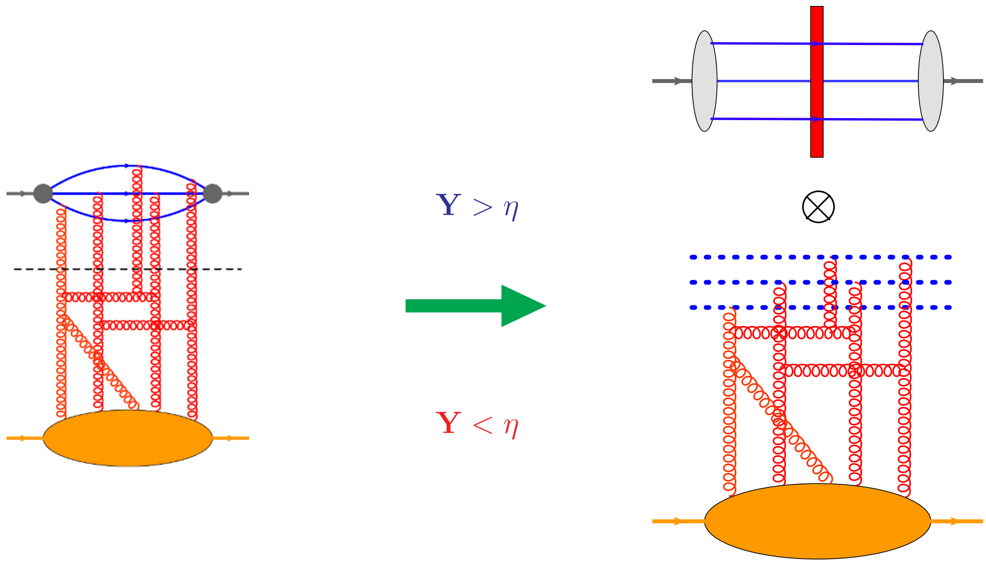

Figure 1. Rapidity factorization. The impact factors with $Y > \eta$ are given by diagrams in the shock-wave background. Wilson-line operators with $Y < \eta$ are denoted by dotted lines.

of the gluons in our Feynman diagrams, the background field can be taken in the form of a shock wave thanks to Lorentz contraction. To derive the expression of a quark (or a gluon) propagator in this shock-wave background we represent the propagator as a path integral over various trajectories, each of them weighed with the gauge factor $\mathrm{Pexp}(ig\int dx_\mu A^\mu)$ ordered along the propagation path. Now, since the shock wave is very thin, the quark (or gluon) does not have time to deviate in the transverse direction. Therefore, its trajectory inside the shock wave can be approximated by a segment of the straight line. Moreover, since there is no external field outside the shock wave, the integral over the segment of straight line can be formally extended to $\pm\infty$ limits yielding the Wilson-line gauge factor

$$\begin{aligned} U_x^\eta &= \mathrm{Pexp}\Big[ig\int_{-\infty}^{\infty} du\; p_1^\mu A_\mu^\sigma(up_1 + x_\perp)\Big], \\ A_\mu^\eta(x) &= \int d^4k\; \theta(e^\eta - |\alpha_k|)e^{ik\cdot x}A_\mu(k). \end{aligned} \tag{2.1}$$

Here

$$U(\vec{r},\eta) = Pe^{ig\int_{-\infty}^{+\infty} A_\eta^-(r^+,\vec{r})dr^+}, \tag{2.2}$$

where A_η^- is the external shock wave field built from only slow gluons

$$A_\eta^- = \int \frac{d^4p}{(2\pi)^4} e^{-ipz} A^-(p)\, \theta(e^\eta - p^+). \tag{2.3}$$

(Our light-cone conventions are listed in the appendix A). The propagation of a quark (or gluon) in the shock-wave background is then described by the free propagation to the point of interaction with the shock wave, the Wilson line U at the interaction point, and the free propagation to the final point.

Thus, the result of the integration over the rapidities $Y > \eta$ gives the proton impact factor proportional to the product of two proton wavefunctions integrated over the longitudinal momenta. This impact factor is multiplied by a "color tripole" — 3QWL operator made of three (light-like) Wilson lines with rapidities up to η:

$$B_{mnl} \equiv \varepsilon^{i'j'h'} \varepsilon_{ijh} U^i_{mi'} U^j_{nj'} U^h_{lh'} \equiv U_m \cdot U_n \cdot U_l, \tag{2.4}$$

where $U_i \equiv U(\vec{r}_i, \eta)$. As discussed in refs. [1, 20], these Wilson lines should be connected by appropriate gauge links at infinity making operator (2.4) and similar many-Wilson-line operators below gauge invariant. It should be noted that the proton impact factor is non-perturbative, so at this point it can be calculated only using some models of proton wavefunctions.

At the second step the integrals over the gluons with the rapidity $Y < \eta$ give the matrix element of the triple Wilson-line operator B_{123} between the target states. The "rapidity cutoff" η is arbitrary (between the rapidities of the projectile and the target) but it is convenient to choose it in such a way that the impact factor does not scale with energy. So all the energy dependence is shifted to the matrix element of the triple Wilson-line operator (see the discussion in ref. [31, 32]). In the leading order the rapidity evolution of this operator was calculated in ref. [25, 26] while in this paper we present the result for the NLO evolution.

3 NLO evolution of triple-Wilson-line operator

In this section we outline the calculation of the NLO kernel for the rapidity evolution of 3QWL operator (2.4). In accordance with general logic of high-energy OPE in order to find the evolution of the Wilson-line operators with respect to the rapidity cutoff we consider the matrix element of operators with the rapidities up to η_1 and we integrate over the region of rapidities $\Delta\eta = \eta_1 - \eta_2$ (where $\eta_1 > \eta_2 > Y_{\text{target}}$). Since particles with different rapidities perceive each other as Wilson lines, the result of the integration will be $\Delta\eta$ times the kernel of the rapidity evolution times the Wilson lines with rapidities up to η_2. As we discussed in section 2, it is convenient to use the background-field formalism where the gluons with rapidities $Y < \eta_2$ form a narrow shockwave. As $\Delta\eta \to 0$, we get the evolution equation

$$\frac{\partial}{\partial \eta} \langle B_{123} \rangle = \langle K_{LO} \otimes B_{123} \rangle + \langle K_{NLO} \otimes B_{123} \rangle. \tag{3.1}$$

Here we explicitly write the $\langle \ldots \rangle$ brackets, which denote that the calculation was performed in the shockwave background. We will often omit them to avoid overloading the notation.

The typical leading-order diagrams are shown in figure 2 and it is clear that at this order the evolution equation for the 3-line operator can be restored from the evolution of the two-line operators since all the interactions are either pairwise (figure 2b) or self-interactions (figure 2a). The result for the LO evolution has the form [25] (In this paper we set $N_c = 3$ explicitly)

$$\frac{\partial B_{123}}{\partial \eta} = \frac{\alpha_s 3}{4\pi^2} \int d\vec{r}_0 \tag{3.2}$$

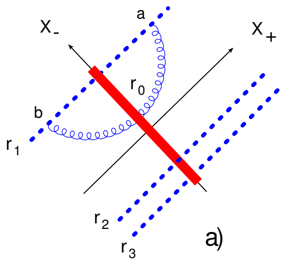

Figure 2. Leading-order diagrams.

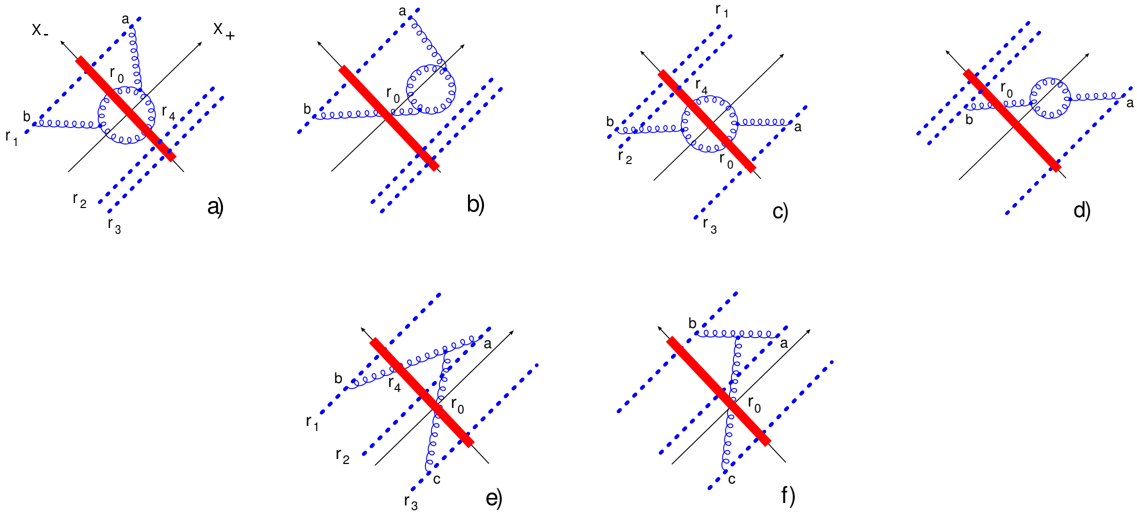

Figure 3. Typical NLO diagrams.

$$\times \left[\frac{\vec{r}_{12}^{\,2}}{\vec{r}_{01}^{\,2}\vec{r}_{02}^{\,2}} \left(-B_{123} + \frac{1}{6}(B_{100}B_{320} + B_{200}B_{310} - B_{300}B_{210}) \right) + (1 \leftrightarrow 3) + (2 \leftrightarrow 3) \right],$$

where $\vec{r}_1, \vec{r}_2, \vec{r}_3$ are the coordinates of the quark Wilson lines within the 3QWL and $\vec{r}_0$ is the coordinate of the gluon Wilson line coming from the intersection with the shock wave; $\vec{r}_{ij} = \vec{r}_i - \vec{r}_j$. The notation $(i \leftrightarrow j)$ stands for the permutation. It means that we have to change $\vec{r}_i \leftrightarrow \vec{r}_j$ and $U_i \leftrightarrow U_j$. As a result performing $(1 \leftrightarrow 3)$, we change $B_{100} \to B_{300}$, $B_{320} \to B_{120}$, etc.

The typical diagrams in the next-to-leading order are shown in figure 3 where $\vec{r}_0$ and $\vec{r}_4$ are the coordinates of intersections with the shock wave. It is clear that at the NLO in addition to the self-interaction (figure 3a) and the pairwise-interaction (figure 3b) diagrams we have also the triple-interaction diagrams of figure 3c type. It should be emphasized that for the self-interaction and the pairwise diagrams one can use the results of ref. [20] while the triple-interaction diagrams were already calculated in ref. [26]. In this paper we combine these results to get a concise expression for the evolution of three-Wilson-line operator (2.4). The building blocks for our work are taken from ref. [20] and the relation of our notations to those of ref. [20] is presented in appendix A.

$$\frac{\partial\,(U_1)_i^j}{\partial\eta} = \frac{\alpha_s^2}{4\pi^4}\int d\vec{r}_4 d\vec{r}_0 \Big\{U_4^{dd'}(U_0^{ee'} - U_4^{ee'})\Big(G_9[if^{ade}(t^aU_1\{t^{d'}t^{e'}\})_i^j - if^{ad'e'}(\{t^dt^e\}U_1t^a)_i^j]$$
$$+ 2G_3 f^{ade}f^{bd'e'}(t^aU_1t^b)_i^j\Big)$$
$$+ \frac{4n_f}{\vec{r}_{04}{}^4}\left\{\frac{(\vec{r}_{14}\vec{r}_{01})}{\vec{r}_{14}{}^2 - \vec{r}_{01}{}^2}\ln\left(\frac{\vec{r}_{14}{}^2}{\vec{r}_{01}{}^2}\right) + 1\right\}(t^aU_1t^b)_i^j tr(t^aU_4t^b(U_0^\dagger - U_4^\dagger))\Big\}$$
$$+ \frac{\alpha_s^2 N_c}{4\pi^3}\int \frac{d\vec{r}_4}{\vec{r}_{14}{}^4}(U_4^{ab} - U_1^{ab})(t^aU_1t^b)_i^j$$
$$\times\left\{\frac{11}{3}\ln\left(\frac{\vec{r}_{14}^{\,2}\mu^2}{4e^{2\psi(1)}}\right) + \frac{67}{9} - \frac{\pi^2}{3} - \frac{n_f}{N_c}\left[\frac{2}{3}\ln\left(\frac{\vec{r}_{14}^{\,2}\mu^2}{4e^{2\psi(1)}}\right) + \frac{10}{9}\right]\right\}, \tag{3.3}$$

where G_3 is defined in (3.17), and G_9 is defined in (3.25); n_f is the number of the quark flavours and μ^2 is the renormalization scale in the $\overline{MS}$-scheme.

$$\frac{\partial\,(U_1)_i^j\,(U_2)_k^l}{\partial\eta} = \frac{\alpha_s^2}{8\pi^4}\int d\vec{r}_4 d\vec{r}_0\,(\mathcal{A}_1 + \mathcal{A}_2 + \mathcal{A}_3) + \frac{\alpha_s^2}{8\pi^3}\int d\vec{r}_4\,(\mathcal{B}_1 + N_c\mathcal{B}_2)\,. \tag{3.4}$$

$$\mathcal{A}_1 = [(t^aU_1)_i^j(U_2t^b)_k^l + (t^aU_2)_k^l(U_1t^b)_i^j]\Big[(f^{ade}f^{bd'e'}U_4^{dd'}(U_0^{ee'} - U_4^{ee'})4(H_3 + H_4 + (1\leftrightarrow 2)))$$
$$+ 4n_f\left(\frac{1}{\vec{r}_{04}{}^4}\left\{\frac{(\vec{r}_{14}\vec{r}_{01})}{\vec{r}_{14}{}^2 - \vec{r}_{01}{}^2}\ln\left(\frac{\vec{r}_{14}{}^2}{\vec{r}_{01}{}^2}\right) + 1\right\} + \frac{L_{12}^q}{2} + (1\leftrightarrow 2)\right)tr(t^aU_4t^b(U_0^\dagger - U_4^\dagger))\Big], \tag{3.5}$$

where H_3 is defined in (3.34), and H_4 is defined in (3.35) and

$$L_{12}^q = \frac{1}{\vec{r}_{04}{}^4}\left\{\frac{\vec{r}_{02}{}^2\vec{r}_{14}{}^2 + \vec{r}_{01}{}^2\vec{r}_{24}{}^2 - \vec{r}_{04}{}^2\vec{r}_{12}{}^2}{2(\vec{r}_{02}{}^2\vec{r}_{14}{}^2 - \vec{r}_{01}{}^2\vec{r}_{24}{}^2)}\ln\left(\frac{\vec{r}_{02}{}^2\vec{r}_{14}{}^2}{\vec{r}_{01}{}^2\vec{r}_{24}{}^2}\right) - 1\right\}. \tag{3.6}$$

$$\mathcal{A}_2 = 8(U_4 - U_1)^{dd'}(U_0 - U_2)^{ee'}\Big\{i[f^{ad'e'}(t^dU_1t^a)_i^j(t^eU_2)_k^l - f^{ade}(t^aU_1t^{d'})_i^j(U_2t^{e'})_k^l](H_3 - H_2)$$
$$- i[f^{ad'e'}(t^dU_1)_i^j(t^eU_2t^a)_k^l - f^{ade}(U_1t^{d'})_i^j(t^aU_2t^{e'})_k^l](H_2(1\leftrightarrow 2) + H_3(1\leftrightarrow 2))\Big\}, \tag{3.7}$$

where H_2 is defined in (3.33), and $H_i(1\leftrightarrow 2) \equiv H_i|_{\vec{r}_1\leftrightarrow\vec{r}_2}$.

$$\mathcal{A}_3 = 8U_4^{dd'}\Big\{i[f^{ad'e'}(U_1t^a)_i^j(t^dt^eU_2)_k^l - f^{ade}(t^aU_1)_i^j(U_2t^{e'}t^{d'})_k^l](H_1 + H_3(1\leftrightarrow 2))(U_0 - U_2)^{ee'}$$
$$+ i[f^{ad'e'}(t^dt^eU_1)_i^j(U_2t^a)_k^l - f^{ade}(U_1t^{e'}t^{d'})_i^j(t^aU_2)_k^l](H_1(1\leftrightarrow 2) + H_3)(U_0 - U_1)^{ee'}\Big\}, \tag{3.8}$$

where H_1 is defined in (3.32).

$$\mathcal{B}_1 = 2\ln\left(\frac{\vec{r}_{14}^{\,2}}{\vec{r}_{12}^{\,2}}\right)\ln\left(\frac{\vec{r}_{24}^{\,2}}{\vec{r}_{12}^{\,2}}\right)$$
$$\times\left\{(U_4 - U_1)^{ab} i[f^{bde}(t^aU_1t^d)_i^j(U_2t^e)_k^l + f^{ade}(t^eU_1t^b)_i^j(t^dU_2)_k^l]\left(\frac{(\vec{r}_{14}\vec{r}_{24})}{\vec{r}_{14}^{\,2}\vec{r}_{24}^{\,2}} - \frac{1}{\vec{r}_{14}^{\,2}}\right)\right.$$
$$\left. + (U_4 - U_2)^{ab} i[f^{bde}(U_1t^e)_i^j(t^aU_2t^d)_k^l + f^{ade}(t^dU_1)_i^j(t^eU_2t^b)_k^l]\left(\frac{(\vec{r}_{14}\vec{r}_{24})}{\vec{r}_{14}^{\,2}\vec{r}_{24}^{\,2}} - \frac{1}{\vec{r}_{24}^{\,2}}\right)\right\}. \tag{3.9}$$

$$\mathcal{B}_2 = (2U_4 - U_1 - U_2)^{ab}\,[(t^aU_1)^j_i(U_2t^b)^l_k + (U_1t^b)^j_i(t^aU_2)^l_k]$$
$$\times\left\{\frac{(\vec r_{14}\vec r_{24})}{\vec r_{14}^{\,2}\vec r_{24}^{\,2}}\left[\left(\frac{11}{3}-\frac{2}{3}\frac{n_f}{N_c}\right)\ln\left(\frac{\vec r_{12}^{\,2}\mu^2}{4e^{2\psi(1)}}\right)+\frac{67}{9}-\frac{\pi^2}{3}-\frac{10}{9}\frac{n_f}{N_c}\right]\right.$$
$$\left.+\left(\frac{11}{3}-\frac{2}{3}\frac{n_f}{N_c}\right)\left(\frac{1}{2\vec r_{14}^{\,2}}\ln\left(\frac{\vec r_{24}^{\,2}}{\vec r_{12}^{\,2}}\right)+\frac{1}{2\vec r_{24}^{\,2}}\ln\left(\frac{\vec r_{24}^{\,2}}{\vec r_{12}^{\,2}}\right)\right)\right\}, \tag{3.10}$$

where n_f is the number of the quark flavours and μ^2 is the renormalization scale in the $\overline{MS}$-scheme.

$$\frac{\partial}{\partial\eta}(U_1)^j_i\,(U_2)^l_k\,(U_3)^n_m$$
$$= i\frac{\alpha_s^2}{\pi^4}\int d\vec r_4 d\vec r_0\left\{f^{cde}[(t^aU_1)^j_i\left(t^bU_2\right)^l_k(U_3t^c)^n_m\,(U_4-U_1)^{ad}(U_0-U_2)^{be}\right.$$
$$-(U_1t^a)^j_i\left(U_2t^b\right)^l_k(t^cU_3)^n_m\,(U_4-U_1)^{da}(U_0-U_2)^{eb}](H_5(1\leftrightarrow 3)+H_6(1\leftrightarrow 3))$$
$$+f^{ade}[(U_1t^a)^j_i\left(t^bU_2\right)^l_k(t^cU_3)^n_m\,(U_4-U_3)^{cd}(U_0-U_2)^{be}$$
$$-(t^aU_1)^j_i\left(U_2t^b\right)^l_k(U_3t^c)^n_m\,(U_4-U_3)^{dc}(U_0-U_2)^{eb}](H_5+H_6)$$
$$+f^{bde}[(t^aU_1)^j_i\left(U_2t^b\right)^l_k(t^cU_3)^n_m\,(U_4-U_1)^{ad}(U_0-U_3)^{ce}$$
$$-(U_1t^a)^j_i\left(t^bU_2\right)^l_k(U_3t^c)^n_m\,(U_4-U_1)^{da}(U_0-U_3)^{ec}]$$
$$\times(H_5(1\to 2\to 3\to 1)+H_6(1\to 2\to 3\to 1)), \tag{3.11}$$

where H_5 is defined in (3.37), and H_6 is defined in (3.38) and $H_i(1\to 2\to 3\to 1)\equiv H_i|_{\vec r_1\to\vec r_2,\vec r_2\to\vec r_3,\vec r_3\to\vec r_1}$.

It is obvious that expressions (3.3)–(3.11) do not have UV singularities since they have subtractions like $U_4 - U_i$ and $U_0 - U_i$, which make them convergent. To construct the evolution equation for the 3QWL operator we have to convolute (3.3)–(3.11) according to (2.4) and simplify the result. The final evolution equation will consist of 3 parts: the part with 2 integrations w.r.t. $\vec r_0$ and $\vec r_4$, the part with one integration w.r.t. $\vec r_0$ and the purely virtual part. The Wilson line structure in the latter part depends neither on $\vec r_0$ nor on $\vec r_4$. Therefore it seems that we can take both integrals in it. However, we can not do so since this part cancels the UV singularities in the former two parts. In fact, the purely virtual part is unambiguously determined by the first two parts. Indeed, it can be proportional to only one color structure B_{123} and it must cancel the other parts if we switch off the shockwave, i.e. when $U_i = 1$. Hence in our paper we will calculate only the former two parts of the evolution equation which come from the diagrams with one and two gluon intersections of the shockwave and then we restore the virtual part from the aforesaid conditions. Throughout the paper (except for section 7 where we discuss linearization of the already constructed UV-safe equation) we work with the parts of the kernels (integrands) and we do not integrate w.r.t. $\vec r_0$ and $\vec r_4$. Therefore we do not need these parts to be UV finite. On constructing all of them we restore the virtual part, which automatically makes the whole kernel UV-convergent.

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Let us start with the self- and the pairwise-interactions of the type shown in figure 3 a-d. At $N_c = 3$ one can use the SU(3) identities

$$U_4^{ba} = 2tr(t^b U_4 t^a U_4^\dagger), \quad (t^a)_i^j (t^a)_k^l = \frac{1}{2}\delta_i^l \delta_k^j - \frac{1}{6}\delta_i^j \delta_k^l \tag{3.12}$$

to rewrite (3.3)–(3.11) only through the Wilson lines in the fundamental representation. We will consider the gluon part without subtractions now. For the contribution of the the states with 2 gluons crossing the shockwave it reads

$$\langle K_{NLO} \otimes (U_1)_{i_1}^{i_3} (U_2)_{j_1}^{j_3} \rangle|_{2g} = -\frac{\alpha_s^2}{8\pi^4} \int d\vec{r}_0 d\vec{r}_4 \, \mathbf{G}_{12}, \tag{3.13}$$

where

$$\begin{aligned}
\mathbf{G}_{12} = G_1 &\left\{ \left(U_1 U_0^\dagger U_4\right)_{i_1}^{i_3} \left(U_0 U_4^\dagger U_2\right)_{j_1}^{j_3} + \left(U_4 U_0^\dagger U_1\right)_{i_1}^{i_3} \left(U_2 U_4^\dagger U_0\right)_{j_1}^{j_3} \right\} \\
&+ G_2 (U_1)_{i_1}^{i_3} \left\{ \left(U_0 U_4^\dagger U_2 U_0^\dagger U_4\right)_{j_1}^{j_3} + \left(U_4 U_0^\dagger U_2 U_4^\dagger U_0\right)_{j_1}^{j_3} \right\} \\
&+ G_3 (U_2)_{j_1}^{j_3} \left\{ \left(U_0 U_4^\dagger U_1 U_0^\dagger U_4\right)_{i_1}^{i_3} + \left(U_4 U_0^\dagger U_1 U_4^\dagger U_0\right)_{i_1}^{i_3} \right\} \\
&+ G_4 \left\{ (U_4)_{j_1}^{i_3} \left(U_1 U_0^\dagger U_2\right)_{i_1}^{j_3} + (U_4)_{i_1}^{j_3} \left(U_2 U_0^\dagger U_1\right)_{j_1}^{i_3} \right\} tr(U_0 U_4^\dagger) \\
&+ G_5 (U_2)_{j_1}^{j_3} (U_4)_{i_1}^{i_3} tr(U_0 U_4^\dagger) tr(U_0^\dagger U_1) \\
&+ G_6 \left((U_0)_{i_1}^{i_3} \left\{ \left(U_4 U_0^\dagger U_1 U_4^\dagger U_2\right)_{j_1}^{j_3} + \left(U_2 U_4^\dagger U_1 U_0^\dagger U_4\right)_{j_1}^{j_3} \right\} \right. \\
&\left. - \left\{ (U_4)_{j_1}^{i_3} \left(U_0 U_4^\dagger U_2\right)_{i_1}^{j_3} + (U_4)_{i_1}^{j_3} \left(U_2 U_4^\dagger U_0\right)_{j_1}^{i_3} \right\} tr(U_0^\dagger U_1) \right) \\
&+ G_7 (U_1)_{i_1}^{i_3} (U_4)_{j_1}^{j_3} tr(U_0 U_4^\dagger) tr(U_0^\dagger U_2) \\
&+ G_8 \left((U_4)_{j_1}^{j_3} \left\{ \left(U_0 U_4^\dagger U_2 U_0^\dagger U_1\right)_{i_1}^{i_3} + \left(U_1 U_0^\dagger U_2 U_4^\dagger U_0\right)_{i_1}^{i_3} \right\} \right. \\
&\left. - \left\{ (U_0)_{i_1}^{j_3} \left(U_4 U_0^\dagger U_1\right)_{j_1}^{i_3} + (U_0)_{j_1}^{i_3} \left(U_1 U_0^\dagger U_4\right)_{i_1}^{j_3} \right\} tr(U_4^\dagger U_2) \right).
\end{aligned} \tag{3.14}$$

Note that as discussed above and in ref. [20], one can present some of the terms with one intersection in the two-intersection form with an additional integration over $\vec{r}_4$ (3.4). In doing so, some U_4 and U_0 factors in eq. (3.14) are replaced by $U_4 - U_i$ and $U_0 - U_i$ ($i = 1, 2$ or 3). Such subtractions make this contribution explicitly convergent at $\vec{r}_{0,4} = \vec{r}_i$. We do not write these subtraction terms here since it is easier to make the subtraction after the color convolution. The functions have the form

$$\begin{aligned}
G_1 = -&\left(\frac{\vec{r}_{04}^{\,2} - 2\vec{r}_{02}^{\,2}}{2\vec{r}_{02}^{\,2}\vec{r}_{04}^{\,2}\left(\vec{r}_{24}^{\,2} - \vec{r}_{02}^{\,2}\right)} - \frac{\vec{r}_{04}^{\,2}\vec{r}_{12}^{\,2} + \vec{r}_{02}^{\,2}\left(\vec{r}_{12}^{\,2} - \vec{r}_{14}^{\,2}\right) + \left(\vec{r}_{01}^{\,2} + \vec{r}_{02}^{\,2} - \vec{r}_{12}^{\,2}\right)\vec{r}_{24}^{\,2}}{2\vec{r}_{01}^{\,2}\vec{r}_{02}^{\,2}\vec{r}_{04}^{\,2}\vec{r}_{24}^{\,2}} \right. \\
&\left. + \frac{1}{\vec{r}_{01}^{\,2}\vec{r}_{24}^{\,2} - \vec{r}_{02}^{\,2}\vec{r}_{14}^{\,2}} \left[\frac{2\vec{r}_{12}^{\,2}}{\vec{r}_{04}^{\,2}} - \frac{\vec{r}_{12}^{\,4}}{2\vec{r}_{01}^{\,2}\vec{r}_{24}^{\,2}} - \frac{\vec{r}_{14}^{\,2}}{\vec{r}_{04}^{\,2}} - \frac{\left(\vec{r}_{02}^{\,2} - \vec{r}_{04}^{\,2}\right)\left(\vec{r}_{14}^{\,2} - \vec{r}_{01}^{\,2}\right)\vec{r}_{24}^{\,2}}{\vec{r}_{04}^{\,4}\left(\vec{r}_{24}^{\,2} - \vec{r}_{02}^{\,2}\right)} \right] \right)
\end{aligned}$$

$$\times \ln\left(\frac{\vec{r}_{02}{}^2}{\vec{r}_{24}{}^2}\right) - \frac{1}{2\vec{r}_{04}{}^4} + (0 \leftrightarrow 4, 1 \leftrightarrow 2)\,. \tag{3.15}$$

$$G_2 = \left(\frac{1}{(\vec{r}_{02}{}^2 - \vec{r}_{24}{}^2)}\left[\left(\frac{1}{\vec{r}_{04}{}^4} + \frac{1}{2\vec{r}_{02}{}^2\vec{r}_{24}{}^2}\right)\frac{(\vec{r}_{02}{}^2 + \vec{r}_{24}{}^2)}{2} - \frac{2}{\vec{r}_{04}{}^2}\right] - \frac{\vec{r}_{02}{}^2 - \vec{r}_{24}{}^2}{4\vec{r}_{02}{}^2\vec{r}_{04}{}^2\vec{r}_{24}{}^2}\right)$$
$$\times \ln\left(\frac{\vec{r}_{02}{}^2}{\vec{r}_{24}{}^2}\right) - \frac{1}{\vec{r}_{04}{}^4}. \tag{3.16}$$

$$G_3 = G_2|_{1\leftrightarrow 2}. \tag{3.17}$$

$$G_4 = \left(\frac{(\vec{r}_{02}{}^2 - \vec{r}_{24}{}^2)(\vec{r}_{02}{}^2\vec{r}_{14}{}^2 - \vec{r}_{01}{}^2\vec{r}_{24}{}^2)}{2\vec{r}_{01}{}^2\vec{r}_{02}{}^2\vec{r}_{14}{}^2\vec{r}_{24}{}^2\vec{r}_{04}{}^2} - \frac{\vec{r}_{02}{}^2 + \vec{r}_{24}{}^2}{2\vec{r}_{02}{}^2\vec{r}_{24}{}^2\vec{r}_{04}{}^2} + \frac{1}{2\vec{r}_{01}{}^2\vec{r}_{14}{}^2} + \frac{\vec{r}_{12}{}^4}{2\vec{r}_{01}{}^2\vec{r}_{02}{}^2\vec{r}_{14}{}^2\vec{r}_{24}{}^2}\right.$$
$$\left. + \left(\frac{(\vec{r}_{01}{}^2 - \vec{r}_{14}{}^2)(\vec{r}_{02}{}^2 - \vec{r}_{24}{}^2)}{2\vec{r}_{01}{}^2\vec{r}_{02}{}^2\vec{r}_{04}{}^2\vec{r}_{14}{}^2\vec{r}_{24}{}^2} - \frac{\vec{r}_{02}{}^2 + \vec{r}_{24}{}^2}{2\vec{r}_{01}{}^2\vec{r}_{02}{}^2\vec{r}_{14}{}^2\vec{r}_{24}{}^2}\right)\vec{r}_{12}{}^2\right)\ln\left(\frac{\vec{r}_{01}{}^2}{\vec{r}_{14}{}^2}\right) - G_1. \tag{3.18}$$

$$G_5 = \frac{2}{\vec{r}_{04}{}^4} + \left(\frac{1}{\vec{r}_{01}{}^2 - \vec{r}_{14}{}^2}\left[\frac{4}{\vec{r}_{04}{}^2} - \frac{\vec{r}_{01}{}^2 + \vec{r}_{14}{}^2}{\vec{r}_{04}{}^4} - \frac{1}{\vec{r}_{01}{}^2}\right] - \frac{1}{\vec{r}_{01}{}^2\vec{r}_{04}{}^2}\right)\ln\left(\frac{\vec{r}_{01}{}^2}{\vec{r}_{14}{}^2}\right). \tag{3.19}$$

$$G_6 = \left(\frac{\vec{r}_{12}{}^2 - \vec{r}_{24}{}^2}{2\vec{r}_{01}{}^2\vec{r}_{14}{}^2\vec{r}_{24}{}^2} + \frac{\vec{r}_{14}{}^2(\vec{r}_{24}{}^2 - \vec{r}_{02}{}^2) + \vec{r}_{01}{}^2(\vec{r}_{14}{}^2 - \vec{r}_{12}{}^2 + \vec{r}_{24}{}^2)}{2\vec{r}_{01}{}^2\vec{r}_{04}{}^2\vec{r}_{14}{}^2\vec{r}_{24}{}^2}\right)\ln\left(\frac{\vec{r}_{01}{}^2}{\vec{r}_{14}{}^2}\right). \tag{3.20}$$

$$G_7 = G_5|_{1\leftrightarrow 2}, \quad G_8 = G_6|_{1\leftrightarrow 2, 0\leftrightarrow 4}. \tag{3.21}$$

In all these expressions the notation $i \leftrightarrow j$ stands for the permutation. As before, it means that we have to change $\vec{r}_i \leftrightarrow \vec{r}_j$. After the convolution with $\varepsilon^{i_1 j_1 h}\varepsilon_{i_3 j_3 h'}(U_3)_h^{h'}$, (3.14) gives the contribution of the 2-gluon states to the evolution of the 3QWL operator $U_1 \cdot U_2 \cdot U_3$ describing the total interaction of Wilson lines 1 and 2, leaving Wilson line 3 intact.

$$\begin{aligned}
&\mathbf{G}_{12}\varepsilon^{i_1 j_1 h}\varepsilon_{i_3 j_3 h'}(U_3)_h^{h'} \\
&\quad = G_1\left(\left(U_0 U_4{}^\dagger U_2\right)\cdot\left(U_1 U_0{}^\dagger U_4\right) + \left(U_2 U_4{}^\dagger U_0\right)\cdot\left(U_4 U_0{}^\dagger U_1\right)\right)\cdot U_3 \\
&\quad + \left[G_2\left(U_0 U_4{}^\dagger U_2 U_0{}^\dagger U_4 + U_4 U_0{}^\dagger U_2 U_4{}^\dagger U_0\right)\cdot U_1\cdot U_3 + (1\leftrightarrow 2)\right] \\
&\quad - G_4 tr\left(U_0 U_4{}^\dagger\right)\left(U_1 U_0{}^\dagger U_2 + U_2 U_0{}^\dagger U_1\right)\cdot U_3\cdot U_4 \\
&\quad + \left[G_5\, U_2\cdot U_3\cdot U_4\; tr\left(U_0{}^\dagger U_1\right) tr\left(U_0 U_4{}^\dagger\right) + (1\leftrightarrow 2)\right] \\
&\quad + \left[G_6\left(tr\left(U_0{}^\dagger U_1\right)\left(U_0 U_4{}^\dagger U_2 + U_2 U_4{}^\dagger U_0\right)\cdot U_3\cdot U_4\right.\right. \\
&\quad + \left.\left.\left(U_2 U_4{}^\dagger U_1 U_0{}^\dagger U_4 + U_4 U_0{}^\dagger U_1 U_4{}^\dagger U_2\right)\cdot U_0\cdot U_3\right) + (1\leftrightarrow 2, 0\leftrightarrow 4)\right].
\end{aligned} \tag{3.22}$$

One can also write

$$\langle K_{NLO}\otimes (U_1^\dagger)_{j_1}^{j_3}\rangle|_{2g} = -\frac{\alpha_s^2}{8\pi^4}\int d\vec{r}_0 d\vec{r}_4\; \mathbf{G}_{1^\dagger}, \tag{3.23}$$

where

$$\begin{aligned}
\mathbf{G}_{1^\dagger} = G_3 &\left(U_4{}^\dagger U_0 U_1{}^\dagger U_4 U_0{}^\dagger + U_0{}^\dagger U_4 U_1{}^\dagger U_0 U_4{}^\dagger\right. \\
&\left. - tr\left(U_1{}^\dagger U_4\right) tr\left(U_4{}^\dagger U_0\right) U_0{}^\dagger - tr\left(U_0{}^\dagger U_4\right) tr\left(U_1{}^\dagger U_0\right) U_4{}^\dagger\right)_{j_1}^{j_3}
\end{aligned}$$

$$+ G_9 \left(tr \left({U_1}^\dagger U_4 \right) tr \left({U_4}^\dagger U_0 \right) {U_0}^\dagger - tr \left({U_0}^\dagger U_4 \right) tr \left({U_1}^\dagger U_0 \right) {U_4}^\dagger \right)^{j_3}_{j_1}, \tag{3.24}$$

$$G_9 = \frac{{\vec{r}_{01}}^2 - {\vec{r}_{04}}^2 + {\vec{r}_{14}}^2}{4{\vec{r}_{01}}^2{\vec{r}_{04}}^2{\vec{r}_{14}}^2} \ln \left(\frac{{\vec{r}_{01}}^2}{{\vec{r}_{14}}^2} \right). \tag{3.25}$$

and then take the convolution

$$\begin{aligned}
\mathbf{G}_{\langle 1^\dagger \rangle 3} = \mathbf{G}_{1^\dagger} (U_3)^{j_1}_{j_3} = G_3 & \left(tr \left({U_0}^\dagger U_3 {U_4}^\dagger U_0 {U_1}^\dagger U_4 \right) + tr \left({U_0}^\dagger U_4 {U_1}^\dagger U_0 {U_4}^\dagger U_3 \right) \right. \\
& \left. - tr \left({U_0}^\dagger U_3 \right) tr \left({U_1}^\dagger U_4 \right) tr \left({U_4}^\dagger U_0 \right) - tr \left({U_0}^\dagger U_4 \right) tr \left({U_1}^\dagger U_0 \right) tr \left({U_4}^\dagger U_3 \right) \right) \\
& + G_9 \Big(tr \Big({U_0}^\dagger U_3 \Big) tr \Big({U_1}^\dagger U_4 \Big) tr \Big({U_4}^\dagger U_0 \Big) - tr \Big({U_0}^\dagger U_4 \Big) tr \Big({U_1}^\dagger U_0 \Big) tr \Big({U_4}^\dagger U_3 \Big) \Big).
\end{aligned} \tag{3.26}$$

