Integrable structure, W-symmetry and AGT relation

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ABSTRACT: In these notes we consider integrable structure of the conformal field theory with the algebra of symmetries $\mathcal{A} = W_n \otimes H$, where W_n is $W-$algebra and H is Heisenberg algebra. We found the system of commuting Integrals of Motion with relatively simple properties. In particular, this system has very simple spectrum and the matrix elements of special primary operators between its eigenstates have nice factorized form coinciding exactly with the contribution of the bifundamental multiplet to the Nekrasov partition function for U(n) gauge theories.

To the memory of Alyosha Zamolodchikov

Contents

1 Introduction

In the seminal paper [1] the relation between U(2) four-dimensional supersymmetric quiver gauge theories and two-dimensional conformal field theory with Virasoro symmetry has been proposed (AGT relation). In [2] the construction of [1] has been generalized to the relation between U(n) gauge theories and conformal field theories with W_n symmetry. The most non-trivial part of the statement in [1, 2] is the relation between the conformal blocks and the instanton part of the Nekrasov partition function [3, 4]. This relation has been further checked in some particular cases in [5–7] and was proved in [8–11] for the special values of the parameters. In our paper [12] we suggested a proof of AGT relation for linear quiver gauge theories with U(2) gauge groups. The main ingredient of [12] was the construction of special basis of states in the highest weight representations of the conformal algebra such that the matrix elements of primary operators in this basis have particularly simple factorized form which has natural interpretation in gauge theory. It was shown in [12] that these states are the eigenvectors for the infinite system of mutually commuting quantities (Integrals of Motion) related to Benjamin-Ono$_2$ integrable hierarchy [13–15]. This paper is devoted to generalization of our construction to the case of W_n algebras. The logic of the paper is backward to [12]. We start by defining the set of Integrals of Motion in the universal enveloping of algebra $\mathcal{A} = W_n \otimes \mathsf{H}$, where H is Heisenberg algebra and

then find its eigenstates. The matrix elements of appropriate primary fields between these states have again completely factorized form and coincide with the function called Z_{bif}.

We note that while in U(2) case the AGT relation gives just another possible algorithm for computation of the conformal blocks comparing to the traditional one [16] (see also [17]) for higher rank groups it seems to be the only way which can be accomplished. The standard bootstrap approach to two-dimensional CFT [16] requires the knowledge of structure constants of the operator product expansion as well as of the conformal blocks. The situation in W_n theory is conceptually more complicated than in the theory with Virasoro symmetry [18]. In particular, the conformal blocks are not fixed by the conformal and W-invariances. However, in some particular cases the bootstrap program can be completed similarly to what was done in [16]. We consider the $k-$point conformal block on a sphere of special primary fields

(1.1)

Namely, we took the charges α_1 and α_k to be arbitrary while all the charges $\alpha_2 \ldots \alpha_{k-1}$ to be proportional to the first fundamental weight ω_1 of the Lie algebra sl_n. In all the intermediate channels we took arbitrary fields and parametrized them in terms of momentums P_j. In this particular case the conformal block (1.1) is completely fixed by the W-invariance. At the same time as it was proposed by Wyllard [2] the conformal block in exactly this setup can be related to the instanton part of the Nekrasov partition function for the linear quiver gauge theory with the gauge group

$$\underbrace{\mathrm{U}(n) \otimes \cdots \otimes \mathrm{U}(n)}_{k-3}$$

and with $n-$ fundamental, $n-$ anti-fundamental and $k-4$ bifundamental matter hypermultiplets. The exact relation can be formulated as follows. We use the projective invariance and fix the positions of primary fields in (1.1) as $z_1 = 0$, $z_{k-1} = 1$ and $z_k = \infty$. It is also convenient to choose

$$z_{i+1} = q_i q_{i+1} \ldots q_{k-3} \quad \text{for} \quad 1 \le i \le k-3,$$

then the conformal block corresponding to the picture (1.1) is a power series expansion

$$\mathbb{F}(q) = 1 + \sum_{\vec{j}} q_1^{j_1} q_2^{j_2} \ldots q_{k-3}^{j_{k-3}} \, \mathbb{F}_{\vec{j}}, \tag{1.2}$$

where sum goes over all set of positive integers $\vec{j} = (j_1, \ldots, j_{k-3})$ and the coefficients $\mathbb{F}_{\vec{j}}$ are some rational functions of α_1, α_k, a_j and the central charge c. The AGT relation claims that the function

$$\mathbb{Z}(q) \stackrel{\text{def}}{=} \prod_{j=1}^{k-3} \prod_{m=j}^{k-3} (1 - q_j \ldots q_m)^{a_{j+1}(Q - a_{m+2}/n)} \, \mathbb{F}(q) = 1 + \sum_{\vec{j}} q_1^{j_1} q_2^{j_2} \ldots q_{k-3}^{j_{k-3}} \, \mathbb{Z}_{\vec{j}}, \tag{1.3}$$

coincides the instanton part of the Nekrasov partition function for corresponding gauge theory. The Nekrasov partition function is known in explicit terms (see (4.25)) and the main ingredient is the contribution of the bifundamental hypermultiplet Z_{bif}. In [12] we constructed a basis of states in highest weight representation of algebra $\mathcal{A} = \mathsf{Vir} \otimes \mathsf{H}$ such that the matrix elements of special primary operators between these states are given exactly by Z_{bif}. Here we give the generalization of this construction for W_n algebra.

The plan of the paper is the following. In section 2 we give short review of W_n algebras, then we define the system of commuting Integrals of Motion in the universal enveloping of algebra $\mathcal{A} = W_n \otimes \mathsf{H}$ and then we define primary fields which have factorized matrix elements between eigenstates of this system of IM's. In section 3 we discuss free field representation of W_n algebras and reduce the problem of computation of certain class of the matrix elements to the problem of computing Selberg like contour integrals with insertion of two Jack polynomials. The computation of these integrals is performed in appendix A. In section 4 we discuss the factorization properties of the eigenstates which allow to compute the most general matrix elements.

2 W_3 algebra, integrals of motion and matrix elements

2.1 W_3 algebra

In this paper we consider in details the case of W_3 algebra. We start by reminding basic definitions. The chiral part of the algebra of symmetries in this case consists of two currents of the spin two $T(x)$ and three $W(z)$

$$T(z) = \sum_{n=-\infty}^{\infty} \frac{L_n}{z^{n+2}} \qquad \text{and} \qquad W(z) = \sum_{n=-\infty}^{\infty} \frac{W_n}{z^{n+3}}. \tag{2.1}$$

The Laurent components L_k and W_k form closed W_3 algebra with the commutation relations [19, 20]

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3-n)\delta_{n,-m}, \tag{2.2a}$$

$$[L_n, W_m] = (2n-m)W_{n+m}, \tag{2.2b}$$

$$[W_n, W_m] = \frac{c}{3 \cdot 5!}(n^2-1)(n^2-4)n\delta_{n,-m} + \frac{16}{22+5c}(n-m)\Lambda_{n+m} +$$
$$+ (n-m)\left(\frac{1}{15}(n+m+2)(n+m+3) - \frac{1}{6}(n+2)(m+2)\right)L_{n+m}, \tag{2.2c}$$

here

$$\Lambda_n = \sum_{k=-\infty}^{\infty} :L_k L_{n-k}: +\frac{1}{5}x_n L_n,$$
$$x_{2l} = (1+l)(1-l) \qquad x_{2l+1} = (2+l)(1-l).$$

Since the Cartan subalgebra of (2.2) is two dimensional it is convenient to introduce vector notations. Namely, let e_1 and e_2 be the simple roots of the Lie algebra sl_3 then it is convenient to parameterize the central charge as

$$c = 2 + 12\mathcal{Q}^2 = 2 + 24Q^2, \quad \text{where} \quad \mathcal{Q} = Q\rho, \quad Q = b + \frac{1}{b}, \tag{2.3}$$

and ρ is the Weyl vector $\rho = e_1 + e_2$. We also parameterize the primary fields V_α by the vector parameter α. The operator product expansions of the holomorphic currents (2.1) with the primary fields V_α has the form

$$\begin{aligned}
T(\xi)V_\alpha(z) &= \frac{\Delta(\alpha)V_\alpha(z)}{(\xi - z)^2} + \frac{\partial V_\alpha(z)}{(\xi - z)} + \dots \\
W(\xi)V_\alpha(z) &= \frac{w(\alpha)V_\alpha(z)}{(\xi - z)^3} + \frac{W_{-1}V_\alpha(z)}{(\xi - z)^2} + \frac{W_{-2}V_\alpha(z)}{(\xi - z)} + \dots
\end{aligned} \tag{2.4}$$

here

$$\Delta(\alpha) = \frac{(2\mathcal{Q} - \alpha, \alpha)}{2} \tag{2.4a}$$

is the conformal dimension and

$$w(\alpha) = i\sqrt{\frac{48}{22 + 5c}} (\alpha - \mathcal{Q}, h_1)(\alpha - \mathcal{Q}, h_2)(\alpha - \mathcal{Q}, h_3) \tag{2.4b}$$

is the quantum number associated to the $W(z)$ current where h_k are wights of the first fundamental representation.

2.2 Integrals of motion

The computation of the conformal block is equivalent to the problem of computing the matrix elements of primary operator $V_{a\omega_1}$ between CFT states. In [12] we showed that this exercise can be facilitated if one introduces additional bosonic field. In fact in [12] we considered the conformal field theory with Virasoro symmetry and showed that in the universal enveloping of algebra $\mathcal{A} = \mathsf{Vir} \otimes H$ there exists a system of mutually commuting quantities (Integrals of Motion) such that matrix elements of certain primary operators between its eigenstates have particularly simple factorized form. Now we consider the generalization of our construction to the case of W_3 algebra. Working with $W-$ algebras it is more convenient to renormalize our additional bosonic field ϕ. This field does not have zero mode and on a circle can be expanded in the series

$$\phi(y) = \phi_+(y) + \phi_-(y) = \sum_{k>0} \frac{ia_k}{k}e^{iky} + \sum_{k<0} \frac{ia_k}{k}e^{iky}$$

where operators a_k have the commutation relations

$$[a_k, a_l] = k\delta_{k+l}.$$

The integral of motion I announced in [12] in this case we can be written in the form

$$I_3 = \sqrt{\frac{8}{3}} \left(6iQ\sqrt{\frac{3}{8}} \sum_{k>0} ka_{-k}a_k + \sum_{k\neq 0} a_{-k}L_k \pm \frac{\sqrt{4 + 15Q^2}}{4}W_0 + \frac{1}{3} \sum_{i+j+k=0} a_ia_ja_k \right). \tag{2.5}$$

This integral commutes with the total energy

$$I_2 = L_0 + 2\sum_{k=1} a_{-k}a_k,$$

and also produces an infinite hierarchy of Integrals of Motions. Consider the eigenvectors of this operator at the first level

$$|\Psi_P\rangle = (U_1 W_{-1} + U_2 L_{-1} + U_3 a_{-1})|\Theta_P\rangle, \tag{2.6}$$

where $|\Theta_P\rangle$ is the vacua state such that $L_n|\Theta_P\rangle = W_n|\Theta_P\rangle = 0$ for $n > 0$, $L_0|\Theta_P\rangle = \Delta|\Theta_P\rangle$ and $W_0|\Theta_P\rangle = w_0|\Theta_P\rangle$. It is easy to calculate the matrix for the operator (2.5). We introduce the notations

$$r = 1/4\sqrt{4 + 15Q^2}, \quad \kappa = \frac{2}{4 + 15Q^2}, \quad w_0 = i\sqrt{\frac{6}{4 + 15Q^2}}\, x_1 x_2 x_3$$

where

$$x_j = (h_j, P). \tag{2.7}$$

(h_i are the weights of fundamental representation of sl_3 with highest weight ω_1). Then taking into account the commutation relations for W_3 algebra (2.2) our matrix can be written in the form: $M = \frac{2}{\sqrt{3}}M'$ where

$$M' = \begin{pmatrix} 0 & 2r & 0 \\ 2r\kappa(\Delta - 3/4Q^2) & 0 & 1 \\ 3w_0 & 2\Delta & iQ\frac{3}{2\sqrt{3}} \end{pmatrix}$$

The eigenvalues of this matrix can be easily calculated ($\Delta = Q^2 - P^2/2 = Q^2 + x_1 x_2 + x_1 x_3 + x_2 x_3$) and are equal

$$\lambda_i = i(Q - 2x_i) \tag{2.8}$$

They trivially transform under the Weyl reflections $x_i \to x_j$ so it is enough to find only one eigenvector. For eigenvalue $\lambda_3 = i(Q - 2x_3)$ it is

$$U_1 = 1, \quad U_2 = i\sqrt{\frac{3}{2}}\frac{(Q - 2x_3)}{\sqrt{4 + 15Q^2}}, \quad U_3 = -\frac{\sqrt{2}(Q - x_3 + x_1)(Q - x_3 + x_2)}{\sqrt{4 + 15Q^2}}. \tag{2.9}$$

It is easy to see that if we take $P = \omega_1 y - \mathcal{Q}$ where $\mathcal{Q} = Q\rho$ (ρ is the Weil vector) we derive that $U_3 = 0$ and our vector Ψ coincides with the null vector

$$\left(W_{-1} - \frac{3w_0}{2\Delta}L_{-1}\right)|\Theta_P\rangle.$$

We note that the sign before the operator W_0 in the integral I_3 can be chosen arbitrary. The change of the sign leads to the conjugated representation: $h_j \to -h_{4-j}$. Here we consider the case corresponding to the choice $+$ in (2.5). It is more convenient to renormalize the operators W_k in the definition of eigenvectors Ψ_P. We introduce the operators

$$W_k = i\sqrt{\frac{6}{4 + 15Q^2}}\hat{W}_k, \tag{2.10}$$

and normalize eigenvectors at the level k by the condition

$$|\Psi_P\rangle = (\hat{W}_{-1}^k + \dots)|\Theta_P\rangle, \tag{2.11}$$

then at the first level we will have

$$|\Psi_P\rangle = \left(\hat{W}_{-1} + \frac{1}{2}(Q - 2x_3)L_{-1} + \frac{i(Q + x_{13})(Q + x_{23})}{\sqrt{3}}a_{-1}\right)|\Theta_P\rangle \tag{2.12}$$

where $x_{ij} = x_i - x_j$. We note that the integral (2.5) is anti-Hermitian under the conjugation in the algebra $\mathcal{A} = W_3 \otimes H$

$$W_{-k}^+ = -W_k, \qquad \hat{W}_{-k}^+ = \hat{W}_k, \qquad L_{-k}^+ = L_k, \qquad a_{-k}^+ = -a_k, \tag{2.13}$$

and hence the left eigenstates can be defined by the conjugation of the right ones. Moreover, the states corresponding to different eigenvalues are orthogonal to each other. We can calculate the norm of the vector (2.12). It is equal

$$N^2 = x_{23}x_{13}(Q + x_{23})(Q + x_{13}) \tag{2.14}$$

where $x_{ij} = x_i - x_j$.

At the second level we have nine vectors

$$|\Psi'_P\rangle = \Big(\hat{W}_{-1}^2 + U_2\hat{W}_{-2} + U_3L_{-1}\hat{W}_{-1} + U_4L_{-2} + U_5L_{-1}^2 + U_6\hat{W}_{-1}a_{-1} +$$
$$U_7L_{-1}a_{-1} + U_8a_{-1}^2 + U_9a_{-2}\Big)|\Theta_P\rangle$$

Now the matrix will be more complicated. Let $M = \frac{2}{\sqrt{3}}M_1$ then the matrix M_1 is

$$\begin{pmatrix}
0 & 0 & 2r & 0 & 0 & 0 & 0 & 0 & 0 \\
2r\kappa X & 0 & 0 & 4r & 2r & 0 & 0 & 0 & 0 \\
4r\kappa Y & 0 & 0 & 0 & 4r & 1 & 0 & 0 & 0 \\
6r\kappa w_0 & 4r\kappa\Delta & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\
0 & 2r\kappa & 2r\kappa X & 0 & 0 & 0 & 1 & 0 & 0 \\
6w_0 & 4 & 2\Delta + 2 & 0 & 0 & i3q & 2r & 0 & 0 \\
6\kappa X & 0 & 3w_0 & 3 & 4\Delta + 2 & 2r\kappa X & i3q & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 6w_0 & 2\Delta & i6q & 1 \\
10\kappa\Delta X & 6w_0 & 9w_0 & 4\Delta + \frac{c}{2} & 6\Delta & 0 & 0 & 1 & i12q
\end{pmatrix}$$

where $X = (\Delta - \frac{3Q^2}{4})$, $Y = (\Delta - \frac{3Q^2-2}{4})$ and $q = Q\frac{\sqrt{3}}{2}$. It is possible to calculate the eigenvalues of this matrix. They have rather simple form linear in x_j

$$\lambda_j^{(1)} = i(2Q + 2x_j), \qquad \lambda_j^{(2)} = 2i(Q + b - 2x_j), \qquad \lambda_j^{(3)} = 2i(Q + 1/b - 2x_j)$$

We can find all the eigenvectors corresponding to these eigenvalues. We give here the eigenvectors for more symmetrical case corresponding to $\lambda_j^{(1)}$. Then the eigenvector for $\lambda_3^{(1)} = i(2Q + 2x_3)$ has a form

$$U_2 = -\frac{1}{2}(Q + 2x_3), U_3 = (Q + x_3), \quad U_4 = -\frac{(Q + x_{32})(Q + x_{31})}{3},$$

$$U_5 = \frac{(Q - 2x_2)(Q - 2x_3)}{4}, \quad U_6 = \frac{i(1 + Q^2 + 3Qx_3 + x_{12}^2)}{\sqrt{3}},$$

$$U_7 = \frac{i(2Q^3 + 5Q^2x_3 + Q(x_3^2 + 8x_2x_1 - 1) - 2x_3(1 + x_{12}^2))}{\sqrt{2}},$$

$$U_8 = U_4(Q^2 - 1 - 4x_1^2 - x_{12}^2), \quad U_9 = iQ\sqrt{3}U_4. \tag{2.15}$$

and the norm of this vector is

$$N^2 = x_{13}x_{23}(Q + x_{31})(Q + x_{32})(x_{12} + b)(x_{12} - b)(x_{12} - 1/b)(x_{12} + 1/b)$$

The eigenvectors corresponding to $\lambda_i^{(2)}$ and $\lambda_i^{(3)}$ are even more simple. For eigenvalue $\lambda_3^{(2)} = 2i(Q + b - 2x_3)$ they have the following form in the normalization $U_1 = 1$

$$U_2 = -\frac{1}{2}(Q(1 + 2b^2) - 2b^2(x_3(2Q + b - 2x_3) - x_1x_2)), \quad U_3 = (Q + b - 2x_3)$$

$$U_4 = -\frac{b(Q + 2b - 3x_3)(Q + x_{23})(Q + x_{13})}{3}, \quad U_5 = \frac{(Q - 2x_3)(Q + 2b - 2x_3)}{4},$$

$$U_6 = \frac{i2(Q + b + x_{23})(Q + b + x_{13})}{\sqrt{3}},$$

$$U_7 = i\sqrt{2}(Q - 2x_3)(Q + b + x_{13})(Q + b + x_{23}),$$

$$U_8 = -\frac{(Q + x_{13})(Q + x_{23})(Q + b + x_{13})(Q + b + x_{23})}{3}, \quad U_9 = i\sqrt{3}bU_8 \tag{2.16}$$

The norm of this vector has a form

$$N^2 = 2b(Q - 2b)(Q + x_{13})(Q + x_{23})(Q + b + x_{13})(Q + b + x_{23})(b + x_{13})$$
$$\times (b + x_{23})x_{13}x_{23}$$

2.3 Matrix elements

For calculation of matrix elements of our vectors we introduce the vertex operator V_a

$$V_a = V_a^H \cdot V_a^W \tag{2.17}$$

where

$$V_a^H = \exp\left(\phi_-(a - 3Q)/\sqrt{3}\right) \exp\left(\phi_+ a/\sqrt{3}\right) \tag{2.18}$$

and V_a^W is the primary field of the W_3 algebra with the charge $\alpha = a\omega_1$. This field has the usual commutation relations with the generators of W-algebra

$$[L_m, V_a^W(z)] = z^m\left((m + 1)\Delta(a\omega_1) + z\partial_z\right)V_a^W(z),$$

$$[W_m, V_a^W(z)] = z^m\left(\frac{(m + 1)(m + 2)}{2}w(a\omega_1) + z(m + 1)W_{-1} + z^2W_{-2}\right)V_a^W(z), \tag{2.19a}$$

and

$$[a_n V_a^H(z)] = -i\sqrt{\frac{1}{3}}\, a\, V_a^H(z) z^n, n < 0$$

(2.19b)

$$[a_n V_a^H(z)] = i\sqrt{\frac{1}{3}}(3Q - a)\, V_a^H(z) z^n, n > 0$$

The fields $W_{-1}V_a^W(z)$ and $W_{-2}V_a^W(z)$ in (2.19a) are some descendant fields which in general can not be related to the primary field $V_a^W(z)$ by some differential operator. However, in a special situation which we are considering here, namely, the weight of primary field $V_a^W(z)$ is proportional to the first fundamental weight, the matrix elements can be computed with the help of (2.19) and conformal Ward identities (see [21] for details).