For the elements of SU(3) group one has the identity

$$\varepsilon^{ijh} \varepsilon_{i'j'h'} (U_1)^{i'}_i (U_1)^{j'}_j = 2(U_1^\dagger)^h_{h'}, \quad U_1 \cdot U_1 \cdot U_3 = 2tr(U_1^\dagger U_3), \tag{3.27}$$

Taking $\vec{r}_2 = \vec{r}_1$ in (3.22) one can check that it is related to (3.26) using the above identity along with (B.1) and (B.3). Taking the conjugate of $\mathbf{G}_{1^\dagger}$, one gets

$$\langle K_{NLO} \otimes (U_1)^{j'}_j \rangle |_{2g} = -\frac{\alpha_s^2}{8\pi^4} \int d\vec{r}_0 d\vec{r}_4 \; \mathbf{G}_1, \tag{3.28}$$

where

$$\begin{aligned}
\mathbf{G}_1 = G_3 & \left(U_4 U_0^\dagger U_1 U_4^\dagger U_0 + U_0 U_4^\dagger U_1 U_0^\dagger U_4 \right. \\
& \left. - tr(U_1 U_4^\dagger) tr(U_4 U_0^\dagger) U_0 - tr(U_0 U_4^\dagger) tr(U_1 U_0^\dagger) U_4 \right)^{j'}_j \\
& + G_9 \left(tr(U_1 U_4^\dagger) tr(U_4 U_0^\dagger) U_0 - tr(U_0 U_4^\dagger) tr(U_1 U_0^\dagger) U_4 \right)^{j'}_j,
\end{aligned} \tag{3.29}$$

The contribution of the evolution of a single line U_1 to the evolution of the 3QWL related to the diagrams with 2 gluons crossing the shockwave reads

$$\begin{aligned}
\mathbf{G}_{\langle 1 \rangle 23} = \mathbf{G}_1 \varepsilon^{ijh} \varepsilon_{i'j'h'} (U_2)^{i'}_i (U_3)^{h'}_h = G_3 & \left(U_4 {U_0}^\dagger U_1 {U_4}^\dagger U_0 + U_0 U_4^\dagger U_1 U_0^\dagger U_4 \right. \\
& \left. - tr(U_1 U_4^\dagger) tr(U_4 U_0^\dagger) U_0 - tr \left({U_0}^\dagger U_1 \right) tr \left({U_4}^\dagger U_0 \right) U_4 \right) \cdot U_2 \cdot U_3 \\
& + G_9 \left(tr(U_1 U_4^\dagger) tr(U_4 U_0^\dagger) U_0 - tr \left({U_0}^\dagger U_1 \right) tr \left({U_4}^\dagger U_0 \right) U_4 \right) \cdot U_2 \cdot U_3.
\end{aligned} \tag{3.30}$$

The connected contribution of the evolution of lines 1 and 2 has the form

$$\begin{aligned}
\mathbf{G}_{\langle 12 \rangle 3} = & \frac{1}{2} [H_1 - (1 \leftrightarrow 2)] \\
& \times \left[\left(U_0 {U_4}^\dagger U_2 \right) \cdot \left(U_1 {U_0}^\dagger U_4 \right) \cdot U_3 - \left(U_0 {U_4}^\dagger U_1 \right) \cdot \left(U_2 {U_0}^\dagger U_4 \right) \cdot U_3 - (4 \leftrightarrow 0) \right] \\
& + H_2 \left[tr \left({U_0}^\dagger U_1 \right) \left(U_0 {U_4}^\dagger U_2 + U_2 {U_4}^\dagger U_0 \right) \cdot U_3 \cdot U_4 \right. \\
& \left. - \left(U_2 {U_0}^\dagger U_1 {U_4}^\dagger U_0 + U_0 {U_4}^\dagger U_1 {U_0}^\dagger U_2 \right) \cdot U_3 \cdot U_4 - (4 \leftrightarrow 0) \right] \\
& + H_3 \left[tr \left({U_0}^\dagger U_1 \right) \left(U_0 {U_4}^\dagger U_2 + U_2 {U_4}^\dagger U_0 \right) \cdot U_3 \cdot U_4 \right.
\end{aligned}$$

$$+\left(U_2U_0{}^\dagger U_1U_4{}^\dagger U_0+U_0U_4{}^\dagger U_1U_0{}^\dagger U_2\right)\cdot U_3\cdot U_4+(4\leftrightarrow 0)\Big]$$
$$+H_4[tr\left(U_0U_4{}^\dagger\right)\left(U_1U_0{}^\dagger U_2+U_2U_0{}^\dagger U_1\right)\cdot U_3\cdot U_4$$
$$+\left(U_0U_4{}^\dagger U_1\right)\cdot\left(U_2U_0{}^\dagger U_4\right)\cdot U_3+\left(U_0U_4{}^\dagger U_2\right)\cdot\left(U_1U_0{}^\dagger U_4\right)\cdot U_3+(4\leftrightarrow 0)]$$
$$+H_1[tr\left(U_4U_0{}^\dagger\right)\left(U_1U_4{}^\dagger U_2+U_2U_4{}^\dagger U_1\right)\cdot U_0\cdot U_3-(4\leftrightarrow 0)]+(1\leftrightarrow 2). \quad (3.31)$$

Here

$$H_1=\frac{1}{8}\left[\frac{\left(\vec r_{02}{}^2-\vec r_{12}{}^2\right)\left(\vec r_{14}{}^2\left(\vec r_{02}{}^2-\vec r_{24}{}^2\right)+\vec r_{04}{}^2\left(\vec r_{24}{}^2-\vec r_{12}{}^2\right)\right)}{\vec r_{01}{}^2\vec r_{02}{}^2\vec r_{04}{}^2\vec r_{14}{}^2\vec r_{24}{}^2}\right.$$
$$\left.+\frac{\vec r_{12}{}^2-\vec r_{14}{}^2-\vec r_{24}{}^2}{\vec r_{04}{}^2\vec r_{14}{}^2\vec r_{24}{}^2}+\frac{\vec r_{24}{}^2-\vec r_{12}{}^2-\vec r_{14}{}^2}{\vec r_{02}{}^2\vec r_{04}{}^2\vec r_{14}{}^2}\right]\ln\left(\frac{\vec r_{01}{}^2}{\vec r_{14}{}^2}\right). \quad (3.32)$$
$$H_2=\frac{1}{8}\left[\frac{\vec r_{12}{}^2}{\vec r_{01}{}^2\vec r_{14}{}^2\vec r_{02}{}^2}+\frac{\vec r_{12}{}^2}{\vec r_{01}{}^2\vec r_{14}{}^2\vec r_{24}{}^2}+\frac{\vec r_{01}{}^2-\vec r_{02}{}^2}{\vec r_{01}{}^2\vec r_{04}{}^2\vec r_{24}{}^2}+2\frac{\vec r_{14}{}^2-\vec r_{04}{}^2+\vec r_{01}{}^2}{\vec r_{01}{}^2\vec r_{04}{}^2\vec r_{14}{}^2}\right.$$
$$\left.-\frac{\vec r_{12}{}^2}{\vec r_{04}{}^2\vec r_{14}{}^2\vec r_{24}{}^2}+\frac{\vec r_{01}{}^2-\vec r_{12}{}^2}{\vec r_{01}{}^2\vec r_{04}{}^2\vec r_{02}{}^2}-\frac{\vec r_{24}{}^2}{\vec r_{04}{}^2\vec r_{14}{}^2\vec r_{02}{}^2}\right]\ln\left(\frac{\vec r_{01}{}^2}{\vec r_{14}{}^2}\right). \quad (3.33)$$
$$H_3=\frac{1}{8}\left[\frac{\vec r_{01}{}^2-\vec r_{02}{}^2}{\vec r_{01}{}^2\vec r_{04}{}^2\vec r_{24}{}^2}+\frac{\vec r_{12}{}^2}{\vec r_{01}{}^2\vec r_{14}{}^2\vec r_{24}{}^2}-\frac{\vec r_{12}{}^2}{\vec r_{04}{}^2\vec r_{14}{}^2\vec r_{24}{}^2}\right.$$
$$\left.+\frac{\vec r_{12}{}^2}{\vec r_{01}{}^2\vec r_{04}{}^2\vec r_{02}{}^2}+\frac{\vec r_{24}{}^2-\vec r_{14}{}^2}{\vec r_{04}{}^2\vec r_{14}{}^2\vec r_{02}{}^2}-\frac{\vec r_{12}{}^2}{\vec r_{01}{}^2\vec r_{14}{}^2\vec r_{02}{}^2}\right]\ln\left(\frac{\vec r_{01}{}^2}{\vec r_{14}{}^2}\right). \quad (3.34)$$
$$H_4=\frac{-1}{4\vec r_{04}{}^4}-\frac{1}{8}\left[\frac{\vec r_{12}{}^2(\vec r_{14}{}^2-\vec r_{01}{}^2)(\vec r_{02}{}^2+\vec r_{24}{}^2)}{\vec r_{01}{}^2\vec r_{02}{}^2\vec r_{04}{}^2\vec r_{14}{}^2\vec r_{24}{}^2}+\frac{\vec r_{12}{}^2}{\vec r_{01}{}^2\vec r_{14}{}^2\vec r_{24}{}^2}-\frac{\vec r_{12}{}^4}{\vec r_{01}{}^2\vec r_{02}{}^2\vec r_{14}{}^2\vec r_{24}{}^2}\right.$$
$$+\frac{\vec r_{24}{}^2+\vec r_{02}{}^2-\vec r_{14}{}^2}{\vec r_{02}{}^2\vec r_{04}{}^2\vec r_{14}{}^2}+\frac{\vec r_{01}{}^2-\vec r_{02}{}^2-\vec r_{24}{}^2}{\vec r_{01}{}^2\vec r_{04}{}^2\vec r_{24}{}^2}$$
$$+\frac{1}{\vec r_{01}{}^2-\vec r_{14}{}^2}\left(\frac{\vec r_{12}{}^2-\vec r_{02}{}^2}{\vec r_{01}{}^2\vec r_{02}{}^2}-\frac{\vec r_{12}{}^2+\vec r_{02}{}^2}{\vec r_{02}{}^2\vec r_{14}{}^2}-\frac{4\vec r_{14}{}^2}{\vec r_{04}{}^4}+\frac{8}{\vec r_{04}{}^2}\right)$$
$$\left.+\frac{1}{\vec r_{01}{}^2\vec r_{24}{}^2-\vec r_{02}{}^2\vec r_{14}{}^2}\left(\frac{2\vec r_{12}{}^4}{\vec r_{02}{}^2\vec r_{14}{}^2}+\frac{4\vec r_{02}{}^2\vec r_{14}{}^2}{\vec r_{04}{}^4}-\frac{8\vec r_{12}{}^2}{\vec r_{04}{}^2}\right)\right]\ln\left(\frac{\vec r_{01}{}^2}{\vec r_{14}{}^2}\right). \quad (3.35)$$

The fully connected "triple" contribution corresponding to the diagrams in figure 3 e can be taken from (3.11) or [26] and transformed to the form

$$\mathbf{G}_{\langle 123\rangle}=H_5\Big[\left(U_0U_4{}^\dagger U_3\right)\cdot\left(U_1U_0{}^\dagger U_2\right)\cdot U_4-\left(U_0U_4{}^\dagger U_2\right)\cdot\left(U_1U_0{}^\dagger U_3\right)\cdot U_4$$
$$+\left(U_2U_0{}^\dagger U_1\right)\cdot\left(U_3U_4{}^\dagger U_0\right)\cdot U_4-\left(U_2U_0{}^\dagger U_4\right)\cdot\left(U_3U_4{}^\dagger U_1\right)\cdot U_0+(4\leftrightarrow 0)\Big]$$
$$+H_6[\left(U_0U_4{}^\dagger U_2\right)\cdot\left(U_1U_0{}^\dagger U_3\right)\cdot U_4+\left(U_0U_4{}^\dagger U_3\right)\cdot\left(U_1U_0{}^\dagger U_2\right)\cdot U_4$$
$$+\left(U_2U_0{}^\dagger U_1\right)\cdot\left(U_3U_4{}^\dagger U_0\right)\cdot U_4-\left(U_2U_0{}^\dagger U_4\right)\cdot\left(U_3U_4{}^\dagger U_1\right)\cdot U_0-(4\leftrightarrow 0)]$$
$$+(1\leftrightarrow 2)+(1\leftrightarrow 3). \quad (3.36)$$

where

$$H_5=\frac{1}{8}\left[\frac{\vec r_{13}{}^2\vec r_{02}{}^2}{\vec r_{01}{}^2\vec r_{03}{}^2\vec r_{04}{}^2\vec r_{24}{}^2}-\frac{\vec r_{12}{}^2\vec r_{13}{}^2}{\vec r_{01}{}^2\vec r_{03}{}^2\vec r_{14}{}^2\vec r_{24}{}^2}+\frac{\vec r_{12}{}^2}{\vec r_{01}{}^2\vec r_{14}{}^2\vec r_{24}{}^2}+\frac{\vec r_{13}{}^2}{\vec r_{01}{}^2\vec r_{03}{}^2\vec r_{14}{}^2}+\frac{\vec r_{24}{}^2}{\vec r_{04}{}^2\vec r_{14}{}^2\vec r_{02}{}^2}\right.$$

$$
\begin{aligned}
&+\frac{\vec r_{12}^{\,2}\vec r_{13}^{\,2}}{\vec r_{01}^{\,2}\vec r_{14}^{\,2}\vec r_{34}^{\,2}\vec r_{02}^{\,2}}-\frac{\vec r_{03}^{\,2}\vec r_{12}^{\,2}}{\vec r_{01}^{\,2}\vec r_{04}^{\,2}\vec r_{34}^{\,2}\vec r_{02}^{\,2}}+\frac{\vec r_{14}^{\,2}-\vec r_{34}^{\,2}}{\vec r_{03}^{\,2}\vec r_{04}^{\,2}\vec r_{14}^{\,2}}-\frac{\vec r_{12}^{\,2}}{\vec r_{04}^{\,2}\vec r_{14}^{\,2}\vec r_{24}^{\,2}}+\frac{\vec r_{03}^{\,2}-\vec r_{01}^{\,2}}{\vec r_{01}^{\,2}\vec r_{04}^{\,2}\vec r_{34}^{\,2}}\\
&-\frac{\vec r_{13}^{\,2}\vec r_{24}^{\,2}}{\vec r_{04}^{\,2}\vec r_{14}^{\,2}\vec r_{34}^{\,2}\vec r_{02}^{\,2}}+\frac{\vec r_{12}^{\,2}\vec r_{34}^{\,2}}{\vec r_{03}^{\,2}\vec r_{04}^{\,2}\vec r_{14}^{\,2}\vec r_{24}^{\,2}}-\frac{\vec r_{13}^{\,2}}{\vec r_{01}^{\,2}\vec r_{14}^{\,2}\vec r_{34}^{\,2}}+\frac{\vec r_{13}^{\,2}}{\vec r_{04}^{\,2}\vec r_{14}^{\,2}\vec r_{34}^{\,2}}+\frac{\vec r_{12}^{\,2}}{\vec r_{01}^{\,2}\vec r_{04}^{\,2}\vec r_{02}^{\,2}}\\
&+\frac{\vec r_{03}^{\,2}-\vec r_{23}^{\,2}}{\vec r_{03}^{\,2}\vec r_{04}^{\,2}\vec r_{24}^{\,2}}-\frac{\vec r_{12}^{\,2}}{\vec r_{01}^{\,2}\vec r_{14}^{\,2}\vec r_{02}^{\,2}}+\frac{\vec r_{23}^{\,2}-\vec r_{34}^{\,2}}{\vec r_{04}^{\,2}\vec r_{34}^{\,2}\vec r_{02}^{\,2}}-\frac{\vec r_{02}^{\,2}}{\vec r_{01}^{\,2}\vec r_{04}^{\,2}\vec r_{24}^{\,2}}-\frac{\vec r_{13}^{\,2}}{\vec r_{01}^{\,2}\vec r_{03}^{\,2}\vec r_{04}^{\,2}}\Bigg]\ln\left(\frac{\vec r_{01}^{\,2}}{\vec r_{14}^{\,2}}\right).
\end{aligned}
\tag{3.37}
$$

and

$$
\begin{aligned}
H_6=\frac{1}{8}&\Bigg[\frac{\vec r_{12}^{\,2}\vec r_{13}^{\,2}}{\vec r_{01}^{\,2}\vec r_{03}^{\,2}\vec r_{14}^{\,2}\vec r_{24}^{\,2}}-\frac{\vec r_{13}^{\,2}\vec r_{02}^{\,2}}{\vec r_{01}^{\,2}\vec r_{03}^{\,2}\vec r_{04}^{\,2}\vec r_{24}^{\,2}}-\frac{\vec r_{12}^{\,2}\vec r_{34}^{\,2}}{\vec r_{03}^{\,2}\vec r_{04}^{\,2}\vec r_{14}^{\,2}\vec r_{24}^{\,2}}\\
&-\frac{\vec r_{03}^{\,2}\vec r_{12}^{\,2}}{\vec r_{01}^{\,2}\vec r_{04}^{\,2}\vec r_{34}^{\,2}\vec r_{02}^{\,2}}-\frac{\vec r_{13}^{\,2}\vec r_{24}^{\,2}}{\vec r_{04}^{\,2}\vec r_{14}^{\,2}\vec r_{34}^{\,2}\vec r_{02}^{\,2}}+\frac{\vec r_{12}^{\,2}\vec r_{13}^{\,2}}{\vec r_{01}^{\,2}\vec r_{14}^{\,2}\vec r_{34}^{\,2}\vec r_{02}^{\,2}}+\frac{\vec r_{03}^{\,2}-\vec r_{01}^{\,2}}{\vec r_{01}^{\,2}\vec r_{04}^{\,2}\vec r_{34}^{\,2}}+\frac{\vec r_{13}^{\,2}-\vec r_{01}^{\,2}}{\vec r_{01}^{\,2}\vec r_{03}^{\,2}\vec r_{04}^{\,2}}\\
&+\frac{\vec r_{23}^{\,2}-\vec r_{03}^{\,2}}{\vec r_{03}^{\,2}\vec r_{04}^{\,2}\vec r_{24}^{\,2}}+2\frac{\vec r_{04}^{\,2}-\vec r_{14}^{\,2}-\vec r_{01}^{\,2}}{\vec r_{01}^{\,2}\vec r_{14}^{\,2}\vec r_{04}^{\,2}}+\frac{\vec r_{24}^{\,2}-\vec r_{14}^{\,2}}{\vec r_{04}^{\,2}\vec r_{14}^{\,2}\vec r_{02}^{\,2}}+\frac{\vec r_{02}^{\,2}}{\vec r_{01}^{\,2}\vec r_{04}^{\,2}\vec r_{24}^{\,2}}+\frac{\vec r_{34}^{\,2}}{\vec r_{03}^{\,2}\vec r_{04}^{\,2}\vec r_{14}^{\,2}}\\
&+\frac{\vec r_{12}^{\,2}}{\vec r_{04}^{\,2}\vec r_{14}^{\,2}\vec r_{24}^{\,2}}-\frac{\vec r_{13}^{\,2}}{\vec r_{01}^{\,2}\vec r_{14}^{\,2}\vec r_{34}^{\,2}}+\frac{\vec r_{13}^{\,2}}{\vec r_{04}^{\,2}\vec r_{14}^{\,2}\vec r_{34}^{\,2}}-\frac{\vec r_{12}^{\,2}}{\vec r_{01}^{\,2}\vec r_{14}^{\,2}\vec r_{24}^{\,2}}-\frac{\vec r_{13}^{\,2}}{\vec r_{01}^{\,2}\vec r_{03}^{\,2}\vec r_{14}^{\,2}}\\
&+\frac{\vec r_{12}^{\,2}}{\vec r_{01}^{\,2}\vec r_{04}^{\,2}\vec r_{02}^{\,2}}-\frac{\vec r_{12}^{\,2}}{\vec r_{01}^{\,2}\vec r_{14}^{\,2}\vec r_{02}^{\,2}}+\frac{\vec r_{23}^{\,2}}{\vec r_{04}^{\,2}\vec r_{34}^{\,2}\vec r_{02}^{\,2}}\Bigg]\ln\left(\frac{\vec r_{01}^{\,2}}{\vec r_{14}^{\,2}}\right).
\end{aligned}
\tag{3.38}
$$

The connection of our notations with the notations in [20] is given in the appendix A.

4 Construction of the kernel: gluon part

Taking the contributions of the self-interaction of Wilson lines (3.30) along with the "pair-wise" (3.31) and "triple" (3.36) connected contributions from the previous section one can write the full contribution to the evolution of the 3QWL with two gluons intersecting the shockwave in the form

$$
\langle K_{NLO}\otimes B_{123}\rangle|_{2g}=\langle K_{NLO}\otimes U_1\cdot U_2\cdot U_3\rangle|_{2g}=-\frac{\alpha_s^2}{8\pi^4}\int d\vec r_0 d\vec r_4\ \mathbf{G}, \tag{4.1}
$$

where

$$
\mathbf{G}=\mathbf{G}_{\langle 1\rangle 23}+\mathbf{G}_{1\langle 2\rangle 3}+\mathbf{G}_{12\langle 3\rangle}+\mathbf{G}_{\langle 12\rangle 3}+\mathbf{G}_{1\langle 23\rangle}+\mathbf{G}_{\langle 13\rangle 2}+\mathbf{G}_{\langle 123\rangle}, \tag{4.2}
$$

Here $\langle\dots\rangle$ stands for the connected contribution, i.e. $\mathbf{G}_{\langle 1\rangle 23}$ gives the contribution of the evolution of line 1 (3.30) with lines 2 and 3 being spectators, $\mathbf{G}_{\langle 12\rangle 3}$ corresponds to the connected contribution of the evolution of lines 1 and 2 (3.31) with line 3 being intact, and $\mathbf{G}_{\langle 123\rangle}$ stands for the fully connected contribution (3.36). All the rest can be obtained from them by $1\leftrightarrow 2\leftrightarrow 3$ transformation, i.e. by all 5 possible permutations of $\vec r_1$, $\vec r_2$, and $\vec r_3$, which assumes the permutations of $U_1\equiv U(\vec r_1)$, U_2, and U_3 as well.

There are several useful SU(3) identities, which help to reduce the number of color structures. They are listed in the appendix B. First, we use (B.5) to get rid of the structure

$$
\left(U_0 U_4{}^{\dagger} U_3 U_0{}^{\dagger} U_4\right)\cdot U_1\cdot U_2+(0\leftrightarrow 4) \tag{4.3}
$$

and 2 other structures $\left(U_0U_4{}^\dagger U_3U_0{}^\dagger U_4\right)\cdot U_1\cdot U_2$ goes into after the $1\leftrightarrow 2\leftrightarrow 3$ transformations. Again, $i\leftrightarrow j$ stands for the the permutation. It means that we have to change $\vec{r}_i\leftrightarrow\vec{r}_j$ and $U_i\leftrightarrow U_j$. Second, we use (B.6) to eliminate 6 such contributions antisymmetric w.r.t. $0\leftrightarrow 4$ exchange as

$$\left(U_2U_0{}^\dagger U_1U_4{}^\dagger U_0+U_0U_4{}^\dagger U_1U_0{}^\dagger U_2\right)\cdot U_3\cdot U_4-(4\leftrightarrow 0). \tag{4.4}$$

Next we use (B.7) to express 6 structures like

$$\left(U_2U_4{}^\dagger U_1U_0{}^\dagger U_4+U_4U_0{}^\dagger U_1U_4{}^\dagger U_2\right)\cdot U_0\cdot U_3 \tag{4.5}$$

and their symmetric counterparts w.r.t. $0\leftrightarrow 4$ exchange through other structures. After that we can cancel 3 structures of the form

$$U_2\cdot U_3\cdot U_4\ tr\left(U_0{}^\dagger U_1\right)tr\left(U_0U_4{}^\dagger\right)-U_2\cdot U_3\cdot U_0\ tr\left(U_4{}^\dagger U_1\right)tr\left(U_4U_0{}^\dagger\right). \tag{4.6}$$

using (B.8), and, by means of (B.9), discard the 3 nonconformal terms proportional to

$$tr\left(U_0U_4{}^\dagger\right)\left(U_1U_0{}^\dagger U_2+U_2U_0{}^\dagger U_1\right)\cdot U_3\cdot U_4-(4\leftrightarrow 0) \tag{4.7}$$

and the 2 structures they go into after the $1\leftrightarrow 2\leftrightarrow 3$ transformations. Finally, we get

$$\begin{aligned}\mathbf{G}=&\{(L_{12}+\tilde{L}_{12})\left(U_0U_4{}^\dagger U_2\right)\cdot\left(U_1U_0{}^\dagger U_4\right)\cdot U_3+L_{12}tr\left(U_0U_4{}^\dagger\right)\left(U_1U_0{}^\dagger U_2\right)\cdot U_3\cdot U_4\\&+(M_{13}-M_{12}-M_{23}+M_2^{13})\left[\left(U_0U_4{}^\dagger U_3\right)\cdot\left(U_2U_0{}^\dagger U_1\right)+\left(U_1U_0{}^\dagger U_2\right)\cdot\left(U_3U_4{}^\dagger U_0\right)\right]\cdot U_4\\&+(\text{all 5 permutations}\,1\leftrightarrow 2\leftrightarrow 3)\}+(0\leftrightarrow 4).\end{aligned} \tag{4.8}$$

Here

$$L_{ij}\equiv L(\vec{r}_i,\vec{r}_j),\quad \tilde{L}_{ij}\equiv\tilde{L}(\vec{r}_i,\vec{r}_j),\quad M_{ij}\equiv M(\vec{r}_i,\vec{r}_j),\quad M_i^{jk}\equiv M(\vec{r}_i,\vec{r}_j,\vec{r}_k), \tag{4.9}$$

$$\begin{aligned}L_{12}&=H_3+H_4-\frac{1}{2}G_3+(1\leftrightarrow 2)\\&=\left[\frac{1}{\vec{r}_{01}{}^2\vec{r}_{24}{}^2-\vec{r}_{02}{}^2\vec{r}_{14}{}^2}\left(-\frac{\vec{r}_{12}{}^4}{8}\left(\frac{1}{\vec{r}_{01}{}^2\vec{r}_{24}{}^2}+\frac{1}{\vec{r}_{02}{}^2\vec{r}_{14}{}^2}\right)+\frac{\vec{r}_{12}{}^2}{\vec{r}_{04}{}^2}-\frac{\vec{r}_{02}{}^2\vec{r}_{14}{}^2+\vec{r}_{01}{}^2\vec{r}_{24}{}^2}{4\vec{r}_{04}{}^4}\right)\right.\\&\quad\left.+\frac{\vec{r}_{12}{}^2}{8\vec{r}_{04}{}^2}\left(\frac{1}{\vec{r}_{02}{}^2\vec{r}_{14}{}^2}-\frac{1}{\vec{r}_{01}{}^2\vec{r}_{24}{}^2}\right)\right]\ln\left(\frac{\vec{r}_{01}{}^2\vec{r}_{24}{}^2}{\vec{r}_{14}{}^2\vec{r}_{02}{}^2}\right)+\frac{1}{2\vec{r}_{04}{}^4}.\end{aligned} \tag{4.10}$$

Again, here $i\leftrightarrow j$ stands for the the permutation. It means that we have to change $\vec{r}_i\leftrightarrow\vec{r}_j$ and $U_i\leftrightarrow U_j$.

$$\begin{aligned}\tilde{L}_{12}&=H_1+H_2-\frac{1}{2}G_9-(1\leftrightarrow 2)\\&=\frac{\vec{r}_{12}{}^2}{8}\left[\frac{\vec{r}_{12}{}^2}{\vec{r}_{01}{}^2\vec{r}_{02}{}^2\vec{r}_{14}{}^2\vec{r}_{24}{}^2}-\frac{1}{\vec{r}_{01}{}^2\vec{r}_{04}{}^2\vec{r}_{24}{}^2}-\frac{1}{\vec{r}_{02}{}^2\vec{r}_{04}{}^2\vec{r}_{14}{}^2}\right]\ln\left(\frac{\vec{r}_{01}{}^2\vec{r}_{24}{}^2}{\vec{r}_{14}{}^2\vec{r}_{02}{}^2}\right).\end{aligned} \tag{4.11}$$

$$M_{12}=\frac{1}{2}\left\{H_1+H_2-\frac{1}{2}G_9+(1\leftrightarrow 2)\right\}$$

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$$= \frac{\vec{r}_{12}{}^2}{16}\left[\frac{\vec{r}_{12}{}^2}{\vec{r}_{01}{}^2\vec{r}_{02}{}^2\vec{r}_{14}{}^2\vec{r}_{24}{}^2} - \frac{1}{\vec{r}_{01}{}^2\vec{r}_{04}{}^2\vec{r}_{24}{}^2} - \frac{1}{\vec{r}_{02}{}^2\vec{r}_{04}{}^2\vec{r}_{14}{}^2}\right]\ln\left(\frac{\vec{r}_{01}{}^2\vec{r}_{02}{}^2}{\vec{r}_{14}{}^2\vec{r}_{24}{}^2}\right). \quad (4.12)$$

$$\begin{aligned} M_2^{13} &= H_2(1 \leftrightarrow 2) + H_2(1 \to 2 \to 3 \to 1) + H_3(1 \to 2 \to 3 \to 1) \\ &\quad - H_3(1 \leftrightarrow 2) + H_5(1 \leftrightarrow 2) + H_6(1 \leftrightarrow 2) - G_9(1 \leftrightarrow 2) \quad (4.13) \\ &= \frac{1}{4}\left(\frac{\vec{r}_{12}{}^2\vec{r}_{23}{}^2}{\vec{r}_{01}{}^2\vec{r}_{02}{}^2\vec{r}_{24}{}^2\vec{r}_{34}{}^2} - \frac{\vec{r}_{14}{}^2\vec{r}_{23}{}^2}{\vec{r}_{01}{}^2\vec{r}_{04}{}^2\vec{r}_{24}{}^2\vec{r}_{34}{}^2} - \frac{\vec{r}_{03}{}^2\vec{r}_{12}{}^2}{\vec{r}_{01}{}^2\vec{r}_{02}{}^2\vec{r}_{04}{}^2\vec{r}_{34}{}^2} + \frac{\vec{r}_{13}{}^2}{\vec{r}_{01}{}^2\vec{r}_{04}{}^2\vec{r}_{34}{}^2}\right)\ln\left(\frac{\vec{r}_{02}{}^2}{\vec{r}_{24}{}^2}\right). \end{aligned}$$

Here $H_i(1 \to 2 \to 3 \to 1) \equiv H_i|_{\vec{r}_1\to\vec{r}_2, \vec{r}_2\to\vec{r}_3, \vec{r}_3\to\vec{r}_1}$. In the dipole limit, i.e. when the coordinates of 2 quarks in the 3QWL coincide these functions obey the identities

$$M_2^{13}|_{\vec{r}_1\to\vec{r}_3} = \frac{\vec{r}_{23}{}^2}{4}\left(\frac{\vec{r}_{23}{}^2}{\vec{r}_{03}{}^2\vec{r}_{02}{}^2\vec{r}_{24}{}^2\vec{r}_{34}{}^2} - \frac{1}{\vec{r}_{03}{}^2\vec{r}_{04}{}^2\vec{r}_{24}{}^2} - \frac{1}{\vec{r}_{02}{}^2\vec{r}_{04}{}^2\vec{r}_{34}{}^2}\right)\ln\left(\frac{\vec{r}_{02}{}^2}{\vec{r}_{24}{}^2}\right). \quad (4.14)$$

$$\left(M_{13} - M_{12} - M_{23} + M_2^{13}\right)|_{\vec{r}_1\to\vec{r}_3} = \tilde{L}_{23}. \quad (4.15)$$

$$\left(M_{13} - M_{12} - M_{23} + M_2^{13}\right)|_{\vec{r}_1\to\vec{r}_2} = \left(M_{13} - M_{12} - M_{23} + M_2^{13}\right)|_{\vec{r}_3\to\vec{r}_2} = 0. \quad (4.16)$$

Using these identities and (B.1) with $l = 3$, we get the dipole result

$$\begin{aligned} \mathbf{G}|_{\vec{r}_1\to\vec{r}_3} =& 4(L_{32} + \tilde{L}_{32})tr\left(U_0{}^\dagger U_4\right)tr\left(U_3{}^\dagger U_0\right)tr\left(U_4{}^\dagger U_2\right) \\ &- 4L_{32}tr\left(U_0{}^\dagger U_2U_4{}^\dagger U_0U_3{}^\dagger U_4\right) + (0 \leftrightarrow 4). \end{aligned} \quad (4.17)$$

This expression is twice the corresponding part of the BK kernel for $tr(U_2U_3^\dagger)$.

The only UV divergent term in (4.8) is the term proportional to L_{12}. This term has the same coordinate structure as the corresponding term in the dipole kernel. Therefore we can do the same subtraction as in the dipole case. Using (B.3), we get

$$\begin{aligned} &\left(U_0U_4{}^\dagger U_2\right)\cdot\left(U_1U_0{}^\dagger U_4\right)\cdot U_3 + tr\left(U_0U_4{}^\dagger\right)\left(U_1U_0{}^\dagger U_2\right)\cdot U_3\cdot U_4 + (2 \leftrightarrow 1)|_{\vec{r}_0\to\vec{r}_4} \\ &= 3[tr\left(U_1U_4{}^\dagger\right)U_2\cdot U_3 + tr\left(U_2U_4{}^\dagger\right)U_1\cdot U_3 - tr\left(U_3U_4{}^\dagger\right)U_1\cdot U_2]\cdot U_4 - U_1\cdot U_2\cdot U_3 \\ &= \frac{3}{2}[B_{144}B_{234} + B_{244}B_{134} - B_{344}B_{124}] - B_{123}, \end{aligned} \quad (4.18)$$

Therefore we can separate the result into the UV finite and divergent parts

$$\langle K_{NLO} \otimes B_{123}\rangle|_{2g} = -\frac{\alpha_s^2}{8\pi^4}\int d\vec{r}_0 d\vec{r}_4\ \mathbf{G}_{\text{finite}} - \frac{\alpha_s^2}{8\pi^3}\int d\vec{r}_0\ \mathbf{G}_{UV}, \quad (4.19)$$

where

$$\begin{aligned} \mathbf{G}_{\text{finite}} =& \{\tilde{L}_{12}\left(U_0U_4{}^\dagger U_2\right)\cdot\left(U_1U_0{}^\dagger U_4\right)\cdot U_3 \\ &+ L_{12}\left[\left(U_0U_4{}^\dagger U_2\right)\cdot\left(U_1U_0{}^\dagger U_4\right)\cdot U_3 + tr\left(U_0U_4{}^\dagger\right)\left(U_1U_0{}^\dagger U_2\right)\cdot U_3\cdot U_4\right. \\ &\left.- \frac{3}{4}[B_{144}B_{234} + B_{244}B_{134} - B_{344}B_{124}] + \frac{1}{2}B_{123}\right] \\ &+ (M_{13} - M_{12} - M_{23} + M_2^{13})\left[\left(U_0U_4{}^\dagger U_3\right)\cdot\left(U_2U_0{}^\dagger U_1\right) + \left(U_1U_0{}^\dagger U_2\right)\cdot\left(U_3U_4{}^\dagger U_0\right)\right]\cdot U_4 \end{aligned}$$

$$+\,(\text{all 5 permutations}\, 1\leftrightarrow 2\leftrightarrow 3)\}+(0\leftrightarrow 4). \tag{4.20}$$

And $\mathbf{G}_{UV}$ is included into the term describing the contribution to the kernel with one gluon crossing the shockwave [20], which is proportional to the first coefficient of the β-function. We take this contribution from (3.3) and (3.10). For the pure gluon field $\beta_0=\frac{11}{3}$

$$\langle \tilde{K}_{NLO}\otimes B_{123}\rangle|^{\beta}_{1g}=\left[-\frac{\alpha_s^2}{(2\pi)^3}\frac{11}{2}\int d\vec{r}_0\left[\ln\left(\frac{\vec{r}_{01}^{\,2}}{\vec{r}_{02}^{\,2}}\right)\left(\frac{1}{\vec{r}_{02}^{\,2}}-\frac{1}{\vec{r}_{01}^{\,2}}\right)-\frac{\vec{r}_{12}^{\,2}}{\vec{r}_{01}^{\,2}\vec{r}_{02}^{\,2}}\ln\left(\frac{\vec{r}_{12}^{\,2}}{\tilde{\mu}_g^2}\right)\right.\right.$$
$$\left.+\frac{1}{\vec{r}_{02}^{\,2}}\ln\left(\frac{\vec{r}_{02}^{\,2}}{\tilde{\mu}_g^2}\right)+\frac{1}{\vec{r}_{01}^{\,2}}\ln\left(\frac{\vec{r}_{01}^{\,2}}{\tilde{\mu}_g^2}\right)\right]$$
$$\left.\times\left(U_0\cdot U_3\cdot(U_2U_0^\dagger U_1)+U_0\cdot U_3\cdot(U_1U_0^\dagger U_2)+\frac{2}{3}U_1\cdot U_2\cdot U_3\right)+(1\leftrightarrow 3)+(2\leftrightarrow 3)\right]$$
$$+\left[\frac{\alpha_s^2}{(2\pi)^3}11\int\frac{d\vec{r}_0}{\vec{r}_{01}^{\,2}}\ln\left(\frac{\vec{r}_{01}^{\,2}}{\tilde{\mu}_g^2}\right)\left(U_0\cdot U_2\cdot U_3 tr(U_1U_0^\dagger)-\frac{1}{3}U_1\cdot U_2\cdot U_3\right)+(1\leftrightarrow 3)+(1\leftrightarrow 2)\right], \tag{4.21}$$

where

$$\frac{11}{3}\ln\frac{1}{\tilde{\mu}_g^2}=\frac{11}{3}\ln\left(\frac{\mu^2}{4e^{2\psi(1)}}\right)+\frac{67}{9}-\frac{\pi^2}{3}, \tag{4.22}$$

where the $\overline{MS}$ scheme is used. After some algebra one obtains

$$\langle \tilde{K}_{NLO}\otimes B_{123}\rangle|^{\beta}_{1g}=-\frac{\alpha_s^2}{(2\pi)^3}\frac{11}{6}\int d\vec{r}_0\left[\ln\left(\frac{\vec{r}_{01}^{\,2}}{\vec{r}_{02}^{\,2}}\right)\left(\frac{1}{\vec{r}_{02}^{\,2}}-\frac{1}{\vec{r}_{01}^{\,2}}\right)-\frac{\vec{r}_{12}^{\,2}}{\vec{r}_{01}^{\,2}\vec{r}_{02}^{\,2}}\ln\left(\frac{\vec{r}_{12}^{\,2}}{\tilde{\mu}_g^2}\right)\right] \tag{4.23}$$
$$\times\left(\frac{3}{2}(B_{100}B_{230}+B_{200}B_{130}-B_{300}B_{210})-B_{123}\right)+(1\leftrightarrow 3)+(2\leftrightarrow 3).$$

It also has the correct dipole limit

$$\langle \tilde{K}_{NLO}\otimes B_{122}\rangle|^{\beta}_{1g}=-\frac{\alpha_s^2}{(2\pi)^3}\frac{11}{3}\int d\vec{r}_0\left[\ln\left(\frac{\vec{r}_{01}^{\,2}}{\vec{r}_{02}^{\,2}}\right)\left(\frac{1}{\vec{r}_{02}^{\,2}}-\frac{1}{\vec{r}_{01}^{\,2}}\right)-\frac{\vec{r}_{12}^{\,2}}{\vec{r}_{01}^{\,2}\vec{r}_{02}^{\,2}}\ln\left(\frac{\vec{r}_{12}^{\,2}}{\tilde{\mu}_g^2}\right)\right]$$
$$\times\left(\frac{3}{2}B_{100}B_{220}-B_{122}\right). \tag{4.24}$$

and matches the BFKL kernel [33].