Every vector $|\Psi_{\vec{\nu},P}\rangle$ in the highest weight representation of the algebra $\mathcal{A} = W_3 \otimes H$ can be numerated by three Young diagrams $\vec{\nu} = (\nu_1, \nu_2, \nu_3)$. The vectors at the first level can be denoted as $|\Psi_{(1,0,0),P}\rangle$, $|\Psi_{(1,0,0),P}\rangle$, $|\Psi_{(0,0,1),P}\rangle$, at the second level $|\Psi_{(2,0,0),P}\rangle$, $|\Psi_{(0,2,0),P}\rangle$, $|\Psi_{(0,0,2),P}\rangle$, $|\Psi_{(\{11\},0,0),P}\rangle$, $|\Psi_{(0,\{11\},0),P}\rangle$, $|\Psi_{(0,0,\{11\}),P}\rangle$, $|\Psi_{(1,1,0),P}\rangle$, $|\Psi_{(1,0,1),P}\rangle$, $|\Psi_{(0,1,1),P}\rangle$. For example the vector (2.16)) is $|\Psi_{0,0,\{11\}}\rangle$ and the vector (2.15) is $|\Psi_{1,1,0}\rangle$.

For calculation of the multipoint conformal blocks (1.1) we need the matrix elements

$$F_{\vec{\nu}'}^{\vec{\nu}}(a, P, P') = \frac{\left\langle \Psi_{\vec{\nu},P^*} | V_a | \Psi_{\vec{\nu}',P'} \right\rangle}{\left\langle \Psi_{\vec{0},P^*} | V_a | \Psi_{\vec{0},P'} \right\rangle},$$

(2.20)

and also the matrix element

$$\frac{\left\langle \Psi_{\vec{0},P} | V_a | \Psi_{\vec{\nu}'P'} \right\rangle}{\left\langle \Psi_{\vec{0},P} | V_a | \Psi_{\vec{0},P'} \right\rangle},$$

here P^* denotes the usual conjugation: $\Delta(P^*) = \Delta(P)$, $w(P^*) = -w(P)$. The last element differs from $F_{\vec{\nu}'}^{\vec{0}}(a, P, P')$ by $P \to -P$, or $x_i \to -x_i$.

It is possible to calculate some matrix elements at the first and second levels. At the first level they are

$$F_{100}^{\vec{0}}(a, P, P') = -\prod_{j=1}^{3}(a/3 - x'_1 + x_j) = g_1(x, a, x'_1)$$

$$F_{\vec{0}}^{100}(a, P, P') = -\prod_{j=1}^{3}(a/3 - (Q - x_1 + x'_j)) = g_2(x_1, a, x')$$

$$F_{100}^{001}(a, P, P') = (x'_1 - x_1 - a/3)(x'_1 - x_2 - a/3)(x'_1 - x_3 + b - a/3) \times$$

$$(x'_1 - x_3 + 1/b - a/3)(Q + x'_3 - x_3 - a/3)(Q + x'_2 - x_3 - a/3) \quad (2.21)$$

All other matrix elements can be derived by simple transformations. At the second level we calculated elements which appear in the four point conformal block (see below)

$$F_{\{11\}00}^{\vec{0}}(a, P, P') = \prod_{j=1}^{3}(a/3 - x_1' + x_j)(a/3 - x_1' + x_j + b) = g_{\{1,1\}}(x, a, x_1', b),$$

$$F_{\vec{0}}^{00\{11\}}(a, P, P') = \prod_{j=1}^{3}(a/3 - (Q - x_3 + x_j'))(a/3 - (Q + b - x_3 + x_j')) = g^{\{1,1\}}(x_3, a, x', b),$$

$$F_{110}^{\vec{0}}(a, P, P') = \prod_{j=1}^{3}(a/3 - x_1' + x_j)(a/3 - x_2' + x_j) = g_{1,1}(x, a, x_1', x_2'),$$

$$F_{\vec{0}}^{110}(a, P, P') = \prod_{j=1}^{3}(a/3 - (Q - x_1 + x_j'))(a/3 - (Q - x_2 + x_j')) = g^{1,1}(x_1, x_2, a, x')$$

$$(2.22)$$

All other matrix elements having one empty triple of Young diagrams can be derived from these by simple transformations. For example, $\{11\} \to 2$ corresponds to $b \to 1/b$. We note that all the matrix elements calculated above can be written as

$$F_{\vec{\nu}'}^{\vec{\nu}}(a, P, P') = \prod_{i,j=1}^{3} \prod_{s\in\nu_i'}(Q - E_{\nu_i',\nu_j}(x_j - x_i'|s) - a/3) \prod_{t\in\nu_j}(E_{\nu_j,\nu_i'}(x_i' - x_j|t) - a/3) \quad (2.23)$$

where

$$E_{\lambda,\mu}(x|s) = x - bl_\mu(s) + (a_\lambda(s) + 1)/b,$$

where $a_\lambda(s)$ and $l_\mu(s)$ are correspondingly the arm length of the square s in the partition λ and the leg length of the square s in the partition μ.

It is natural to assume that in W_n- case the expression for the matrix elements will have the similar form (here $\vec{\nu}$ is the set of n Young diagrams $\vec{\nu} = (\nu_1, \ldots, \nu_n)$)

$$F_{\vec{\nu}'}^{\vec{\nu}}(a, P, P') = \prod_{i,j=1}^{n} \prod_{s\in\nu_i'}(Q - E_{\nu_i',\nu_j}(x_j - x_i'|s) - a/n) \prod_{t\in\nu_j}(E_{\nu_j,\nu_i'}(x_i' - x_j|t) - a/n). \quad (2.24)$$

We return to this equation later. We note that (2.24) coincides exactly with the contribution of the bifundamental multiplet to the Nekrasov partition function [22–24]

$$Z_{\mathbf{bif}}(\vec{a}, \vec{\nu}; \vec{b}, \vec{\nu}'; m), \qquad (2.25)$$

where $\vec{a} = (-x_1, -x_2, \ldots, -x_n)$, $\vec{b} = (-x_1', -x_2', \ldots, -x_n')$, $m = a/n$ and deformation parameters are taken $\epsilon_1 = b$, $\epsilon_2 = 1/b$.

2.4 Four-point conformal block

In order to check our construction we consider four point conformal block of the type (1.1)

$$\langle V_1(\infty)V_2(1)V_3(z)V_4(0)\rangle \qquad (2.26)$$

where fields V_1 and V_4 are the primary fields of $W-$ algebra with arbitrary parameters α_1, α_4 and fields V_2, V_3 have the parameters α_i proportional to ω_1 or ω_2. In this case it is possible to calculate the conformal block

$$\mathrm{F}_{1,2,3,4}(\alpha) = z^{\Delta_\alpha - \Delta_3 - \Delta_4}(1 + f_1 z + f_2 z^2 + \dots) \tag{2.27}$$

The first coefficient can be easily calculated and is equal

$$
\begin{aligned}
f_1 = {} & \frac{(\Delta_\alpha + \Delta_1 - \Delta_2)(\Delta_\alpha + \Delta_3 - \Delta_4)}{2\Delta_\alpha} + \\
& + \left(\frac{w_3 - w_\alpha}{2} - w_4 + \frac{3w_3}{2\Delta_3}(\Delta_\alpha - \Delta_4) + \frac{3w_\alpha}{2\Delta_a}(\Delta_4 - \Delta_3) \right) \times \\
& \left(\frac{-w_2 - w_\alpha}{2} - w_1 + \frac{3w_2}{2\Delta_2}(\Delta_1 - \Delta_a) + \frac{3w_\alpha}{2\Delta_\alpha}(\Delta_2 - \Delta_1) \right) \times \\
& \left(\Delta_\alpha \left[\frac{4}{4 + 15Q^2}(\Delta_a + \frac{1}{5}) - \frac{1}{5} \right] - \frac{9w_\alpha^2}{2\Delta_\alpha} \right)^{-1} \tag{2.28}
\end{aligned}
$$

We can calculate independently this function using our combinatorial expansion. In these notes we consider the case when $\alpha_2 = a_2\omega_1$ and $\alpha_3 = a_3\omega_1$. In this case we should have in mind that (due to the free field exponents in the vertex operator (2.17)) it gives us the expansion for the function $\mathrm{F}' = (1-z)^{a_2(Q-a_3/3)}\mathrm{F}$. For example, for the first coefficient of function $\mathrm{F}_{1,2,3,4}(\alpha)$ we have the following expression in terms of functions g_i (2.21)

$$f_1 = \sum_{k=1}^{3} \frac{g_1(-x, a_2, x'_k)g_2(x'_k, a_3, y)}{N_k^2} + a_2(Q - a_3/3) \tag{2.29}$$

where $x = \alpha_1 - \mathcal{Q}$, $y = \alpha_4 - \mathcal{Q}$, $x'_k = (h_k, \alpha - \mathcal{Q})$,

$$N_1^2 = x'_{12}x'_{13}(Q - x'_{12})(Q - x'_{13})$$

and all other N_k can be derived by the cyclic permutations $1 \to 2 \to 3 \to 1 \dots$. It can be checked that expression (2.28) coincides with (2.29). Similarly, we can obtain

$$
\begin{aligned}
f_2 = {} & \sum_{i<j} \frac{g_{1,1}(-x, a, x'_i, x'_j)g^{1,1}(x'_{4-i}, x'_{4-j}, a, y)}{N_{i,j}^2} + \\
& + \sum_{k=1}^{3} \left(\frac{g_{\{1,1\}}(-x, a, x'_k, b)g^{\{1,1\}}(x_{4-k}, a, y, b)}{(N_k(b))^2} + \frac{g_{\{1,1\}}(-x, a, x'_k, b^{-1})g^{\{1,1\}}(x'_{4-k}, a, y, b^{-1})}{(N_k(b^{-1}))^2} \right) + \\
& + a_2(Q - a_3/3)f_1 + \frac{a_2(Q - a_3/3)(a_2(Q - a_3/3) - 1)}{2}, \quad \tag{2.30}
\end{aligned}
$$

where (together with cyclic permutations)

$$
\begin{aligned}
(N_{1,2})^2 &= x_{13}x_{23}(Q + x_{31})(Q + x_{32})(x_{12} + b)(x_{12} - b)(x_{12} - 1/b)(x_{12} + 1/b), \\
(N_1(b))^2 &= 2b(Q - 2b)(Q + x_{13})(Q + x_{23})(Q + b + x_{13})(Q + b + x_{23}) \times \\
& \quad \times (b + x_{13})(b + x_{23})x_{13}x_{23}, \tag{2.31}
\end{aligned}
$$

and show that it coincides with the corresponding coefficient in (2.27).