There are also diagrams with one gluon intersecting the shockwave which are not proportional to the β-function. They can be taken from (3.3)–(3.11). However, they were already calculated in eq. (5.27) in [26]:

$$\langle \tilde{K}_{NLO}\otimes B_{123}\rangle|_{1g}=\frac{\alpha_s^2}{(2\pi)^3}\int d\vec{r}_0\left[\frac{(\vec{r}_{10}\vec{r}_{20})}{\vec{r}_{10}^{\,2}\vec{r}_{20}^{\,2}}-\frac{(\vec{r}_{30}\vec{r}_{20})}{\vec{r}_{30}^{\,2}\vec{r}_{20}^{\,2}}\right]\ln\frac{\vec{r}_{30}^{\,2}}{\vec{r}_{31}^{\,2}}\ln\frac{\vec{r}_{10}^{\,2}}{\vec{r}_{31}^{\,2}}(B_{100}B_{320}-B_{300}B_{210})$$
$$+\frac{\alpha_s^2}{(2\pi)^3}\int d\vec{r}_0\left[\frac{1}{\vec{r}_{10}^{\,2}}-\frac{(\vec{r}_{30}\vec{r}_{10})}{\vec{r}_{30}^{\,2}\vec{r}_{10}^{\,2}}\right]\ln\frac{\vec{r}_{30}^{\,2}}{\vec{r}_{31}^{\,2}}\ln\frac{\vec{r}_{10}^{\,2}}{\vec{r}_{31}^{\,2}}\left(B_{123}-\frac{1}{2}[3B_{100}B_{320}+B_{300}B_{120}-B_{200}B_{130}]\right)$$
$$+\frac{\alpha_s^2}{(2\pi)^3}\int d\vec{r}_0\left[\frac{(\vec{r}_{10}\vec{r}_{30})}{\vec{r}_{10}^{\,2}\vec{r}_{30}^{\,2}}-\frac{1}{\vec{r}_{30}^{\,2}}\right]\ln\frac{\vec{r}_{30}^{\,2}}{\vec{r}_{31}^{\,2}}\ln\frac{\vec{r}_{10}^{\,2}}{\vec{r}_{31}^{\,2}}\left(\frac{1}{2}[3B_{300}B_{120}+B_{100}B_{320}-B_{200}B_{130}]-B_{123}\right)$$
$$+(2\leftrightarrow 1)+(2\leftrightarrow 3). \tag{4.25}$$

This term has the correct dipole limit (see (5.28) in [26]).

Thus, the real part of the whole kernel reads

$$\langle K_{NLO} \otimes B_{123} \rangle|_{\rm real} = -\frac{\alpha_s^2}{8\pi^4} \int d\vec{r}_0 d\vec{r}_4 \, \mathbf{G}_{\rm finite} - \frac{\alpha_s^2}{8\pi^3} \int d\vec{r}_0 \, \mathbf{G}_{\rm real}, \tag{4.26}$$

where

$$\begin{aligned}
\mathbf{G}_{\rm real} = & -\frac{1}{2}\left[\frac{(\vec{r}_{10}\vec{r}_{20})}{\vec{r}_{10}^{\,2}\vec{r}_{20}^{\,2}} - \frac{(\vec{r}_{30}\vec{r}_{20})}{\vec{r}_{30}^{\,2}\vec{r}_{20}^{\,2}}\right] \ln\frac{\vec{r}_{30}^{\,2}}{\vec{r}_{31}^{\,2}} \ln\frac{\vec{r}_{10}^{\,2}}{\vec{r}_{31}^{\,2}} \left(B_{100}B_{320} - B_{300}B_{210}\right) \\
& - \left[\frac{1}{\vec{r}_{10}^{\,2}} - \frac{(\vec{r}_{30}\vec{r}_{10})}{\vec{r}_{30}^{\,2}\vec{r}_{10}^{\,2}}\right] \ln\frac{\vec{r}_{30}^{\,2}}{\vec{r}_{31}^{\,2}} \ln\frac{\vec{r}_{10}^{\,2}}{\vec{r}_{31}^{\,2}} \left(B_{123} - \frac{1}{2}\left[3B_{100}B_{320} + B_{300}B_{120} - B_{200}B_{130}\right]\right) \\
& + \frac{11}{12}\left[\ln\left(\frac{\vec{r}_{01}^{\,2}}{\vec{r}_{02}^{\,2}}\right)\left(\frac{1}{\vec{r}_{02}^{\,2}} - \frac{1}{\vec{r}_{01}^{\,2}}\right) - \frac{\vec{r}_{12}^{\,2}}{\vec{r}_{01}^{\,2}\vec{r}_{02}^{\,2}} \ln\left(\frac{\vec{r}_{12}^{\,2}}{\tilde{\mu}_g^2}\right)\right] \\
& \times \left(\frac{3}{2}(B_{100}B_{230} + B_{200}B_{130} - B_{300}B_{210}) - B_{123}\right) \\
& + (\text{all 5 permutations}\, 1 \leftrightarrow 2 \leftrightarrow 3),
\end{aligned} \tag{4.27}$$

and $\mathbf{G}_{\rm finite}$ is defined in (4.20). If we put $\vec{r}_2 = \vec{r}_3$ here, we get the dipole result (see (100) in [18])

$$\begin{aligned}
\mathbf{G}_{\rm real}|_{\vec{r}_2=\vec{r}_3} = & \left\{\frac{11}{3}\left[\ln\left(\frac{\vec{r}_{01}^{\,2}}{\vec{r}_{02}^{\,2}}\right)\left(\frac{1}{\vec{r}_{02}^{\,2}} - \frac{1}{\vec{r}_{01}^{\,2}}\right) - \frac{\vec{r}_{12}^{\,2}}{\vec{r}_{01}^{\,2}\vec{r}_{02}^{\,2}} \ln\left(\frac{\vec{r}_{12}^{\,2}}{\tilde{\mu}_g^2}\right)\right] + 2\frac{\vec{r}_{12}^{\,2}}{\vec{r}_{20}^{\,2}\vec{r}_{10}^{\,2}} \ln\frac{\vec{r}_{20}^{\,2}}{\vec{r}_{21}^{\,2}} \ln\frac{\vec{r}_{10}^{\,2}}{\vec{r}_{21}^{\,2}}\right\} \\
& \times \left(\frac{3}{2}B_{100}B_{220} - B_{122}\right).
\end{aligned} \tag{4.28}$$

Finally, from the condition that the kernel must vanish without the shockwave (if all the $B = 6$) and that the virtual contribution is proportional to B_{123}, we get the total kernel

$$\langle K_{NLO} \otimes B_{123} \rangle = -\frac{\alpha_s^2}{8\pi^4} \int d\vec{r}_0 d\vec{r}_4 \, \mathbf{G}_{\rm finite} - \frac{\alpha_s^2}{8\pi^3} \int d\vec{r}_0 \, \mathbf{G}', \tag{4.29}$$

where $\mathbf{G}_{\rm finite}$ is defined in (4.20) and

$$\begin{aligned}
\mathbf{G}' = & \frac{1}{2}\left[\frac{\vec{r}_{13}^{\,2}}{\vec{r}_{10}^{\,2}\vec{r}_{30}^{\,2}} - \frac{\vec{r}_{32}^{\,2}}{\vec{r}_{30}^{\,2}\vec{r}_{20}^{\,2}}\right] \ln\frac{\vec{r}_{20}^{\,2}}{\vec{r}_{21}^{\,2}} \ln\frac{\vec{r}_{10}^{\,2}}{\vec{r}_{21}^{\,2}} \left(B_{100}B_{320} - B_{200}B_{310}\right) \\
& - \frac{\vec{r}_{12}^{\,2}}{\vec{r}_{10}^{\,2}\vec{r}_{20}^{\,2}} \ln\frac{\vec{r}_{10}^{\,2}}{\vec{r}_{12}^{\,2}} \ln\frac{\vec{r}_{20}^{\,2}}{\vec{r}_{12}^{\,2}} \left(9B_{123} - \frac{1}{2}\left[2\left(B_{100}B_{320} + B_{200}B_{130}\right) - B_{300}B_{120}\right]\right) \\
& + \frac{11}{6}\left[\ln\left(\frac{\vec{r}_{01}^{\,2}}{\vec{r}_{02}^{\,2}}\right)\left(\frac{1}{\vec{r}_{02}^{\,2}} - \frac{1}{\vec{r}_{01}^{\,2}}\right) - \frac{\vec{r}_{12}^{\,2}}{\vec{r}_{01}^{\,2}\vec{r}_{02}^{\,2}} \ln\left(\frac{\vec{r}_{12}^{\,2}}{\tilde{\mu}_g^2}\right)\right] \\
& \times \left(\frac{3}{2}(B_{100}B_{230} + B_{200}B_{130} - B_{300}B_{210}) - 9B_{123}\right) + (1 \leftrightarrow 3) + (2 \leftrightarrow 3).
\end{aligned} \tag{4.30}$$

It differs from (4.27) by the coefficients of B_{123}'s which turn into 9's to accommodate the condition that the kernel must vanish without the shockwave.

5 Construction of the kernel: quark part

One can take the quark contribution to the NLO evolution of 3QWL from (3.3)–(3.4). The contribution with 2 quarks intersecting the shockwave without subtraction reads

$$\langle K_{NLO} \otimes B_{123} \rangle|_{2g}^{q} = -\frac{\alpha_s^2 n_f}{8\pi^4} \int d\vec{r}_0 d\vec{r}_4 \, \mathbf{G}^q. \tag{5.1}$$

Here

$$\begin{aligned}
\mathbf{G}^q = & \Bigg[\bigg((U_1 U_0{}^\dagger U_4 + U_4 U_0{}^\dagger U_1) \cdot U_2 \cdot U_3 - 3 U_2 \cdot U_3 \cdot U_4 tr(U_0{}^\dagger U_1) - \frac{1}{3} U_1 \cdot U_2 \cdot U_3 tr(U_0{}^\dagger U_4) \bigg) \\
& \times \frac{2}{3} \frac{1}{\vec{r}_{04}{}^4} \left\{ \frac{(\vec{r}_{14}\vec{r}_{01})}{\vec{r}_{14}{}^2 - \vec{r}_{01}{}^2} \ln \left(\frac{\vec{r}_{14}{}^2}{\vec{r}_{01}{}^2} \right) + 1 \right\} + (1 \leftrightarrow 2) + (1 \leftrightarrow 3) \Bigg] \\
& + \Bigg[\bigg(\frac{1}{3} (U_1 U_0{}^\dagger U_4 + U_4 U_0{}^\dagger U_1) \cdot U_2 \cdot U_3 - \frac{1}{9} U_1 \cdot U_2 \cdot U_3 tr(U_0{}^\dagger U_4) + (U_1 U_0{}^\dagger U_2) \cdot U_3 \cdot U_4 \\
& + \, (1 \leftrightarrow 2) \bigg) \left(\frac{1}{\vec{r}_{04}{}^4} \left\{ \frac{(\vec{r}_{14}\vec{r}_{01})}{\vec{r}_{14}{}^2 - \vec{r}_{01}{}^2} \ln \left(\frac{\vec{r}_{14}{}^2}{\vec{r}_{01}{}^2} \right) + 1 \right\} + \frac{L_{12}^q}{2} + (1 \leftrightarrow 2) \right) \\
& + \, (1 \leftrightarrow 3) + (2 \leftrightarrow 3) \Bigg],
\end{aligned} \tag{5.2}$$

where L_{12}^q is defined in (3.6). Using identity (B.15) one can see that this contribution is conformally invariant, indeed

$$\begin{aligned}
\mathbf{G}^q = & \frac{1}{2} \Bigg\{ \bigg(\frac{1}{3} (U_1 U_0{}^\dagger U_4 + U_4 U_0{}^\dagger U_1) \cdot U_2 \cdot U_3 - \frac{1}{9} U_1 \cdot U_2 \cdot U_3 tr(U_0{}^\dagger U_4) \\
& + \, (U_1 U_0{}^\dagger U_2) \cdot U_3 \cdot U_4 + (1 \leftrightarrow 2) \bigg) + (0 \leftrightarrow 4) \Bigg\} L_{12}^q + (1 \leftrightarrow 3) + (2 \leftrightarrow 3).
\end{aligned} \tag{5.3}$$

In the dipole limit $\vec{r}_3 \to \vec{r}_2$, one has

$$\begin{aligned}
\mathbf{G}^q|_{\vec{r}_3 \to \vec{r}_2} = & \frac{2}{3} L_{12}^q \left\{ \frac{1}{3} tr(U_0{}^\dagger U_4) tr(U_2{}^\dagger U_1) + 3 tr(U_0{}^\dagger U_1) tr(U_2{}^\dagger U_4) \right. \\
& \left. - \, tr(U_0{}^\dagger U_1 U_2{}^\dagger U_4) - tr(U_0{}^\dagger U_4 U_2{}^\dagger U_1) + (0 \leftrightarrow 4) \right\},
\end{aligned} \tag{5.4}$$

which is twice the corresponding part of the BK kernel [16], and therefore one can do the same subtraction as in the BK case

$$\mathbf{G}^q = \mathbf{G}^q_{\text{finite}} + \mathbf{G}^q_{UV}, \tag{5.5}$$

where

$$\begin{aligned}
\mathbf{G}^q_{\text{finite}} = & \frac{1}{2} \Bigg\{ \bigg(\frac{1}{3} (U_1 U_0{}^\dagger U_4 + U_4 U_0{}^\dagger U_1) \cdot U_2 \cdot U_3 - \frac{1}{9} B_{123} tr(U_0{}^\dagger U_4) + (U_1 U_0{}^\dagger U_2) \cdot U_3 \cdot U_4 \\
& + \, \frac{1}{6} B_{123} - \frac{1}{4} (B_{013} B_{002} + B_{001} B_{023} - B_{012} B_{003}) + (1 \leftrightarrow 2) \bigg) + (0 \leftrightarrow 4) \Bigg\} L_{12}^q \\
& + (1 \leftrightarrow 3) + (2 \leftrightarrow 3).
\end{aligned} \tag{5.6}$$

Similarly to the gluon case one can take $\mathbf{G}^q_{UV}$ contribution with one gluon crossing the shockwave from the terms proportional to β-function in (3.3) and (3.10). One can restore this contribution from the gluon part via the substitutions

$$\frac{11}{3} \to \beta = \left(\frac{11}{3} - \frac{2}{3}\frac{n_f}{3}\right), \tag{5.7}$$

$$\frac{11}{3}\ln\frac{1}{\tilde{\mu}_g^2} = \frac{11}{3}\ln\left(\frac{\mu^2}{4e^{2\psi(1)}}\right) + \frac{67}{9} - \frac{\pi^2}{3} \to \beta\ln\frac{1}{\tilde{\mu}^2}, \tag{5.8}$$

where

$$\beta\ln\frac{1}{\tilde{\mu}^2} = \left(\frac{11}{3} - \frac{2}{3}\frac{n_f}{3}\right)\ln\left(\frac{\mu^2}{4e^{2\psi(1)}}\right) + \frac{67}{9} - \frac{\pi^2}{3} - \frac{10}{9}\frac{n_f}{3}. \tag{5.9}$$

As a result, the full kernel in QCD reads

$$\langle K_{NLO} \otimes B_{123}\rangle = -\frac{\alpha_s^2}{8\pi^4}\int d\vec{r}_0 d\vec{r}_4\ \left(\mathbf{G}_{\text{finite}} + n_f \mathbf{G}^q_{\text{finite}}\right) - \frac{\alpha_s^2}{8\pi^3}\int d\vec{r}_0\ \mathbf{G}^{q\prime}, \tag{5.10}$$

where n_f is the number of quark flavours,

$$\begin{aligned}\mathbf{G}^{q\prime} =& \frac{1}{2}\left[\frac{\vec{r}_{13}^{\,2}}{\vec{r}_{10}^{\,2}\vec{r}_{30}^{\,2}} - \frac{\vec{r}_{32}^{\,2}}{\vec{r}_{30}^{\,2}\vec{r}_{20}^{\,2}}\right]\ln\frac{\vec{r}_{20}^{\,2}}{\vec{r}_{21}^{\,2}}\ln\frac{\vec{r}_{10}^{\,2}}{\vec{r}_{21}^{\,2}}\left(B_{100}B_{320} - B_{200}B_{310}\right)\\ &- \frac{\vec{r}_{12}^{\,2}}{\vec{r}_{10}^{\,2}\vec{r}_{20}^{\,2}}\ln\frac{\vec{r}_{10}^{\,2}}{\vec{r}_{12}^{\,2}}\ln\frac{\vec{r}_{20}^{\,2}}{\vec{r}_{12}^{\,2}}\left(9B_{123} - \frac{1}{2}\left[2\left(B_{100}B_{320} + B_{200}B_{130}\right) - B_{300}B_{120}\right]\right)\\ &+ \frac{\beta}{2}\left[\ln\left(\frac{\vec{r}_{01}^{\,2}}{\vec{r}_{02}^{\,2}}\right)\left(\frac{1}{\vec{r}_{02}^{\,2}} - \frac{1}{\vec{r}_{01}^{\,2}}\right) - \frac{\vec{r}_{12}^{\,2}}{\vec{r}_{01}^{\,2}\vec{r}_{02}^{\,2}}\ln\left(\frac{\vec{r}_{12}^{\,2}}{\tilde{\mu}^2}\right)\right]\\ &\times\left(\frac{3}{2}(B_{100}B_{230} + B_{200}B_{130} - B_{300}B_{210}) - 9B_{123}\right) + (1\leftrightarrow 3) + (2\leftrightarrow 3). \end{aligned}\tag{5.11}$$

Here $\mathbf{G}_{\text{finite}}$ is defined in (4.20) and $\mathbf{G}^q_{\text{finite}}$ is defined in (5.6).

6 Evolution equation for composite 3QWL operator

In this section we consider only the gluon part of the kernel since the quark one is quasi-conformal. To construct composite conformal operators we will use the prescription [19] (see also ref. [34])

$$O^{\text{conf}} = O + \frac{1}{2}\frac{\partial O}{\partial\eta}\Bigg|_{\frac{\vec{r}_{mn}^{\,2}}{\vec{r}_{im}^{\,2}\vec{r}_{in}^{\,2}} \to \frac{\vec{r}_{mn}^{\,2}}{\vec{r}_{im}^{\,2}\vec{r}_{in}^{\,2}}\ln\left(\frac{\vec{r}_{mn}^{\,2}a}{\vec{r}_{im}^{\,2}\vec{r}_{in}^{\,2}}\right)}, \tag{6.1}$$

where a is an arbitrary constant. For the conformal 3QWL operator we have the following ansatz

$$\begin{aligned}B_{123}^{\text{conf}} =& B_{123} + \frac{\alpha_s 3}{8\pi^2}\int d\vec{r}_4\left[\frac{\vec{r}_{12}^{\,2}}{\vec{r}_{41}^{\,2}\vec{r}_{42}^{\,2}}\ln\left(\frac{a\vec{r}_{12}^{\,2}}{\vec{r}_{41}^{\,2}\vec{r}_{42}^{\,2}}\right)\right.\\ &\left.\times\left(-B_{123} + \frac{1}{6}(B_{144}B_{324} + B_{244}B_{314} - B_{344}B_{214})\right) + (1\leftrightarrow 3) + (2\leftrightarrow 3)\right]. \end{aligned}\tag{6.2}$$

If we put $\vec r_2 = \vec r_3$, then

$$B^{\rm conf}_{122} = B_{122} + \frac{\alpha_s 3}{4\pi^2}\int d\vec r_4 \frac{\vec r^{\,2}_{12}}{\vec r^{\,2}_{41}\vec r^{\,2}_{42}} \ln\left(\frac{a\vec r^{\,2}_{12}}{\vec r^{\,2}_{41}\vec r^{\,2}_{42}}\right)\Big(- B_{122} + \frac{1}{6}B_{144}B_{224}\Big),$$

or

$$tr(U_1U_2^\dagger)^{\rm conf} = tr(U_1U_2^\dagger) + \frac{\alpha_s}{4\pi^2}\int d\vec r_4 \frac{\vec r^{\,2}_{12}}{\vec r^{\,2}_{41}\vec r^{\,2}_{42}} \ln\left(\frac{a\vec r^{\,2}_{12}}{\vec r^{\,2}_{41}\vec r^{\,2}_{42}}\right)(tr(U_1U_4^\dagger)tr(U_4U_2^\dagger) - 3tr(U_1U_2^\dagger)), \tag{6.3}$$

which is exactly the composite dipole operator of [19]. Using the SU(3) identity (B.3) one can rewrite (6.2) as

$$\begin{aligned} B^{\rm conf}_{123} = B_{123} + \frac{\alpha_s}{8\pi^2}\int d\vec r_4 \Bigg[&\frac{\vec r^{\,2}_{12}}{\vec r^{\,2}_{41}\vec r^{\,2}_{42}} \ln\left(\frac{a\vec r^{\,2}_{12}}{\vec r^{\,2}_{41}\vec r^{\,2}_{42}}\right)((U_2U_4^\dagger U_1 + U_1U_4^\dagger U_2)\cdot U_4\cdot U_3 - 2B_{123}) \\ &+ (1\leftrightarrow 3) + (2\leftrightarrow 3)\Bigg]. \end{aligned} \tag{6.4}$$

For the operator $(-B_{123} + \frac{1}{6}(B_{144}B_{324} + B_{244}B_{314} - B_{344}B_{214}))$ we get

$$\begin{aligned} &\Big(- 3B_{123} + \frac{1}{2}(B_{144}B_{324} + B_{244}B_{314} - B_{344}B_{214})\Big)^{\rm conf} \\ &= \Big(- 3B_{123} + \frac{1}{2}(B_{144}B_{324} + B_{244}B_{314} - B_{344}B_{214})\Big) \\ &\quad + \frac{\alpha_s}{8\pi^2}\int d\vec r_0 \left(A_{34}\frac{\vec r^{\,2}_{34}}{\vec r^{\,2}_{03}\vec r^{\,2}_{04}} \ln\left(\frac{\vec r^{\,2}_{34}a}{\vec r^{\,2}_{03}\vec r^{\,2}_{04}}\right) + A_{13}\frac{\vec r^{\,2}_{13}}{\vec r^{\,2}_{03}\vec r^{\,2}_{01}} \ln\left(\frac{\vec r^{\,2}_{13}a}{\vec r^{\,2}_{03}\vec r^{\,2}_{01}}\right) + A_{23}\frac{\vec r^{\,2}_{23}}{\vec r^{\,2}_{03}\vec r^{\,2}_{02}} \ln\left(\frac{\vec r^{\,2}_{23}a}{\vec r^{\,2}_{03}\vec r^{\,2}_{02}}\right)\right. \\ &\quad \left. + A_{14}\frac{\vec r^{\,2}_{14}}{\vec r^{\,2}_{01}\vec r^{\,2}_{04}} \ln\left(\frac{\vec r^{\,2}_{14}a}{\vec r^{\,2}_{01}\vec r^{\,2}_{04}}\right) + A_{24}\frac{\vec r^{\,2}_{24}}{\vec r^{\,2}_{02}\vec z^{\,2}_{04}} \ln\left(\frac{\vec r^{\,2}_{24}a}{\vec r^{\,2}_{02}\vec z^{\,2}_{04}}\right) + A_{12}\frac{\vec r^{\,2}_{12}}{\vec r^{\,2}_{01}\vec r^{\,2}_{02}} \ln\left(\frac{\vec r^{\,2}_{12}a}{\vec r^{\,2}_{01}\vec r^{\,2}_{02}}\right)\right), \end{aligned} \tag{6.5}$$

where the functions A were calculated in appendix C (C.4)–(C.8) according to prescription (6.1). As a result, the evolution equation for $B^{\rm conf}_{123}$ turns into

$$\begin{aligned} \frac{\partial B^{\rm conf}_{123}}{\partial\eta} = \frac{\alpha_s 3}{4\pi^2}\int d\vec r_4 \Bigg[&\frac{\vec r^{\,2}_{12}}{\vec r^{\,2}_{41}\vec r^{\,2}_{42}}\Big(- B_{123} + \frac{1}{6}(B_{144}B_{324} + B_{244}B_{314} - B_{344}B_{214})\Big)^{\rm conf} \\ &+ (1\leftrightarrow 3) + (2\leftrightarrow 3)\Bigg] - \frac{\alpha_s^2}{8\pi^4}\int d\vec r_0 d\vec r_4\ \mathbf{G}_{\rm finite} - \frac{\alpha_s^2}{8\pi^3}\int d\vec r_0\ \mathbf{G}' \\ &- \frac{\alpha_s}{4\pi^2}\frac{\alpha_s}{8\pi^2}\int d\vec r_4 d\vec r_0 \Bigg[\frac{\vec r^{\,2}_{12}}{\vec r^{\,2}_{41}\vec r^{\,2}_{42}}\left(A_{34}\frac{\vec r^{\,2}_{34}}{\vec r^{\,2}_{03}\vec r^{\,2}_{04}} \ln\left(\frac{\vec r^{\,2}_{34}a}{\vec r^{\,2}_{03}\vec r^{\,2}_{04}}\right) + A_{13}\frac{\vec r^{\,2}_{13}}{\vec z^{\,2}_{03}\vec r^{\,2}_{01}} \ln\left(\frac{\vec r^{\,2}_{13}a}{\vec z^{\,2}_{03}\vec r^{\,2}_{01}}\right)\right. \\ &+ A_{23}\frac{\vec r^{\,2}_{23}}{\vec r^{\,2}_{03}\vec r^{\,2}_{02}} \ln\left(\frac{\vec r^{\,2}_{23}a}{\vec r^{\,2}_{03}\vec r^{\,2}_{02}}\right) + A_{14}\frac{\vec r^{\,2}_{14}}{\vec r^{\,2}_{01}\vec r^{\,2}_{04}} \ln\left(\frac{\vec r^{\,2}_{14}a}{\vec r^{\,2}_{01}\vec r^{\,2}_{04}}\right) + A_{24}\frac{\vec r^{\,2}_{24}}{\vec r^{\,2}_{02}\vec z^{\,2}_{04}} \ln\left(\frac{\vec r^{\,2}_{24}a}{\vec r^{\,2}_{02}\vec z^{\,2}_{04}}\right) \\ &\left. + A_{12}\frac{\vec r^{\,2}_{12}}{\vec r^{\,2}_{01}\vec r^{\,2}_{02}} \ln\left(\frac{\vec r^{\,2}_{12}a}{\vec r^{\,2}_{01}\vec r^{\,2}_{02}}\right)\right) + (1\leftrightarrow 3) + (2\leftrightarrow 3)\Bigg] \\ &+ \frac{\alpha_s}{8\pi^2}\frac{\alpha_s}{4\pi^2}\int d\vec r_4 d\vec r_0 \left[\frac{\vec r^{\,2}_{12}}{\vec r^{\,2}_{41}\vec z^{\,2}_{42}} \ln\left(\frac{a\vec r^{\,2}_{12}}{\vec r^{\,2}_{41}\vec r^{\,2}_{42}}\right)\right. \\ &\times \left(A_{34}\frac{\vec r^{\,2}_{34}}{\vec r^{\,2}_{03}\vec z^{\,2}_{04}} + A_{13}\frac{\vec r^{\,2}_{13}}{\vec r^{\,2}_{03}\vec z^{\,2}_{01}} + A_{23}\frac{\vec r^{\,2}_{23}}{\vec r^{\,2}_{03}\vec r^{\,2}_{02}} + A_{14}\frac{\vec r^{\,2}_{14}}{\vec r^{\,2}_{01}\vec r^{\,2}_{04}} + A_{24}\frac{\vec r^{\,2}_{24}}{\vec r^{\,2}_{02}\vec r^{\,2}_{04}} + A_{12}\frac{\vec r^{\,2}_{12}}{\vec z^{\,2}_{01}\vec r^{\,2}_{02}}\right) \end{aligned}$$

$$+\ (1\leftrightarrow 3)+(2\leftrightarrow 3)\Big]\,. \tag{6.6}$$

After simplification one obtains

$$\frac{\partial B_{123}^{\rm conf}}{\partial\eta}=\frac{\alpha_s 3}{4\pi^2}\int d\vec r_4\left[\frac{\vec r_{12}^{\,2}}{\vec r_{41}^{\,2}\vec r_{42}^{\,2}}(-B_{123}+\frac{1}{6}(B_{144}B_{324}+B_{244}B_{314}-B_{344}B_{214}))^{\rm conf}\right.$$
$$\left.+\ (1\leftrightarrow 3)+(2\leftrightarrow 3)\right]-\frac{\alpha_s^2}{8\pi^4}\int d\vec r_0 d\vec r_4\ \mathbf{G}_{\rm finite}-\frac{\alpha_s^2}{8\pi^3}\int d\vec r_0\ \mathbf{G}'$$
$$-\frac{\alpha_s^2}{32\pi^4}\int d\vec r_4 d\vec r_0\left[\frac{\vec r_{12}^{\,2}}{\vec r_{41}^{\,2}\vec r_{42}^{\,2}}\left(A_{34}\frac{\vec r_{34}^{\,2}}{\vec r_{03}^{\,2}\vec r_{04}^{\,2}}\ln\left(\frac{\vec r_{34}^{\,2}\vec r_{41}^{\,2}\vec r_{42}^{\,2}}{\vec r_{03}^{\,2}\vec r_{04}^{\,2}\vec r_{12}^{\,2}}\right)+A_{13}\frac{\vec r_{13}^{\,2}}{\vec r_{03}^{\,2}\vec r_{01}^{\,2}}\ln\left(\frac{\vec r_{13}^{\,2}\vec r_{41}^{\,2}\vec r_{42}^{\,2}}{\vec r_{03}^{\,2}\vec r_{01}^{\,2}\vec r_{12}^{\,2}}\right)\right.\right.$$
$$+A_{23}\frac{\vec r_{23}^{\,2}}{\vec r_{03}^{\,2}\vec r_{02}^{\,2}}\ln\left(\frac{\vec r_{23}^{\,2}\vec r_{41}^{\,2}\vec r_{42}^{\,2}}{\vec r_{03}^{\,2}\vec r_{02}^{\,2}\vec r_{12}^{\,2}}\right)+A_{14}\frac{\vec r_{14}^{\,2}}{\vec r_{01}^{\,2}\vec r_{04}^{\,2}}\ln\left(\frac{\vec r_{14}^{\,4}\vec r_{42}^{\,2}}{\vec r_{01}^{\,2}\vec r_{04}^{\,2}\vec r_{12}^{\,2}}\right)+A_{12}\frac{\vec r_{12}^{\,2}}{\vec r_{01}^{\,2}\vec z_{02}^{\,2}}\ln\left(\frac{\vec r_{41}^{\,2}\vec r_{42}^{\,2}}{\vec r_{01}^{\,2}\vec r_{02}^{\,2}}\right)$$
$$\left.\left.+\ A_{24}\frac{\vec r_{24}^{\,2}}{\vec r_{02}^{\,2}\vec r_{04}^{\,2}}\ln\left(\frac{\vec r_{24}^{\,4}\vec r_{41}^{\,2}}{\vec r_{02}^{\,2}\vec r_{04}^{\,2}\vec r_{12}^{\,2}}\right)\right)+(1\leftrightarrow 3)+(2\leftrightarrow 3)\right]. \tag{6.7}$$

Now we can symmetrize the last 3 lines of this expression w.r.t. $0\leftrightarrow 4$ transformation, i.e.

$$A_{ij}F(\vec r\ldots)\to[A_{ij}F(\vec r\ldots)]^{\rm sym} \tag{6.8}$$
$$=\frac{[A_{ij}+A_{ij}\,(0\leftrightarrow 4)]\,[F+F\,(0\leftrightarrow 4)]+[A_{ij}-A_{ij}\,(0\leftrightarrow 4)]\,[F-F\,(0\leftrightarrow 4)]}{4}.$$

Again, $i\leftrightarrow j$ stands for the the permutation. It means that we have to change $\vec r_i\leftrightarrow\vec r_j$ and $U_i\leftrightarrow U_j$. Next, one can use (B.9) to show that all the nonconformal terms have the SU(3) coefficients independent either of $\vec r_4$ or of $\vec r_0$.