3 Free field representation of W_n algebra and Selberg integrals

3.1 Bosonization of W_n algebra

We consider here the free field representation of W_n- algebra [20, 25] mainly studying this representation for W_3 current which was denoted as W. This current plays the important role because it generates all other currents in the OPE of this current with itself and with new currents. We define the free massless chiral $n-1$ component field $\varphi(z) = (\varphi_1(z), \ldots, \varphi_{n-1}(z))$ normalized by the condition

$$\langle \varphi_i(z) \varphi_j(z') \rangle = -\log(z - z') \delta_{ij}. \tag{3.1}$$

and introduce $n-1$ holomorphic currents $W_k(z)$ via the Miura transformation

$$\prod_{i=0}^{n} : (Q\partial + h_{n-i}\partial\varphi) := \sum_{k=0}^{n} W_{n-k}(z) (Q\partial)^k \tag{3.2}$$

where symbol :: denotes the Wick ordering and vectors h_k are the weights of the first fundamental representation of the Lie algebra $sl(n)$ with the highest weight ω_1

$$h_k = \omega_1 - e_1 - \cdots - e_{k-1}, \quad h_i h_j = 1 - \frac{1}{n}\delta_{ij}. \tag{3.3}$$

In particular, it follows from eq. (3.2) that $W_0(z) = 1$, $W_1(z) = 0$ and the current

$$W_2(z) = T(z) = -\frac{1}{2}(\partial\varphi)^2 + (\mathcal{Q}, \partial^2\varphi) = -\frac{1}{2}(\partial\varphi)^2 + Q(\rho, \partial^2\varphi) \tag{3.4}$$

where ρ is the Weil vector, form the Virasoro algebra with the central charge

$$c = n - 1 + 12\mathcal{Q}^2 = (n-1)\left(1 + n(n+1)Q^2\right)$$

The fields $W_k(z)$ are not conformal primary fields, however their OPEs calculated explicitly in [26] contain only bilinear combinations of W-currents. It is easy to check that the field

$$\hat{W}_3(z) = W_3(z) - \frac{(n-2)Q}{2}\partial T(z) \tag{3.5}$$

will be Virasoro primary field with respect to the stress energy tensor (3.4). The transformation of other fields $W_k(z)$ to the basis of conformal primaries $\widetilde{W}_k(z)$ is more complicated (see [27]). The transformation $W_k(z) \to \widetilde{W}_k(z)$ exists and is invertible but we will not need it here. As all the generators of $W-$algebra depend only on fields $(h_k, \partial\varphi)$ and their derivatives it is convenient to introduce the notation

$$v = \partial\varphi(z), \quad u_k(z) = (h_k, \partial\varphi(z)) = (h_k, v) \tag{3.6}$$

Sometimes it is convenient to choose the currents $\hat{W}_k(z)$ which transform as $\hat{W}_k(z) \to (-)^k \hat{W}_k(z)$ under the natural conjugation $e_i \to e_{n-i}$, $\omega_i \to \omega_{n-i}$, $h_i \to -h_{n+1-i}$. For this we can define the fields $W_k^*(z)$ which can be derived from the fields $W_k(z)$ if in their expression in terms of $\partial^i u_k$ we make a substitution $u_k \to -u_{n+1-k}$. Then the fields

$\hat{W}_k(z) = \frac{1}{2}(W_k(z) + (-1)^k W_k^*)$ will have the proper transformation under the conjugation. It is easy to check that the field $\hat{W}_3(z)$ defined in such a way coincides with the primary field defined by eq. (3.5).

The explicit expression for $\hat{W}_3(z)$ in the case $n = 3$ has a form

$$\hat{W}_3(z) =: u_1 u_2 u_3 + \frac{Q}{2}\left(u_3 \partial u_1 - u_1 \partial u_3 + \partial((u_1 - u_3)\, u_2) - \partial^2 u_2\right) : . \tag{3.7}$$

Later we will not use the symbol :: having in mind that all operators are Wick ordered. It is easy to see from eqs. (3.2) and (3.5) that the $\hat{W}_3(z)$ current in general case can be written as (up to total derivatives)

$$\hat{W}_3(z) = \frac{1}{3}\sum_{i=1}^{n} u_i^3 + Q(u_n(\rho_{n-1}, \partial v) + u_{n-1}(\rho_{n-2}, \partial v) + \cdots + u_3 \partial u_1) + \partial(\dots) \tag{3.8}$$

where $\rho_k = (k-1)h_1 + (k-2)h_2 + \cdots + h_{k-1}$. In particular $\rho_n = \rho$, $\rho_2 = h_1$. In eq. (3.8) we also used the identity

$$\sum_{i>j>k}^{n} u_i u_j u_k = \frac{1}{3}\sum_{i=1}^{n} u_i^3. \tag{3.9}$$

The usual normalization for W_3−current $W_3(z) = d_n^{(3)}\hat{W}_3(z)$ is defined by the condition

$$W_3(z)W_3(z') = \frac{c}{3}\frac{1}{(z-z')^6} + \frac{2}{(z-z')^4}T(z') + O\left(\frac{1}{(z-z')^3}\right), \tag{3.10}$$

which defines the coefficient $d_n^{(3)} = i\sqrt{\frac{2n}{(n-2)(4+n(n+2)Q^2)}}$

$$W_3(z) = i\sqrt{\frac{2n}{(n-2)(4+n(n+2)Q^2)}}\hat{W}_3(z). \tag{3.11}$$

The important role in the representations of $W-$ algebras play the exponential fields

$$V_\alpha(z) = e^{(\alpha, \varphi(z))}.$$

They are the primary fields of $W-$ algebra i.e.

$$\hat{W}_k(z)V_\alpha(z') = \frac{w_k(\alpha)}{(z-z')^k}V_\alpha(z') + O\left(\frac{1}{(z-z')^{k-1}}\right).$$

The set of Weyl invariant functions of $P = Q - \alpha$, $w_k(\alpha)$ for $k = 2, \ldots, n$ characterize the highest weight representations of $W-$ algebras. In particular

$$w_2(\alpha) = \Delta(\alpha) = \frac{1}{2}(\alpha, Q - \alpha) = \frac{1}{2}(Q^2 - \sum_i^n x_i^2), \qquad w_3(\alpha) = \frac{1}{3}\sum_i^n x_i^3,$$

where $x_j = (h_j, \alpha - Q) = (h_j, P)$. Function $w_3(a)$ is related to the function $w_3(\alpha)$ defined by (2.4b) as

$$w_3(a\omega_1) = d_n^{(3)}w_3(a).$$

All elements of the enveloping of $W-$ algebra commute with the screening charges

$$S_i = \int_C dz \mathrm{V}_{e_ib}(z), \quad \widetilde{S}_i = \int_C dz \mathrm{V}_{e_{i/b}}(z) \tag{3.12}$$

where C is a closed contour. This property permits to calculate some correlation functions in $W-$ invariant CFT [21, 28, 29].

In Hamiltonian approach it is useful to consider the theory on an infinite cylinder of circumference 2π i.e. to make the conformal transformation $z = e^{-iy}$ where $y = y_1 + iy_2$. The coordinate y_2 along the cylinder can be interpreted as an imaginary time and $y_1 \sim y_1 + 2\pi$ as a space coordinate. The fields $T(z)$ and Virasoro primary fields $\hat{\mathrm{W}}_3(z)$ under this change of variables transform as

$$T(y) = \frac{c}{24} - z^2(y)T(z), \quad \hat{\mathrm{W}}_3(y) = (-iz(y))^3 \hat{\mathrm{W}}_3(z) \tag{3.13}$$

The fields $\hat{\mathrm{W}}_k(z)$ for $k > 3$ defined above in general are not Virasoro primary fields and their transformation has more complicated form. In principle, we can choose the basis in the $W-$ algebra of conformal primary fields $\widetilde{\mathrm{W}}_k(z)$ which transform in a standard way under the conformal transformations. Here we will be mainly interested in fields $T(z)$ and $\hat{\mathrm{W}}_3(x)$. We note that after this transformation the fields $\hat{\mathrm{W}}_k(y)$ with $k > 3$ will still form a $W-$ algebra, but their representation in terms of fields v, contrary to the primary fields, will be modified. We consider the time slice with fixed y_2. Then all the fields $\hat{\mathrm{W}}_k(y)$ can be expanded in Fourier series

$$\hat{\mathrm{W}}_k(y) = \sum_{m=-\infty}^{\infty} \hat{\mathrm{W}}_k^{(m)} e^{imy}$$

The space of states of $W-$ algebra $\mathcal{A}$ is classified by its highest weight representations.

$$\mathcal{A} = \oplus_P \mathcal{A}_P$$

Each space $\mathcal{A}_P$ contains a primary state $|\Theta_P\rangle$ which is annihilated by all positive Fourier modes of operators $\hat{\mathrm{W}}_k$ and satisfies

$$\hat{\mathrm{W}}_k^{(m)}|\Theta_P\rangle = 0, \quad m > 0,$$
$$\hat{\mathrm{W}}_k^{(0)}|\Theta_P\rangle = \mathbf{w}_k(P)|\Theta_P\rangle$$

The free massless field v(y) and fields $u_i(y)$ can be expanded in Fourier series at the space $\mathcal{A}_P$ as

$$\mathrm{v}(y) = i\mathrm{P} - \sum_{m \neq 0} \mathrm{b}^{(m)} e^{imy}, \quad u_i = ix_i - \sum_{m \neq 0} c_i^{(m)} e^{imy} \tag{3.14}$$

where the components of the vector $\mathrm{b}^{(m)} = (\mathrm{b}_1^{(m)}, \ldots, \mathrm{b}_{n-1}^{(m)})$ and the operators $c_i^{(m)} = (h_i, \mathrm{b}^{(m)})$ have the commutation relations

$$[\mathrm{b}_r^{(m)}, \mathrm{b}_q^{(l)}] = m\delta_{m+l,0}\delta_{r,q}, \quad [c_i^{(m)}, c_j^{(l)}] = m\delta_{m+l,0}\left(\delta_{i,,j} - \frac{1}{n}\right). \tag{3.15}$$