To get rid of the non-conformal terms, first we add the symmetrized last 3 lines of (6.7) to the nonconformal part of $\mathbf{G}_{\rm finite}$ (4.20), define the result as $\tilde{\mathbf{G}}$ (6.9) and work with it to avoid rewriting the conformally invariant parts of (6.7). Taking into account (B.3), (B.9), and (B.13), we can write the result as

$$-\frac{\alpha_s^2}{8\pi^4}\int d\vec r_0 d\vec r_4\tilde{\mathbf{G}}=-\frac{\alpha_s^2}{8\pi^4}\int d\vec r_0 d\vec r_4\left[\left\{(M_{13}-M_{12}-M_{23}+M_2^{13})[(U_0U_4{}^\dagger U_3)\cdot(U_2U_0{}^\dagger U_1)\cdot U_4\right.\right.$$
$$\left.\left.+\ (U_1U_0{}^\dagger U_2)\cdot(U_3U_4{}^\dagger U_0)\cdot U_4]+(\text{all 5 permutations}\,1\leftrightarrow 2\leftrightarrow 3)\right\}+(0\leftrightarrow 4)\right]$$
$$-\frac{\alpha_s^2}{32\pi^4}\int d\vec r_4 d\vec r_0\left[\frac{\vec r_{12}^{\,2}}{\vec r_{41}^{\,2}\vec r_{42}^{\,2}}\left(A_{34}\frac{\vec r_{34}^{\,2}}{\vec r_{03}^{\,2}\vec r_{04}^{\,2}}\ln\left(\frac{\vec r_{34}^{\,2}\vec r_{41}^{\,2}\vec r_{42}^{\,2}}{\vec r_{03}^{\,2}\vec r_{04}^{\,2}\vec r_{12}^{\,2}}\right)\right.\right.$$
$$+\ A_{23}\frac{\vec r_{23}^{\,2}}{\vec r_{03}^{\,2}\vec r_{02}^{\,2}}\ln\left(\frac{\vec r_{23}^{\,2}\vec r_{41}^{\,2}\vec r_{42}^{\,2}}{\vec r_{03}^{\,2}\vec r_{02}^{\,2}\vec r_{12}^{\,2}}\right)+A_{14}\frac{\vec r_{14}^{\,2}}{\vec r_{01}^{\,2}\vec r_{04}^{\,2}}\ln\left(\frac{\vec r_{14}^{\,4}\vec r_{42}^{\,2}}{\vec r_{01}^{\,2}\vec r_{04}^{\,2}\vec r_{12}^{\,2}}\right)+A_{12}\frac{\vec r_{12}^{\,2}}{\vec r_{01}^{\,2}\vec z_{02}^{\,2}}\ln\left(\frac{\vec r_{41}^{\,2}\vec r_{42}^{\,2}}{\vec r_{01}^{\,2}\vec r_{02}^{\,2}}\right)$$
$$\left.\left.+\ A_{13}\frac{\vec r_{13}^{\,2}}{\vec r_{03}^{\,2}\vec r_{01}^{\,2}}\ln\left(\frac{\vec r_{13}^{\,2}\vec r_{41}^{\,2}\vec r_{42}^{\,2}}{\vec r_{03}^{\,2}\vec r_{01}^{\,2}\vec r_{12}^{\,2}}\right)+A_{24}\frac{\vec r_{24}^{\,2}}{\vec r_{02}^{\,2}\vec r_{04}^{\,2}}\ln\left(\frac{\vec r_{24}^{\,4}\vec r_{41}^{\,2}}{\vec r_{02}^{\,2}\vec r_{04}^{\,2}\vec r_{12}^{\,2}}\right)\right)+(1\leftrightarrow 3)+(2\leftrightarrow 3)\right]^{\rm sym}. \tag{6.9}$$

After simplification

$$-\frac{\alpha_s^2}{8\pi^4}\int d\vec r_0 d\vec r_4\tilde{\mathbf{G}}$$
$$=-\frac{\alpha_s^2}{8\pi^4}\int d\vec r_0 d\vec r_4\left\{\left(\frac{\vec r_{12}{}^4}{8\vec r_{01}{}^2\vec r_{02}{}^2\vec r_{14}{}^2\vec r_{24}{}^2}\ln\left(\frac{\vec r_{14}{}^2\vec r_{24}{}^2}{\vec r_{01}{}^2\vec r_{02}{}^2}\right)\left(\frac{1}{2}B_{003}B_{012}-2B_{001}B_{023}\right)\right.\right.$$

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$$
\begin{aligned}
&+\frac{\vec{r}_{12}{}^2}{8\vec{r}_{01}{}^2\vec{r}_{02}{}^2}\left(\frac{\vec{r}_{13}{}^2}{\vec{r}_{14}{}^2\vec{r}_{34}{}^2}\ln\left(\frac{\vec{r}_{12}{}^2\vec{r}_{14}{}^2\vec{r}_{34}{}^2}{\vec{r}_{01}{}^2\vec{r}_{02}{}^2\vec{r}_{13}{}^2}\right)+\frac{\vec{r}_{03}{}^2}{\vec{r}_{04}{}^2\vec{r}_{34}{}^2}\ln\left(\frac{\vec{r}_{01}{}^2\vec{r}_{02}{}^2\vec{r}_{03}{}^2}{\vec{r}_{04}{}^2\vec{r}_{12}{}^2\vec{r}_{34}{}^2}\right)\right)\\
&\times\left(B_{003}B_{012}-B_{001}B_{023}\right)\\
&+\frac{\vec{r}_{12}{}^2}{8\vec{r}_{02}{}^2\vec{r}_{04}{}^2\vec{r}_{14}{}^2}\left[\ln\left(\frac{\vec{r}_{01}{}^4\vec{r}_{02}{}^2}{\vec{r}_{04}{}^2\vec{r}_{12}{}^2\vec{r}_{14}{}^2}\right)(2B_{003}B_{012}-2B_{002}B_{013}-3B_{001}B_{023}+4B_{123})\right.\\
&+\ln\left(\frac{\vec{r}_{04}{}^2\vec{r}_{12}{}^2}{\vec{r}_{01}{}^2\vec{r}_{24}{}^2}\right)\left(-tr\left(U_0{}^\dagger U_4\right)\left(U_1U_4{}^\dagger U_2+U_2U_4{}^\dagger U_1\right)\cdot U_0\cdot U_3\right.\\
&\left.+\,2\left(U_1\cdot U_2\cdot U_3-\left(U_0U_4{}^\dagger U_1\right)\cdot\left(U_2U_0{}^\dagger U_4\right)\cdot U_3\right)\right)\\
&+\ln\left(\frac{\vec{r}_{01}{}^4\vec{r}_{02}{}^2\vec{r}_{34}{}^4}{\vec{r}_{03}{}^4\vec{r}_{04}{}^2\vec{r}_{12}{}^2\vec{r}_{14}{}^2}\right)\left(tr\left(U_4{}^\dagger U_1\right)\left(\left(U_2U_0{}^\dagger U_4\right)\cdot U_0\cdot U_3+\left(U_4U_0{}^\dagger U_2\right)\cdot U_0\cdot U_3\right)\right.\\
&\left.\left.+\;\left(U_0U_4{}^\dagger U_1U_0{}^\dagger U_2\right)\cdot U_3\cdot U_4+\left(U_2U_0{}^\dagger U_1U_4{}^\dagger U_0\right)\cdot U_3\cdot U_4\right)\right]\\
&+\left(\frac{\vec{r}_{12}{}^2\vec{r}_{13}{}^2}{16\vec{r}_{01}{}^2\vec{r}_{02}{}^2\vec{r}_{14}{}^2\vec{r}_{34}{}^2}\ln\left(\frac{\vec{r}_{01}{}^2\vec{r}_{12}{}^2\vec{r}_{34}{}^2}{\vec{r}_{02}{}^2\vec{r}_{13}{}^2\vec{r}_{14}{}^2}\right)+\frac{\vec{r}_{03}{}^2\vec{r}_{12}{}^2}{8\vec{r}_{01}{}^2\vec{r}_{02}{}^2\vec{r}_{04}{}^2\vec{r}_{34}{}^2}\ln\left(\frac{\vec{r}_{02}{}^2\vec{r}_{03}{}^2\vec{r}_{14}{}^4}{\vec{r}_{01}{}^2\vec{r}_{04}{}^2\vec{r}_{12}{}^2\vec{r}_{34}{}^2}\right)\right)\\
&\times\left(\left(U_0U_4{}^\dagger U_3\right)\cdot\left(U_1U_0{}^\dagger U_2\right)\cdot U_4+\left(U_2U_0{}^\dagger U_1\right)\cdot\left(U_3U_4{}^\dagger U_0\right)\cdot U_4\right.\\
&\left.-\left(U_2U_0{}^\dagger U_4\right)\cdot\left(U_3U_4{}^\dagger U_1\right)\cdot U_0-\left(U_1U_4{}^\dagger U_3\right)\cdot\left(U_4U_0{}^\dagger U_2\right)\cdot U_0\right)\\
&\left.+\;(\text{all 5 permutations } 1\leftrightarrow 2\leftrightarrow 3)\right)+(0\leftrightarrow 4)\Big\}\,. \qquad (6.10)
\end{aligned}
$$

Indeed, in this expression all the nonconformal terms have the SU(3) coefficients independent either of $\vec{r}_4$ or of $\vec{r}_0$. In principle one can integrate them w.r.t. $\vec{r}_4$ or $\vec{r}_0$ and add to eq. (4.30). However, it is easier to transform (4.30) using integral (116) from [19] in the symmetric form

$$
\begin{aligned}
\frac{\vec{r}^{\,2}_{12}}{\vec{r}^{\,2}_{10}\vec{r}^{\,2}_{20}}\ln\frac{\vec{r}^{\,2}_{10}}{\vec{r}^{\,2}_{12}}\ln\frac{\vec{r}^{\,2}_{20}}{\vec{r}^{\,2}_{12}}&=2\pi\zeta\left(3\right)\left(\delta\left(\vec{r}_{10}\right)+\delta\left(\vec{r}_{20}\right)\right)\\
&+\frac{\vec{r}^{\,2}_{12}}{\vec{r}^{\,2}_{10}\vec{r}^{\,2}_{20}}\int\frac{d\vec{r}_4}{2\pi}\left(\frac{\vec{r}^{\,2}_{20}}{\vec{r}^{\,2}_{04}\vec{r}^{\,2}_{24}}+\frac{\vec{r}^{\,2}_{10}}{\vec{r}^{\,2}_{04}\vec{r}^{\,2}_{14}}-\frac{\vec{r}^{\,2}_{12}}{\vec{r}^{\,2}_{24}\vec{r}^{\,2}_{14}}\right)\ln\left(\frac{\vec{r}^{\,2}_{10}\vec{r}^{\,2}_{20}}{\vec{r}^{\,2}_{14}\vec{r}^{\,2}_{24}}\right). \qquad (6.11)
\end{aligned}
$$

We get then

$$
\begin{aligned}
\mathbf{G}'=&\frac{1}{2}\left[\frac{\vec{r}^{\,2}_{13}\vec{r}^{\,2}_{20}}{\vec{r}^{\,2}_{30}\vec{r}^{\,2}_{12}}-\frac{\vec{r}^{\,2}_{32}\vec{r}^{\,2}_{10}}{\vec{r}^{\,2}_{30}\vec{r}^{\,2}_{12}}\right]\frac{\vec{r}^{\,2}_{12}}{\vec{r}^{\,2}_{10}\vec{r}^{\,2}_{20}}\int\frac{d\vec{r}_4}{2\pi}\left(\frac{\vec{r}^{\,2}_{20}}{\vec{r}^{\,2}_{04}\vec{r}^{\,2}_{24}}+\frac{\vec{r}^{\,2}_{10}}{\vec{r}^{\,2}_{04}\vec{r}^{\,2}_{14}}-\frac{\vec{r}^{\,2}_{12}}{\vec{r}^{\,2}_{24}\vec{r}^{\,2}_{14}}\right)\ln\left(\frac{\vec{r}^{\,2}_{10}\vec{r}^{\,2}_{20}}{\vec{r}^{\,2}_{14}\vec{r}^{\,2}_{24}}\right)\\
&\times\left(B_{100}B_{320}-B_{200}B_{310}\right)\\
&+\left[\frac{\vec{r}^{\,2}_{13}\vec{r}^{\,2}_{20}}{\vec{r}^{\,2}_{30}\vec{r}^{\,2}_{12}}-\frac{\vec{r}^{\,2}_{32}\vec{r}^{\,2}_{10}}{\vec{r}^{\,2}_{30}\vec{r}^{\,2}_{12}}\right]\zeta\left(3\right)\pi\left(\delta\left(\vec{r}_{10}\right)+\delta\left(\vec{r}_{20}\right)\right)\left(B_{100}B_{320}-B_{200}B_{310}\right)\\
&-\frac{\vec{r}^{\,2}_{12}}{\vec{r}^{\,2}_{10}\vec{r}^{\,2}_{20}}\int\frac{d\vec{r}_4}{2\pi}\left(\frac{\vec{r}^{\,2}_{20}}{\vec{r}^{\,2}_{04}\vec{r}^{\,2}_{24}}+\frac{\vec{r}^{\,2}_{10}}{\vec{r}^{\,2}_{04}\vec{r}^{\,2}_{14}}-\frac{\vec{r}^{\,2}_{12}}{\vec{r}^{\,2}_{24}\vec{r}^{\,2}_{14}}\right)\ln\left(\frac{\vec{r}^{\,2}_{10}\vec{r}^{\,2}_{20}}{\vec{r}^{\,2}_{14}\vec{r}^{\,2}_{24}}\right)\\
&\times\left(9B_{123}-\frac{1}{2}\left[2\left(B_{100}B_{320}+B_{200}B_{130}\right)-B_{300}B_{120}\right]\right)\\
&-2\pi\zeta\left(3\right)\left(\delta\left(\vec{r}_{10}\right)+\delta\left(\vec{r}_{20}\right)\right)\left(9B_{123}-\frac{1}{2}\left[2\left(B_{100}B_{320}+B_{200}B_{130}\right)-B_{300}B_{120}\right]\right)\\
&+\frac{11}{6}\left[\ln\left(\frac{\vec{r}^{\,2}_{01}}{\vec{r}^{\,2}_{02}}\right)\left(\frac{1}{\vec{r}^{\,2}_{02}}-\frac{1}{\vec{r}^{\,2}_{01}}\right)-\frac{\vec{r}^{\,2}_{12}}{\vec{r}^{\,2}_{01}\vec{r}^{\,2}_{02}}\ln\left(\frac{\vec{r}^{\,2}_{12}}{\tilde{\mu}^2}\right)\right]
\end{aligned}
$$

$$\times\left(\frac{3}{2}(B_{100}B_{230}+B_{200}B_{130}-B_{300}B_{210})-9B_{123}\right)+(1\leftrightarrow 3)+(2\leftrightarrow 3). \tag{6.12}$$

One can write it as

$$\begin{aligned}\mathbf{G}'=&\frac{1}{2}\left[\frac{\vec r_{13}^{\,2}\vec r_{20}^{\,2}}{\vec r_{30}^{\,2}\vec r_{12}^{\,2}}-\frac{\vec r_{32}^{\,2}\vec r_{10}^{\,2}}{\vec r_{30}^{\,2}\vec r_{12}^{\,2}}\right]\frac{\vec r_{12}^{\,2}}{\vec r_{10}^{\,2}\vec r_{20}^{\,2}}\int\frac{d\vec r_4}{2\pi}\left(\frac{\vec r_{20}^{\,2}}{\vec r_{04}^{\,2}\vec r_{24}^{\,2}}+\frac{\vec r_{10}^{\,2}}{\vec r_{04}^{\,2}\vec r_{14}^{\,2}}-\frac{\vec r_{12}^{\,2}}{\vec r_{24}^{\,2}\vec r_{14}^{\,2}}\right)\ln\left(\frac{\vec r_{10}^{\,2}\vec r_{20}^{\,2}}{\vec r_{14}^{\,2}\vec r_{24}^{\,2}}\right)\\&\times(B_{100}B_{320}-B_{200}B_{310})-\frac{\vec r_{12}^{\,2}}{\vec r_{10}^{\,2}\vec r_{20}^{\,2}}\int\frac{d\vec r_4}{2\pi}\left(\frac{\vec r_{20}^{\,2}}{\vec r_{04}^{\,2}\vec r_{24}^{\,2}}+\frac{\vec r_{10}^{\,2}}{\vec r_{04}^{\,2}\vec r_{14}^{\,2}}-\frac{\vec r_{12}^{\,2}}{\vec r_{24}^{\,2}\vec r_{14}^{\,2}}\right)\ln\left(\frac{\vec r_{10}^{\,2}\vec r_{20}^{\,2}}{\vec r_{14}^{\,2}\vec r_{24}^{\,2}}\right)\\&\times\left(9B_{123}-\frac{1}{2}\left[2\left(B_{100}B_{320}+B_{200}B_{130}\right)-B_{300}B_{120}\right]\right)\\&+\frac{11}{6}\left[\ln\left(\frac{\vec r_{01}^{\,2}}{\vec r_{02}^{\,2}}\right)\left(\frac{1}{\vec r_{02}^{\,2}}-\frac{1}{\vec r_{01}^{\,2}}\right)-\frac{\vec r_{12}^{\,2}}{\vec r_{01}^{\,2}\vec r_{02}^{\,2}}\ln\left(\frac{\vec r_{12}^{\,2}}{\tilde\mu^2}\right)\right]\\&\times\left(\frac{3}{2}(B_{100}B_{230}+B_{200}B_{130}-B_{300}B_{210})-9B_{123}\right)+(1\leftrightarrow 3)+(2\leftrightarrow 3).\end{aligned}\tag{6.13}$$

Next, we symmetrize the previous expression w.r.t. $0\leftrightarrow 4$ exchange and combine it with (4.20), (6.7), and (6.10) to obtain the NLO kernel for the composite 3QWL operator $B_{123}^{\rm conf}$

$$\begin{aligned}\langle K_{NLO}\otimes B_{123}^{\rm conf}\rangle=&-\frac{\alpha_s^2}{8\pi^3}\int d\vec r_0\mathbf{G}'-\frac{\alpha_s^2}{8\pi^4}\int d\vec r_0 d\vec r_4\tilde{\mathbf{G}}\\&-\frac{\alpha_s^2}{8\pi^4}\int d\vec r_0 d\vec r_4\ \left(\left\{\tilde L_{12}\left(U_0U_4{}^\dagger U_2\right)\cdot\left(U_1U_0{}^\dagger U_4\right)\cdot U_3\right.\right.\\&+L_{12}\left[\left(U_0U_4{}^\dagger U_2\right)\cdot\left(U_1U_0{}^\dagger U_4\right)\cdot U_3+tr\left(U_0U_4{}^\dagger\right)\left(U_1U_0{}^\dagger U_2\right)\cdot U_3\cdot U_4\right.\\&\left.-\frac{3}{4}[B_{144}B_{234}+B_{244}B_{134}-B_{344}B_{124}]+\frac{1}{2}B_{123}\right]\\&\left.\left.+\ \ (\text{all 5 permutations}\,1\leftrightarrow 2\leftrightarrow 3)\right\}+(0\leftrightarrow 4)\right).\end{aligned}\tag{6.14}$$

Using (B.9) to get rid of the terms like

$$\left(U_0U_4{}^\dagger U_1U_0{}^\dagger U_2\right)\cdot U_3\cdot U_4+\left(U_2U_0{}^\dagger U_1U_4{}^\dagger U_0\right)\cdot U_3\cdot U_4 \tag{6.15}$$

it can be transformed to

$$\begin{aligned}\langle K_{NLO}\otimes B_{123}^{\rm conf}\rangle=&-\frac{\alpha_s^2}{8\pi^4}\int d\vec r_0 d\vec r_4\ \left(\left\{\tilde L_{12}^C\left(U_0U_4{}^\dagger U_2\right)\cdot\left(U_1U_0{}^\dagger U_4\right)\cdot U_3\right.\right.\\&+L_{12}^C\left[\left(U_0U_4{}^\dagger U_2\right)\cdot\left(U_1U_0{}^\dagger U_4\right)\cdot U_3+tr\left(U_0U_4{}^\dagger\right)\left(U_1U_0{}^\dagger U_2\right)\cdot U_3\cdot U_4\right.\\&\left.-\frac{3}{4}[B_{144}B_{234}+B_{244}B_{134}-B_{344}B_{124}]+\frac{1}{2}B_{123}\right]\\&+M_{12}^C\left[\left(U_0U_4{}^\dagger U_3\right)\cdot\left(U_2U_0{}^\dagger U_1\right)\cdot U_4+\left(U_1U_0{}^\dagger U_2\right)\cdot\left(U_3U_4{}^\dagger U_0\right)\cdot U_4\right]\\&\left.+\ Z_{12}B_{003}B_{012}+\ (\text{all 5 permutations}\,1\leftrightarrow 2\leftrightarrow 3)\right\}+(0\leftrightarrow 4)\Big)\\&-\frac{\alpha_s^2}{8\pi^3}\int d\vec r_0\left(\frac{11}{6}\left[\ln\left(\frac{\vec r_{01}^{\,2}}{\vec r_{02}^{\,2}}\right)\left(\frac{1}{\vec r_{02}^{\,2}}-\frac{1}{\vec r_{01}^{\,2}}\right)-\frac{\vec r_{12}^{\,2}}{\vec r_{01}^{\,2}\vec r_{02}^{\,2}}\ln\left(\frac{\vec r_{12}^{\,2}}{\tilde\mu^2}\right)\right]\right.\end{aligned}$$

$$\times\left(\frac{3}{2}(B_{100}B_{230}+B_{200}B_{130}-B_{300}B_{210})-9B_{123}\right)+(1\leftrightarrow 3)+(2\leftrightarrow 3)\Big). \qquad (6.16)$$

Here

$$L^C_{12}=L_{12}+\frac{\vec r_{12}{}^2}{4\vec r_{01}{}^2\vec r_{04}{}^2\vec r_{24}{}^2}\ln\left(\frac{\vec r_{02}{}^2\vec r_{14}{}^2}{\vec r_{04}{}^2\vec r_{12}{}^2}\right)+\frac{\vec r_{12}{}^2}{4\vec r_{02}{}^2\vec r_{04}{}^2\vec r_{14}{}^2}\ln\left(\frac{\vec r_{01}{}^2\vec r_{24}{}^2}{\vec r_{04}{}^2\vec r_{12}{}^2}\right), \qquad (6.17)$$

$$\tilde L^C_{12}=\tilde L_{12}+\frac{\vec r_{12}{}^2}{4\vec r_{01}{}^2\vec r_{04}{}^2\vec r_{24}{}^2}\ln\left(\frac{\vec r_{02}{}^2\vec r_{14}{}^2}{\vec r_{04}{}^2\vec r_{12}{}^2}\right)-\frac{\vec r_{12}{}^2}{4\vec r_{02}{}^2\vec r_{04}{}^2\vec r_{14}{}^2}\ln\left(\frac{\vec r_{01}{}^2\vec r_{24}{}^2}{\vec r_{04}{}^2\vec r_{12}{}^2}\right), \qquad (6.18)$$

$$\begin{aligned}M^C_{12}=&\frac{\vec r_{12}{}^2}{16\vec r_{02}{}^2\vec r_{04}{}^2\vec r_{14}{}^2}\ln\left(\frac{\vec r_{01}{}^2\vec r_{02}{}^2\vec r_{34}{}^4}{\vec r_{03}{}^4\vec r_{14}{}^2\vec r_{24}{}^2}\right)+\frac{\vec r_{12}{}^2}{16\vec r_{01}{}^2\vec r_{04}{}^2\vec r_{24}{}^2}\ln\left(\frac{\vec r_{03}{}^4\vec r_{04}{}^4\vec r_{12}{}^4\vec r_{24}{}^2}{\vec r_{01}{}^2\vec r_{02}{}^6\vec r_{14}{}^2\vec r_{34}{}^4}\right)\\
&+\frac{\vec r_{23}{}^2}{16\vec r_{02}{}^2\vec r_{04}{}^2\vec r_{34}{}^2}\ln\left(\frac{\vec r_{01}{}^4\vec r_{03}{}^2\vec r_{24}{}^6\vec r_{34}{}^2}{\vec r_{02}{}^2\vec r_{04}{}^4\vec r_{14}{}^4\vec r_{23}{}^4}\right)+\frac{\vec r_{23}{}^2}{16\vec r_{03}{}^2\vec r_{04}{}^2\vec r_{24}{}^2}\ln\left(\frac{\vec r_{02}{}^2\vec r_{03}{}^2\vec r_{14}{}^4}{\vec r_{01}{}^4\vec r_{24}{}^2\vec r_{34}{}^2}\right)\\
&+\frac{\vec r_{13}{}^2}{16\vec r_{03}{}^2\vec r_{04}{}^2\vec r_{14}{}^2}\ln\left(\frac{\vec r_{02}{}^4\vec r_{14}{}^2\vec r_{34}{}^2}{\vec r_{01}{}^2\vec r_{03}{}^2\vec r_{24}{}^4}\right)+\frac{\vec r_{13}{}^2}{16\vec r_{01}{}^2\vec r_{04}{}^2\vec r_{34}{}^2}\ln\left(\frac{\vec r_{02}{}^4\vec r_{14}{}^2\vec r_{34}{}^2}{\vec r_{01}{}^2\vec r_{03}{}^2\vec r_{24}{}^4}\right)\\
&+\frac{\vec r_{03}{}^2\vec r_{12}{}^2}{8\vec r_{01}{}^2\vec r_{02}{}^2\vec r_{04}{}^2\vec r_{34}{}^2}\ln\left(\frac{\vec r_{01}{}^2\vec r_{03}{}^2\vec r_{24}{}^4}{\vec r_{02}{}^2\vec r_{04}{}^2\vec r_{12}{}^2\vec r_{34}{}^2}\right)+\frac{\vec r_{23}{}^2\vec r_{12}{}^2}{8\vec r_{01}{}^2\vec r_{02}{}^2\vec r_{24}{}^2\vec r_{34}{}^2}\ln\left(\frac{\vec r_{02}{}^2\vec r_{12}{}^2\vec r_{34}{}^2}{\vec r_{01}{}^2\vec r_{23}{}^2\vec r_{24}{}^2}\right)\\
&+\frac{\vec r_{14}{}^2\vec r_{23}{}^2}{8\vec r_{01}{}^2\vec r_{04}{}^2\vec r_{24}{}^2\vec r_{34}{}^2}\ln\left(\frac{\vec r_{01}{}^2\vec r_{04}{}^2\vec r_{23}{}^2\vec r_{24}{}^2}{\vec r_{02}{}^4\vec r_{14}{}^2\vec r_{34}{}^2}\right), \qquad (6.19)\end{aligned}$$

$$\begin{aligned}Z_{12}=&\frac{\vec r_{12}{}^2}{8\vec r_{01}{}^2\vec r_{02}{}^2}\left[\left(\frac{\vec r_{03}{}^2}{\vec r_{04}{}^2\vec r_{34}{}^2}-\frac{\vec r_{02}{}^2}{\vec r_{04}{}^2\vec r_{24}{}^2}\right)\ln\left(\frac{\vec r_{02}{}^2\vec r_{14}{}^2}{\vec r_{04}{}^2\vec r_{12}{}^2}\right)\right.\\
&\left.+\frac{\vec r_{01}{}^2}{\vec r_{04}{}^2\vec r_{14}{}^2}\ln\left(\frac{\vec r_{02}{}^2\vec r_{34}{}^2}{\vec r_{03}{}^2\vec r_{24}{}^2}\right)+\frac{\vec r_{13}{}^2}{\vec r_{14}{}^2\vec r_{34}{}^2}\ln\left(\frac{\vec r_{03}{}^2\vec r_{12}{}^2}{\vec r_{02}{}^2\vec r_{13}{}^2}\right)\right]-(1\leftrightarrow 3), \qquad (6.20)\end{aligned}$$

and L_{12} and $\tilde L_{12}$ are the elements of the nonconformal kernel defined in (4.10) and (4.11). Checking that L^C_{12}, $\tilde L^C_{12}$, M^C_{12}, and Z_{12} have integrable singularities at $\vec r_4=\vec r_0$ and that L^C_{12}, $\tilde L^C_{12}$, and Z_{12} have integrable singularities at $\vec r_4=\vec r_{1,2,3}$ is straightforward. To prove that all the terms with M^C have safe behavior at $\vec r_4=\vec r_{1,2,3}$ one has to use SU(3) identity (B.14).

Now one can see that the NLO kernel for the evolution equation for the composite 3QWL operator $B^{\rm conf}_{123}$ (6.2) is quasi-conformal if one expresses the LO kernel in terms of composite operator (6.5).

The term with Z can be integrated w.r.t. $\vec r_4$. The integral

$$\int\frac{d\vec r_4}{\pi}Z_{12}=\frac{\vec r_{32}{}^2}{8\vec r_{03}{}^2\vec r_{02}{}^2}\ln^2\left(\frac{\vec r_{32}{}^2\vec r_{10}{}^2}{\vec r_{13}{}^2\vec r_{20}{}^2}\right)-\frac{\vec r_{12}{}^2}{8\vec r_{01}{}^2\vec r_{02}{}^2}\ln^2\left(\frac{\vec r_{12}{}^2\vec r_{30}{}^2}{\vec r_{13}{}^2\vec r_{20}{}^2}\right). \qquad (6.21)$$

was calculated in the appendix D.

Finally, the kernel reads

$$\begin{aligned}\langle K_{NLO}\otimes B^{\rm conf}_{123}\rangle=&-\frac{\alpha_s^2}{8\pi^4}\int d\vec r_0 d\vec r_4\ \Big(\Big\{\tilde L^C_{12}\left(U_0U_4{}^\dagger U_2\right)\cdot\left(U_1U_0{}^\dagger U_4\right)\cdot U_3\\
&+L^C_{12}\Big[\left(U_0U_4{}^\dagger U_2\right)\cdot\left(U_1U_0{}^\dagger U_4\right)\cdot U_3+tr\left(U_0U_4{}^\dagger\right)\left(U_1U_0{}^\dagger U_2\right)\cdot U_3\cdot U_4\\
&-\frac{3}{4}[B_{144}B_{234}+B_{244}B_{134}-B_{344}B_{124}]+\frac{1}{2}B_{123}\Big]\\
&+M^C_{12}\Big[\left(U_0U_4{}^\dagger U_3\right)\cdot\left(U_2U_0{}^\dagger U_1\right)\cdot U_4+\left(U_1U_0{}^\dagger U_2\right)\cdot\left(U_3U_4{}^\dagger U_0\right)\cdot U_4\Big]\end{aligned}$$

$$+\ \text{(all 5 permutations}\, 1 \leftrightarrow 2 \leftrightarrow 3)\Big\} + (0 \leftrightarrow 4)\Big)$$
$$- \frac{\alpha_s^2}{8\pi^3}\int d\vec{r}_0 \left(\frac{11}{6}\left[\ln\left(\frac{\vec{r}_{01}^{\,2}}{\vec{r}_{02}^{\,2}}\right)\left(\frac{1}{\vec{r}_{02}^{\,2}} - \frac{1}{\vec{r}_{01}^{\,2}}\right) - \frac{\vec{r}_{12}^{\,2}}{\vec{r}_{01}^{\,2}\vec{r}_{02}^{\,2}}\ln\left(\frac{\vec{r}_{12}^{\,2}}{\tilde{\mu}^2}\right)\right]\right.$$
$$\times \left(\frac{3}{2}(B_{100}B_{230} + B_{200}B_{130} - B_{300}B_{210}) - 9B_{123}\right) + (1 \leftrightarrow 3) + (2 \leftrightarrow 3)\Big)$$
$$- \frac{\alpha_s^2}{32\pi^3}\int d\vec{r}_0 \left(B_{003}B_{012}\left[\frac{\vec{r}_{32}{}^2}{\vec{r}_{03}{}^2\vec{r}_{02}{}^2}\ln^2\left(\frac{\vec{r}_{32}{}^2\vec{r}_{10}{}^2}{\vec{r}_{13}{}^2\vec{r}_{20}{}^2}\right) - \frac{\vec{r}_{12}{}^2}{\vec{r}_{01}{}^2\vec{r}_{02}{}^2}\ln^2\left(\frac{\vec{r}_{12}{}^2\vec{r}_{30}{}^2}{\vec{r}_{13}{}^2\vec{r}_{20}{}^2}\right)\right]\right.$$
$$\left. +\ \text{(all 5 permutations}\, 1 \leftrightarrow 2 \leftrightarrow 3)\right). \tag{6.22}$$

In the quark-diquark limit $\vec{r}_3 \to \vec{r}_2$ one has

$$\Big\{M_{12}^C\Big[\left(U_0{U_4}^\dagger U_3\right)\cdot\left(U_2{U_0}^\dagger U_1\right)\cdot U_4 + \left(U_1{U_0}^\dagger U_2\right)\cdot\left(U_3{U_4}^\dagger U_0\right)\cdot U_4\Big]$$
$$+\ \text{(all 5 permutations}\, 1 \leftrightarrow 2 \leftrightarrow 3)\Big\} + (0 \leftrightarrow 4)$$
$$\to 2\tilde{L}_{12}^C\Big[tr\left({U_0}^\dagger U_4\right)\left(tr\left({U_2}^\dagger U_0{U_4}^\dagger U_1\right) + tr\left({U_2}^\dagger U_1{U_4}^\dagger U_0\right)\right)$$
$$+\, 2tr\left({U_0}^\dagger U_1\right)tr\left({U_2}^\dagger U_4\right)tr\left({U_4}^\dagger U_0\right) - (0 \leftrightarrow 4)\Big], \tag{6.23}$$
$$\Big\{\tilde{L}_{12}^C\left(U_0{U_4}^\dagger U_2\right)\cdot\left(U_1{U_0}^\dagger U_4\right)\cdot U_3 + \text{(all 5 permutations}\, 1 \leftrightarrow 2 \leftrightarrow 3)\Big\} + (0 \leftrightarrow 4)$$
$$\to 2\tilde{L}_{12}^C\Big[tr\left({U_4}^\dagger U_0\right)\left(tr\left({U_0}^\dagger U_1{U_2}^\dagger U_4\right) + tr\left({U_0}^\dagger U_4{U_2}^\dagger U_1\right)\right) - (0 \leftrightarrow 4)\Big], \tag{6.24}$$
$$L_{12}^C\Big[\left(U_0{U_4}^\dagger U_2\right)\cdot\left(U_1{U_0}^\dagger U_4\right)\cdot U_3 + tr\left(U_0{U_4}^\dagger\right)\left(U_1{U_0}^\dagger U_2\right)\cdot U_3\cdot U_4 + \frac{1}{2}B_{123}$$
$$- \frac{3}{4}[B_{144}B_{234} + B_{244}B_{134} - B_{344}B_{124}] + \text{(all 5 permutations}\, 1 \leftrightarrow 2 \leftrightarrow 3)\Big] + (0 \leftrightarrow 4)$$
$$\to 4L_{12}^C\Big[tr\left({U_2}^\dagger U_1\right) - 3tr\left({U_0}^\dagger U_1\right)tr\left({U_2}^\dagger U_0\right) + tr\left({U_0}^\dagger U_1\right)tr\left({U_2}^\dagger U_4\right)tr\left({U_4}^\dagger U_0\right)$$
$$- tr\left({U_0}^\dagger U_1{U_4}^\dagger U_0{U_2}^\dagger U_4\right) + (0 \leftrightarrow 4)\Big]. \tag{6.25}$$

We get

$$\langle K_{NLO}\otimes B_{122}^{\rm conf}\rangle = -\frac{\alpha_s^2}{2\pi^4}\int d\vec{r}_0 d\vec{r}_4\ \Big(\Big\{\left(\tilde{L}_{12}^C + L_{12}^C\right)tr\left({U_0}^\dagger U_1\right)tr\left({U_2}^\dagger U_4\right)tr\left({U_4}^\dagger U_0\right)$$
$$+\ L_{12}^C\Big[tr\left({U_2}^\dagger U_1\right) - 3tr\left({U_0}^\dagger U_1\right)tr\left({U_2}^\dagger U_0\right) - tr\left({U_0}^\dagger U_1{U_4}^\dagger U_0{U_2}^\dagger U_4\right)\Big]\Big\} + (0 \leftrightarrow 4)\Big)$$
$$- \frac{3\alpha_s^2}{2\pi^3}\int d\vec{r}_0\frac{11}{6}\left[\ln\left(\frac{\vec{r}_{01}^{\,2}}{\vec{r}_{02}^{\,2}}\right)\left(\frac{1}{\vec{r}_{02}^{\,2}} - \frac{1}{\vec{r}_{01}^{\,2}}\right) - \frac{\vec{r}_{12}^{\,2}}{\vec{r}_{01}^{\,2}\vec{r}_{02}^{\,2}}\ln\left(\frac{\vec{r}_{12}^{\,2}}{\tilde{\mu}^2}\right)\right]$$
$$\times\left(tr\left({U_0}^\dagger U_1\right)tr\left({U_2}^\dagger U_0\right) - 3tr\left({U_2}^\dagger U_1\right)\right) \tag{6.26}$$

which is twice the gluon part of the BK kernel (see (67) in [19]).

7 Linearization

In the 3-gluon approximation

$$B_{003}B_{012} \overset{3g}{=} 6B_{003} + 6B_{012} - 36. \tag{7.1}$$

We use the following identity to linearize the color structures in (6.22).

$$
\begin{aligned}
&(U_0U_4^\dagger U_2)\cdot(U_1U_0^\dagger U_4)\cdot U_3+(1\leftrightarrow 2,0\leftrightarrow 4)\\
&=(U_0U_4^\dagger - E)(U_2-U_4)\cdot(U_1-U_0)U_0^\dagger U_4\cdot U_3+U_4U_0^\dagger(U_1-U_0)\cdot(U_2-U_4)(U_4^\dagger U_0-E)\cdot U_3\\
&+U_0\cdot(U_1U_0^\dagger U_4+U_4U_0^\dagger U_1)\cdot U_3+(U_0U_4^\dagger U_2+U_2U_4^\dagger U_0)\cdot U_4\cdot U_3\\
&+(U_2-U_4)\cdot(U_1-U_0)(U_0^\dagger U_4-E)\cdot U_3+(U_4U_0^\dagger-E)(U_1-U_0)\cdot(U_2-U_4)\cdot U_3\\
&+2(U_2-U_4)\cdot(U_1-U_0)\cdot U_3-2U_0\cdot U_4\cdot U_3.
\end{aligned}
\tag{7.2}
$$

Here E is the identity matrix. In the 3-gluon approximation the previous expression reads

$$
\begin{aligned}
&(U_0U_4^\dagger U_2)\cdot(U_1U_0^\dagger U_4)\cdot U_3+(1\leftrightarrow 2,0\leftrightarrow 4)\\
&\stackrel{3g}{=}(U_0-U_4)(U_2-U_4)\cdot(U_1-U_0)\cdot E+(U_1-U_0)\cdot(U_2-U_4)(U_0-U_4)\cdot E\\
&+U_0\cdot(U_1U_0^\dagger U_4+U_4U_0^\dagger U_1)\cdot U_3+(U_0U_4^\dagger U_2+U_2U_4^\dagger U_0)\cdot U_4\cdot U_3\\
&+(U_2-U_4)\cdot(U_1-U_0)(U_4-U_0)\cdot E+(U_4-U_0)(U_1-U_0)\cdot(U_2-U_4)\cdot E\\
&+2(U_2-U_4)\cdot(U_1-U_0)\cdot U_3-2U_0\cdot U_4\cdot U_3.
\end{aligned}
\tag{7.3}
$$

Using identity (B.3) and the fact that in the 3-gluon approximation

$$
\begin{aligned}
&((U_0-U_4)(U_2-U_4)+(U_2-U_4)(U_0-U_4))\cdot(U_1-U_0)\cdot E\\
&\stackrel{3g}{=}-(U_2-U_4)\cdot(U_0-U_4)\cdot(U_1-U_0),
\end{aligned}
\tag{7.4}
$$

we get

$$
\begin{aligned}
&(U_0U_4^\dagger U_2)\cdot(U_1U_0^\dagger U_4)\cdot U_3+(1\leftrightarrow 2,0\leftrightarrow 4)\\
&\stackrel{3g}{=}-B_{134}+\frac{1}{2}(B_{100}B_{340}+B_{400}B_{130}-B_{300}B_{140})\\
&-B_{023}+\frac{1}{2}(B_{044}B_{234}+B_{244}B_{034}-B_{344}B_{024})\\
&+2(U_2-U_4)\cdot(U_1-U_0)\cdot U_3-2U_0\cdot U_4\cdot U_3\\
&=B_{123}-3B_{134}+\frac{1}{2}(B_{100}B_{340}+B_{400}B_{130}-B_{300}B_{140})+(1\leftrightarrow 2,0\leftrightarrow 4)\\
&\stackrel{3g}{=}B_{123}+3(B_{100}+B_{340}+B_{400}+B_{130}-B_{300}-B_{140}-B_{134}-6)+(1\leftrightarrow 2,0\leftrightarrow 4).
\end{aligned}
\tag{7.5}
$$

As a result, the coefficient of $\tilde{L}_{12}^C$ in (6.22) reads

$$
\begin{aligned}
&(\left(U_0{U_4}^\dagger U_2\right)\cdot\left(U_1{U_0}^\dagger U_4\right)\cdot U_3+(1\leftrightarrow 2,0\leftrightarrow 4))-(0\leftrightarrow 4)\\
&\stackrel{3g}{=}(3B_{001}+6B_{130}-(1\leftrightarrow 2))-(0\leftrightarrow 4).
\end{aligned}
\tag{7.6}
$$

Using integrals (114) and (125) from [19],

$$
\int d\vec{r}_4\tilde{L}_{12}=\frac{\pi^2}{2}\zeta\left(3\right)\left(\delta\left(\vec{r}_{10}\right)-\delta\left(\vec{r}_{20}\right)\right),
\tag{7.7}
$$

and

$$\int d\vec{r}_4\left[\frac{\vec{r}_{12}{}^2}{4\vec{r}_{01}{}^2\vec{r}_{04}{}^2\vec{r}_{24}{}^2}\ln\left(\frac{\vec{r}_{02}{}^2\vec{r}_{14}{}^2}{\vec{r}_{04}{}^2\vec{r}_{12}{}^2}\right)-\frac{\vec{r}_{12}{}^2}{4\vec{r}_{02}{}^2\vec{r}_{04}{}^2\vec{r}_{14}{}^2}\ln\left(\frac{\vec{r}_{01}{}^2\vec{r}_{24}{}^2}{\vec{r}_{04}{}^2\vec{r}_{12}{}^2}\right)\right]$$
$$=\pi^2\zeta(3)\left(\delta(\vec{r}_{10})-\delta(\vec{r}_{20})\right), \tag{7.8}$$

one obtains

$$\int d\vec{r}_4\tilde{L}_{12}^C=\frac{3}{2}\pi^2\zeta(3)\left(\delta(\vec{r}_{10})-\delta(\vec{r}_{20})\right), \tag{7.9}$$

and

$$-\frac{\alpha_s^2}{8\pi^4}\int d\vec{r}_0 d\vec{r}_4\ \left(\left\{\tilde{L}_{12}^C\left(U_0U_4{}^\dagger U_2\right)\cdot\left(U_1U_0{}^\dagger U_4\right)\cdot U_3\right.\right.$$
$$\left.\left.+\ \text{(all 5 permutations}\, 1\leftrightarrow 2\leftrightarrow 3)\right\}+(0\leftrightarrow 4)\right)$$
$$\stackrel{3g}{=}-\frac{\alpha_s^2}{8\pi^4}\int d\vec{r}_0(3B_{001}+6B_{130}-(1\leftrightarrow 2))3\pi^2\zeta(3)\left(\delta(\vec{r}_{10})-\delta(\vec{r}_{20})\right)+(1\leftrightarrow 3)+(2\leftrightarrow 3)$$
$$=-\frac{9\alpha_s^2}{8\pi^2}\zeta(3)\left(36+B_{131}+B_{133}+B_{121}+B_{212}+B_{232}+B_{233}-12B_{231}\right)). \tag{7.10}$$

The second structure reads

$$\left(U_1U_0{}^\dagger U_2+U_2U_0{}^\dagger U_1\right)\cdot U_3\cdot U_4$$
$$=(U_1-U_0)U_0{}^\dagger(U_2-U_0))\cdot U_3\cdot(U_4-U_0)+(U_2-U_0)U_0{}^\dagger(U_1-U_0))\cdot U_3\cdot(U_4-U_0)$$
$$+2(U_2+U_1-U_0)\cdot U_3\cdot(U_4-U_0)+\left(U_1U_0{}^\dagger U_2+U_2U_0{}^\dagger U_1\right)\cdot U_3\cdot U_0. \tag{7.11}$$

Again, applying identity (B.3) and equality (7.4) one gets in the 3-gluon approximation

$$\left(U_1U_0{}^\dagger U_2+U_2U_0{}^\dagger U_1\right)\cdot U_3\cdot U_4\stackrel{3g}{=}-(U_1-U_0)\cdot(U_2-U_0)\cdot(U_4-U_0)$$
$$+2(U_2+U_1-U_0)\cdot U_3\cdot(U_4-U_0)-B_{123}+\frac{1}{2}\left(B_{100}B_{230}+B_{200}B_{130}-B_{300}B_{120}\right). \tag{7.12}$$

Finally, the coefficient of L_{12}^C in (6.22) reads

$$\left[\left(U_0U_4{}^\dagger U_2\right)\cdot\left(U_1U_0{}^\dagger U_4\right)\cdot U_3+tr\left(U_0U_4{}^\dagger\right)\left(U_1U_0{}^\dagger U_2\right)\cdot U_3\cdot U_4\right.$$
$$\left.-\frac{3}{4}[B_{144}B_{234}+B_{244}B_{134}-B_{344}B_{124}]+\frac{1}{2}B_{123}+(1\leftrightarrow 2)\right]+(0\leftrightarrow 4)$$
$$\stackrel{3g}{=}9(B_{044}+B_{004}-12) \tag{7.13}$$

and therefore

$$-\frac{\alpha_s^2}{8\pi^4}\int d\vec{r}_0 d\vec{r}_4\ \left(\left\{L_{12}^C\left[\left(U_0U_4{}^\dagger U_2\right)\cdot\left(U_1U_0{}^\dagger U_4\right)\cdot U_3+tr\left(U_0U_4{}^\dagger\right)\left(U_1U_0{}^\dagger U_2\right)\cdot U_3\cdot U_4\right.\right.\right.$$
$$\left.-\frac{3}{4}[B_{144}B_{234}+B_{244}B_{134}-B_{344}B_{124}]+\frac{1}{2}B_{123}\right]$$
$$\left.\left.+\ \text{(all 5 permutations}\, 1\leftrightarrow 2\leftrightarrow 3)\right\}+(0\leftrightarrow 4)\right)$$

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$$\stackrel{3g}{=} -\frac{9\alpha_s^2}{8\pi^4}\int d\vec{r}_0 d\vec{r}_4 (L_{12}^C + L_{13}^C + L_{23}^C)(B_{044} + B_{004} - 12) \ . \tag{7.14}$$

The third structure reads

$$\begin{aligned}
&(U_2U_0^\dagger U_1)\cdot U_4\cdot(U_0U_4^\dagger U_3) + (U_1U_0^\dagger U_2)\cdot U_4\cdot(U_3U_4^\dagger U_0)\\
&= U_1\cdot U_4\cdot(U_3U_4^\dagger U_0 + U_0U_4^\dagger U_3) + (U_1U_0^\dagger U_2 + U_2U_0^\dagger U_1)\cdot U_4\cdot U_0\\
&\quad + (U_2-U_0)\cdot U_4\cdot((U_0-U_4)U_4^\dagger(U_3-U_4) + (U_3-U_4)U_4^\dagger(U_0-U_4))\\
&\quad + ((U_1-U_0)U_0^\dagger(U_2-U_0))\cdot U_4\cdot((U_3-U_4)U_4^\dagger U_0)\\
&\quad + ((U_2-U_0)U_0^\dagger(U_1-U_0))\cdot U_4\cdot(U_0U_4^\dagger(U_3-U_4))\\
&\quad + 2(U_2-U_0)\cdot U_4\cdot(U_3-U_4) - 2U_1\cdot U_4\cdot U_0.
\end{aligned}\tag{7.15}$$

Using (B.3) and (7.4) we get

$$\begin{aligned}
&(U_2U_0^\dagger U_1)\cdot U_4\cdot(U_0U_4^\dagger U_3) + (U_1U_0^\dagger U_2)\cdot U_4\cdot(U_3U_4^\dagger U_0)\\
&\stackrel{3g}{=} -B_{013} + \frac{1}{2}(B_{344}B_{014} + B_{044}B_{134} - B_{144}B_{034}) - B_{124}\\
&- (U_2-U_0)\cdot(U_1-3U_4)\cdot(U_3-U_4) - 2U_1\cdot U_4\cdot U_0 + \frac{1}{2}(B_{100}B_{240} + B_{200}B_{140} - B_{004}B_{012})\\
&\stackrel{3g}{=} 3(B_{010} - B_{441} + B_{020} - B_{442} - B_{040} + 2B_{440} - B_{120} + B_{140}\\
&+ B_{341} + B_{240} - 2B_{340} + B_{342} + B_{443}) - B_{231} - 36.
\end{aligned}\tag{7.16}$$

As a result, one obtains

$$\begin{aligned}
&-\frac{\alpha_s^2}{8\pi^4}\int d\vec{r}_0 d\vec{r}_4 \ \left(\left\{M_{12}^C\left[\left(U_0U_4{}^\dagger U_3\right)\cdot\left(U_2U_0{}^\dagger U_1\right)\cdot U_4 + \left(U_1U_0{}^\dagger U_2\right)\cdot\left(U_3U_4{}^\dagger U_0\right)\cdot U_4\right]\right.\right.\\
&+ \ \left.\left.(\text{all 5 permutations}\, 1\leftrightarrow 2\leftrightarrow 3)\right\} + (0\leftrightarrow 4)\right)\\
&\stackrel{3g}{=} -\frac{3\alpha_s^2}{32\pi^4}\int d\vec{r}_0 d\vec{r}_4 \ \left(\frac{3}{2}F_0(B_{040} - B_{044}) + \left\{\frac{3}{2}F_{140}B_{140} + F_{100}B_{100} + F_{230}B_{230}\right.\right.\\
&+ \ \left.\left.(0\leftrightarrow 4)\right\} + (\text{all 5 permutations}\, 1\leftrightarrow 2\leftrightarrow 3)\right) .
\end{aligned}\tag{7.17}$$

where

$$\begin{aligned}
F_0 =& \frac{\vec{r}_{12}{}^2}{2\vec{r}_{14}{}^2\vec{r}_{24}{}^2}\left(\frac{\vec{r}_{24}{}^2}{\vec{r}_{02}{}^2\vec{r}_{04}{}^2}\ln\left(\frac{\vec{r}_{01}{}^2\vec{r}_{02}{}^2\vec{r}_{34}{}^4}{\vec{r}_{14}{}^2\vec{r}_{24}{}^2\vec{r}_{03}{}^4}\right) - \frac{\vec{r}_{13}{}^2}{\vec{r}_{01}{}^2\vec{r}_{03}{}^2}\ln\left(\frac{\vec{r}_{01}{}^2\vec{r}_{13}{}^2\vec{r}_{24}{}^2}{\vec{r}_{03}{}^2\vec{r}_{12}{}^2\vec{r}_{14}{}^2}\right)\right.\\
&\left. + \frac{2\vec{r}_{34}{}^2}{\vec{r}_{03}{}^2\vec{r}_{04}{}^2}\ln\left(\frac{\vec{r}_{01}{}^2\vec{r}_{02}{}^2\vec{r}_{34}{}^2}{\vec{r}_{03}{}^2\vec{r}_{04}{}^2\vec{r}_{12}{}^2}\right)\right) - (0\leftrightarrow 4).
\end{aligned}\tag{7.18}$$

$$\begin{aligned}
F_{140} =& \frac{\vec{r}_{12}{}^2}{\vec{r}_{02}{}^2\vec{r}_{04}{}^2\vec{r}_{14}{}^2}\ln\left(\frac{\vec{r}_{02}{}^2\vec{r}_{04}{}^2\vec{r}_{12}{}^2\vec{r}_{34}{}^4}{\vec{r}_{03}{}^4\vec{r}_{14}{}^2\vec{r}_{24}{}^4}\right)\\
&-\frac{\vec{r}_{01}{}^2\vec{r}_{23}{}^2}{\vec{r}_{02}{}^2\vec{r}_{03}{}^2\vec{r}_{04}{}^2\vec{r}_{14}{}^2}\ln\left(\frac{\vec{r}_{01}{}^2\vec{r}_{24}{}^2\vec{r}_{34}{}^2}{\vec{r}_{04}{}^2\vec{r}_{14}{}^2\vec{r}_{23}{}^2}\right) - \frac{\vec{r}_{23}{}^2\vec{r}_{12}{}^2}{\vec{r}_{02}{}^2\vec{r}_{03}{}^2\vec{r}_{14}{}^2\vec{r}_{24}{}^2}\ln\left(\frac{\vec{r}_{02}{}^2\vec{r}_{14}{}^2\vec{r}_{23}{}^2}{\vec{r}_{03}{}^2\vec{r}_{12}{}^2\vec{r}_{24}{}^2}\right)\\
&+\frac{\vec{r}_{23}{}^2}{\vec{r}_{03}{}^2\vec{r}_{04}{}^2\vec{r}_{24}{}^2}\ln\left(\frac{\vec{r}_{02}{}^2\vec{r}_{34}{}^2}{\vec{r}_{04}{}^2\vec{r}_{23}{}^2}\right) + \frac{\vec{r}_{02}{}^2\vec{r}_{13}{}^2}{\vec{r}_{01}{}^2\vec{r}_{03}{}^2\vec{r}_{04}{}^2\vec{r}_{24}{}^2}\ln\left(\frac{\vec{r}_{01}{}^2\vec{r}_{02}{}^2\vec{r}_{34}{}^4}{\vec{r}_{03}{}^2\vec{r}_{04}{}^2\vec{r}_{13}{}^2\vec{r}_{24}{}^2}\right) .
\end{aligned}\tag{7.19}$$

$$F_{100} = \frac{\vec r_{23}{}^2}{2\vec r_{03}{}^2\vec r_{04}{}^2\vec r_{24}{}^2}\ln\left(\frac{\vec r_{01}{}^8\vec r_{04}{}^2\vec r_{23}{}^2\vec r_{24}{}^4\vec r_{34}{}^2}{\vec r_{02}{}^6\vec r_{03}{}^4\vec r_{14}{}^8}\right) - \frac{\vec r_{02}{}^2\vec r_{13}{}^2}{\vec r_{01}{}^2\vec r_{03}{}^2\vec r_{04}{}^2\vec r_{24}{}^2}\ln\left(\frac{\vec r_{01}{}^2\vec r_{04}{}^2\vec r_{13}{}^2\vec r_{24}{}^2}{\vec r_{02}{}^2\vec r_{03}{}^2\vec r_{14}{}^4}\right)$$
$$-\frac{\vec r_{34}{}^2\vec r_{12}{}^2}{2\vec r_{03}{}^2\vec r_{04}{}^2\vec r_{14}{}^2\vec r_{24}{}^2}\ln\left(\frac{\vec r_{01}{}^8\vec r_{02}{}^4\vec r_{24}{}^2\vec r_{34}{}^6}{\vec r_{03}{}^6\vec r_{04}{}^6\vec r_{12}{}^6\vec r_{14}{}^2}\right) - \frac{\vec r_{12}{}^2}{2\vec r_{02}{}^2\vec r_{04}{}^2\vec r_{14}{}^2}\ln\left(\frac{\vec r_{01}{}^4\vec r_{02}{}^2\vec r_{34}{}^4}{\vec r_{03}{}^4\vec r_{04}{}^2\vec r_{12}{}^2\vec r_{14}{}^2}\right)$$
$$+\frac{\vec r_{23}{}^2\vec r_{12}{}^2}{2\vec r_{02}{}^2\vec r_{03}{}^2\vec r_{14}{}^2\vec r_{24}{}^2}\ln\left(\frac{\vec r_{02}{}^2\vec r_{14}{}^2\vec r_{23}{}^2}{\vec r_{03}{}^2\vec r_{12}{}^2\vec r_{24}{}^2}\right) + \frac{\vec r_{12}{}^2}{\vec r_{01}{}^2\vec r_{04}{}^2\vec r_{24}{}^2}\ln\left(\frac{\vec r_{02}{}^2\vec r_{14}{}^2}{\vec r_{04}{}^2\vec r_{12}{}^2}\right)$$
$$+\frac{\vec r_{13}{}^2\vec r_{12}{}^2}{\vec r_{01}{}^2\vec r_{03}{}^2\vec r_{14}{}^2\vec r_{24}{}^2}\ln\left(\frac{\vec r_{01}{}^2\vec r_{13}{}^2\vec r_{24}{}^2}{\vec r_{03}{}^2\vec r_{12}{}^2\vec r_{14}{}^2}\right) + \frac{\vec r_{01}{}^2\vec r_{23}{}^2}{2\vec r_{02}{}^2\vec r_{03}{}^2\vec r_{04}{}^2\vec r_{14}{}^2}\ln\left(\frac{\vec r_{01}{}^2\vec r_{24}{}^2\vec r_{34}{}^2}{\vec r_{04}{}^2\vec r_{14}{}^2\vec r_{23}{}^2}\right). \quad (7.20)$$

$$F_{230} = \frac{\vec r_{02}{}^2\vec r_{13}{}^2}{2\vec r_{01}{}^2\vec r_{03}{}^2\vec r_{04}{}^2\vec r_{24}{}^2}\ln\left(\frac{\vec r_{01}{}^2\vec r_{04}{}^2\vec r_{13}{}^2\vec r_{24}{}^2}{\vec r_{02}{}^2\vec r_{03}{}^2\vec r_{14}{}^4}\right) - \frac{\vec r_{23}{}^2}{2\vec r_{03}{}^2\vec r_{04}{}^2\vec r_{24}{}^2}\ln\left(\frac{\vec r_{01}{}^4\vec r_{04}{}^4\vec r_{23}{}^4\vec r_{24}{}^2}{\vec r_{02}{}^6\vec r_{03}{}^2\vec r_{14}{}^4\vec r_{34}{}^2}\right)$$
$$+\frac{\vec r_{34}{}^2\vec r_{12}{}^2}{2\vec r_{03}{}^2\vec r_{04}{}^2\vec r_{14}{}^2\vec r_{24}{}^2}\ln\left(\frac{\vec r_{01}{}^4\vec r_{02}{}^8\vec r_{14}{}^2\vec r_{34}{}^6}{\vec r_{03}{}^6\vec r_{04}{}^6\vec r_{12}{}^6\vec r_{24}{}^2}\right) + \frac{\vec r_{12}{}^2}{2\vec r_{02}{}^2\vec r_{04}{}^2\vec r_{14}{}^2}\ln\left(\frac{\vec r_{01}{}^2\vec r_{02}{}^4\vec r_{04}{}^2\vec r_{12}{}^2\vec r_{34}{}^8}{\vec r_{03}{}^8\vec r_{14}{}^4\vec r_{24}{}^6}\right)$$
$$-\frac{\vec r_{23}{}^2\vec r_{12}{}^2}{\vec r_{02}{}^2\vec r_{03}{}^2\vec r_{14}{}^2\vec r_{24}{}^2}\ln\left(\frac{\vec r_{02}{}^2\vec r_{14}{}^2\vec r_{23}{}^2}{\vec r_{03}{}^2\vec r_{12}{}^2\vec r_{24}{}^2}\right) - \frac{\vec r_{12}{}^2}{2\vec r_{01}{}^2\vec r_{04}{}^2\vec r_{24}{}^2}\ln\left(\frac{\vec r_{02}{}^2\vec r_{14}{}^2}{\vec r_{04}{}^2\vec r_{12}{}^2}\right)$$
$$-\frac{\vec r_{13}{}^2\vec r_{12}{}^2}{2\vec r_{01}{}^2\vec r_{03}{}^2\vec r_{14}{}^2\vec r_{24}{}^2}\ln\left(\frac{\vec r_{01}{}^2\vec r_{13}{}^2\vec r_{24}{}^2}{\vec r_{03}{}^2\vec r_{12}{}^2\vec r_{14}{}^2}\right) - \frac{\vec r_{01}{}^2\vec r_{23}{}^2}{\vec r_{02}{}^2\vec r_{03}{}^2\vec r_{04}{}^2\vec r_{14}{}^2}\ln\left(\frac{\vec r_{01}{}^2\vec r_{24}{}^2\vec r_{34}{}^2}{\vec r_{04}{}^2\vec r_{14}{}^2\vec r_{23}{}^2}\right). \quad (7.21)$$

One can integrate F_{100} and F_{230} w.r.t. $\vec r_4$. The integrals are given in appendix D (D.22) and (D.31).

The color structure in the quark part of the kernel can be linearized using (7.12)

$$\frac{1}{2}\left\{\left(\frac{1}{3}(U_1U_0{}^\dagger U_4 + U_4U_0{}^\dagger U_1)\cdot U_2\cdot U_3 - \frac{1}{9}B_{123}tr(U_0{}^\dagger U_4) + (U_1U_0{}^\dagger U_2)\cdot U_3\cdot U_4\right.\right.$$
$$\left.\left.+\ \frac{1}{6}B_{123} - \frac{1}{4}(B_{013}B_{002} + B_{001}B_{023} - B_{012}B_{003}) + (1\leftrightarrow 2)\right) + (0\leftrightarrow 4)\right\}$$
$$\overset{3g}{\equiv} \frac{1}{6}\,(12 - B_{004} - B_{044} + 2\,(2B_{014} - B_{001} - B_{144})$$
$$+ 2\,(2B_{024} - B_{002} - B_{244}) - 4\,(2B_{034} - B_{344} - B_{003})). \quad (7.22)$$

and therefore

$$-\frac{\alpha_s^2 n_f}{8\pi^4}\int d\vec r_0 d\vec r_4\ \mathbf{G}^q \overset{3g}{\equiv} -\frac{\alpha_s^2 n_f}{48\pi^4}\int d\vec r_0 d\vec r_4\ \ \{(12 - B_{004} - B_{044})\,(L^q_{12} + L^q_{13} + L^q_{23})$$
$$+ 2\,(2B_{014} - B_{001} - B_{144})\,(L^q_{12} + L^q_{13} - 2L^q_{32}) + 2\,(2B_{024} - B_{002} - B_{244})\,(L^q_{12} + L^q_{23} - 2L^q_{31})$$
$$+ 2\,(2B_{034} - B_{344} - B_{003})\,(L^q_{32} + L^q_{13} - 2L^q_{12})\}\,. \quad (7.23)$$

Finally, we get the linearized kernel in the form

$$\langle K_{NLO}\otimes B^{\rm conf}_{123}\rangle \overset{3g}{\equiv} \frac{27\alpha_s^2}{4\pi^2}\zeta(3)(3 - \delta_{23} - \delta_{13} - \delta_{21})(B_{123} - 6)$$
$$-\frac{9\alpha_s^2}{8\pi^4}\int d\vec r_0 d\vec r_4\left(L^C_{12} + L^C_{13} + L^C_{23} - \frac{n_f}{54}(L^q_{12} + L^q_{13} + L^q_{23})\right)(B_{044} + B_{004} - 12)$$
$$-\frac{\alpha_s^2 n_f}{24\pi^4}\int d\vec r_0 d\vec r_4\ \ \{(2B_{014} - B_{001} - B_{144})\,(L^q_{12} + L^q_{13} - 2L^q_{32}) + (1\leftrightarrow 3) + (1\leftrightarrow 2)\}$$
$$-\frac{9\alpha_s^2}{64\pi^4}\int d\vec r_0 d\vec r_4\ \ (F_0(B_{040} - B_{044}) + \{F_{140} + (0\leftrightarrow 4)\}\,B_{140} + (\text{all 5 perm.}\ 1\leftrightarrow 2\leftrightarrow 3))$$

$$
\begin{aligned}
&-\frac{9\alpha_s^2}{64\pi^3}\int d\vec r_0\left(\tilde F_{100}B_{100}+\tilde F_{230}B_{230}+(1\leftrightarrow 3)+(1\leftrightarrow 2)\right)\\
&-\frac{9\alpha_s^2}{16\pi^3}\int d\vec r_0\left(\beta\left[\ln\left(\frac{\vec r_{01}^{\,2}}{\vec r_{02}^{\,2}}\right)\left(\frac{1}{\vec r_{02}^{\,2}}-\frac{1}{\vec r_{01}^{\,2}}\right)-\frac{\vec r_{12}^{\,2}}{\vec r_{01}^{\,2}\vec r_{02}^{\,2}}\ln\left(\frac{\vec r_{12}^{\,2}}{\tilde\mu^2}\right)\right]\right.\\
&\left.\times\left(B_{100}+B_{230}+B_{200}+B_{130}-B_{300}-B_{210}-B_{123}-6\right)+(1\leftrightarrow 3)+(2\leftrightarrow 3)\right). \qquad (7.24)
\end{aligned}
$$

Here $\delta_{ij}=1$, if $\vec r_i=\vec r_j$ and $\delta_{ij}=0$ otherwise; $\tilde\mu^2$ and β are defined in (5.9); F_0 and F_{140} are defined in (7.18) and (7.19); L_{12}^q is defined in (3.6) and L_{12}^C is defined in (6.17) and

$$
\begin{aligned}
\tilde F_{100}&=\left(\frac{\vec r_{12}{}^2}{\vec r_{01}{}^2\vec r_{02}{}^2}-\frac{\vec r_{13}{}^2}{\vec r_{01}{}^2\vec r_{03}{}^2}-\frac{2\vec r_{23}{}^2}{\vec r_{02}{}^2\vec r_{03}{}^2}\right)\ln^2\left(\frac{\vec r_{02}{}^2\vec r_{13}{}^2}{\vec r_{01}{}^2\vec r_{23}{}^2}\right)+\frac{\vec r_{23}{}^2}{2\vec r_{02}{}^2\vec r_{03}{}^2}\ln^2\left(\frac{\vec r_{03}{}^2\vec r_{12}{}^2}{\vec r_{02}{}^2\vec r_{13}{}^2}\right)\\
&\quad+\tilde S_{123}I\left(\frac{\vec r_{12}{}^2}{\vec r_{01}{}^2\vec r_{02}{}^2},\frac{\vec r_{13}{}^2}{\vec r_{01}{}^2\vec r_{03}{}^2},\frac{\vec r_{23}{}^2}{\vec r_{02}{}^2\vec r_{03}{}^2}\right)+(2\leftrightarrow 3), \qquad (7.25)\\
\tilde F_{230}&=\left(\frac{2\vec r_{12}{}^2}{\vec r_{01}{}^2\vec r_{02}{}^2}-\frac{\vec r_{23}{}^2}{2\vec r_{02}{}^2\vec r_{03}{}^2}\right)\ln^2\left(\frac{\vec r_{03}{}^2\vec r_{12}{}^2}{\vec r_{02}{}^2\vec r_{13}{}^2}\right)+\left(\frac{\vec r_{13}{}^2}{\vec r_{01}{}^2\vec r_{03}{}^2}-\frac{\vec r_{12}{}^2}{\vec r_{01}{}^2\vec r_{02}{}^2}\right)\ln^2\left(\frac{\vec r_{02}{}^2\vec r_{13}{}^2}{\vec r_{01}{}^2\vec r_{23}{}^2}\right)\\
&\quad-\tilde S_{123}I\left(\frac{\vec r_{12}{}^2}{\vec r_{01}{}^2\vec r_{02}{}^2},\frac{\vec r_{13}{}^2}{\vec r_{01}{}^2\vec r_{03}{}^2},\frac{\vec r_{23}{}^2}{\vec r_{02}{}^2\vec r_{03}{}^2}\right)+(2\leftrightarrow 3). \qquad (7.26)
\end{aligned}
$$

The functions $\tilde S_{123}$ and I are defined in appendix D (D.15) and (D.11).

Now, if we consider the dipole limit $\vec r_3=\vec r_2$ and take into account that in this limit

$$
\begin{aligned}
\tilde F_{100}|_{\vec r_3=\vec r_2}=\tilde F_{200}|_{\vec r_3=\vec r_2}=\tilde F_{300}|_{\vec r_3=\vec r_2}&=0, &(7.27)\\
\tilde F_{230}|_{\vec r_3=\vec r_2}=\tilde F_{130}|_{\vec r_3=\vec r_2}=\tilde F_{210}|_{\vec r_3=\vec r_2}&=0 &(7.28)\\
(F_0+(\text{all 5 permutations}\,1\leftrightarrow 2\leftrightarrow 3))|_{\vec r_3=\vec r_2}&=-16\tilde L_{12}^C, &(7.29)\\
((F_{140}+(0\leftrightarrow 4))+(2\leftrightarrow 3))\,|_{\vec r_3=\vec r_2}&=0, &(7.30)\\
((F_{340}+(0\leftrightarrow 4))+(2\leftrightarrow 1))\,|_{\vec r_3=\vec r_2}&=0, &(7.31)\\
((F_{240}+(0\leftrightarrow 4))+(1\leftrightarrow 3))\,|_{\vec r_3=\vec r_2}&=0, &(7.32)
\end{aligned}
$$

we obtain the linearized BK kernel in the 3-gluon approximation whose C-even part is the BFKL kernel [33]

$$
\begin{aligned}
\langle K_{NLO}\otimes B_{122}^{\rm conf}\rangle &\overset{\rm 3g}{=}-\frac{9\alpha_s^2}{4\pi^4}\int d\vec r_0 d\vec r_4\left(L_{12}^C-\frac{n_f}{54}L_{12}^q\right)(B_{044}+B_{004}-12)\\
&-\frac{\alpha_s^2 n_f}{12\pi^4}\int d\vec r_0 d\vec r_4\left\{(2B_{014}-B_{001}-B_{144})-(2B_{024}-B_{002}-B_{244})\right\}L_{12}^q\\
&+\frac{27\alpha_s^2}{2\pi^2}\zeta(3)(B_{122}-6)-\frac{9\alpha_s^2}{4\pi^4}\int d\vec r_0 d\vec r_4\ \tilde L_{12}^C(B_{044}-B_{040})\\
&-\frac{9\alpha_s^2}{8\pi^3}\beta\int d\vec r_0\left[\ln\left(\frac{\vec r_{01}^{\,2}}{\vec r_{02}^{\,2}}\right)\left(\frac{1}{\vec r_{02}^{\,2}}-\frac{1}{\vec r_{01}^{\,2}}\right)-\frac{\vec r_{12}^{\,2}}{\vec r_{01}^{\,2}\vec r_{02}^{\,2}}\ln\left(\frac{\vec r_{12}^{\,2}}{\tilde\mu^2}\right)\right](B_{100}+B_{220}-B_{122}-6). \qquad (7.33)
\end{aligned}
$$

Let us compare this kernel for $B_{122}=2tr(U_1U_2^\dagger)$ with the linearized BK kernel in the 2-gluon approximation from [19]. One can see that their C-even parts coincide as they are fixed by the BFKL kernel [33]. However the 2-gluon approximation is not enough to figure out the correct C-odd part of the kernel. Only the 3-gluon approximation (7.33) allows one to write it. It is easy to see that even for the color dipole the C-odd part of the kernel in the 3-gluon approximation can not be expressed through dipoles only. One necessarily needs to introduce the 3QWL operators as is clear from the second line of this expression. One can check it by direct calculation via (7.12), indeed

$$\begin{aligned}
&12\left\{tr(U_2^\dagger t^a U_1 t^b)tr(U_4^\dagger t^a U_0 t^b)-(0\to 4)\right\}+(0\leftrightarrow 4)\\
&=\left\{\frac{1}{3}tr(U_0U_4^\dagger)tr(U_2{}^\dagger U_1)+3tr(U_4{}^\dagger U_1)tr(U_2{}^\dagger U_0)\right.\\
&-\left. tr(U_0U_4^\dagger U_1U_2{}^\dagger)-tr(U_0U_2{}^\dagger U_1U_4^\dagger)-(0\to 4)\right\}+(0\leftrightarrow 4)\\
&=\left\{\frac{1}{12}B_{044}B_{122}+\frac{3}{4}B_{144}B_{022}-\frac{1}{2}tr(U_1U_2{}^\dagger U_0+U_0U_2{}^\dagger U_1)\cdot U_4\cdot U_4-(0\to 4)\right\}+(0\leftrightarrow 4)\\
&\overset{3g}{=}\frac{1}{2}\left\{12-B_{044}-B_{004}+2(2B_{014}-B_{001}-B_{144})-2(2B_{024}-B_{002}-B_{244})\right\}. \qquad (7.34)
\end{aligned}$$

As in [25] to separate the C-even and C-odd contributions we introduce C-even (pomeron) and C-odd (odderon) Green functions

$$B^+_{123}=B_{123}+B_{\bar{1}\bar{2}\bar{3}}-12, \qquad (7.35)$$

and

$$B^-_{123}=B_{123}-B_{\bar{1}\bar{2}\bar{3}}, \qquad (7.36)$$

where $B_{\bar{1}\bar{2}\bar{3}}$ is the 3-antiquark Wilson loop operator

$$B_{\bar{1}\bar{2}\bar{3}}=U_1^\dagger\cdot U_2^\dagger\cdot U_3^\dagger. \qquad (7.37)$$

The NLO kernel for the C-even Green function in the 3-gluon approximation reads

$$\begin{aligned}
&\langle K_{NLO}\otimes B^{+conf}_{123}\rangle\overset{3g}{=}-\frac{9\alpha_s^2}{4\pi^4}\int d\vec r_0 d\vec r_4(L^C_{12}+L^C_{13}+L^C_{23}-\frac{n_f}{54}(L^q_{12}+L^q_{13}+L^q_{23}))B^+_{044}\\
&+\frac{27\alpha_s^2}{4\pi^2}\zeta(3)(3-\delta_{23}-\delta_{13}-\delta_{21})B^+_{123}\\
&-\frac{9\alpha_s^2}{64\pi^4}\int d\vec r_0 d\vec r_4\;\left(\{F_{140}+(0\leftrightarrow 4)\}\,B^+_{140}+(\text{all 5 permutations}\,1\leftrightarrow 2\leftrightarrow 3)\right)\\
&-\frac{9\alpha_s^2}{64\pi^3}\int d\vec r_0\left(\tilde F_{100}B^+_{100}+\tilde F_{230}B^+_{230}+(1\leftrightarrow 3)+(1\leftrightarrow 2)\right)\\
&-\frac{9\alpha_s^2}{16\pi^3}\int d\vec r_0\left(\beta\left[\ln\left(\frac{\vec r_{01}^{\,2}}{\vec r_{02}^{\,2}}\right)\left(\frac{1}{\vec r_{02}^{\,2}}-\frac{1}{\vec r_{01}^{\,2}}\right)-\frac{\vec r_{12}^{\,2}}{\vec r_{01}^{\,2}\vec r_{02}^{\,2}}\ln\left(\frac{\vec r_{12}^{\,2}}{\tilde\mu^2}\right)\right]\right.\\
&\times\left.\left(B^+_{100}+B^+_{230}+B^+_{200}+B^+_{130}-B^+_{300}-B^+_{210}-B^+_{123}\right)+(1\leftrightarrow 3)+(2\leftrightarrow 3)\right). \qquad (7.38)
\end{aligned}$$

In the 3-gluon approximation we can use the identity [25]

$$B^+_{123}\overset{3g}{=}\frac{1}{2}(B^+_{133}+B^+_{211}+B^+_{322}), \qquad (7.39)$$

which kills all the terms in the third line in (7.24) in the C-even case. It is easy to see that the same identity holds true for the "conformal prescription" (6.1)

$$B^{+conf}_{123}\overset{3g}{=}\frac{1}{2}(B^{+conf}_{133}+B^{+conf}_{211}+B^{+conf}_{322}) \qquad (7.40)$$

and we get

$$\langle K_{NLO}\otimes B^{+conf}_{123}\rangle \stackrel{3g}{=} \frac{1}{2}\langle K_{NLO}\otimes(B^{+conf}_{133}+B^{+conf}_{211}+B^{+conf}_{322})\rangle. \tag{7.41}$$

This equality imposes the following constraints

$$0 = \{F_{140}+(0\leftrightarrow 4)\}+(\text{all 5 permutations}\, 1\leftrightarrow 2\leftrightarrow 3), \tag{7.42}$$

$$0 = \int d\vec{r}_0 \tilde{F}_{230}, \tag{7.43}$$

$$0 = \int \frac{d\vec{r}_4}{\pi}\ (\{F_{140}+(0\leftrightarrow 4)\}+(2\leftrightarrow 3))+\tilde{F}_{100}+\frac{1}{2}\tilde{F}_{230}|_{1\leftrightarrow 3}+\frac{1}{2}\tilde{F}_{230}|_{1\leftrightarrow 2}. \tag{7.44}$$

Constraint (7.42) follows from the definition of F_{140} (7.19) directly whereas constraint (7.43) holds since thanks to conformal invariance

$$\int d\vec{r}_0 \tilde{F}_{230} = \int d\vec{r}_0 \tilde{F}_{230}|_{2=3} = 0. \tag{7.45}$$

Using (7.25) and (7.26) one can rewrite constraint (7.44) as

$$\int \frac{d\vec{r}_4}{\pi}\ (\{F_{140}+(0\leftrightarrow 4)\}+(2\leftrightarrow 3)) = \frac{\vec{r}_{23}{}^2}{2\vec{r}_{02}{}^2\vec{r}_{03}{}^2}\left(\ln^2\left(\frac{\vec{r}_{03}{}^2\vec{r}_{12}{}^2}{\vec{r}_{01}{}^2\vec{r}_{23}{}^2}\right)+\ln^2\left(\frac{\vec{r}_{02}{}^2\vec{r}_{13}{}^2}{\vec{r}_{01}{}^2\vec{r}_{23}{}^2}\right)\right)$$
$$-\frac{1}{2}\left(\frac{\vec{r}_{12}{}^2}{\vec{r}_{01}{}^2\vec{r}_{02}{}^2}+\frac{\vec{r}_{13}{}^2}{\vec{r}_{01}{}^2\vec{r}_{03}{}^2}\right)\ln^2\left(\frac{\vec{r}_{02}{}^2\vec{r}_{13}{}^2}{\vec{r}_{03}{}^2\vec{r}_{12}{}^2}\right). \tag{7.46}$$

The calculation of the integral and proof of this identity is given in the appendix D.