3.2 Integrals of motion and Jack polynomials

We consider now the expression for the integral (2.5) and its generalization for the case of arbitrary n in terms of the fields u_k (operators $c_i^{(m)}$) and ϕ' where field ϕ', has no zero mode and possesses the expansion

$$\phi'(y) = -\sum_{k=-\infty}^{\infty} a_k e^{iky}. \tag{3.16}$$

For these purposes we need the expression for the generators L_k of the Virasoro algebra and operator $W_3^{(0)} = i d_n \frac{1}{2\pi} \int_0^{2\pi} \hat{W}_3(y_1) dy_1$. The explicit expression for the generators L_k

$$T(y) = \frac{c}{24} - \sum_{m=-\infty}^{\infty} L_k e^{iky},$$

where

$$L_{k\neq 0} = \frac{1}{2\pi} \int_0^{2\pi} \left(\frac{1}{2} \sum_{i=1}^n u_i^2 - Q(n-i)u_i' \right) e^{-iky} =$$

$$= \frac{1}{2} \sum_{j=1}^n \sum_{m\neq k,0} c_j^{(m)} c_j^{(k-m)} + i \sum_{j=1}^n (kQ(n-j) - x_j) c_j^{(k)}, \tag{3.17}$$

$$L_0 = \sum_{j=1}^n \left(-\frac{1}{2} x_j^2 + \sum_{m>0}^{\infty} c_j^{(m)} c_j^{(m)} \right).$$

To write the explicit expression for $\hat{W}_3^{(0)}$ (or $W_3^{(0)}$) operator it is convenient to introduce the fields $\widetilde{u}_k$, $\widetilde{v}$, which do not have the zero modes

$$u_k(y) = ix_k + \widetilde{u}_k(y), \quad v(y) = i\hat{P} + \widetilde{v}(y).$$

Sometimes we will also use the notations

$$\widetilde{u}_k^{(+)}(y), \quad \phi'_+(y), \quad (\widetilde{u}_k^{(-)}(y), \phi'_-(y)),$$

where $u_k(y) = \widetilde{u}_k^{(+)}(y) + \widetilde{u}_k^{(-)}(y)$ and fields $\widetilde{u}_k^{(+)}(y)$, $\phi'_+(y)$ and $(\widetilde{u}_k^{(-)}(y), \phi'_-(y))$ contain only positive (negative) Fourier components of the fields $u_k(y)$ and $\phi'(y)$. For example, the first and second terms in the integral $I_3^{(n)}$ (3.20) can be written as

$$\frac{n\sqrt{n}Q}{2} i \sum_{k=1}^{\infty} k a_{-k} a_k = \frac{n\sqrt{n}Q}{2} \frac{1}{2\pi} \int_0^{2\pi} dy_1 \phi'_-(y_1) \phi''_+(y_1).$$

$$\sum_{k=-\infty}^{\infty} k a_k L_{-k} = -\frac{1}{2\pi} \int_0^{2\pi} dy_1 \phi'(y_1) T(y_1) =$$

$$-\frac{1}{2\pi} \int_0^{2\pi} dy_1 \phi' \sum_{k=1}^n \frac{1}{2} \left(\widetilde{u}_k^2 + 2(ix_k - Q(n-k))\widetilde{u}_k' \right) \tag{3.18}$$

The operator $\hat{W}_3^{(0)}$ in these notation has a form (3.8), (3.13)

$$i\hat{W}_3^{(0)} = \int_0^{2\pi} dy_1 \left(\frac{1}{3}\sum_{k=1}^{n}\left((ix_i^3 - 3ix_i\widetilde{u}_k^2 - \widetilde{u}_k^3) - Q\widetilde{u}_{n+1-k}\rho_{n-k}\widetilde{v}'(y_1)\right)\right) \qquad (3.19)$$

where

$$(\rho_{n-k}, \widetilde{v}'(y_1)) = (n-k-1)\widetilde{u}_1' + (n-k-2)\widetilde{u}_2' + \cdots + \widetilde{u}_{n-k-1}', \quad (\rho_1, \widetilde{v}'(y_1)) = 0.$$

We should have in mind that all our expressions are Wick ordered i.e. the fields $\widetilde{u}_k^{(+)}(y)$ are always placed at the right from the fields $\widetilde{u}_k^{(-)}(y)$. In terms of fields $\widetilde{u}_k, \phi'$ the generalization of integral I_3 (2.5) the operator $I_3^{(n)} = \frac{2}{\sqrt{n}}\mathrm{I}_3^{(n)}$ can be written as

$$\mathrm{I}_3^{(n)} = \frac{\sqrt{4+n(n+2)Q^2}\sqrt{n-2}}{2^{3/2}}W_3^{(0)} + \frac{1}{2\pi}\int_0^{2\pi} dy_1 \left(\frac{n\sqrt{n}Q}{2}\phi_-'\phi_+'' - \phi'T - \frac{\phi'^3}{6}\right) =$$

$$= i\frac{\sqrt{n}}{2}\hat{W}_3^{(0)} + \frac{1}{2\pi}\int_0^{2\pi} dy_1 \left(\frac{n\sqrt{n}Q}{2}\phi_-'\phi_+'' - \phi'T - \frac{\phi'^3}{6}\right). \qquad (3.20)$$

To study the integral $\mathrm{I}_3^{(n)}$ we introduce the fields

$$\Phi_k(y) = \Phi_k^{(+)}(y) + \Phi_k^{(-)}(y) = \frac{\phi' + \sqrt{n}\widetilde{u}_k}{\sqrt{n}} = -\sum_{m\neq 0}A_k^{(m)}e^{imy}$$

where operators $A_k^{(m)} = \frac{a_m + \sqrt{n}c_k^{(m)}}{\sqrt{n}}$ have the commutation relations

$$[A_i^{(m)}, A_j^{(k)}] = m\delta_0^{m+k}\delta_j^i$$

Our operator $\mathrm{I}_3^{(n)}$ acts in the Fock space $\mathcal{F}_P$ generated by these operators from the vacuum state $|P\rangle$, which is annihilated by all the operators $\Phi_i^{(+)}$ and satisfies the equation $\mathrm{P}|P\rangle = P|P\rangle$. Now we show that the integral $\mathrm{I}_3^{(n)}$ has in this space the invariant subspace $F_{n,M}$ generated by the operators $A_n^{(m)}$ with negative m of the form

$$F_{n,M} = \sum_k \sum_{m_1+\cdots+m_k=M} C^{(m_1,\ldots,m_k)}A_n^{(-m_1)}\ldots A_n^{(-m_k)}|P\rangle$$

which is annihilated by all operators $\Phi_i^{(-)}$ with $i \neq n$.

We start with terms which depend on x_i. We will not take into account the constant cubic term $\frac{i}{3}\sum_1^n x_i^3$. It follows from the eqs. (3.18), (3.19) and (3.20) that these terms are

$$\frac{-i}{2\pi}\int_0^{2\pi}\sum_{k=1}^{n}\frac{x_k}{2}\left(2\phi'\widetilde{u}_k + \sqrt{n}\widetilde{u}_k^2\right)dy_1 = \frac{-i}{2\pi}\int_0^{2\pi}\sum_{k=1}^{n}\frac{x_k}{2}\left(\phi'^2/\sqrt{n} + 2\phi'\widetilde{u}_k + \sqrt{n}\widetilde{u}_k^2\right)dy_1 =$$

$$\frac{-i}{2\pi}\int_0^{2\pi}\sum_{k=1}^{n}\frac{x_k\sqrt{n}}{2}\Phi_k^{\;2} = \frac{-i}{2\pi}\int_0^{2\pi}\sum_{k=1}^{n}x_k\sqrt{n}\Phi_k^{(-)}\Phi_k^{(+)}.$$

We note that this operator acts on $F_{n,M}$ as Casimir with eigenvalue $-ix_n\sqrt{n}M$.

It easy to see that the cubic terms in the integral $I_3^{(n)}$ can be written as

$$\frac{-1}{2\pi}\int\limits_0^{2\pi}\frac{dy_1}{6}\left(\phi'^3 + \sum_{k=1}^n(3\phi'\widetilde{u}_k^2 + \sqrt{n}\widetilde{u}_k^3)\right) = \frac{-1}{2\pi}\int\limits_0^{2\pi}\sum_{k=1}^n\frac{dy_1\sqrt{n}}{6}\Phi_k^3 =$$

$$= \frac{-1}{2\pi}\int\limits_0^{2\pi}\frac{dy_1\sqrt{n}}{2}\sum_{k=1}^n\left(\Phi_k^{(-)2}\Phi_k^{(+)} + \Phi_k^{(-)}\Phi_k^{(+)2}\right)$$

We see that the space $F_{n,M}$ is invariant under the action of this operators.

To study the quadratic terms it is convenient to introduce the notation $\Phi_0 = \Phi_1 + \cdots + \Phi_{n-1} = \frac{(n-1)\phi' - \sqrt{n}u_n}{\sqrt{n}}$. Field Φ_0 commutes with Φ_n and $\phi' = \frac{1}{\sqrt{n}}(\Phi_0 + \Phi_n)$. The quadratic terms have the form

$$\frac{Q}{2\pi}\int\limits_0^{2\pi}dy_1\left(\frac{\sqrt{n}}{2}(\Phi_0^{(-)} + \Phi_n^{(-)})\partial(\Phi_0^{(+)} + \Phi_n^{(+)}) + \phi'\rho\widetilde{v}' - \sum_1^n\frac{\sqrt{n}}{2}u_{n-k+1}\rho_{(n-k)}\widetilde{v}'\right) \quad (3.21)$$

After simple transformations neglecting the total derivatives we arrive to the following expression for (3.21)

$$\frac{Q}{2\pi}\int\limits_0^{2\pi}dy_1\left(\frac{\sqrt{n}}{2}(\Phi_0^{(-)} + \Phi_n^{(-)})\partial(\Phi_0^{(+)} + \Phi_n^{(+)}) + \frac{\sqrt{n}}{2}\Phi_0'\Phi_n + S_{n-1}\right) \quad (3.22)$$

where the term S_{n-1} annihilates the vectors in $F_{n,M}$ and has a form

$$S_{n-1} = \sum_{i>j}^{n-1}\frac{\sqrt{n}}{2}\Phi_i\Phi_j'.$$

The "unwanted" term in the first term of integrand in eq. (3.22) is $\Phi_0^{(-)}\partial\Phi_n^{(+)}$ but it cancels after integration. The first two terms in the integrand in eq. (3.22) can be written as

$$\frac{\sqrt{n}}{2}\left(\Phi_n^{(-)}\partial\Phi_n^{(+)} + \Phi_0^{(-)}\partial\Phi_0^{(+)} + \Phi_n^{(-)}\partial\Phi_0^{(+)} + \Phi_0^{(-)}\partial\Phi_n^{(+)} + \Phi_n^{(-)}\partial\Phi_0^{(+)} + \partial\Phi_0^{(-)}\Phi_n^{(+)}\right).$$

We see that the terms $\Phi_0^{(-)}\partial\Phi_n^{(+)} + \partial\Phi_0^{(-)}\Phi_n^{(+)}$ form the total derivative and cancel after integration. If we take into account the terms $\Phi_0^{(-)}\partial\Phi_0^{(+)}$, $2\Phi_n^{(-)}\partial\Phi_0^{(+)}$ and the term S_{n-1} the quadratic terms in the integrand can be written in the form

$$\frac{\sqrt{n}}{2}\left(\sum_{k=1}^n\Phi_k^{(-)}\partial\Phi_k^{(+)} + 2\sum_{i>j}^n\Phi_i^{(-)}\partial\Phi_j^{(+)}\right).$$

Taking into account all terms the expression for the integral $I_3^{(n)} = \frac{2}{\sqrt{n}}\mathrm{I}_3^{(n)}$ can be written as

$$I_3^{(n)} = \frac{1}{2\pi}\int\limits_0^{2\pi}dy_1\left(\sum_{k=1}^n(Q\Phi_k^{(-)}\partial\Phi_k^{(+)} - 2ix_k\Phi_k^{(-)}\Phi_k^{(+)} - \frac{1}{3}\Phi_k^3) + 2Q\sum_{i>j}^n\Phi_i^{(-)}\partial\Phi_j^{(+)}\right)$$

$$(3.23)$$

We note that under Hermitian conjugation, which acts as $i\Phi_j \to i\Phi_{n+1-j}$ the first three terms change the sign and the last term changes also the order in the sum over i, j. It means that the left invariant subspace will be the space $F_{1,M}^*$.

It is worth to mention that the operator $I_3^{(n)}$ possesses nice representation in terms of differential operators. Namely, consider the quotient of the integral $I_3^{(n)}$ on invariant subspace $F_{n,M}$

$$\hat{I}_3^{(n)} = \frac{1}{2\pi} \int\limits_0^{2\pi} dy_1 \left(Q\Phi_n^{(-)}\partial\Phi_n^{(+)} - 2ix_n\Phi_n^{(-)}\Phi_n^{(+)} - \frac{1}{3}\Phi_n^3 \right), \tag{3.24}$$

and replace

$$A_n^{-k} = -ibp_k, \qquad A_n^k = \frac{i}{b}k\frac{\partial}{\partial p_k}, \tag{3.25}$$

where $p_k = p_k(z)$ are power-sum symmetric polynomials in infinitely many variables z_j

$$p_k(z) = \sum z_j^k.$$

When acting on a space of symmetric functions the operator $\hat{I}_3^{(n)}$ can be rewritten as a differential operator

$$\hat{I}_3^{(n)} = ib^{-1} \left(\mathcal{H}_2 + \frac{b^2}{2}\mathcal{H}_1^2 - bx_n\mathcal{H}_1 \right), \tag{3.26}$$

where $\mathcal{H}_1$ and $\mathcal{H}_2$ are first two Calogero-Sutherland Hamiltonians

$$\begin{aligned}
\mathcal{H}_1 &= \sum z_i\frac{\partial}{\partial z_i}, \\
\mathcal{H}_2 &= \sum \left(z_i\frac{\partial}{\partial z_i} \right)^2 + g\sum_{i<j} \frac{z_i + z_j}{z_i - z_j} \left(z_i\frac{\partial}{\partial z_i} - z_j\frac{\partial}{\partial z_j} \right),
\end{aligned} \tag{3.27}$$

with $g = -b^2$. Similarly, we can shown that the operator $I_3^{(n)}$ restricted on a left invariant subspace $F_{1,M}^*$ can also be rewritten as a Calogero-Sutherland Hamiltonian. One can also show that higher integrals of motion are reducible to higher CS Hamiltonians. All this allows to prove that the right and the left eigenvectors of this operator can be expressed through the Jack polynomials associated with Young diagrams $\vec{\nu} = (0, \dots 0, \nu_n)$ and $\mu = (\mu_1, 0, \dots 0)$. Namely, the states $|P'\rangle_{\vec{\lambda}}$ and $_{\vec{\mu}}\langle P^*|$ corresponding to these diagrams can be defined as

$$\begin{aligned}
|P'\rangle_{\vec{\nu}} &= \Omega_{\nu_n}(P')\mathrm{J}_{\nu_n}^{(1/g)}(y)|P'\rangle \\
_{\vec{\mu}}\langle P^*| &= \Omega_{\mu_1}(P)\langle P^*|\mathrm{J}_{\mu_1}^{(1/g)}(x)
\end{aligned} \tag{3.28}$$

where $g = -b^2$,

$$A_n^{(-m)} = -ibp_m(y), \quad A_1^{(m)} = ibp_m(x)$$

and $\mathrm{J}_\nu^{(1/g)}(x)$ is the Jack polynomial associated with the diagram (partition) ν normalized as

$$\mathrm{J}_\nu^{(1/g)}(x) = |\nu|!m_{[1\dots1]}(x) + \dots$$

where $m_{[\nu_1 \ldots]}$ is the monomial symmetric polynomial. The factors $\Omega_{\nu_k}(P)$ ($k = 1, n$) are defined by

$$\Omega_{\nu_k}(P) = (-b)^{|\nu_k|} \prod_{l \neq k} \prod_{(i,j) \in \nu_k} (x_{lk} + ib + jb^{-1}) \tag{3.29}$$

where index i runs horizontally and j vertically over diagram ν_k and $x_{lk} = x_l - x_k$.

Before proceed further we note that if we take parameters x_k in eq. (3.23) $|x_k| \gg |Q|$ we can neglect the first and last quadratic terms in the integral of motion $I_3^{(n)}$. In this limit the eigenvectors of $I_3^{(n)}$ will not depend on b and can be written as ("asymptotic freedom")

$$|P\rangle_{\vec{\nu}} = \prod_{i=1}^n \mathrm{J}_{\nu_i}^{(1)}(A_i^{(-1)}, A_i^{(-2)} \ldots)|P\rangle \tag{3.30}$$

This asymptotic gives the indication for the appearance of Young diagram in the classification of eigenstates of $I_3^{(n)}$ which can be considered as finite momentum deformation of the states (3.30). The eigenvalues of operator $I_3^{(n)}$ will have the asymptotics $\mathfrak{Q}_{\vec{\nu}}^{(3)} = \sum_{j=1}^n -2i|\nu_j||x_i$ which is the sum over all partitions ν_j. It is possible calculate the first correction in Q to the spectrum and find that it is also the sum over all partitions. If we take $|P\rangle_{\vec{\nu}}$ in the form (3.28) we will have $\mathfrak{Q}_{\vec{\nu}}^{(3)} = q^{(3)}(\nu_n, x_n)$ where

$$q^{(3)}(\nu, x) = i\left(-2|\nu|x + \frac{1}{b}\sum_l \nu_l(\nu_l + (2l-1)b^2)\right). \tag{3.31}$$

Our integral (3.20) does not depend on x_i. This dependence appear after bosonization. There exist $n!$ different bosonizations corresponding to different Weyl transformations of the vector P. This group acts as the group of permutation S_n of the parameters x_k. It is always possible to choose the transformation of the Weyl group $\hat{s}$ such that the operator $\mathfrak{Q}^{(3)}$ written in terms of operators $A_k^{(-m)}(\hat{s})$, which possess rather complicated expansion in terms of operators A_j (all $A_k^{(-m)}(\hat{s})$ have canonical commutation relations and hence connected by canonical transformation) will have the invariant subspace $F_{i,M}$ with arbitrary i and eigenstate (3.28) with eigenvalue $q^{(3)}(\nu_i, x_i)$. All this gives us the reasons to assume that operator $\mathfrak{Q}^{(3)}$ similar to $sl(2)$ case [12] has the spectrum

$$\mathfrak{Q}_{\vec{\nu}}^{(3)} = \sum_{j=1}^n q^{(3)}(x_j, \nu_j). \tag{3.32}$$

Of course this property is valid for the first integral $I_2^{(n)} = L_0 + \sum_{k>0} a_{-k}a_k$. It can be proved that it is valid for all higher integrals $\mathfrak{Q}^{(s)}$, $s = 4, 5, \ldots$.

3.3 Free-field representation of the matrix elements and Selberg integrals

The important property of operators A_1 and A_n is that they commute and have simple commutation relations with vertex operator which for W_n case has a form

$$\mathcal{V}(z) = e^{(a-Qn)\phi_-(z)/\sqrt{n}} e^{a\phi_+(z)/\sqrt{n}} V_{a\omega_1}(z). \tag{3.33}$$

Namely, operators A_n commute with $\mathcal{V}$ and $[A_1^{(m)}, \mathcal{V}(1)] = i(a - Q)\mathcal{V}(1)$.
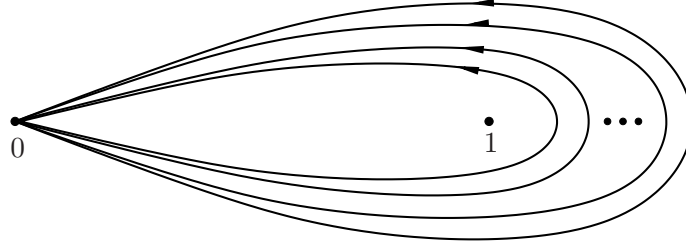
Figure 1. Contours of integration in the integral (3.39).