The NLO kernel for the C-odd Green function in the 3-gluon approximation reads

$$\begin{aligned}
&\langle K_{NLO}\otimes B^{-conf}_{123}\rangle \stackrel{3g}{=} \frac{27\alpha_s^2}{4\pi^2}\zeta(3)(3-\delta_{23}-\delta_{13}-\delta_{21})B^{-}_{123}\\
&-\frac{\alpha_s^2 n_f}{24\pi^4}\int d\vec{r}_0 d\vec{r}_4\ \left\{\left(2B^{-}_{014}-B^{-}_{001}-B^{-}_{144}\right)\left(L^q_{12}+L^q_{13}-2L^q_{32}\right)+(1\leftrightarrow 3)+(1\leftrightarrow 2)\right\}\\
&-\frac{9\alpha_s^2}{64\pi^4}\int d\vec{r}_0 d\vec{r}_4\ \left(2F_0B^{-}_{040}+\{F_{140}+(0\leftrightarrow 4)\}\,B^{-}_{140}+(\text{all 5 permutations}\,1\leftrightarrow 2\leftrightarrow 3)\right)\\
&-\frac{9\alpha_s^2}{64\pi^3}\int d\vec{r}_0\left(\tilde{F}_{100}B^{-}_{100}+\tilde{F}_{230}B^{-}_{230}+(1\leftrightarrow 3)+(1\leftrightarrow 2)\right)\\
&-\frac{9\alpha_s^2}{16\pi^3}\int d\vec{r}_0\left(\beta\left[\ln\left(\frac{\vec{r}_{01}^{\,2}}{\vec{r}_{02}^{\,2}}\right)\left(\frac{1}{\vec{r}_{02}^{\,2}}-\frac{1}{\vec{r}_{01}^{\,2}}\right)-\frac{\vec{r}_{12}^{\,2}}{\vec{r}_{01}^{\,2}\vec{r}_{02}^{\,2}}\ln\left(\frac{\vec{r}_{12}^{\,2}}{\tilde{\mu}^2}\right)\right]\right.\\
&\left.\times\left(B^{-}_{100}+B^{-}_{230}+B^{-}_{200}+B^{-}_{130}-B^{-}_{300}-B^{-}_{210}-B^{-}_{123}\right)+(1\leftrightarrow 3)+(2\leftrightarrow 3)\right).
\end{aligned} \tag{7.47}$$

8 Results

In this section we list the main results of the paper. Taking LO equation (3.2) and using (4.30) we can write the NLO evolution equation for the 3QWL operator as

$$\frac{\partial B_{123}}{\partial\eta} = \frac{\alpha_s(\mu^2)}{8\pi^2}\int d\vec{r}_0\Big[(B_{100}B_{320}+B_{200}B_{310}-B_{300}B_{210}-6B_{123})$$

$$
\begin{aligned}
&\times\left\{\frac{\vec r_{12}^{\,2}}{\vec r_{01}^{\,2}\vec r_{02}^{\,2}}-\frac{3\alpha_s}{4\pi}\beta\left[\ln\left(\frac{\vec r_{01}^{\,2}}{\vec r_{02}^{\,2}}\right)\left(\frac{1}{\vec r_{02}^{\,2}}-\frac{1}{\vec r_{01}^{\,2}}\right)-\frac{\vec r_{12}^{\,2}}{\vec r_{01}^{\,2}\vec r_{02}^{\,2}}\ln\left(\frac{\vec r_{12}^{\,2}}{\tilde\mu^2}\right)\right]\right\}\\
&-\frac{\alpha_s}{\pi}\ln\frac{\vec r_{20}^{\,2}}{\vec r_{21}^{\,2}}\ln\frac{\vec r_{10}^{\,2}}{\vec r_{21}^{\,2}}\left\{\frac{1}{2}\left[\frac{\vec r_{13}^{\,2}}{\vec r_{10}^{\,2}\vec r_{30}^{\,2}}-\frac{\vec r_{32}^{\,2}}{\vec r_{30}^{\,2}\vec r_{20}^{\,2}}\right](B_{100}B_{320}-B_{200}B_{310})\right.\\
&\left.-\frac{\vec r_{12}^{\,2}}{\vec r_{10}^{\,2}\vec r_{20}^{\,2}}\left(9B_{123}-\frac{1}{2}\left[2\left(B_{100}B_{320}+B_{200}B_{130}\right)-B_{300}B_{120}\right]\right)\right\}+(1\leftrightarrow3)+(2\leftrightarrow3)\Big]\\
&-\frac{\alpha_s^2n_f}{16\pi^4}\int d\vec r_0 d\vec r_4\left[\left\{\left(\frac{1}{3}(U_1U_0{}^\dagger U_4+U_4U_0{}^\dagger U_1)\cdot U_2\cdot U_3-\frac{1}{9}B_{123}tr(U_0{}^\dagger U_4)\right.\right.\right.\\
&+(U_1U_0{}^\dagger U_2)\cdot U_3\cdot U_4+\frac{1}{6}B_{123}-\frac{1}{4}(B_{013}B_{002}+B_{001}B_{023}-B_{012}B_{003})\\
&\left.\left.+\ \ (1\leftrightarrow2)\right)+(0\leftrightarrow4)\right\}L^q_{12}+(1\leftrightarrow3)+(2\leftrightarrow3)\Big]\\
&-\frac{\alpha_s^2}{8\pi^4}\int d\vec r_0 d\vec r_4\ \Big[\{\tilde L_{12}\left(U_0U_4{}^\dagger U_2\right)\cdot\left(U_1U_0{}^\dagger U_4\right)\cdot U_3\\
&+L_{12}\Big[\left(U_0U_4{}^\dagger U_2\right)\cdot\left(U_1U_0{}^\dagger U_4\right)\cdot U_3+tr\left(U_0U_4{}^\dagger\right)\left(U_1U_0{}^\dagger U_2\right)\cdot U_3\cdot U_4\\
&-\frac{3}{4}[B_{144}B_{234}+B_{244}B_{134}-B_{344}B_{124}]+\frac{1}{2}B_{123}\Big]\\
&+(M_{13}-M_{12}-M_{23}+M_2^{13})\Big[\left(U_0U_4{}^\dagger U_3\right)\cdot\left(U_2U_0{}^\dagger U_1\right)+\left(U_1U_0{}^\dagger U_2\right)\cdot\left(U_3U_4{}^\dagger U_0\right)\Big]\cdot U_4\\
&+(\text{all 5 permutations}\,1\leftrightarrow2\leftrightarrow3)\}+\ (0\leftrightarrow4)\Big]\,.
\end{aligned}\tag{8.1}
$$

Here the functions $L_{12},\tilde L_{12},M_{12},M_2^{13}$ are defined in (4.10)–(4.13), L^q_{12} is defined in (3.6), the $\overline{MS}$ renormalization scale μ^2 is related to scale $\tilde\mu^2$ through (5.9),

$$
\beta=\left(\frac{11}{3}-\frac{2}{3}\frac{n_f}{3}\right).\tag{8.2}
$$

As we mentioned above, all the expressions in this paper are written in the $\overline{MS}$ renormalization scheme.

The evolution equation for the composite 3QWL operator $B_{123}^{\rm conf}$ (6.2)

$$
\begin{aligned}
B_{123}^{\rm conf}=B_{123}+\frac{\alpha_s3}{8\pi^2}\int d\vec r_4&\left[\frac{\vec r_{12}^{\,2}}{\vec r_{41}^{\,2}\vec r_{42}^{\,2}}\ln\left(\frac{a\vec r_{12}^{\,2}}{\vec r_{41}^{\,2}\vec r_{42}^{\,2}}\right)\right.\\
&\left.\times\left(-B_{123}+\frac{1}{6}(B_{144}B_{324}+B_{244}B_{314}-B_{344}B_{214})\right)+(1\leftrightarrow3)+(2\leftrightarrow3)\right]
\end{aligned}\tag{8.3}
$$

follows from (6.22)

$$
\begin{aligned}
&\frac{\partial B_{123}^{\rm conf}}{\partial\eta}=\frac{\alpha_s\left(\mu^2\right)}{8\pi^2}\int d\vec r_0\,\Big[\left((B_{100}B_{320}+B_{200}B_{310}-B_{300}B_{210})-6B_{123}\right)^{\rm conf}\\
&\times\left(\frac{\vec r_{12}^{\,2}}{\vec r_{01}^{\,2}\vec r_{02}^{\,2}}-\frac{3\alpha_s}{4\pi}\beta\left[\ln\left(\frac{\vec r_{01}^{\,2}}{\vec r_{02}^{\,2}}\right)\left(\frac{1}{\vec r_{02}^{\,2}}-\frac{1}{\vec r_{01}^{\,2}}\right)-\frac{\vec r_{12}^{\,2}}{\vec r_{01}^{\,2}\vec r_{02}^{\,2}}\ln\left(\frac{\vec r_{12}^{\,2}}{\tilde\mu^2}\right)\right]\right)+(1\leftrightarrow3)+(2\leftrightarrow3)\Big]\\
&-\frac{\alpha_s^2}{32\pi^3}\int d\vec r_0\left(B_{003}B_{012}\left[\frac{\vec r_{32}{}^2}{\vec r_{03}{}^2\vec r_{02}{}^2}\ln^2\left(\frac{\vec r_{32}{}^2\vec r_{10}{}^2}{\vec r_{13}{}^2\vec r_{20}{}^2}\right)-\frac{\vec r_{12}{}^2}{\vec r_{01}{}^2\vec r_{02}{}^2}\ln^2\left(\frac{\vec r_{12}{}^2\vec r_{30}{}^2}{\vec r_{13}{}^2\vec r_{20}{}^2}\right)\right]\right.\\
&\left.+\ (\text{all 5 permutations}\,1\leftrightarrow2\leftrightarrow3)\right)
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{\alpha_s^2 n_f}{16\pi^4} \int d\vec{r}_0 d\vec{r}_4 \Bigg[\Bigg\{\Bigg(\frac{1}{3}(U_1 U_0{}^\dagger U_4 + U_4 U_0{}^\dagger U_1) \cdot U_2 \cdot U_3 - \frac{1}{9} B_{123} tr(U_0{}^\dagger U_4) \\
&+ (U_1 U_0{}^\dagger U_2) \cdot U_3 \cdot U_4 + \frac{1}{6} B_{123} - \frac{1}{4}(B_{013}B_{002} + B_{001}B_{023} - B_{012}B_{003}) \\
&+ \ (1 \leftrightarrow 2)\Bigg) + (0 \leftrightarrow 4)\Bigg\} L_{12}^q + (1 \leftrightarrow 3) + (2 \leftrightarrow 3)\Bigg] \\
&- \frac{\alpha_s^2}{8\pi^4} \int d\vec{r}_0 d\vec{r}_4 \ \Bigg(\Bigg\{\tilde{L}_{12}^C \left(U_0 U_4{}^\dagger U_2\right) \cdot \left(U_1 U_0{}^\dagger U_4\right) \cdot U_3 \\
&+ L_{12}^C \Bigg[\left(U_0 U_4{}^\dagger U_2\right) \cdot \left(U_1 U_0{}^\dagger U_4\right) \cdot U_3 + tr\left(U_0 U_4{}^\dagger\right) \left(U_1 U_0{}^\dagger U_2\right) \cdot U_3 \cdot U_4 \\
&- \frac{3}{4}[B_{144}B_{234} + B_{244}B_{134} - B_{344}B_{124}] + \frac{1}{2} B_{123}\Bigg] \\
&+ M_{12}^C \Big[\left(U_0 U_4{}^\dagger U_3\right) \cdot \left(U_2 U_0{}^\dagger U_1\right) \cdot U_4 + \left(U_1 U_0{}^\dagger U_2\right) \cdot \left(U_3 U_4{}^\dagger U_0\right) \cdot U_4\Big] \\
&+ \ (\text{all 5 permutations}\, 1 \leftrightarrow 2 \leftrightarrow 3)\Bigg\} + (0 \leftrightarrow 4)\Bigg) . \qquad (8.4)
\end{aligned}
$$

Here the composite operator $([B_{100}B_{320} + B_{200}B_{310} - B_{300}B_{210}] - 6B_{123})^{\text{conf}}$ is defined in (6.5) according to the prescription (6.1) and the functions $L_{12}^C, \tilde{L}_{12}^C, M_{12}^C$ are defined in (6.17)–(6.19).

The equation for composite 3QWL operator B_{123}^{conf} (6.2) linearized in the 3-gluon approximation is the result of (7.24) and (C.13)

$$
\begin{aligned}
&\frac{\partial B_{123}^{\text{conf}}}{\partial \eta} \overset{3g}{=} \frac{3\alpha_s\left(\mu^2\right)}{4\pi^2} \int d\vec{r}_0 \Bigg[(B_{100}^{\text{conf}} + B_{320}^{\text{conf}} + B_{200}^{\text{conf}} + B_{310}^{\text{conf}} - B_{300}^{\text{conf}} - B_{210}^{\text{conf}} - B_{123}^{\text{conf}} - 6) \\
&\times \left(\frac{\vec{r}_{12}^{\,2}}{\vec{r}_{01}^{\,2}\vec{r}_{02}^{\,2}} - \frac{3\alpha_s}{4\pi}\beta\left[\ln\left(\frac{\vec{r}_{01}^{\,2}}{\vec{r}_{02}^{\,2}}\right)\left(\frac{1}{\vec{r}_{02}^{\,2}} - \frac{1}{\vec{r}_{01}^{\,2}}\right) - \frac{\vec{r}_{12}^{\,2}}{\vec{r}_{01}^{\,2}\vec{r}_{02}^{\,2}} \ln\left(\frac{\vec{r}_{12}^{\,2}}{\tilde{\mu}^2}\right)\right]\right) + (1 \leftrightarrow 3) + (2 \leftrightarrow 3)\Bigg] \\
&- \frac{9\alpha_s^2}{64\pi^3} \int d\vec{r}_0 \left(\tilde{F}_{100}B_{100} + \tilde{F}_{230}B_{230} + (1 \leftrightarrow 3) + (1 \leftrightarrow 2)\right) \\
&+ \frac{27\alpha_s^2}{4\pi^2} \zeta(3)(3 - \delta_{23} - \delta_{13} - \delta_{21})(B_{123} - 6) \\
&- \frac{9\alpha_s^2}{8\pi^4} \int d\vec{r}_0 d\vec{r}_4 \left(L_{12}^C + L_{13}^C + L_{23}^C - \frac{n_f}{54}(L_{12}^q + L_{13}^q + L_{23}^q)\right)(B_{044} + B_{004} - 12) \\
&- \frac{\alpha_s^2 n_f}{24\pi^4} \int d\vec{r}_0 d\vec{r}_4 \ \{(2B_{014} - B_{001} - B_{144})(L_{12}^q + L_{13}^q - 2L_{32}^q) + (1 \leftrightarrow 3) + (1 \leftrightarrow 2)\} \\
&- \frac{9\alpha_s^2}{64\pi^4} \int d\vec{r}_0 d\vec{r}_4 \ (\{F_0 B_{040} + F_{140}B_{140} + (0 \leftrightarrow 4)\} + (\text{all 5 permutations}\, 1 \leftrightarrow 2 \leftrightarrow 3)) . \qquad (8.5)
\end{aligned}
$$

Here $\delta_{ij} = 1$, if $\vec{r}_i = \vec{r}_j$ and $\delta_{ij} = 0$ otherwise; the functions F_0 and F_{140} are defined in (7.18) and (7.19); $\tilde{F}_{100}$ and $\tilde{F}_{230}$ are defined in (7.25)–(7.26).

The linearized equation for C-even composite 3QWL Green function is the consequence of (7.38) and (C.13)

$$
\begin{aligned}
&\frac{\partial B_{123}^{+conf}}{\partial \eta} \overset{3g}{=} \frac{3\alpha_s\left(\mu^2\right)}{4\pi^2} \int d\vec{r}_0 \Big[\Big(B_{100}^{+conf} + B_{320}^{+conf} + B_{200}^{+conf} + B_{310}^{+conf} \\
&- B_{300}^{+conf} - B_{210}^{+conf} - B_{123}^{+conf}\Big)
\end{aligned}
$$

$$\times\left(\frac{\vec r_{12}^{\,2}}{\vec r_{01}^{\,2}\vec r_{02}^{\,2}}-\frac{3\alpha_s}{4\pi}\beta\left[\ln\left(\frac{\vec r_{01}^{\,2}}{\vec r_{02}^{\,2}}\right)\left(\frac{1}{\vec r_{02}^{\,2}}-\frac{1}{\vec r_{01}^{\,2}}\right)-\frac{\vec r_{12}^{\,2}}{\vec r_{01}^{\,2}\vec r_{02}^{\,2}}\ln\left(\frac{\vec r_{12}^{\,2}}{\tilde\mu^2}\right)\right]\right)+(1\leftrightarrow3)+(2\leftrightarrow3)\Bigg]$$

$$-\frac{9\alpha_s^2}{4\pi^4}\int d\vec r_0 d\vec r_4\left(L_{12}^C+L_{13}^C+L_{23}^C-\frac{n_f}{54}(L_{12}^q+L_{13}^q+L_{23}^q)\right)(B_{044}+B_{004}-12)$$

$$-\frac{9\alpha_s^2}{64\pi^3}\int d\vec r_0\left(\tilde F_{100}B_{100}^{+}+\tilde F_{230}B_{230}^{+}+(1\leftrightarrow3)+(1\leftrightarrow2)\right)$$

$$+\frac{27\alpha_s^2}{4\pi^2}\zeta(3)(3-\delta_{23}-\delta_{13}-\delta_{21})B_{123}^{+}$$

$$-\frac{9\alpha_s^2}{64\pi^4}\int d\vec r_0 d\vec r_4\ \left(\{F_{140}+(0\leftrightarrow4)\}\,B_{140}^{+}+(\text{all 5 permutations }1\leftrightarrow2\leftrightarrow3)\right). \tag{8.6}$$

The linearized equations for C-odd composite 3QWL Green function is the consequence of (7.47) and (C.13)

$$\frac{\partial B_{123}^{-conf}}{\partial\eta}\overset{3g}{=}\frac{3\alpha_s(\mu^2)}{4\pi^2}\int d\vec r_0\left[\left(B_{100}^{-conf}+B_{320}^{-conf}+B_{200}^{-conf}+B_{310}^{-conf}\right.\right.$$

$$\left.-B_{300}^{-conf}-B_{210}^{-conf}-B_{123}^{-conf}\right)$$

$$\times\left(\frac{\vec r_{12}^{\,2}}{\vec r_{01}^{\,2}\vec r_{02}^{\,2}}-\frac{3\alpha_s}{4\pi}\beta\left[\ln\left(\frac{\vec r_{01}^{\,2}}{\vec r_{02}^{\,2}}\right)\left(\frac{1}{\vec r_{02}^{\,2}}-\frac{1}{\vec r_{01}^{\,2}}\right)-\frac{\vec r_{12}^{\,2}}{\vec r_{01}^{\,2}\vec r_{02}^{\,2}}\ln\left(\frac{\vec r_{12}^{\,2}}{\tilde\mu^2}\right)\right]\right)+(1\leftrightarrow3)+(2\leftrightarrow3)\Bigg]$$

$$-\frac{\alpha_s^2 n_f}{24\pi^4}\int d\vec r_0 d\vec r_4\ \left\{\left(2B_{014}^{-}-B_{001}^{-}-B_{144}^{-}\right)\left(L_{12}^q+L_{13}^q-2L_{32}^q\right)+(1\leftrightarrow3)+(1\leftrightarrow2)\right\}$$

$$-\frac{9\alpha_s^2}{64\pi^3}\int d\vec r_0\left(\tilde F_{100}B_{100}^{-}+\tilde F_{230}B_{230}^{-}+(1\leftrightarrow3)+(1\leftrightarrow2)\right)$$

$$+\frac{27\alpha_s^2}{4\pi^2}\zeta(3)(3-\delta_{23}-\delta_{13}-\delta_{21})B_{123}^{-}$$

$$-\frac{9\alpha_s^2}{64\pi^4}\int d\vec r_0 d\vec r_4\ \left(2F_0B_{040}^{-}+\{F_{140}+(0\leftrightarrow4)\}\,B_{140}^{-}+(\text{all 5 permutations }1\leftrightarrow2\leftrightarrow3)\right). \tag{8.7}$$

From these expressions one can see that terms with L_{ij}, L_{ij}^C, which comprise the BFKL kernels, contribute only to the evolution of the C-even part of the Green function while terms with $F_0, \tilde L_{ij}, \tilde L_{ij}^C$ contribute only to the evolution of the C-odd one.

The BK equation for the color dipole $B_{122}=2tr(U_1U_2^\dagger)$ in the 3-gluon approximation reads (see (7.33))

$$\frac{\partial B_{122}^{\rm conf}}{\partial\eta}\overset{3g}{=}\frac{3\alpha_s(\mu^2)}{2\pi^2}\int d\vec r_0(B_{100}^{\rm conf}+B_{220}^{\rm conf}-B_{122}^{\rm conf}-6)$$

$$\times\left(\frac{\vec r_{12}^{\,2}}{\vec r_{01}^{\,2}\vec r_{02}^{\,2}}-\frac{3\alpha_s}{4\pi}\beta\left[\ln\left(\frac{\vec r_{01}^{\,2}}{\vec r_{02}^{\,2}}\right)\left(\frac{1}{\vec r_{02}^{\,2}}-\frac{1}{\vec r_{01}^{\,2}}\right)-\frac{\vec r_{12}^{\,2}}{\vec r_{01}^{\,2}\vec r_{02}^{\,2}}\ln\left(\frac{\vec r_{12}^{\,2}}{\tilde\mu^2}\right)\right]\right)+\frac{27\alpha_s^2}{2\pi^2}\zeta(3)(B_{122}-6)$$

$$-\frac{9\alpha_s^2}{4\pi^4}\int d\vec r_0 d\vec r_4\left(L_{12}^C-\frac{n_f}{54}L_{12}^q\right)(B_{044}+B_{004}-12)-\frac{9\alpha_s^2}{4\pi^4}\int d\vec r_0 d\vec r_4\ \tilde L_{12}^C(B_{044}-B_{040})$$

$$-\frac{\alpha_s^2 n_f}{12\pi^4}\int d\vec r_0 d\vec r_4\ \{(2B_{014}-B_{001}-B_{144})-(2B_{024}-B_{002}-B_{244})\}\,L_{12}^q. \tag{8.8}$$

As is clear from the last line, the evolution of the color dipole in the 3-gluon approximation depends on the 3QWL operators which have nondipole structure. The BK equation for the

C-even part of the color dipole operator $B^{+}_{122} = 2tr(U_1U_2^\dagger) + 2tr(U_1^\dagger U_2) - 6$ in the 3-gluon approximation is the same as in the 2-gluon one (BFKL)

$$\frac{\partial B^{+conf}_{122}}{\partial\eta} \overset{3g}{=} \frac{3\alpha_s(\mu^2)}{2\pi^2}\int d\vec{r}_0 (B^{+conf}_{100} + B^{+conf}_{220} - B^{+conf}_{122}) \times \left(\frac{\vec{r}_{12}^{\,2}}{\vec{r}_{01}^{\,2}\vec{r}_{02}^{\,2}} - \frac{3\alpha_s}{4\pi}\beta\left[\ln\left(\frac{\vec{r}_{01}^{\,2}}{\vec{r}_{02}^{\,2}}\right)\left(\frac{1}{\vec{r}_{02}^{\,2}} - \frac{1}{\vec{r}_{01}^{\,2}}\right) - \frac{\vec{r}_{12}^{\,2}}{\vec{r}_{01}^{\,2}\vec{r}_{02}^{\,2}}\ln\left(\frac{\vec{r}_{12}^{\,2}}{\tilde{\mu}^2}\right)\right]\right) - \frac{9\alpha_s^2}{2\pi^4}\int d\vec{r}_0 d\vec{r}_4 \left(L^C_{12} - \frac{n_f}{54}L^q_{12}\right)B^{+}_{044} + \frac{27\alpha_s^2}{2\pi^2}\zeta(3)B^{+}_{122}. \tag{8.9}$$

At the same time the BK equation for the C-odd part of the color dipole operator $B^{-}_{122} = 2tr(U_1U_2^\dagger) - 2tr(U_1^\dagger U_2)$ in the 3-gluon approximation reads

$$\frac{\partial B^{-conf}_{122}}{\partial\eta} \overset{3g}{=} \frac{3\alpha_s(\mu^2)}{2\pi^2}\int d\vec{r}_0 (B^{-conf}_{100} + B^{-conf}_{220} - B^{-conf}_{122}) \tag{8.10}$$
$$\times \left(\frac{\vec{r}_{12}^{\,2}}{\vec{r}_{01}^{\,2}\vec{r}_{02}^{\,2}} - \frac{3\alpha_s}{4\pi}\beta\left[\ln\left(\frac{\vec{r}_{01}^{\,2}}{\vec{r}_{02}^{\,2}}\right)\left(\frac{1}{\vec{r}_{02}^{\,2}} - \frac{1}{\vec{r}_{01}^{\,2}}\right) - \frac{\vec{r}_{12}^{\,2}}{\vec{r}_{01}^{\,2}\vec{r}_{02}^{\,2}}\ln\left(\frac{\vec{r}_{12}^{\,2}}{\tilde{\mu}^2}\right)\right]\right) - \frac{9\alpha_s^2}{2\pi^4}\int d\vec{r}_0 d\vec{r}_4\, \tilde{L}^C_{12}B^{-}_{044}$$
$$+ \frac{27\alpha_s^2}{2\pi^2}\zeta(3)B^{-}_{122} - \frac{\alpha_s^2 n_f}{12\pi^4}\int d\vec{r}_0 d\vec{r}_4\ \left\{(2B^{-}_{014} - B^{-}_{001} - B^{-}_{144}) - (2B^{-}_{024} - B^{-}_{002} - B^{-}_{244})\right\} L^q_{12}.$$

This equation contains the nondipole 3QWL operators in its quark part.

The question how to choose the operator basis for the evolution equation is nontrivial. We tried to find the basis with the minimal number of the operators. The part of the kernel with one integration does not present a problem since all the operators in it can be reduced to products of the 3QWLs B. Next, the quark contribution to the part of the kernel with 2 integrations can be reduced to one operator up to 3 permutations (8.1), (8.4). It is obviously a minimal choice here. The corresponding gluon contribution depends on 3 operators up to permutations (8.1), (8.4): $((U_0U_4{}^\dagger U_2)\cdot(U_1U_0{}^\dagger U_4)\cdot U_3 + tr\,(U_0U_4{}^\dagger)\,(U_1U_0{}^\dagger U_2)\cdot U_3\cdot U_4 + (1\leftrightarrow 2)) - (0\to 4) + (0\leftrightarrow 4)$, $((U_0U_4{}^\dagger U_2)\cdot(U_1U_0{}^\dagger U_4)\cdot U_3 - (1\leftrightarrow 2)) - (0\leftrightarrow 4)$, and $\left[(U_0U_4{}^\dagger U_3)\cdot(U_2U_0{}^\dagger U_1) + (U_1U_0{}^\dagger U_2)\cdot(U_3U_4{}^\dagger U_0)\right]\cdot U_4$. There are 3 operators of the first type, 3 of the second type, and 12 ones of the third type.

The operators of the first and the second type are independent because of the different symmetry w.r.t. $(0\leftrightarrow 4)$ and $(i\leftrightarrow j)$ permutations, where $i,j = 1,2,3$. In the dipole limit the part of the kernel containing the operators of the third type reduces to the antisymmetric w.r.t. $(0\leftrightarrow 4)$ structure (6.23). If this part of the kernel could be expressed through the operators of the first two types, it would reduce to the same operator in the dipole limit as the part of the kernel containing the operators of the second type (6.24). Plainly, (6.23) and (6.24) depend on different operators, which can not be expressed through each other. There remains a question how many of the 12 third type operators are independent. In fact they are not all independent. They obey identity (B.14), which ensures the UV-safety of evolution equation (6.22) as we discussed above. Hence, we could rewrite the evolution equation using only any 11 of the 12 third type operators. However, such rearrangement makes the equation much more cumbersome and blurs its symmetry w.r.t. permutations. Therefore we left all the 12 operators in the final formulae. Using identities (B.1)–(B.4) with $l = 0,1,2,3,4$ we were unable to express 11 third type operators via one another.

Reduction of the number of the operators used in the evolution equation simplifies the equation and helps to solve it. The best example of this fact is the LO equation for the 3QWL. By construction it contains several different operators [25]

$$\frac{\partial B_{123}}{\partial\eta} = \frac{\alpha_s}{2\pi^2}\int d\vec{r}_4 \left[\left\{\frac{1}{\vec{r}_{41}^{\,2}}(tr(U_1U_4^\dagger)B_{423} - 3B_{123}) + (1\leftrightarrow 2) + (1\leftrightarrow 3)\right\}\right.$$
$$\left. + \left\{\frac{\vec{r}_{41}\vec{r}_{42}}{\vec{r}_{41}^{\,2}\vec{r}_{42}^{\,2}}\left(2B_{123}^{\eta} - \left(U_2U_4^\dagger U_1 + U_1U_4^\dagger U_2\right)\cdot U_4\cdot U_3\right) + (1\leftrightarrow 3) + (2\leftrightarrow 3)\right\}\right].$$

After application of identity (B.3) it turns into (3.2). The latter equation depends only on products of B and has the closed form. In the NLO the reduction of the color structures helped to find the quasi-conformal form and simplified the equation, which will aid numerical solution.

9 Conclusions

In this paper we constructed the NLO evolution equation for the "color triple" - three-quark Wilson loop operator $\varepsilon^{i'j'h'}\varepsilon_{ijh}U^i_{1i'}U^j_{2j'}U^h_{3h'}$. As in the case of the color dipole evolution, for the "rigid cutoff" $Y < \eta$ of the Wilson lines the kernel of this equation has non-conformal terms not related to renormalization. We have constructed the composite 3QWL operator (6.2) obeying the NLO evolution equation with the quasi-conformal kernel. We linearized the quasi-conformal equation in the 3-gluon approximation. It is worth noting that our results have correct dipole limit in the case when the coordinates of the two lines coincide. We also constructed the 3-gluon approximation of the BK equation and showed that it contains non-dipole 3QWL operators (8.8), (8.10).

The 3QWL operator may have many phenomenological applications. First, it is a natural SU(3) model for a baryon Green function in the Regge limit. Also, it is the irreducible operator describing C-odd (odderon) exchange. For example as shown in the appendix E, the odderon part of the quadrupole operator $tr(U_1U_2^\dagger U_3U_4^\dagger)$ in the 3-gluon approximation in SU(3) can be decomposed into a sum of 3QWLs

$$2tr(U_1U_2^\dagger U_3U_4^\dagger) - 2tr(U_4U_3^\dagger U_2U_1^\dagger) \overset{3g}{=} B^-_{144} + B^-_{322} - B^-_{433} - B^-_{211} + B^-_{124} + B^-_{234} - B^-_{123} - B^-_{134}.$$

Moreover, even the NLO evolution equation for the dipole C-odd Green function in the 3-gluon approximation (8.10) in QCD can not be written without the introduction of the 3QWL operator.

The evolution equation for the C-odd part of the 3QWL operator is the generalization of the BKP equation for odderon exchange to the saturation regime. However, it is valid for the colorless object, i.e. for the function $B^-_{ijk} = B^-(\vec{r}_i,\vec{r}_j,\vec{r}_k)$, which vanishes as $\vec{r}_i = \vec{r}_j = \vec{r}_k$. The linear approximation of the equation for the C-odd part of the 3QWL should be equivalent to the NLO BKP for odderon exchange acting in the space of such functions. One may try to restore the full NLO BKP kernel from our result via the technique similar to the one developed for the 2-point operators in [35].

The result for the evolution of the 3QWL operator was also presented in [30] (which was put on arXiv the same day as our paper). As we mentioned above, both evolution

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kernels reproduce NLO BK in the dipole limit $\vec{r}_1 \to \vec{r}_2$ and survive other checks. However, the result of [30], as it is written, differs from our kernel since ref. [30] has much larger basis of operators in the evolution equation, possibly because not all SU(3)-relations were taken into account. For example, in the r.h.s. of eq. (4.25) in [30], in addition to our basis, there are operators $tr(U_1U_4^\dagger)tr(U_4U_0^\dagger)U_2 \cdot U_3 \cdot U_0$, $tr(U_1U_0^\dagger)(U_2U_4^\dagger U_0) \cdot U_3 \cdot U_4$, and $U_0 \cdot U_3 \cdot (U_4U_0^\dagger U_1U_4^\dagger U_2 + U_2U_4^\dagger U_1U_0^\dagger U_4)$ which can be eliminated using our relations (B.6)–(B.13). Since simplifying the kernel by reducing the number of basic operators was one of the most tedious parts of our calculation, we believe that the detailed comparison to the result of ref. [30] is beyond the scope of the present paper.

Acknowledgments

A. V. G. acknowledges support of the Russian Fund for Basic Research grants 13-02-01023, 15-32-20548, and 15-52-15055, and president grant MK-7551.2015.2. The work of I. B. was supported by the contract DE-AC05-06OR23177 under which the Jefferson Science Associates, LLC operate the Thomas Jefferson National Accelerator Facility, and by the grant DEFG02-97ER41028. The authors are grateful to S. Caron-Huot and M. Lublinsky for valuable discussions.

A Notations

First, let us describe our notations. We introduce the light cone vectors n_1 and n_2

$$n_1 = (1, 0, 0, 1)\,, \quad n_2 = \frac{1}{2}\,(1, 0, 0, -1)\,, \quad n_1^+ = n_2^- = n_1 n_2 = 1 \tag{A.1}$$

so that for any vector p we have

$$p^+ = p_- = pn_2 = \frac{1}{2}\left(p^0 + p^3\right), \qquad p_+ = p^- = pn_1 = p^0 - p^3, \tag{A.2}$$

$$p = p^+ n_1 + p^- n_2 + p_\perp, \qquad\qquad p^2 = 2p^+p^- - \vec{p}^{\,2}, \tag{A.3}$$

$$p\,k = p^\mu k_\mu = p^+k^- + p^-k^+ - \vec{p}\vec{k} = p_+k_- + p_-k_+ - \vec{p}\vec{k}. \tag{A.4}$$

The index convention is $a_i^j b_j^k = (ab)_i^k$.

Second, let us present the connection of our notation with the notation of [20]. In that paper the quark and gluon coordinates were denoted z_i. So we have to change

$$z_{1,2,3,4} \leftrightarrow \vec{r}_{1,2,3,4}, \quad z_5 \leftrightarrow \vec{r}_0. \tag{A.5}$$

Assuming this substitution for self-interaction we get

$$G_3 = \frac{I_1}{2z_{45}^2} \ln \frac{z_{14}^2}{z_{15}^2} - \frac{1}{z_{45}^4}, \qquad G_9 = -\frac{(z_{14}, z_{15})}{2z_{45}^2 z_{14}^2 z_{15}^2} \ln \frac{z_{14}^2}{z_{15}^2}. \tag{A.6}$$

For pairwise interaction we obtain

$$8H_1 = 2\mathcal{J}_{1245} \ln \frac{z_{14}^2}{z_{15}^2}, \tag{A.7}$$

$$8H_2 = -2(J_{1245} + J_{1254}) \ln \frac{z_{14}^2}{z_{15}^2}, \tag{A.8}$$

$$8H_3 = 2(J_{1245} - J_{1254}) \ln \frac{z_{14}^2}{z_{15}^2}, \tag{A.9}$$

$$8H_4 = \ln \frac{z_{14}^2}{z_{15}^2} \, (2L - 2J_{1245} + 2J_{1254}) - \frac{2}{z_{45}^4}, \tag{A.10}$$

or

$$8(H_3 + H_4) = 2L \ln \frac{z_{14}^2}{z_{15}^2} - \frac{2}{z_{45}^4}. \tag{A.11}$$

Here L is defined as

$$-2K - \frac{8}{z_{45}^4} + \frac{2I_1}{z_{45}^2} + \frac{2I_2}{z_{45}^2} = 2L \ln \frac{z_{14}^2}{z_{15}^2} + (1 \leftrightarrow 2) - \frac{4}{z_{45}^4} = 8(H_3 + H_4 + 1 \leftrightarrow 2). \tag{A.12}$$

For triple interaction we have

$$H_7 \equiv H_5 + H_6 = \frac{1}{2}\mathcal{J}_{32145} \ln \frac{z_{14}^2}{z_{15}^2}, \qquad H_8 \equiv H_5 - H_6 = -\frac{1}{2}\mathcal{J}_{32154} \ln \frac{z_{14}^2}{z_{15}^2}. \tag{A.13}$$

B SU(3) identities

Here we present the list of SU(3) identities used in the paper.

$$U_i \cdot U_j \cdot U_k = (U_i U_l^\dagger) \cdot (U_j U_l^\dagger) \cdot (U_k U_l^\dagger) = (U_l^\dagger U_i) \cdot (U_l^\dagger U_j) \cdot (U_l^\dagger U_k), \tag{B.1}$$

$$\varepsilon^{ijh} \varepsilon_{i'j'h'} (U_1)_i^{i'} (U_1)_j^{j'} = 2(U_1^\dagger)_{h'}^h, \quad U_1 \cdot U_1 \cdot U_3 = 2tr(U_1^\dagger U_3). \tag{B.2}$$

These identities follow from the definition of the group, namely from unitarity and the fact that the determinant of U is 1.

$$\left(U_2 U_4^\dagger U_1 + U_1 U_4^\dagger U_2\right) \cdot U_4 \cdot U_3 = -B_{123} + \frac{1}{2}(B_{144}B_{324} + B_{244}B_{314} - B_{344}B_{214}). \tag{B.3}$$

This identity can be checked using (B.1) with $l = 4$ and then expanding the product of Levi-Civita symbols as

$$\varepsilon_{ijh}\varepsilon^{i'j'h'} = \begin{vmatrix} \delta_i^{i'} & \delta_i^{j'} & \delta_i^{h'} \\ \delta_j^{i'} & \delta_j^{j'} & \delta_j^{h'} \\ \delta_h^{i'} & \delta_h^{j'} & \delta_h^{h'} \end{vmatrix}. \tag{B.4}$$

The next one reads

$$\begin{aligned} 0 = [&\left(U_0 U_4{}^\dagger U_3 U_0{}^\dagger U_4\right) \cdot U_1 \cdot U_2 - U_1 \cdot U_2 \cdot U_4 tr\left(U_0{}^\dagger U_3\right) tr\left(U_4{}^\dagger U_0\right) \\ &+ tr\left(U_0 U_4{}^\dagger\right)\left(U_2 U_0{}^\dagger U_3 + U_3 U_0{}^\dagger U_2\right) \cdot U_1 \cdot U_4 \\ &+ \left(U_0 U_4{}^\dagger U_2\right) \cdot \left(U_3 U_0{}^\dagger U_4\right) \cdot U_1 + \left(U_0 U_4{}^\dagger U_3\right) \cdot \left(U_2 U_0{}^\dagger U_4\right) \cdot U_1 + (1 \leftrightarrow 2)] + (4 \leftrightarrow 0). \end{aligned} \tag{B.5}$$

This identity relates the color structures in $\mathbf{G}_{12\langle 3\rangle}$, $\mathbf{G}_{1\langle 23\rangle}$ and $\mathbf{G}_{\langle 13\rangle 2}$. By $1 \leftrightarrow 2 \leftrightarrow 3$ transformation one can obtain 2 more identities and totally eliminate 3 color structures from $\mathbf{G}_{12\langle 3\rangle}$, $\mathbf{G}_{\langle 1\rangle 23}$, and $\mathbf{G}_{1\langle 2\rangle 3}$.

$$0 = tr\left(U_0{}^\dagger U_2\right)\left(U_0 U_4{}^\dagger U_3 + U_3 U_4{}^\dagger U_0\right) \cdot U_1 \cdot U_4$$

$$
\begin{aligned}
&- \left(U_0 U_4{}^\dagger U_3\right) \cdot \left(U_2 U_0{}^\dagger U_4\right) \cdot U_1 - \left(U_3 U_4{}^\dagger U_0\right) \cdot \left(U_4 U_0{}^\dagger U_2\right) \cdot U_1 \\
&- tr\left(U_4 U_0{}^\dagger\right) \left(U_2 U_4{}^\dagger U_3 + U_3 U_4{}^\dagger U_2\right) \cdot U_0 \cdot U_1 \\
&+ \left(U_3 U_4{}^\dagger U_2 U_0{}^\dagger U_4\right) \cdot U_0 \cdot U_1 + \left(U_4 U_0{}^\dagger U_2 U_4{}^\dagger U_3\right) \cdot U_0 \cdot U_1 \\
&- \left(U_0 U_4{}^\dagger U_3\right) \cdot \left(U_2 U_0{}^\dagger U_1\right) \cdot U_4 - \left(U_1 U_0{}^\dagger U_2\right) \cdot \left(U_3 U_4{}^\dagger U_0\right) \cdot U_4 \\
&+ \left(U_1 U_0{}^\dagger U_4\right) \cdot \left(U_3 U_4{}^\dagger U_2\right) \cdot U_0 + \left(U_2 U_4{}^\dagger U_3\right) \cdot \left(U_4 U_0{}^\dagger U_1\right) \cdot U_0. \qquad \text{(B.6)}
\end{aligned}
$$

This identity relates all color structures in $\mathbf{G}_{1\langle 23\rangle}$ and two structures in $\mathbf{G}_{\langle 123\rangle}$. It goes into 5 different identities after $1 \leftrightarrow 2 \leftrightarrow 3$ transformation, which allows one to get rid of 6 structures.

$$
\begin{aligned}
0 = &\, tr\left(U_0{}^\dagger U_2\right) \left(U_0 U_4{}^\dagger U_1 + U_1 U_4{}^\dagger U_0\right) \cdot U_3 \cdot U_4 \\
&+ \left(U_0 U_4{}^\dagger U_2 U_0{}^\dagger U_1 + U_1 U_0{}^\dagger U_2 U_4{}^\dagger U_0\right) \cdot U_3 \cdot U_4 \\
&- tr\left(U_0 U_4{}^\dagger\right) \left(U_1 U_0{}^\dagger U_2 + U_2 U_0{}^\dagger U_1\right) \cdot U_3 \cdot U_4 \\
&- \left(U_0 U_4{}^\dagger U_1\right) \cdot \left(U_2 U_0{}^\dagger U_4\right) \cdot U_3 - \left(U_0 U_4{}^\dagger U_2\right) \cdot \left(U_1 U_0{}^\dagger U_4\right) \cdot U_3 \\
&- \left(U_1 U_0{}^\dagger U_4\right) \cdot \left(U_3 U_4{}^\dagger U_2\right) \cdot U_0 + \left(U_1 U_4{}^\dagger U_2\right) \cdot \left(U_3 U_0{}^\dagger U_4\right) \cdot U_0 \\
&+ \left(U_2 U_4{}^\dagger U_1\right) \cdot \left(U_4 U_0{}^\dagger U_3\right) \cdot U_0 - \left(U_2 U_4{}^\dagger U_3\right) \cdot \left(U_4 U_0{}^\dagger U_1\right) \cdot U_0 + (4 \leftrightarrow 0). \qquad \text{(B.7)}
\end{aligned}
$$

This identity relates 2 color structures in $\mathbf{G}_{\langle 12\rangle 3}$ and a structure in $\mathbf{G}_{\langle 123\rangle}$. It also goes into 5 different identities after $1 \leftrightarrow 2 \leftrightarrow 3$ transformation, which allows one to get rid of 6 structures.

$$
\begin{aligned}
0 = &\, [U_0 \cdot U_1 \cdot U_2 tr\left(U_0{}^\dagger U_4\right) tr\left(U_4{}^\dagger U_3\right) \\
&- tr\left(U_4 U_0{}^\dagger\right) \left(U_1 U_4{}^\dagger U_3 + U_3 U_4{}^\dagger U_1\right) \cdot U_0 \cdot U_2 + \left(U_0 U_4{}^\dagger U_1\right) \cdot \left(U_3 U_0{}^\dagger U_4\right) \cdot U_2 \\
&+ \left(U_1 U_4{}^\dagger U_0\right) \cdot \left(U_4 U_0{}^\dagger U_3\right) \cdot U_2 + (1 \leftrightarrow 2)] - (4 \leftrightarrow 0). \qquad \text{(B.8)}
\end{aligned}
$$

This identity relates 2 color structures in $\mathbf{G}_{\langle 13\rangle 2}$, 2 color structures in $\mathbf{G}_{1\langle 23\rangle}$ and a structure in $\mathbf{G}_{12\langle 3\rangle}$. It goes into 2 different identities after $1 \leftrightarrow 2 \leftrightarrow 3$ transformation, which allows one to get rid of 3 more structures.

$$
\begin{aligned}
0 = &\, 2tr\left(U_4 U_0{}^\dagger\right) \left(U_2 U_4{}^\dagger U_3 + U_3 U_4{}^\dagger U_2\right) \cdot U_0 \cdot U_1 \\
&+ \left(U_0 U_4{}^\dagger U_1 + U_1 U_4{}^\dagger U_0\right) \cdot \left(U_2 U_0{}^\dagger U_3 + U_3 U_0{}^\dagger U_2\right) \cdot U_4 \\
&+ \left(U_0 U_4{}^\dagger U_2 - U_2 U_4{}^\dagger U_0\right) \cdot \left(U_3 U_0{}^\dagger U_1 - U_1 U_0{}^\dagger U_3\right) \cdot U_4 \\
&+ \left(U_0 U_4{}^\dagger U_3 - U_3 U_4{}^\dagger U_0\right) \cdot \left(U_2 U_0{}^\dagger U_1 - U_1 U_0{}^\dagger U_2\right) \cdot U_4 - (4 \leftrightarrow 0). \qquad \text{(B.9)}
\end{aligned}
$$

This identity relates 3 color structures in $\mathbf{G}_{\langle 132\rangle}$ and a color structure in $\mathbf{G}_{1\langle 23\rangle}$. It goes into 2 different identities after $1 \leftrightarrow 2 \leftrightarrow 3$ transformation, which allows one to get rid of 3 more structures.

All these identities (B.5)–(B.9) can be checked using (B.1) with $l = 1$ and then expanding the product of Levi-Civita symbols via (B.4).

$$0 = 2tr({U_0}^\dagger U_3)\left(U_1{U_4}^\dagger U_2\right)\cdot U_0\cdot U_4 - \left(U_1{U_4}^\dagger U_2\right)\cdot\left(U_3{U_0}^\dagger U_4\right)\cdot U_0$$
$$-\left(U_1{U_4}^\dagger U_2\right)\cdot\left(U_4{U_0}^\dagger U_3\right)\cdot U_0 - 2\left(U_1{U_4}^\dagger U_2\right)\cdot U_3\cdot U_4$$
$$-\left(U_1{U_4}^\dagger U_2{U_0}^\dagger U_3\right)\cdot U_0\cdot U_4 - \left(U_3{U_0}^\dagger U_1{U_4}^\dagger U_2\right)\cdot U_0\cdot U_4 + (1\leftrightarrow 2). \quad \text{(B.10)}$$

This identity can be proved directly using (B.3).

$$0 = 2tr\left({U_0}^\dagger U_4\right)\left(U_1{U_4}^\dagger U_2\right)\cdot U_0\cdot U_3 - tr\left({U_0}^\dagger U_1\right)\left(U_0{U_4}^\dagger U_2 + U_2{U_4}^\dagger U_0\right)\cdot U_3\cdot U_4$$
$$+\left(U_0{U_4}^\dagger U_1\right)\cdot\left(U_2{U_0}^\dagger U_3\right)\cdot U_4 + \left(U_0{U_4}^\dagger U_1\right)\cdot\left(U_2{U_0}^\dagger U_4\right)\cdot U_3$$
$$+\left(U_1{U_4}^\dagger U_0\right)\cdot\left(U_3{U_0}^\dagger U_2\right)\cdot U_4 + \left(U_1{U_4}^\dagger U_0\right)\cdot\left(U_4{U_0}^\dagger U_2\right)\cdot U_3$$
$$-\left(U_1{U_4}^\dagger U_2\right)\cdot\left(U_3{U_0}^\dagger U_4 + U_4{U_0}^\dagger U_3\right)\cdot U_0$$
$$-\left(U_1{U_4}^\dagger U_2{U_0}^\dagger U_4\right)\cdot U_0\cdot U_3 - \left(U_4{U_0}^\dagger U_1{U_4}^\dagger U_2\right)\cdot U_0\cdot U_3 + (1\leftrightarrow 2). \quad \text{(B.11)}$$
$$0 = tr\left({U_0}^\dagger U_2\right)\left(U_0{U_4}^\dagger U_1 + U_1{U_4}^\dagger U_0\right)\cdot U_3\cdot U_4 - 2\left(U_1{U_4}^\dagger U_2 + U_2{U_4}^\dagger U_1\right)\cdot U_3\cdot U_4$$
$$+ U_0\cdot U_3\cdot U_4 tr\left({U_0}^\dagger U_1{U_4}^\dagger U_2\right) + U_0\cdot U_3\cdot U_4 tr\left({U_0}^\dagger U_2{U_4}^\dagger U_1\right)$$
$$-\left(U_1{U_4}^\dagger U_2{U_0}^\dagger U_3\right)\cdot U_0\cdot U_4 - \left(U_3{U_0}^\dagger U_2{U_4}^\dagger U_1\right)\cdot U_0\cdot U_4$$
$$-\left(U_1{U_4}^\dagger U_2{U_0}^\dagger U_4\right)\cdot U_0\cdot U_3 - \left(U_4{U_0}^\dagger U_2{U_4}^\dagger U_1\right)\cdot U_0\cdot U_3$$
$$-\left(U_0{U_4}^\dagger U_1\right)\cdot\left(U_2{U_0}^\dagger U_3\right)\cdot U_4 - \left(U_1{U_4}^\dagger U_0\right)\cdot\left(U_3{U_0}^\dagger U_2\right)\cdot U_4$$
$$-\left(U_0{U_4}^\dagger U_1\right)\cdot\left(U_2{U_0}^\dagger U_4\right)\cdot U_3 - \left(U_1{U_4}^\dagger U_0\right)\cdot\left(U_4{U_0}^\dagger U_2\right)\cdot U_3. \quad \text{(B.12)}$$
$$0 = tr\left({U_4}^\dagger U_1\right)\left(U_2{U_0}^\dagger U_4 + U_4{U_0}^\dagger U_2\right)\cdot U_0\cdot U_3$$
$$+\left(U_0{U_4}^\dagger U_1\right)\cdot\left(U_3{U_0}^\dagger U_2\right)\cdot U_4 + \left(U_1{U_4}^\dagger U_0\right)\cdot\left(U_2{U_0}^\dagger U_3\right)\cdot U_4$$
$$+\left(U_2{U_0}^\dagger U_1{U_4}^\dagger U_0\right)\cdot U_3\cdot U_4 - \left(U_4{U_0}^\dagger U_2{U_4}^\dagger U_1\right)\cdot U_0\cdot U_3 - (1\leftrightarrow 2, 0\leftrightarrow 4). \quad \text{(B.13)}$$

These identities (B.11)–(B.13) also can be checked using (B.1) with $l = 3$ and then expanding the product of Levi-Civita symbols via (B.4).

$$0 = \left(U_2{U_4}^\dagger U_0 - U_0{U_4}^\dagger U_2\right)\cdot\left(U_3{U_0}^\dagger U_1 - U_1{U_0}^\dagger U_3\right)\cdot U_4$$
$$+\left(U_3{U_4}^\dagger U_2 - U_2{U_4}^\dagger U_3\right)\cdot\left(U_4{U_0}^\dagger U_1 - U_1{U_0}^\dagger U_4\right)\cdot U_0$$
$$+\left(U_1{U_0}^\dagger U_2 - U_2{U_0}^\dagger U_1\right)\cdot\left(U_3{U_4}^\dagger U_0 - U_0{U_4}^\dagger U_3\right)\cdot U_4 + (0\leftrightarrow 4). \quad \text{(B.14)}$$

It can be proved using (B.1) with $l = 4$ and (B.4).

$$0 = (U_1{U_0}^\dagger U_4 + U_4{U_0}^\dagger U_1)\cdot U_2\cdot U_3$$

$$+ 2(U_3U_0{}^\dagger U_4 + U_4U_0{}^\dagger U_3)\cdot U_1\cdot U_2 - U_1\cdot U_2\cdot U_3 tr(U_0{}^\dagger U_4)$$
$$+ 3(U_3U_0{}^\dagger U_1 + U_1U_0{}^\dagger U_3)\cdot U_2\cdot U_4 - 3U_1\cdot U_2\cdot U_4 tr(U_0{}^\dagger U_3) + (1\leftrightarrow 2). \tag{B.15}$$

This identity is necessary for calculation of the quark contribution. It can be proved using (B.1) with $l = 2$ and (B.4).

C Construction of conformal 4-point operator

Here we derive the evolution equation for the operator

$$\left(\left(U_2U_4^\dagger U_1 + U_1U_4^\dagger U_2\right)\cdot U_4\cdot U_3 - 2B_{123}\right) = \left(-3B_{123} + \frac{1}{2}(B_{144}B_{324} + B_{244}B_{314} - B_{344}B_{214})\right). \tag{C.1}$$

So one has to find the evolution of the operator $\left(U_1U_4^\dagger U_2\right)\cdot U_4\cdot U_3$ first. It reads

$$\frac{\partial\left(U_1U_4^\dagger U_2\right)\cdot U_4\cdot U_3}{\partial\eta} = \left(U_1U_4^\dagger U_2\right)\cdot U_4\cdot U_3\left(-\frac{\alpha_s}{2\pi^2}\frac{4}{3}2\right)\int d\vec{z}_0\left(\frac{1}{\vec{r}_{10}^{\,2}} + \frac{1}{\vec{r}_{20}^{\,2}} + \frac{1}{\vec{r}_{30}^{\,2}} + \frac{2}{\vec{r}_{40}^{\,2}}\right)$$
$$-\frac{\alpha_s}{\pi^2}\left[\left(t^cU_1U_4^\dagger t^cU_2\right)\cdot U_4\cdot U_3 + \left(U_1t^cU_4^\dagger U_2t^c\right)\cdot U_4\cdot U_3\right]\int d\vec{r}_0\frac{(\vec{r}_{10}\vec{r}_{20})}{\vec{r}_{10}^{\,2}\vec{r}_{20}^{\,2}}$$
$$-\frac{\alpha_s}{\pi^2}\left[\left(t^cU_1U_4^\dagger U_2\right)\cdot U_4\cdot(t^cU_3) + \left(U_1t^cU_4^\dagger U_2\right)\cdot U_4\cdot(U_3t^c)\right]\int d\vec{r}_0\frac{(\vec{r}_{10}\vec{r}_{30})}{\vec{r}_{10}^{\,2}\vec{r}_{30}^{\,2}}$$
$$-\frac{\alpha_s}{\pi^2}\left[\left(U_1U_4^\dagger t^cU_2\right)\cdot U_4\cdot(t^cU_3) + \left(U_1U_4^\dagger U_2t^c\right)\cdot U_4\cdot(U_3t^c)\right]\int d\vec{r}_0\frac{(\vec{r}_{20}\vec{r}_{30})}{\vec{r}_{20}^{\,2}\vec{r}_{30}^{\,2}}$$
$$-\frac{\alpha_s}{\pi^2}\left[\left(t^cU_1U_4^\dagger U_2\right)\cdot(t^cU_4) + \left(U_1t^cU_4^\dagger U_2\right)\cdot(U_4t^c)\right.$$
$$\left.-\left(t^cU_1U_4^\dagger t^cU_2\right)\cdot U_4 - \left(U_1t^ct^cU_4^\dagger U_2\right)\cdot U_4\right]\cdot U_3\int d\vec{r}_0\frac{(\vec{r}_{10}\vec{r}_{40})}{\vec{r}_{10}^{\,2}\vec{r}_{40}^{\,2}}$$
$$-\frac{\alpha_s}{\pi^2}\left[\left(U_1U_4^\dagger t^cU_2\right)\cdot(t^cU_4) + \left(U_1U_4^\dagger U_2t^c\right)\cdot(U_4t^c)\right]$$
$$\left.-\left(U_1U_4^\dagger t^ct^cU_2\right)\cdot U_4 - \left(U_1t^cU_4^\dagger U_2t^c\right)\cdot U_4\right]\cdot U_3\int d\vec{r}_0\frac{(\vec{r}_{20}\vec{r}_{40})}{\vec{r}_{20}^{\,2}\vec{r}_{40}^{\,2}}$$
$$-\frac{\alpha_s}{\pi^2}\left[\left(U_1U_4^\dagger U_2\right)\cdot(t^cU_4)\cdot(t^cU_3) + \left(U_1U_4^\dagger U_2\right)\cdot(U_4t^c)\cdot(U_3t^c)\right.$$
$$\left.-\left(U_1U_4^\dagger t^cU_2\right)\cdot U_4\cdot(t^cU_3) - \left(U_1t^cU_4^\dagger U_2\right)\cdot U_4\cdot(U_3t^c)\right]\int d\vec{r}_0\frac{(\vec{r}_{30}\vec{z}_{40})}{\vec{r}_{30}^{\,2}\vec{r}_{40}^{\,2}}$$
$$+\frac{\alpha_s}{\pi^2}\left[\left(U_1U_4^\dagger t^cU_2\right)\cdot(t^cU_4)\cdot U_3 + \left(U_1t^cU_4^\dagger U_2\right)\cdot(U_4t^c)\cdot U_3\right]\int\frac{d\vec{r}_0}{\vec{r}_{40}^{\,2}}$$
$$+\frac{\alpha_s}{\pi^2}\int d\vec{r}_0U_0^{cd}\left(\frac{\left(U_1U_4^\dagger U_2\right)\cdot(t^cU_4t^d)\cdot U_3}{\vec{r}_{04}^{\,2}} + \frac{\left(U_1t^dU_4^\dagger t^cU_2\right)\cdot U_4\cdot U_3}{\vec{r}_{04}^{\,2}}\right.$$
$$\left.+\frac{\left(t^cU_1t^dU_4^\dagger U_2\right)\cdot U_4\cdot U_3}{\vec{r}_{01}^{\,2}} + \frac{\left(U_1U_4^\dagger t^cU_2t^d\right)\cdot U_4\cdot U_3}{\vec{r}_{02}^{\,2}} + \frac{\left(U_1U_4^\dagger U_2\right)\cdot U_4\cdot(t^cU_3t^d)}{\vec{r}_{03}^{\,2}}\right)$$

$$
\begin{aligned}
&+\frac{\alpha_s}{\pi^2}\int d\vec r_0\frac{(\vec r_{03}\vec z_{02})}{\vec r_{03}^{\,2}\vec r_{02}^{\,2}}U_0^{cd}\Big(\Big(U_1U_4^\dagger t^cU_2\Big)\cdot U_4\cdot(U_3t^d)+\Big(U_1U_4^\dagger U_2t^d\Big)\cdot U_4\cdot(t^cU_3)\Big)\\
&+\frac{\alpha_s}{\pi^2}\int d\vec r_0\frac{(\vec r_{01}\vec z_{02})}{\vec r_{01}^{\,2}\vec r_{02}^{\,2}}U_0^{cd}\Big(\Big(U_1t^dU_4^\dagger t^cU_2\Big)\cdot U_4\cdot U_3+\Big(t^cU_1U_4^\dagger U_2t^d\Big)\cdot U_4\cdot U_3\Big)\\
&+\frac{\alpha_s}{\pi^2}\int d\vec r_0\frac{(\vec r_{01}\vec z_{03})}{\vec r_{01}^{\,2}\vec r_{03}^{\,2}}U_0^{cd}\Big(\Big(U_1t^dU_4^\dagger U_2\Big)\cdot U_4\cdot(t^cU_3)+\Big(t^cU_1U_4^\dagger U_2\Big)\cdot U_4\cdot(U_3t^d)\Big)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\alpha_s}{\pi^2}\int d\vec r_0\frac{(\vec r_{04}\vec z_{01})}{\vec r_{04}^{\,2}\vec r_{01}^{\,2}}U_0^{cd}\Big(\Big(U_1t^dU_4^\dagger U_2\Big)\cdot(t^cU_4)\cdot U_3+\Big(t^cU_1U_4^\dagger U_2\Big)\cdot(U_4t^d)\cdot U_3\\
&-\Big(U_1t^dU_4^\dagger t^cU_2\Big)\cdot U_4\cdot U_3-\Big(t^cU_1t^dU_4^\dagger U_2\Big)\cdot U_4\cdot U_3\Big)\\
&+\frac{\alpha_s}{\pi^2}\int d\vec r_0\frac{(\vec r_{04}\vec z_{02})}{\vec r_{04}^{\,2}\vec r_{02}^{\,2}}U_0^{cd}\Big(\Big(U_1U_4^\dagger U_2t^d\Big)\cdot(t^cU_4)\cdot U_3+\Big(U_1U_4^\dagger t^cU_2\Big)\cdot(U_4t^d)\cdot U_3\\
&-\Big(U_1U_4^\dagger t^cU_2t^d\Big)\cdot U_4\cdot U_3-\Big(U_1t^dU_4^\dagger t^cU_2\Big)\cdot U_4\cdot U_3\Big)\\
&+\frac{\alpha_s}{\pi^2}\int d\vec r_0\frac{(\vec r_{04}\vec z_{03})}{\vec r_{04}^{\,2}\vec r_{03}^{\,2}}U_0^{cd}\Big(\Big(U_1U_4^\dagger U_2\Big)\cdot(t^cU_4)\cdot(U_3t^d)+\Big(U_1U_4^\dagger U_2\Big)\cdot(U_4t^d)\cdot(t^cU_3)\\
&-\Big(U_1U_4^\dagger t^cU_2\Big)\cdot U_4\cdot(U_3t^d)-\Big(U_1t^dU_4^\dagger U_2\Big)\cdot U_4\cdot(t^cU_3)\Big)\\
&-\frac{\alpha_s}{\pi^2}\int\frac{d\vec r_0}{\vec r_{04}^{\,2}}U_0^{cd}\Big(\Big(U_1t^dU_4^\dagger U_2\Big)\cdot(t^cU_4)\cdot U_3+\Big(U_1U_4^\dagger t^cU_2\Big)\cdot(U_4t^d)\cdot U_3\Big). \qquad \text{(C.2)}
\end{aligned}
$$

Using (B.3) and (B.10)–(B.12) after the convolution one gets

$$
\begin{aligned}
&\frac{\partial}{\partial\eta}\Big(\Big(U_2U_4^\dagger U_1+U_1U_4^\dagger U_2\Big)\cdot U_4\cdot U_3-2B_{123}\Big)=\frac{\alpha_s}{4\pi^2}\int d\vec r_0\\
&\times\left(A_{34}\frac{\vec r_{34}^{\,2}}{\vec r_{03}^{\,2}\vec r_{04}^{\,2}}+A_{13}\frac{\vec r_{13}^{\,2}}{\vec r_{03}^{\,2}\vec r_{01}^{\,2}}+A_{23}\frac{\vec r_{23}^{\,2}}{\vec r_{03}^{\,2}\vec r_{02}^{\,2}}+A_{14}\frac{\vec r_{14}^{\,2}}{\vec r_{01}^{\,2}\vec r_{04}^{\,2}}+A_{24}\frac{\vec r_{24}^{\,2}}{\vec r_{02}^{\,2}\vec r_{04}^{\,2}}+A_{12}\frac{\vec r_{12}^{\,2}}{\vec r_{01}^{\,2}\vec r_{02}^{\,2}}\right). \qquad \text{(C.3)}
\end{aligned}
$$

Here

$$
\begin{aligned}
A_{34}=&-2\Big(U_2U_4^\dagger U_1+U_1U_4^\dagger U_2\Big)\cdot U_4\cdot U_3+\Big(U_3U_4^\dagger U_1+U_1U_4^\dagger U_3\Big)\cdot U_4\cdot U_2\\
&+\Big(U_3U_4^\dagger U_2+U_2U_4^\dagger U_3\Big)\cdot U_4\cdot U_1+\Big(U_2U_4{}^\dagger U_1+U_1U_4{}^\dagger U_2\Big)\cdot\Big(U_3U_0{}^\dagger U_4+U_4U_0{}^\dagger U_3\Big)\cdot U_0\\
&-\Big(U_2U_4{}^\dagger U_0\Big)\cdot\Big(U_3U_0{}^\dagger U_1\Big)\cdot U_4-\Big(U_0U_4{}^\dagger U_2\Big)\cdot\Big(U_1U_0{}^\dagger U_3\Big)\cdot U_4\\
&-\Big(U_0U_4{}^\dagger U_1\Big)\cdot\Big(U_2U_0{}^\dagger U_3\Big)\cdot U_4-\Big(U_1U_4{}^\dagger U_0\Big)\cdot\Big(U_3U_0{}^\dagger U_2\Big)\cdot U_4. \qquad \text{(C.4)}\\
A_{13}=&\Big(U_0U_4{}^\dagger U_2\Big)\cdot\Big(U_1U_0{}^\dagger U_3\Big)\cdot U_4+\Big(U_2U_4{}^\dagger U_0\Big)\cdot\Big(U_3U_0{}^\dagger U_1\Big)\cdot U_4\\
&+\Big(U_3U_0{}^\dagger U_1U_4{}^\dagger U_2\Big)\cdot U_0\cdot U_4+\Big(U_2U_4{}^\dagger U_1U_0{}^\dagger U_3\Big)\cdot U_0\cdot U_4\\
&-2\Big(U_1U_0{}^\dagger U_3+U_3U_0{}^\dagger U_1\Big)\cdot U_0\cdot U_2-\Big(U_3U_4{}^\dagger U_2+U_2U_4{}^\dagger U_3\Big)\cdot U_1\cdot U_4\\
&-\Big(U_1U_4{}^\dagger U_2+U_2U_4{}^\dagger U_1\Big)\cdot U_3\cdot U_4+4U_1\cdot U_2\cdot U_3. \qquad \text{(C.5)}\\
A_{14}=&\Big[tr\Big(U_0{}^\dagger U_1\Big)\Big(U_2U_4{}^\dagger U_0+U_0U_4{}^\dagger U_2\Big)+tr\Big(U_4{}^\dagger U_0\Big)\Big(U_1U_0{}^\dagger U_2+U_2U_0{}^\dagger U_1\Big)\Big]\cdot U_3\cdot U_4\\
&+\Big(U_2U_4{}^\dagger U_0\Big)\cdot\Big(U_4U_0{}^\dagger U_1\Big)\cdot U_3+\Big(U_0U_4{}^\dagger U_2\Big)\cdot\Big(U_1U_0{}^\dagger U_4\Big)\cdot U_3
\end{aligned}
$$

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$$- 2U_4 \cdot U_2 \cdot U_3 tr\left(U_4{}^\dagger U_1\right) - 2U_1 \cdot U_2 \cdot U_3 - 4\left(U_1 U_4{}^\dagger U_2 + U_2 U_4{}^\dagger U_1\right) \cdot U_3 \cdot U_4$$
$$+ \left(U_4 U_0{}^\dagger U_1 U_4{}^\dagger U_2\right) \cdot U_0 \cdot U_3 + \left(U_2 U_4{}^\dagger U_1 U_0{}^\dagger U_4\right) \cdot U_0 \cdot U_3. \tag{C.6}$$

$$A_{12} = -2\left(U_1 U_0{}^\dagger U_2 + U_2 U_0{}^\dagger U_1\right) \cdot U_3 \cdot U_0 - tr\left(U_4{}^\dagger U_0\right)\left(U_1 U_0{}^\dagger U_2 + U_2 U_0{}^\dagger U_1\right) \cdot U_3 \cdot U_4$$
$$+ 4U_1 \cdot U_2 \cdot U_3 + 2U_4 \cdot U_2 \cdot U_3 tr\left(U_4{}^\dagger U_1\right) + 2U_4 \cdot U_1 \cdot U_3 tr\left(U_4{}^\dagger U_2\right)$$
$$- U_0 \cdot U_3 \cdot U_4 \left(tr\left(U_0{}^\dagger U_1 U_4{}^\dagger U_2\right) + tr\left(U_0{}^\dagger U_2 U_4{}^\dagger U_1\right)\right). \tag{C.7}$$

$$A_{23} = A_{13}|_{\vec r_1 \leftrightarrow \vec r_2}, \quad A_{24} = A_{14}|_{\vec r_1 \leftrightarrow \vec r_2}. \tag{C.8}$$

Our prescription for the composite conformal operators reads [19] (see also ref. [34])

$$O^{\rm conf} = O + \frac{1}{2}\frac{\partial O}{\partial \eta}\Bigg|_{\frac{\vec r_{mn}^2}{\vec r_{im}^2 \vec r_{\rm in}^2} \to \frac{\vec r_{mn}^2}{\vec r_{im}^2 \vec r_{\rm in}^2}\ln\left(\frac{\vec r_{mn}^2 a}{\vec r_{im}^2 \vec r_{\rm in}^2}\right)}, \tag{C.9}$$

where a is an arbitrary constant. Thus

$$B_{123}^{\rm conf} = B_{123} + \frac{1}{2}\frac{\partial B_{123}}{\partial \eta}\Bigg|_{\frac{\vec r_{mn}^2}{\vec r_{im}^2 \vec r_{\rm in}^2} \to \frac{\vec r_{mn}^2}{\vec r_{im}^2 \vec r_{\rm in}^2}\ln\left(\frac{\vec r_{mn}^2 a}{\vec r_{im}^2 \vec r_{\rm in}^2}\right)}$$
$$= B_{123} + \frac{\alpha_s 3}{8\pi^2}\int d\vec r_0 \left[\frac{\vec r_{12}^{\,2}}{\vec r_{01}^{\,2}\vec r_{02}^{\,2}}\ln\left(\frac{a\vec r_{12}^{\,2}}{\vec r_{01}^{\,2}\vec r_{02}^{\,2}}\right)\right.$$
$$\left.\times\left(-B_{123}+\frac{1}{6}(B_{100}B_{320}+B_{200}B_{310}-B_{300}B_{210})\right)+(1\leftrightarrow 3)+(2\leftrightarrow 3)\right], \tag{C.10}$$

$$\left(-3B_{123} + \frac{1}{2}(B_{144}B_{324} + B_{244}B_{314} - B_{344}B_{214})\right)^{\rm conf}$$
$$= \left(-3B_{123} + \frac{1}{2}(B_{144}B_{324} + B_{244}B_{314} - B_{344}B_{214})\right)$$
$$+ \frac{1}{2}\frac{\partial}{\partial \eta}\left(-3B_{123} + \frac{1}{2}(B_{144}B_{324} + B_{244}B_{314} - B_{344}B_{214})\right)\Bigg|_{\frac{\vec r_{mn}^2}{\vec r_{im}^2 \vec r_{\rm in}^2} \to \frac{\vec r_{mn}^2}{\vec r_{im}^2 \vec r_{\rm in}^2}\ln\left(\frac{\vec r_{mn}^2 a}{\vec r_{im}^2 \vec r_{\rm in}^2}\right)}$$
$$= \left(-3B_{123} + \frac{1}{2}(B_{144}B_{324} + B_{244}B_{314} - B_{344}B_{214})\right)$$
$$+ \frac{\alpha_s}{8\pi^2}\int d\vec r_0 \left(A_{34}\frac{\vec r_{34}^{\,2}}{\vec r_{03}^{\,2}\vec r_{04}^{\,2}}\ln\left(\frac{\vec r_{34}^{\,2}a}{\vec r_{03}^{\,2}\vec r_{04}^{\,2}}\right) + A_{13}\frac{\vec r_{13}^{\,2}}{\vec r_{03}^{\,2}\vec r_{01}^{\,2}}\ln\left(\frac{\vec r_{13}^{\,2}a}{\vec r_{03}^{\,2}\vec r_{01}^{\,2}}\right) + A_{23}\frac{\vec r_{23}^{\,2}}{\vec r_{03}^{\,2}\vec r_{02}^{\,2}}\ln\left(\frac{\vec r_{23}^{\,2}a}{\vec r_{03}^{\,2}\vec r_{02}^{\,2}}\right)\right.$$
$$\left.+ A_{14}\frac{\vec r_{14}^{\,2}}{\vec r_{01}^{\,2}\vec r_{04}^{\,2}}\ln\left(\frac{\vec r_{14}^{\,2}a}{\vec r_{01}^{\,2}\vec r_{04}^{\,2}}\right) + A_{24}\frac{\vec r_{24}^{\,2}}{\vec r_{02}^{\,2}\vec z_{04}^{\,2}}\ln\left(\frac{\vec r_{24}^{\,2}a}{\vec r_{02}^{\,2}\vec z_{04}^{\,2}}\right) + A_{12}\frac{\vec r_{12}^{\,2}}{\vec r_{01}^{\,2}\vec r_{02}^{\,2}}\ln\left(\frac{\vec r_{12}^{\,2}a}{\vec r_{01}^{\,2}\vec r_{02}^{\,2}}\right)\right). \tag{C.11}$$

In the 3-gluon approximation

$$\left(-3B_{123} + \frac{1}{2}(B_{144}B_{324} + B_{244}B_{314} - B_{344}B_{214})\right)$$
$$\stackrel{\rm 3g}{=} 3(-B_{123} + B_{144} + B_{324} + B_{244} + B_{314} - B_{344} - B_{214} - 6). \tag{C.12}$$

Therefore

$$
\begin{aligned}
&\left(-3B_{123}+\frac{1}{2}(B_{144}B_{324}+B_{244}B_{314}-B_{344}B_{214})\right)^{\rm conf}\\
&\overset{\rm 3g}{=} 3(-B_{123}+B_{144}+B_{324}+B_{244}+B_{314}-B_{344}-B_{214}-6)\\
&+\frac{3}{2}\frac{\partial}{\partial\eta}(-B_{123}+B_{144}+B_{324}+B_{244}+B_{314}-B_{344}-B_{214}-6)\Bigg|_{\frac{\vec r_{mn}^{\,2}}{\vec r_{im}^{\,2}\vec r_{in}^{\,2}}\to\frac{\vec r_{mn}^{\,2}}{\vec r_{im}^{\,2}\vec r_{in}^{\,2}}\ln\left(\frac{\vec r_{mn}^{\,2}a}{\vec r_{im}^{\,2}\vec r_{in}^{\,2}}\right)}\\
&=3(-B_{123}^{\rm conf}+B_{144}^{\rm conf}+B_{324}^{\rm conf}+B_{244}^{\rm conf}+B_{314}^{\rm conf}-B_{344}^{\rm conf}-B_{214}^{\rm conf}-6). \qquad \text{(C.13)}
\end{aligned}
$$

D Integrals

Here we describe the calculation of integral (6.21). It reads

$$
\int d\vec r_4 Z_{12} = J_{12} - (1\leftrightarrow 3). \qquad \text{(D.1)}
$$

Here

$$
\begin{aligned}
J_{12} = \frac{\vec r_{12}^{\,2}}{8\vec r_{01}^{\,2}\vec r_{02}^{\,2}}\int d\vec r_4 \Bigg[&\left(\frac{\vec r_{03}^{\,2}}{\vec r_{04}^{\,2}\vec r_{34}^{\,2}}-\frac{\vec r_{02}^{\,2}}{\vec r_{04}^{\,2}\vec r_{24}^{\,2}}\right)\ln\left(\frac{\vec r_{02}^{\,2}\vec r_{14}^{\,2}}{\vec r_{04}^{\,2}\vec r_{12}^{\,2}}\right)\\
&+\frac{\vec r_{01}^{\,2}}{\vec r_{04}^{\,2}\vec r_{14}^{\,2}}\ln\left(\frac{\vec r_{02}^{\,2}\vec r_{34}^{\,2}}{\vec r_{03}^{\,2}\vec r_{24}^{\,2}}\right)+\frac{\vec r_{13}^{\,2}}{\vec r_{14}^{\,2}\vec r_{34}^{\,2}}\ln\left(\frac{\vec r_{03}^{\,2}\vec r_{12}^{\,2}}{\vec r_{02}^{\,2}\vec r_{13}^{\,2}}\right)\Bigg]. \qquad \text{(D.2)}
\end{aligned}
$$

Since the integral is conformally invariant, one can set $\vec r_0 = 0$ and make the inversion, then calculate the integral and then again make the inversion and restore $\vec r_0$.