With these properties we can reduce the calculation of function $F_{\varnothing,\ldots,\varnothing,\nu'}^{\nu,\varnothing,\ldots,\varnothing}(a,P,P')$ for special values of parameters P' and $P^* = -P$ to the $sl(n)$ analog of Selberg integral [30, 31]. Namely, if vectors $P^* = -P, P'$, in eq. (2.20) satisfy the condition

$$-P + P' + \omega_1 a + \sum_{s=1}^{n-1} bk_r e_r = 0 \tag{3.34}$$

where e_k are the simple roots of $sl(n)$ and $k_{r+1} \leq k_r$ are non-negative integers, the calculation of matrix element $F_{(0,\ldots 0,\mu)}^{(\lambda,0,\ldots,0)}(a,P,P')$ can be reduced to the calculation of the integral with screening charges (3.12) S_i. The operators $A_n^{(-m)}$ commute with all screening charges S_i with $i < n-1$, and operators $A_1^{(m)}$ commute with all S_i, $i > 1$. As a result we derive that our matrix element can be represented as the ratio of $sl(n)$ Selberg integrals

$$F_{(0,\ldots 0,\mu)}^{(\lambda,0,\ldots,0)}(a,P,P') = \Omega_\mu(P)\Omega_\lambda(P') \frac{\left\langle \mathrm{J}_\mu^{(1/g)}[p_k+\eta]\mathrm{J}_\lambda^{(1/g)}[p_{-k}] \right\rangle_{\mathrm{Sel}}^{\mathrm{sl}(n)}}{\langle 1 \rangle_{\mathrm{Sel}}^{\mathrm{sl}(n)}} \tag{3.35}$$

where $\eta = (a-Q)/b$, $g = -b^2$ and $\langle \ldots \rangle_{\mathrm{Sel}}^{\mathrm{sl}(n)}$ denotes Selberg average, which can be defined in the following way [30, 31]. We define the functions

$$D(t^{(r)}) = \prod_{1 \leq i < j \leq k_r}(t_j^{(r)} - t_i^{(r)}), \quad D(t^{(r)},t^{(r+1)}) = \prod_{i=1}^{k_r}\prod_{j=1}^{k_{r+1}}(t_i^{(r)} - t_j^{(r+1)}) \tag{3.36}$$

and collection of contours $\mathcal{C}(n) = C_1 \times C_2 \times \ldots C_{k_1+\cdots+k_{n-1}}$, starting at the point 0 and encircling the point 1 and such that C_i is between the interval $[0,1]$ and C_j for $i < j$ as it is shown on figure 1. Let $\mathcal{O}(t^{(1)}|\ldots|t^{(n-1)})$ be symmetric (with respect of $t_a^{(s)} \to t_b^{(s)}$) single valued function. We fix the branch of the function

$$G_n(t^{(1)}|\ldots|t^{(n-1)}) = \prod_{s=1}^{n-1} D(t^{(r)})^{2g} \prod_{i=1}^{k_r}(t_i^{(r)})^{A_r}(1 - t_i^{(r)})^{B_r} \prod_{r=1}^{n-2} D(t^{(r)},t^{(r+1)})^{-g} \tag{3.37}$$

where

$$A_r = -b(\mathcal{Q} + P, e_r) = -1 - b^2 - x_r' + x_{r+1}', \quad g = -b^2$$
$$B_1 = B = -ba, \quad B_2 = B_3 = \cdots = B_{n-1} = 0 \tag{3.38}$$

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in the following way: at the point, where all the variables are real and negative one has $\arg(t_i^{(r)} - t_j^{(r+1)}) = \pi$ for any i, j and $\arg(t_i^{(r)} - t_j^{(r)}) = 0$ for $i < j$. Then

$$\langle \mathcal{O} \rangle_{\text{Sel}}^{\text{sl}(n)} = \int\limits_{\mathcal{C}(n)} \mathcal{O}(t^{(1)} | \ldots | t^{(n-1)}) G(t^{(1)} | \ldots | t^{(n-1)}) d^{k_1} t^{(1)} \ldots d^{k_{n-1}} t^{(n-1)} \tag{3.39}$$

where $d^{k_r} t^{(r)} = dt_1^{(r)} \ldots dt_{k_r}^{(r)}$. As it will be shown in appendix A the ratio (3.35) can be calculated exactly with the expected result that $F_{(0,\ldots 0,\mu)}^{(\lambda,0,\ldots,0)}(a, P, P')$ is given by eq. (2.24).

4 Factorization of the matrix elements

It follows from the previous consideration that the matrix element

$$F_{\varnothing,\ldots,\varnothing,\nu'}^{\nu,\varnothing,\ldots,\varnothing}(a, P, P') = \frac{\langle \Psi_{\nu,\varnothing,\ldots,\varnothing,P^*} | V_a | \Psi_{\varnothing,\ldots,\varnothing,\nu',P'} \rangle}{\left\langle \Psi_{\vec{0},P^*} | V_a | \Psi_{\vec{0},P'} \right\rangle} \tag{4.1}$$

is given by eq. (2.24). Here we use the properties of singular vectors of W_n- algebra to show that all the matrix elements $F_{\vec{\nu}'}^{\vec{\nu}}(a, P, P')$ are given by the same equation. All proofs of the statements that will be done here follow literally to the proofs in the ref. [12] and we will not reproduce them here. Some new phenomena that appears in W_n case is related with consistency (bootstrap) relations very similar to that in the factorized scattering theories. We will not describe here in details the representation theory of W_n-algebra and consider only some special degenerate representations which are important for our consideration.

Consider the zeros of the function

$$\Omega_{\nu_n}(P') = (-b)^{|\nu_n|} \prod_{m \neq n} \prod_{(i,j) \in \nu_n} (x'_{mn} + ib + jb^{-1}).$$

These zeros correspond to the conditions

$$(h_m - h_n, P') = (e_m + e_{m+1} + \cdots + e_{n-1}, P') = (\hat{e}_{m,n-1}, P') = -rb - r'/b \tag{4.2}$$

where $r > 0, r' > 0 \in \nu_n$. When we tend vector P' to these points our vector $|P'\rangle_{\nu_n}$ (3.28) tends to zero in the Fock space generated by the operators $A_n^{(-m)}$. However the state $|\Psi_{\varnothing,\ldots,\varnothing,\nu'P'}\rangle$ does not vanish being expressed in terms of generators $\hat{W}_k^{(-n)}$ and operators a_{-m} applied to the vacuum state $|\Theta_{P'}\rangle$. Instead we have proved that the matrix elements (4.1) are given by eq. (2.24) and do not vanish at these points.

The condition (4.2) means that for these P' in Verma module $|\Theta_{P'}\rangle$ exists a singular vector $\chi_{\hat{P}'}$ at the level rr'

$$|\chi_{P'}\rangle = D_{r,r'} |\Theta_{P'}\rangle = \left((\hat{W}_n^{(-1)})^{rr'} + \ldots \right) |\Theta_{P'}\rangle \tag{4.3}$$

(where omitted terms have degree in $\hat{W}_n^{(-1)}$ at most $rr' - 1$) such that $\hat{W}_l^{(k)} |\chi_{\hat{P}'}\rangle = 0$ for any $k > 0$. We introduce the weights $\omega_i = h_1 + \cdots + h_i$, $(\omega_i, e_j) = \delta_{ij}$ of fundamental representations $\pi(\omega_i)$ of $sl(n)$. Consider, for example, the condition $(\hat{e}_{n-1,n-1}, P') = (e_{n-1}, P') = -rb - r'/b$. The solution to this equation can be written as

$$P'_{r,r'} = -\omega_{n-1}(r_{n-1}b + r'_{n-1}/b) + P'_\perp \tag{4.4}$$

where $(e_{n-1}, P'_\perp) = 0$. For example $P'_\perp = P' + h_n(P, e_{n-1})$. The corresponding singular vector χ can be parametrized by the vector $P'_{r, \hat{t}_{n-1}r'} = P'_{r,r'} + e_{n-1}r'b^{-1}$. Here the symbol $\hat{t}_{n-1}r'$ denotes that we make the Weyl reflection of the vector $-\omega_{n-1}r'/b$ with respect to the plane orthogonal to vector e_{n-1}.[1] We note that solution of arbitrary equation $(\hat{e}_{m,n-1}, P') = -r_m b - r'_m/b$ can be written in the the form (4.4) with numbers r_m, r'_m and with vector $P'_\perp = P' + h_n(P', \hat{e}_{m,n-1})$ which satisfies the condition $(\hat{e}_{m,n-1}, P'_{\perp(m)}) = 0$. The corresponding singular vector at the level $r_m r'_m$ will be parametrized by the vector $P'_{r_m, \hat{s}_m r'_m} = P'_{r_m, r'_m} + \hat{e}_{m,n-1}r'_m b^{-1}$. It easy to check that $x'_m(r_m, r'_m) = x'_n(r_m, r'_m) - (r_m b + r'_m/b)$ and parameters $x'_i(r_m, \hat{s}_m r'_m) = (h_i, P'_{r_m, r'_m})$ are

$$x'_n(r_m, \hat{s}_m r'_m) = \frac{n-1}{n}(r_m b + r'_m/b) - r'_m/b + h_n P'_\perp = x'_n(r_m, r'_m) - r'_m/b$$

$$x'_m(r_m, \hat{s}_m r'_m) = -\frac{1}{n}\left(r_m b + r'_m/b\right) + r'_m/b + h_m P'_\perp = x'_n(r_m, r'_m) - r_m b$$

$$x'_i(r_m, \hat{s}_m r'_m) = x'_i(r_m, r'_m), \quad i \neq n, m. \tag{4.5}$$

For the further consideration we need to calculate the values

$$\mathrm{R}_{r_m, r'_m}(P, P'_{r_m, r'_m}, a) = \frac{\left\langle \Theta_{P^*} | V_a | \mathrm{D}_{r_m, r'_m} \Theta_{P'_m} \right\rangle}{\left\langle \Theta_{P^*} | V_a | \Theta_{P'_m} \right\rangle} \tag{4.6}$$

where $P'_m = P'_{r_m, r'_m}$ and normalization of D_{r_m, r'_m} is defined by (4.3). We note that the numerator and the denominator (in usual normalization) in this ratio vanish when P' tends to P'_{r_m, r'_m}, however their ratio is finite and well defined. This ratio can be calculated in two different ways. The first is based on the known matrix element (4.1) and second on the calculations in conformal Toda theory. In the first way we can take in (4.1) as ν' the rectangular diagram of size $r_m r'_m$ and $\nu = \varnothing$. For such diagram we have

$$|\Psi_{\varnothing,...,\varnothing,\nu',P'_m}\rangle = c_{r_m r'_m}\left(\mathrm{D}_{r_m, r'_m}\right)|\Theta_{P'_m}\rangle$$

because as follows from the results of Feigin and Fuks [32] the vector vanishing in the Fock space is the descendant of the singular vector. In our case we have the singular vector at the level $r_m r'_m$ and hence our vector $|\Psi_{\varnothing,...,\varnothing,\nu',P'_m}\rangle$ is the singular vector at this level. To find the constant $c_{r_m r'_m}$ we note that asymptotics in parameter a of $\mathrm{R}_{r_m, r'_m}(P, P', a)$ is defined by the main term $(\hat{\mathrm{W}}_n^{(-1)})^{r_m r'_m}$ in the operator D_{r_m, r'_m}. It is easy to derive from the the Ward identities that this asymptotics is equal to $\left(\frac{\hat{w}_n(a)}{n-1}\right)^{r_m, r'_m}$ where $\hat{w}_n(a)$ is the eigenvalue of the operator $\hat{\mathrm{W}}_n^{(0)}$ i.e. the main singular term in the OPE of $\hat{\mathrm{W}}_n(z)$ with the field $V_a(z') = \exp(a\omega_1 \varphi(z'))$

$$\hat{\mathrm{W}}_n(z) V_a(z') = \frac{\hat{w}_n(a)}{(z-z')^n} V_a(z') + O((z-z')^{1-n}) \tag{4.7}$$

It follows immediately from eq. (3.2) that the asymptotic of $\hat{w}_n(a)$ is equal $((h_i, h_j) = \delta^i_j - \frac{1}{n})$ to

$$\hat{w}_n(a) \to (-)^n a^n \prod_{i=1}^n (\omega_1 h_i) = (-)^n a^n \frac{(n-1)}{n}(-)^{n-1}\left(\frac{1}{n}\right)^{(n-1)} = -a^n(n-1)\left(\frac{1}{n}\right)^n \tag{4.8}$$

---

[1] We denote by symbol $\hat{t}_k$ the Weyl reflection with respect to the simple root e_k and by symbol $\hat{s}_k$ the Weyl reflection with respect to the root $\hat{e}_{k,n-1}$.