$$
\begin{aligned}
J_{12} &\overset{\vec r_0=0}{\to} \frac{\vec r_{12}^{\,2}}{8\vec r_1^{\,2}\vec r_2^{\,2}}\int dr_4\Bigg[\left(\frac{\vec r_3^{\,2}}{\vec r_4^{\,2}\vec r_{34}^{\,2}}-\frac{\vec r_2^{\,2}}{\vec r_4^{\,2}\vec r_{24}^{\,2}}\right)\ln\left(\frac{\vec r_2^{\,2}\vec r_{14}^{\,2}}{\vec r_4^{\,2}\vec r_{12}^{\,2}}\right)\\
&\quad+\frac{\vec r_1^{\,2}}{\vec r_4^{\,2}\vec r_{14}^{\,2}}\ln\left(\frac{\vec r_2^{\,2}\vec r_{34}^{\,2}}{\vec r_3^{\,2}\vec r_{24}^{\,2}}\right)+\frac{\vec r_{13}^{\,2}}{\vec r_{14}^{\,2}\vec r_{34}^{\,2}}\ln\left(\frac{\vec r_3^{\,2}\vec r_{12}^{\,2}}{\vec r_2^{\,2}\vec r_{13}^{\,2}}\right)\Bigg]\\
&\overset{\rm inversion}{\to}\frac{r_{12}^2}{8}\int dr_4\Bigg[\left(\frac{1}{\vec r_{34}^{\,2}}-\frac{1}{\vec r_{24}^{\,2}}\right)\ln\left(\frac{\vec r_{14}^{\,2}}{\vec r_{12}^{\,2}}\right)+\frac{1}{\vec r_{14}^{\,2}}\ln\left(\frac{\vec r_{34}^{\,2}}{\vec r_{24}^{\,2}}\right)+\frac{\vec r_{13}^{\,2}}{\vec r_{14}^{\,2}\vec r_{34}^{\,2}}\ln\left(\frac{\vec r_{12}^{\,2}}{\vec r_{13}^{\,2}}\right)\Bigg]. \qquad \text{(D.3)}
\end{aligned}
$$

Using the integrals from appendix A in [36] we have

$$
J_{12}\to -\pi\frac{r_{12}^2}{8}\ln^2\left(\frac{\vec r_{13}^{\,2}}{\vec r_{12}^{\,2}}\right). \qquad \text{(D.4)}
$$

After inversion and restoring of $\vec r_0$ we get

$$
J_{12} = \; -\pi\frac{\vec r_{12}^{\,2}}{8\vec r_{01}^{\,2}\vec r_{02}^{\,2}}\ln^2\left(\frac{\vec r_{12}^{\,2}\vec r_{30}^{\,2}}{\vec r_{13}^{\,2}\vec r_{20}^{\,2}}\right). \qquad \text{(D.5)}
$$

Therefore

$$
\int\frac{d\vec r_4}{\pi}Z_{12} = \frac{\vec r_{32}^{\,2}}{8\vec r_{03}^{\,2}\vec r_{02}^{\,2}}\ln^2\left(\frac{\vec r_{32}^{\,2}\vec r_{10}^{\,2}}{\vec r_{13}^{\,2}\vec r_{20}^{\,2}}\right)-\frac{\vec r_{12}^{\,2}}{8\vec r_{01}^{\,2}\vec r_{02}^{\,2}}\ln^2\left(\frac{\vec r_{12}^{\,2}\vec r_{30}^{\,2}}{\vec r_{13}^{\,2}\vec r_{20}^{\,2}}\right). \qquad \text{(D.6)}
$$

Now we will integrate F_{100} (7.20) w.r.t. $\vec{r}_4$. Again we set $\vec{r}_0 = 0$, do inversion, and calculate the integral in the $d = 2+2\epsilon$ dimensional space using the integrals from appendix A in [36] and

$$\int \frac{d^{2+2\epsilon} r_{14}}{\pi^{1+\epsilon}\Gamma(1-\epsilon)} \frac{r_{34}{}^2}{r_{14}{}^2 r_{24}{}^2} = \frac{r_{13}^2 + r_{23}^2 - r_{12}^2}{r_{12}^2}\left(\frac{1}{\epsilon} + \ln\left(r_{12}^2\right)\right) + O(\epsilon). \tag{D.7}$$

We get

$$\begin{aligned}\int \frac{d\vec{r}_4}{\pi} F_{100} + (2\leftrightarrow 3) \to \int \frac{d^d r_4}{\pi}\Bigg(&\frac{r_{12}{}^2}{r_{24}{}^2}\ln\left(\frac{r_{14}{}^2}{r_{12}{}^2}\right) + \frac{r_{23}{}^2 r_{12}{}^2}{2r_{14}{}^2 r_{24}{}^2}\ln\left(\frac{r_{14}{}^2 r_{23}{}^2}{r_{12}{}^2 r_{24}{}^2}\right)\\
&+\frac{r_{13}{}^2 r_{12}{}^2}{r_{14}{}^2 r_{24}{}^2}\ln\left(\frac{r_{13}{}^2 r_{24}{}^2}{r_{12}{}^2 r_{14}{}^2}\right) - \frac{r_{13}{}^2}{r_{24}{}^2}\ln\left(\frac{r_{13}{}^2 r_{24}{}^2}{r_{14}{}^4}\right) + \frac{r_{23}{}^2}{2r_{24}{}^2}\ln\left(\frac{r_{23}{}^2 r_{24}{}^4 r_{34}{}^2}{r_{14}{}^8}\right)\\
&+\frac{r_{23}{}^2}{2r_{14}{}^2}\ln\left(\frac{r_{24}{}^2 r_{34}{}^2}{r_{14}{}^2 r_{23}{}^2}\right) - \frac{r_{34}{}^2 r_{12}{}^2}{2r_{14}{}^2 r_{24}{}^2}\ln\left(\frac{r_{24}{}^2 r_{34}{}^6}{r_{12}{}^6 r_{14}{}^2}\right) - \frac{r_{12}{}^2}{2r_{14}{}^2}\ln\left(\frac{r_{34}{}^4}{r_{12}{}^2 r_{14}{}^2}\right)\Bigg) + (2\leftrightarrow 3)\end{aligned} \tag{D.8}$$

$$\begin{aligned}\overset{d\to 2}{\to}&\left(\frac{3\left(r_{13}{}^2 - r_{12}{}^2\right)}{4} - \frac{r_{23}{}^2}{2}\right)\ln^2\left(\frac{r_{12}{}^2}{r_{23}{}^2}\right) + \left(\frac{3\left(r_{12}{}^2 - r_{13}{}^2\right)}{4} - \frac{r_{23}{}^2}{2}\right)\ln^2\left(\frac{r_{13}{}^2}{r_{23}{}^2}\right)\\
&+\left(\frac{3}{4}r_{23}{}^2 - r_{12}{}^2 - r_{13}{}^2\right)\ln^2\left(\frac{r_{12}{}^2}{r_{13}{}^2}\right) + \frac{3}{2}S_{123} I\left(r_{12}{}^2, r_{13}{}^2, r_{23}{}^2\right).\end{aligned} \tag{D.9}$$

Here

$$S_{123} = r_{12}{}^4 + r_{13}{}^4 + r_{23}{}^4 - 2r_{13}{}^2 r_{12}{}^2 - 2r_{23}{}^2 r_{12}{}^2 - 2r_{13}{}^2 r_{23}{}^2 \tag{D.10}$$

is the Cayley-Menger determinant proportional to the squared area of the triangle with the corners at $r = r_{1,2,3}$, and

$$I(a,b,c) = \int_0^1 \frac{dx}{a(1-x) + bx - cx(1-x)} \ln\left(\frac{a(1-x)+bx}{cx(1-x)}\right) \tag{D.11}$$

$$= \int_0^1\int_0^1\int_0^1 \frac{dx_1 dx_2 dx_3 \delta(1-x_1-x_2-x_3)}{(ax_1 + bx_2 + cx_3)(x_1x_2 + x_1x_3 + x_2x_3)} \tag{D.12}$$

$$= \int_0^1 dx \int_0^1 dz \frac{1}{cx(1-x)z + (b(1-x)+ax)(1-z)}. \tag{D.13}$$

is a symmetric function w.r.t. interchange of its arguments, defined in [37]. Performing inversion and restoring r_0, we get

$$\begin{aligned}\int \frac{d\vec{r}_4}{\pi} F_{100} + (2\leftrightarrow 3) = &\left(\frac{3\vec{r}_{23}{}^2}{4\vec{r}_{02}{}^2\vec{r}_{03}{}^2} - \frac{\vec{r}_{12}{}^2}{\vec{r}_{01}{}^2\vec{r}_{02}{}^2} - \frac{\vec{r}_{13}{}^2}{\vec{r}_{01}{}^2\vec{r}_{03}{}^2}\right)\ln^2\left(\frac{\vec{r}_{03}{}^2\vec{r}_{12}{}^2}{\vec{r}_{02}{}^2\vec{r}_{13}{}^2}\right)\\
&+\left(\frac{3\vec{r}_{12}{}^2}{4\vec{r}_{01}{}^2\vec{r}_{02}{}^2} - \frac{3\vec{r}_{13}{}^2}{4\vec{r}_{01}{}^2\vec{r}_{03}{}^2} - \frac{\vec{r}_{23}{}^2}{2\vec{r}_{02}{}^2\vec{r}_{03}{}^2}\right)\ln^2\left(\frac{\vec{r}_{02}{}^2\vec{r}_{13}{}^2}{\vec{r}_{01}{}^2\vec{r}_{23}{}^2}\right)\\
&+\left(\frac{3\vec{r}_{13}{}^2}{4\vec{r}_{01}{}^2\vec{r}_{03}{}^2} - \frac{3\vec{r}_{12}{}^2}{4\vec{r}_{01}{}^2\vec{r}_{02}{}^2} - \frac{\vec{r}_{23}{}^2}{2\vec{r}_{02}{}^2\vec{r}_{03}{}^2}\right)\ln^2\left(\frac{\vec{r}_{03}{}^2\vec{r}_{12}{}^2}{\vec{r}_{01}{}^2\vec{r}_{23}{}^2}\right)\\
&+\frac{3}{2}\tilde{S}_{123} I\left(\frac{\vec{r}_{12}{}^2}{\vec{r}_{01}{}^2\vec{r}_{02}{}^2}, \frac{\vec{r}_{13}{}^2}{\vec{r}_{01}{}^2\vec{r}_{03}{}^2}, \frac{\vec{r}_{23}{}^2}{\vec{r}_{02}{}^2\vec{r}_{03}{}^2}\right)\end{aligned}$$

$$+ X\left(\frac{\vec{r}_{02}{}^2\vec{r}_{13}{}^2}{\vec{r}_{03}{}^2\vec{r}_{12}{}^2}, \frac{\vec{r}_{02}{}^2\vec{r}_{13}{}^2}{\vec{r}_{01}{}^2\vec{r}_{23}{}^2}\right)\delta(\vec{r}_{20}) + X\left(\frac{\vec{r}_{03}{}^2\vec{r}_{12}{}^2}{\vec{r}_{02}{}^2\vec{r}_{13}{}^2}, \frac{\vec{r}_{03}{}^2\vec{r}_{12}{}^2}{\vec{r}_{01}{}^2\vec{r}_{23}{}^2}\right)\delta(\vec{r}_{30})$$
$$+ Y\left(\frac{\vec{r}_{01}{}^2\vec{r}_{23}{}^2}{\vec{r}_{03}{}^2\vec{r}_{12}{}^2}, \frac{\vec{r}_{01}{}^2\vec{r}_{23}{}^2}{\vec{r}_{02}{}^2\vec{r}_{13}{}^2}\right)\delta\left(\vec{r}_{10}\right). \tag{D.14}$$

Here

$$\tilde{S}_{123} = \left(\frac{\vec{r}_{12}{}^4}{\vec{r}_{01}{}^4\vec{r}_{02}{}^4} + \frac{\vec{r}_{13}{}^4}{\vec{r}_{01}{}^4\vec{r}_{03}{}^4} + \frac{\vec{r}_{23}{}^4}{\vec{r}_{02}{}^4\vec{r}_{03}{}^4} - \frac{2\vec{r}_{13}{}^2\vec{r}_{12}{}^2}{\vec{r}_{01}{}^4\vec{r}_{02}{}^2\vec{r}_{03}{}^2} - \frac{2\vec{r}_{23}{}^2\vec{r}_{12}{}^2}{\vec{r}_{01}{}^2\vec{r}_{02}{}^4\vec{r}_{03}{}^2} - \frac{2\vec{r}_{13}{}^2\vec{r}_{23}{}^2}{\vec{r}_{01}{}^2\vec{r}_{02}{}^2\vec{r}_{03}{}^4}\right) \tag{D.15}$$

and we added the delta-functional contributions, which may be lost via inversion. Thanks to conformal invariance of the integral such contributions may depend only on conformally invariant ratios. We can find the values of the unknown functions X and Y at $\vec{r}_2 = \vec{r}_3, \vec{r}_2 = \vec{r}_1, \vec{r}_1 = \vec{r}_3$. Using (7.9) we have

$$\int \frac{d\vec{r}_4}{\pi} F_{100} + (2 \leftrightarrow 3)|_{\vec{r}_2=\vec{r}_3} = 16 \int \frac{d\vec{r}_4}{\pi} \tilde{L}_{12}^C = 24\pi\zeta(3)[\delta(\vec{r}_{10}) - \delta(\vec{r}_{20})]$$
$$= 2X\left(1,\infty\right)\delta(\vec{r}_{20}) + Y\left(0,0\right)\delta\left(\vec{r}_{10}\right). \tag{D.16}$$

Therefore

$$X\left(1,\infty\right) = -12\pi\zeta\left(3\right), \quad Y\left(0,0\right) = 24\pi\zeta(3) \tag{D.17}$$

and

$$\int \frac{d\vec{r}_4}{\pi} F_{100} + (2 \leftrightarrow 3)|_{\vec{r}_1=\vec{r}_3} = -4 \int \frac{d\vec{r}_4}{\pi} \tilde{L}_{12}^C = -6\pi\zeta(3)[\delta(\vec{r}_{10}) - \delta(\vec{r}_{20})]$$
$$= X\left(0,0\right)\delta(\vec{r}_{20}) + (Y\left(1,\infty\right) + X\left(\infty,1\right))\delta\left(\vec{r}_{10}\right). \tag{D.18}$$

Here again we used (7.9). Therefore

$$X\left(0,0\right) = 6\pi\zeta\left(3\right), \quad Y\left(1,\infty\right) + X\left(\infty,1\right) = -6\pi\zeta(3). \tag{D.19}$$

If $\vec{r}_2 \neq \vec{r}_3, \vec{r}_2 \neq \vec{r}_1, \vec{r}_1 \neq \vec{r}_3$ then the arguments of X and Y are fixed by the integration w.r.t. $\vec{r}_0$

$$X\left(\frac{\vec{r}_{02}{}^2\vec{r}_{13}{}^2}{\vec{r}_{03}{}^2\vec{r}_{12}{}^2}, \frac{\vec{r}_{02}{}^2\vec{r}_{13}{}^2}{\vec{r}_{01}{}^2\vec{r}_{23}{}^2}\right)\delta(\vec{r}_{20}) = X\left(0,0\right)\delta(\vec{r}_{20}) = 6\pi\zeta\left(3\right)\delta(\vec{r}_{20}), \tag{D.20}$$
$$Y\left(\frac{\vec{r}_{01}{}^2\vec{r}_{23}{}^2}{\vec{r}_{03}{}^2\vec{r}_{12}{}^2}, \frac{\vec{r}_{01}{}^2\vec{r}_{23}{}^2}{\vec{r}_{02}{}^2\vec{r}_{13}{}^2}\right)\delta\left(\vec{r}_{10}\right) = Y\left(0,0\right)\delta\left(\vec{r}_{10}\right) = 24\pi\zeta(3)\delta\left(\vec{r}_{10}\right). \tag{D.21}$$

As a result, one can write

$$\int \frac{d\vec{r}_4}{\pi} F_{100} + (2 \leftrightarrow 3) = \left(\frac{3\vec{r}_{23}{}^2}{4\vec{r}_{02}{}^2\vec{r}_{03}{}^2} - \frac{\vec{r}_{12}{}^2}{\vec{r}_{01}{}^2\vec{r}_{02}{}^2} - \frac{\vec{r}_{13}{}^2}{\vec{r}_{01}{}^2\vec{r}_{03}{}^2}\right)\ln^2\left(\frac{\vec{r}_{03}{}^2\vec{r}_{12}{}^2}{\vec{r}_{02}{}^2\vec{r}_{13}{}^2}\right)$$
$$+ \left(\frac{3\vec{r}_{12}{}^2}{4\vec{r}_{01}{}^2\vec{r}_{02}{}^2} - \frac{3\vec{r}_{13}{}^2}{4\vec{r}_{01}{}^2\vec{r}_{03}{}^2} - \frac{\vec{r}_{23}{}^2}{2\vec{r}_{02}{}^2\vec{r}_{03}{}^2}\right)\ln^2\left(\frac{\vec{r}_{02}{}^2\vec{r}_{13}{}^2}{\vec{r}_{01}{}^2\vec{r}_{23}{}^2}\right)$$
$$+ \left(\frac{3\vec{r}_{13}{}^2}{4\vec{r}_{01}{}^2\vec{r}_{03}{}^2} - \frac{3\vec{r}_{12}{}^2}{4\vec{r}_{01}{}^2\vec{r}_{02}{}^2} - \frac{\vec{r}_{23}{}^2}{2\vec{r}_{02}{}^2\vec{r}_{03}{}^2}\right)\ln^2\left(\frac{\vec{r}_{03}{}^2\vec{r}_{12}{}^2}{\vec{r}_{01}{}^2\vec{r}_{23}{}^2}\right)$$

$$+\frac{3}{2}\tilde{S}_{123}I\left(\frac{\vec{r}_{12}{}^2}{\vec{r}_{01}{}^2\vec{r}_{02}{}^2},\frac{\vec{r}_{13}{}^2}{\vec{r}_{01}{}^2\vec{r}_{03}{}^2},\frac{\vec{r}_{23}{}^2}{\vec{r}_{02}{}^2\vec{r}_{03}{}^2}\right)$$
$$+6\pi\zeta\left(3\right)\left(\delta(\vec{r}_{20})+\delta(\vec{r}_{30})\right)+24\pi\zeta(3)\delta\left(\vec{r}_{10}\right)$$
$$-36\pi\zeta\left(3\right)\delta_{23}\delta(\vec{r}_{20})-36\pi\zeta(3)(\delta_{13}+\delta_{12})\delta\left(\vec{r}_{10}\right)+72\pi\zeta(3)\delta_{13}\delta_{12}\delta\left(\vec{r}_{10}\right). \quad \text{(D.22)}$$

Here $\delta_{ij}=1$, if $\vec{r}_i=\vec{r}_j$ and $\delta_{ij}=0$ otherwise. The last term is added since the total contribution at $\vec{r}_1=\vec{r}_2=\vec{r}_3$ is 0.

Now we will integrate F_{230} (7.21) w.r.t. $\vec{r}_4$. Again we set $\vec{r}_0=0$, do inversion, and calculate the integral in the d-dimensional space using the integrals from appendix A in [36] and (D.7). We get

$$\int\frac{d\vec{r}_4}{\pi}F_{230}+(2\leftrightarrow 3)\rightarrow\int\frac{d^dr_4}{\pi}\left(\frac{r_{34}{}^2r_{12}{}^2}{2r_{14}{}^2r_{24}{}^2}\ln\left(\frac{r_{14}{}^2r_{34}{}^6}{r_{24}{}^2r_{12}{}^6}\right)+\frac{r_{12}{}^2}{2r_{14}{}^2}\ln\left(\frac{r_{12}{}^2r_{34}{}^8}{r_{14}{}^4r_{24}{}^6}\right)\right.$$
$$-\frac{r_{12}{}^2}{2r_{24}{}^2}\ln\left(\frac{r_{14}{}^2}{r_{12}{}^2}\right)-\frac{r_{23}{}^2r_{12}{}^2}{r_{14}{}^2r_{24}{}^2}\ln\left(\frac{r_{14}{}^2r_{23}{}^2}{r_{12}{}^2r_{24}{}^2}\right)-\frac{r_{13}{}^2r_{12}{}^2}{2r_{14}{}^2r_{24}{}^2}\ln\left(\frac{r_{13}{}^2r_{24}{}^2}{r_{12}{}^2r_{14}{}^2}\right)$$
$$\left.-\frac{r_{23}{}^2}{r_{14}{}^2}\ln\left(\frac{r_{24}{}^2r_{34}{}^2}{r_{14}{}^2r_{23}{}^2}\right)+\frac{r_{13}{}^2}{2r_{24}{}^2}\ln\left(\frac{r_{13}{}^2r_{24}{}^2}{r_{14}{}^4}\right)-\frac{r_{23}{}^2}{2r_{24}{}^2}\ln\left(\frac{r_{23}{}^4r_{24}{}^2}{r_{14}{}^4r_{34}{}^2}\right)\right)+(2\leftrightarrow 3)$$
$$\stackrel{d\to 2}{\to}\left(\frac{3\left(r_{12}{}^2-r_{13}{}^2\right)}{4}+r_{23}{}^2\right)\ln^2\left(\frac{r_{12}{}^2}{r_{23}{}^2}\right)+\left(\frac{3\left(r_{13}{}^2-r_{12}{}^2\right)}{4}+r_{23}{}^2\right)\ln^2\left(\frac{r_{13}{}^2}{r_{23}{}^2}\right)$$
$$+\left(\frac{r_{12}{}^2+r_{13}{}^2}{2}-\frac{3}{4}r_{23}{}^2\right)\ln^2\left(\frac{r_{12}{}^2}{r_{13}{}^2}\right)-\frac{3}{2}S_{123}I\left(r_{12}{}^2,r_{13}{}^2,r_{23}{}^2\right). \quad \text{(D.23)}$$

Again, inverting and restoring r_0 we have

$$\int\frac{d\vec{r}_4}{\pi}F_{230}+(2\leftrightarrow 3)=\left(\frac{\vec{r}_{12}{}^2}{2\vec{r}_{01}{}^2\vec{r}_{02}{}^2}+\frac{\vec{r}_{13}{}^2}{2\vec{r}_{01}{}^2\vec{r}_{03}{}^2}-\frac{3\vec{r}_{23}{}^2}{4\vec{r}_{02}{}^2\vec{r}_{03}{}^2}\right)\ln^2\left(\frac{\vec{r}_{03}{}^2\vec{r}_{12}{}^2}{\vec{r}_{02}{}^2\vec{r}_{13}{}^2}\right)$$
$$+\left(\frac{3\vec{r}_{13}{}^2}{4\vec{r}_{01}{}^2\vec{r}_{03}{}^2}-\frac{3\vec{r}_{12}{}^2}{4\vec{r}_{01}{}^2\vec{r}_{02}{}^2}+\frac{\vec{r}_{23}{}^2}{\vec{r}_{02}{}^2\vec{r}_{03}{}^2}\right)\ln^2\left(\frac{\vec{r}_{02}{}^2\vec{r}_{13}{}^2}{\vec{r}_{01}{}^2\vec{r}_{23}{}^2}\right)$$
$$+\left(\frac{3\vec{r}_{12}{}^2}{4\vec{r}_{01}{}^2\vec{r}_{02}{}^2}-\frac{3\vec{r}_{13}{}^2}{4\vec{r}_{01}{}^2\vec{r}_{03}{}^2}+\frac{\vec{r}_{23}{}^2}{\vec{r}_{02}{}^2\vec{r}_{03}{}^2}\right)\ln^2\left(\frac{\vec{r}_{03}{}^2\vec{r}_{12}{}^2}{\vec{r}_{01}{}^2\vec{r}_{23}{}^2}\right)$$
$$-\frac{3}{2}\tilde{S}_{123}I\left(\frac{\vec{r}_{12}{}^2}{\vec{r}_{01}{}^2\vec{r}_{02}{}^2},\frac{\vec{r}_{13}{}^2}{\vec{r}_{01}{}^2\vec{r}_{03}{}^2},\frac{\vec{r}_{23}{}^2}{\vec{r}_{02}{}^2\vec{r}_{03}{}^2}\right)$$
$$+\tilde{X}\left(\frac{\vec{r}_{02}{}^2\vec{r}_{13}{}^2}{\vec{r}_{03}{}^2\vec{r}_{12}{}^2},\frac{\vec{r}_{02}{}^2\vec{r}_{13}{}^2}{\vec{r}_{01}{}^2\vec{r}_{23}{}^2}\right)\delta(\vec{r}_{20})+\tilde{X}\left(\frac{\vec{r}_{03}{}^2\vec{r}_{12}{}^2}{\vec{r}_{02}{}^2\vec{r}_{13}{}^2},\frac{\vec{r}_{03}{}^2\vec{r}_{12}{}^2}{\vec{r}_{01}{}^2\vec{r}_{23}{}^2}\right)\delta(\vec{r}_{30})$$
$$+\tilde{Y}\left(\frac{\vec{r}_{01}{}^2\vec{r}_{23}{}^2}{\vec{r}_{03}{}^2\vec{r}_{12}{}^2},\frac{\vec{r}_{01}{}^2\vec{r}_{23}{}^2}{\vec{r}_{02}{}^2\vec{r}_{13}{}^2}\right)\delta\left(\vec{r}_{10}\right). \quad \text{(D.24)}$$

Again, we can find the values of $\tilde{X}$ and $\tilde{Y}$ putting $\vec{r}_2=\vec{r}_3,\vec{r}_2=\vec{r}_1,\vec{r}_1=\vec{r}_3$ in this equation. Indeed via (7.9) we have,

$$\int\frac{d\vec{r}_4}{\pi}F_{230}+(2\leftrightarrow 3)|_{\vec{r}_2=\vec{r}_3}=-8\int\frac{d\vec{r}_4}{\pi}\tilde{L}_{12}^C=-12\pi\zeta(3)[\delta(\vec{r}_{10})-\delta(\vec{r}_{20})]$$
$$=2\tilde{X}\left(1,\infty\right)\delta(\vec{r}_{20})+\tilde{Y}\left(0,0\right)\delta\left(\vec{r}_{10}\right). \quad \text{(D.25)}$$

Therefore

$$\tilde{X}\left(1,\infty\right)=6\pi\zeta\left(3\right),\quad \tilde{Y}\left(0,0\right)=-12\pi\zeta(3). \quad \text{(D.26)}$$

Using (7.9) again, we get

$$\int \frac{d\vec{r}_4}{\pi} F_{230} + (2 \leftrightarrow 3)|_{\vec{r}_1=\vec{r}_3} = 8 \int \frac{d\vec{r}_4}{\pi} \tilde{L}^C_{12} = 12\pi\zeta(3)[\delta(\vec{r}_{10}) - \delta(\vec{r}_{20})]$$
$$= \tilde{X}(0,0)\,\delta(\vec{r}_{20}) + \left(\tilde{X}(\infty,1) + \tilde{Y}(1,\infty)\right)\delta(\vec{r}_{10}). \qquad \text{(D.27)}$$

Therefore

$$\tilde{X}(0,0) = -12\pi\zeta(3), \quad \tilde{Y}(1,\infty) + \tilde{X}(\infty,1) = 12\pi\zeta(3). \qquad \text{(D.28)}$$

If $\vec{r}_2 \neq \vec{r}_3, \vec{r}_2 \neq \vec{r}_1, \vec{r}_1 \neq \vec{r}_3$ then the arguments of $\tilde{X}$ and $\tilde{Y}$ are fixed by the integration w.r.t. $\vec{r}_0$

$$\tilde{X}\left(\frac{\vec{r}_{02}{}^2\vec{r}_{13}{}^2}{\vec{r}_{03}{}^2\vec{r}_{12}{}^2}, \frac{\vec{r}_{02}{}^2\vec{r}_{13}{}^2}{\vec{r}_{01}{}^2\vec{r}_{23}{}^2}\right)\delta(\vec{r}_{20}) = \tilde{X}(0,0)\,\delta(\vec{r}_{20}) = -12\pi\zeta(3)\,\delta(\vec{r}_{20}), \qquad \text{(D.29)}$$
$$\tilde{Y}\left(\frac{\vec{r}_{01}{}^2\vec{r}_{23}{}^2}{\vec{r}_{03}{}^2\vec{r}_{12}{}^2}, \frac{\vec{r}_{01}{}^2\vec{r}_{23}{}^2}{\vec{r}_{02}{}^2\vec{r}_{13}{}^2}\right)\delta(\vec{r}_{10}) = \tilde{Y}(0,0)\,\delta(\vec{r}_{10}) = -12\pi\zeta(3)\delta(\vec{r}_{10}). \qquad \text{(D.30)}$$

Finally,

$$\int \frac{d\vec{r}_4}{\pi} F_{230} + (2 \leftrightarrow 3) = \left(\frac{\vec{r}_{12}{}^2}{2\vec{r}_{01}{}^2\vec{r}_{02}{}^2} + \frac{\vec{r}_{13}{}^2}{2\vec{r}_{01}{}^2\vec{r}_{03}{}^2} - \frac{3\vec{r}_{23}{}^2}{4\vec{r}_{02}{}^2\vec{r}_{03}{}^2}\right)\ln^2\left(\frac{\vec{r}_{03}{}^2\vec{r}_{12}{}^2}{\vec{r}_{02}{}^2\vec{r}_{13}{}^2}\right)$$
$$+ \left(\frac{3\vec{r}_{13}{}^2}{4\vec{r}_{01}{}^2\vec{r}_{03}{}^2} - \frac{3\vec{r}_{12}{}^2}{4\vec{r}_{01}{}^2\vec{r}_{02}{}^2} + \frac{\vec{r}_{23}{}^2}{\vec{r}_{02}{}^2\vec{r}_{03}{}^2}\right)\ln^2\left(\frac{\vec{r}_{02}{}^2\vec{r}_{13}{}^2}{\vec{r}_{01}{}^2\vec{r}_{23}{}^2}\right)$$
$$+ \left(\frac{3\vec{r}_{12}{}^2}{4\vec{r}_{01}{}^2\vec{r}_{02}{}^2} - \frac{3\vec{r}_{13}{}^2}{4\vec{r}_{01}{}^2\vec{r}_{03}{}^2} + \frac{\vec{r}_{23}{}^2}{\vec{r}_{02}{}^2\vec{r}_{03}{}^2}\right)\ln^2\left(\frac{\vec{r}_{03}{}^2\vec{r}_{12}{}^2}{\vec{r}_{01}{}^2\vec{r}_{23}{}^2}\right)$$
$$- \frac{3}{2}\tilde{S}_{123}I\left(\frac{\vec{r}_{12}{}^2}{\vec{r}_{01}{}^2\vec{r}_{02}{}^2}, \frac{\vec{r}_{13}{}^2}{\vec{r}_{01}{}^2\vec{r}_{03}{}^2}, \frac{\vec{r}_{23}{}^2}{\vec{r}_{02}{}^2\vec{r}_{03}{}^2}\right)$$
$$- 12\pi\zeta(3)\left(\delta(\vec{r}_{20}) + \delta(\vec{r}_{30}) + \delta(\vec{r}_{10})\right) + 36\pi\zeta(3)\,\delta_{23}\delta(\vec{r}_{20})$$
$$+ 36\pi\zeta(3)(\delta_{13} + \delta_{12})\delta(\vec{r}_{10}) - 72\pi\zeta(3)\delta_{13}\delta_{12}\delta(\vec{r}_{10}). \qquad \text{(D.31)}$$

Now we will integrate (7.19) and prove equality (7.46). Again we set $\vec{r}_0 = 0$, do inversion, and calculate the integral in the d-dimensional space using the integrals from appendix A in [36] and (D.7). We get

$$\int \frac{d\vec{r}_4}{\pi}\left(\{F_{140} + (0 \leftrightarrow 4)\} + (2 \leftrightarrow 3)\right) \to \int \frac{d^d r_4}{\pi}\left(\frac{r_{12}{}^2}{r_{14}{}^2}\ln\left(\frac{r_{12}{}^2 r_{34}{}^4}{r_{14}{}^2 r_{24}{}^4}\right)\right.$$
$$+ \frac{r_{12}{}^2}{r_{24}{}^2}\ln\left(\frac{r_{12}{}^2 r_{24}{}^2}{r_{34}{}^4}\right) - \frac{r_{23}{}^2 r_{12}{}^2}{r_{14}{}^2 r_{24}{}^2}\ln\left(\frac{r_{14}{}^2 r_{23}{}^2}{r_{12}{}^2 r_{24}{}^2}\right) - \frac{r_{23}{}^2 r_{12}{}^2}{r_{24}{}^2 r_{34}{}^2}\ln\left(\frac{r_{23}{}^2 r_{24}{}^2}{r_{12}{}^2 r_{34}{}^2}\right)$$
$$- \frac{r_{23}{}^2}{r_{14}{}^2}\ln\left(\frac{r_{24}{}^2 r_{34}{}^2}{r_{14}{}^2 r_{23}{}^2}\right) + \frac{r_{13}{}^2}{r_{24}{}^2}\ln\left(\frac{r_{34}{}^4}{r_{13}{}^2 r_{24}{}^2}\right) + \frac{r_{23}{}^2}{r_{24}{}^2}\ln\left(\frac{r_{34}{}^2}{r_{23}{}^2}\right)$$
$$\left. + \frac{r_{23}{}^2}{r_{34}{}^2}\ln\left(\frac{r_{24}{}^2}{r_{23}{}^2}\right) + \frac{r_{13}{}^2 r_{24}{}^2}{r_{14}{}^2 r_{34}{}^2}\ln\left(\frac{r_{14}{}^2 r_{24}{}^2}{r_{13}{}^2 r_{34}{}^2}\right) - \frac{r_{14}{}^2 r_{23}{}^2}{r_{24}{}^2 r_{34}{}^2}\ln\left(\frac{r_{14}{}^2}{r_{23}{}^2}\right)\right) + (2 \leftrightarrow 3)$$
$$\overset{d\to 2}{\to} -\frac{r_{13}{}^2 + r_{12}{}^2}{2}\ln^2\left(\frac{r_{12}{}^2}{r_{13}{}^2}\right) + \frac{1}{2}r_{23}{}^2\left(\ln^2\left(\frac{r_{12}{}^2}{r_{23}{}^2}\right) + \ln^2\left(\frac{r_{13}{}^2}{r_{23}{}^2}\right)\right). \qquad \text{(D.32)}$$

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Inverting and restoring $\vec{r}_0$, we get

$$\int \frac{d\vec{r}_4}{\pi}\left(\{F_{140}+(0\leftrightarrow 4)\}+(2\leftrightarrow 3)\right)=-\frac{1}{2}\left(\frac{\vec{r}_{12}{}^2}{\vec{r}_{01}{}^2\vec{r}_{02}{}^2}+\frac{\vec{r}_{13}{}^2}{\vec{r}_{01}{}^2\vec{r}_{03}{}^2}\right)\ln^2\left(\frac{\vec{r}_{03}{}^2\vec{r}_{12}{}^2}{\vec{r}_{02}{}^2\vec{r}_{13}{}^2}\right)$$
$$+\frac{\vec{r}_{23}{}^2}{2\vec{r}_{02}{}^2\vec{r}_{03}{}^2}\ln^2\left(\frac{\vec{r}_{03}{}^2\vec{r}_{12}{}^2}{\vec{r}_{01}{}^2\vec{r}_{23}{}^2}\right)+\frac{\vec{r}_{23}{}^2}{2\vec{r}_{02}{}^2\vec{r}_{03}{}^2}\ln^2\left(\frac{\vec{r}_{02}{}^2\vec{r}_{13}{}^2}{\vec{r}_{01}{}^2\vec{r}_{23}{}^2}\right). \tag{D.33}$$

This integral has no delta functional contributions since it equals 0 at $\vec{r}_1=\vec{r}_2, \vec{r}_1=\vec{r}_3, \vec{r}_3=\vec{r}_2$.

E Decomposition of C-odd quadrupole operator

Here we demonstrate that the C-odd part of the quadrupole operator $tr(U_1U_2^\dagger U_3U_4^\dagger)$ in the 3-gluon approximation in SU(3) can be decomposed into a sum of 3QWLs. Indeed

$$2tr(U_1U_2^\dagger U_3U_4^\dagger)=\Big((U_1-U_2)(U_2^\dagger-U_3^\dagger)(U_3-U_4)\Big)\cdot U_4\cdot U_4$$
$$-B_{133}+B_{233}+B_{144}-B_{244}+B_{344}+B_{122}-6$$
$$\overset{3g}{\equiv}-(U_1-U_2)(U_2-U_3)(U_3-U_4)\cdot E\cdot E-B_{133}+B_{233}+B_{144}-B_{244}+B_{344}+B_{122}-6. \tag{E.1}$$

Therefore

$$2tr(U_1U_2^\dagger U_3U_4^\dagger)-2tr(U_4U_3^\dagger U_2U_1^\dagger)\overset{3g}{\equiv}-B^-_{133}+B^-_{233}+B^-_{144}-B^-_{244}+B^-_{344}+B^-_{122}$$
$$-\left((U_1-U_2)(U_2-U_3)(U_3-U_4)+(U_3-U_4)(U_2-U_3)(U_1-U_2)\right)\cdot E\cdot E$$
$$\overset{3g}{\equiv}-B^-_{133}+B^-_{233}+B^-_{144}-B^-_{244}+B^-_{344}+B^-_{122}-2(U_1-U_2)\cdot(U_2-U_3)\cdot(U_3-U_4)$$
$$\overset{3g}{\equiv}-B^-_{133}+B^-_{233}+B^-_{144}-B^-_{244}+B^-_{344}+B^-_{122}$$
$$-(U_1-U_2)\cdot(U_2-U_3)\cdot(U_3-U_4)+(U_1^\dagger-U_2^\dagger)\cdot(U_2^\dagger-U_3^\dagger)\cdot(U_3^\dagger-U_4^\dagger)$$
$$=B^-_{144}+B^-_{322}-B^-_{433}-B^-_{211}+B^-_{124}+B^-_{234}-B^-_{123}-B^-_{134}. \tag{E.2}$$

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