The exact expression for $\mathrm{R}_{r_m,r'_m}(P, P', a)$ is given by eq. (2.24)

$$\mathrm{R}_{r_m,r'_m}(P, P'_{r_m,r'_m}, a) = \prod_{j=1}^{n} \prod_{s\in\nu'} \left(Q - E_{\nu',\varnothing_j}(x_j - x'_n(r_m, r'_m)|s) - a/n \right) =$$

$$= \prod_{j=1}^{n} \prod_{k=0}^{r'_m-1} \prod_{l=0}^{r_m-1} (x'_n(r_m, r'_m) - a/n - x_i - bl - k/b) \qquad (4.9)$$

and has the asymptotic $(-\frac{a}{n})^{nr_mr'_m}$. Comparing it with the asymptotic of

$$\left(\frac{\hat{w}_n(a)}{n-1} \right)^{r_m,r'_m} = (-)^{r_m,r'_m}(\frac{a}{n})^{nr_mr'_m},$$

we find that $c_{r_mr'_m} = (-)^{(n-1)r_mr'_m}$. To have no sign factors we accept the factor $c_{r_mr'_m}$ in the normalization of the operator $\mathrm{D}_{r,r'}$ and the vector

$$|\Psi_P^{(\bar{\nu})}\rangle = ((-)^{(n-1)|\bar{\nu}|}((\hat{\mathrm{W}}_n^{(-1)})^{rr'} + \dots))|\Theta_P\rangle.$$

To calculate the same value using conformal Toda theory we note that the state $|\Theta_P\rangle = |\Theta_P^{(w)}\rangle|\theta_F\rangle$ where $|\theta_F\rangle$ is the Fock vacuum annihilated by operators a_k with $k > 0$ and $|\Theta_P^{(w)}\rangle$ is the state annihilated by all $\hat{\mathrm{W}}_l^{(k)}$ ($2 \leq l \leq n$) with $k > 0$. The main objects in conformal Toda theory are the exponential fields $V_\alpha(z, \bar{z}) = e^{\alpha\varphi(z,\bar{z})}$ where $\varphi(z, \bar{z})$ is a Toda field. The primary states of $W_n \otimes \overline{W}_n-$ algebra $|\hat{\Theta}_P\rangle$ are created (up to some factor) by the application of these fields to the Toda vacuum state $|\hat{\Theta}_\varnothing\rangle$ which satisfies the conditions $\hat{\mathrm{W}}_l^{(k)}|\hat{\Theta}_\varnothing\rangle = \overline{\hat{\mathrm{W}}}_l^{(k)}|\hat{\Theta}_\varnothing\rangle = 0$ for $k \geq -l + 1$. It means that the state $|\hat{\Theta}_P\rangle = \lim_{z\to 0} c(\alpha)V_\alpha(z, \bar{z})|\hat{\Theta}_\varnothing\rangle$ ($P = \alpha - \mathcal{Q}$) and the state $\langle\hat{\Theta}_{P^*}| = \lim_{z\to\infty} c(2\mathcal{Q} - \alpha)|z|^{2\Delta(\alpha)}\langle\hat{\Theta}_\varnothing|V_{2\mathcal{Q}-\alpha}(z, \bar{z})$. As the generators of W_n- algebra commute with generators of $\overline{W}_n$ we have in particular that for any P and P'

$$\left(\frac{\langle\Theta_{P^*}|V_a|\mathrm{D}_{r_m,r'_m}\Theta_{P'}\rangle}{\langle\Theta_{P^*}|V_a|\Theta_{P'}\rangle} \right)^2 = \frac{\langle\hat{\Theta}_{P^*}|V_a|\mathrm{D}_{r_m,r'_m}\bar{\mathrm{D}}_{r_m,r'_m}\hat{\Theta}_{P'}\rangle}{\langle\Theta_{P^*}|V_a|\Theta_{P'}\rangle}. \qquad (4.10)$$

The denominator of this ratio is known explicitly [21, 28]. It is equal $c(\alpha)c(2\mathcal{Q}-\alpha)C(2\mathcal{Q}-\alpha, a\omega_1, \alpha')$ where $C(2\mathcal{Q} - \alpha, a\omega_1, \alpha')$ is the three point function in conformal Toda theory

$$C(2\mathcal{Q} - \alpha, a\omega_1, \alpha') = \frac{\mathrm{m}^{(P,\rho)/b}\mathrm{m}^{-(P',\rho)/b}\mathrm{m}^{-a}(\Upsilon(b))^{n-1}\Upsilon(a)\prod_{e>0}\Upsilon(-Pe)\Upsilon(P'e)}{\prod_{i,j}\Upsilon(a/n + Ph_i - P'h_j)} \qquad (4.11)$$

here m is non-essential function of b and Toda cosmological constant. It is convenient to take $c(\alpha) = (\mathrm{m}^{-(P,\rho)/b}\prod_{e>0}\Upsilon(Pe))^{-1}$. Then the field $c(\alpha)V_\alpha$ will be Weyl invariant and at $P' = P'_{r_m,r'_m} + \varepsilon$ where $|\varepsilon| \to 0$

$$\mathrm{D}_{r_m,r'_m}\overline{\mathrm{D}}_{r_m,r'_m}|\hat{\Theta}_{P'}\rangle = (-)^{nr_mr'_m}|\hat{\Theta}_{P'-rm'\hat{e}_{mn}}\rangle + O(|\varepsilon|) \qquad (4.12)$$

If we take into account eq. (4.5) we derive that

$$\left(R_{r_m, r'_m}(P, P'_{r_m, r'_m}, a) \right)^2$$

$$= (-)^{n r_m r'_m} \frac{\prod_i \Upsilon(a/n + x_i - x'_n(r_m, r'_m) + r_m b + r'_m/b) \Upsilon(a/n + x_i - x'_n(r_m, r'_m))}{\prod_i \Upsilon(a/n + x_i - x'_n(r_m, r'_m) + r_m b) \Upsilon(a/n + x_i - x'_n(r_m, r'_m) + r'_m/b)}$$

Function $\Upsilon(x)$ [33] satisfies the functional relations $\Upsilon(x + b) = \gamma(bx) b^{1-2bx} \Upsilon(x), \Upsilon(x + b^{-1}) = \gamma(x/b) b^{-1+2bx} \Upsilon(x)$, where $\gamma(x) = \frac{\Gamma(x)}{\Gamma(1-x)}$. So

$$\left(R_{r_m, r'_m}(P, P'_{r_m, r'_m}, a) \right)^2 =$$

$$= (-)^{n r_m r'_m} b^{2 n r_m r'_m} \prod_i \prod_{k=0}^{r'_m - 1} \frac{\gamma(b^{-1}(a/n + x_i - x'_n(r_m, r'_m) + r_m b + k/b))}{\gamma(b^{-1}(a/n + x_i - x'_n(r_m, r'_m) + r'_m/b))} =$$

$$= \prod_i \prod_{k=0}^{r'_m - 1} \prod_{l=0}^{r_m - 1} (a/n + x_i - x'_n(r_m, r'_m) + bl + k/b)^2 \tag{4.13}$$

It is easy to see that this expression is the square of the r.h.s. of (4.9). As we know the asymptotics of R_{r_m, r'_m} at $a \to \infty$ we can take the proper root in eq. (4.13). We denote as $R_{r, r'}^{(k)}(P, P', a)$ the function

$$R_{r, r'}^{(k)}(P, P', a) = \prod_i^n \prod_{j=0}^{r'-1} \prod_{l=0}^{r-1} (h_k P' - a/n - h_i P - bl - j/b).$$

With the function R_{r_m, r'_m} we can formulate the factorization property for the functions $F_{\vec{\nu}}^{\vec{\lambda}}(P, P', a)$ which states

$$F_{\vec{\nu}}^{\vec{\lambda}}(P, P'_{r_m, r'_m} a) = R_{r_m, r'_m}^{(n)}(P, P'_{r_m, r'_m}, a) F_{\vec{\sigma}}^{\vec{\lambda}}(P, P'_{r_m, \hat{s} r'_m}, a) \tag{4.14}$$

here "vectors" $\vec{\lambda} = (\lambda^{(1)}, \varnothing, \ldots, \varnothing)$, $\nu = (\varnothing, \ldots, \varnothing, \nu^{(n)})$ and "vector" $\vec{\sigma}$ has only two non-zero components $\sigma^{(m)}$ and $\sigma^{(n)}$ where $\sigma^{(n)} = (\nu_1^{(n)} - r'_m, \ldots, \nu_{r_m}^{(n)} - r'_m)$, and $\sigma^{(m)} = (\nu_{r_m+1}^{(n)}, \nu_{r_m+2}^{(n)}, \ldots)$. It is easy to check that this relation holds for the functions (2.24). Really this relation determines the functions $F_{\vec{\sigma}}^{\vec{\lambda}}(P, P', a)$ with arbitrary partitions $\sigma^{(m)}$ and $\sigma^{(n)}$ (see ref. [12] for proof). An example of how the pair of the partitions $\sigma^{(m)}$ and $\sigma^{(n)}$ is defined for given $(r_m, r'_m) \in \nu^{(n)}$ is shown by the following picture
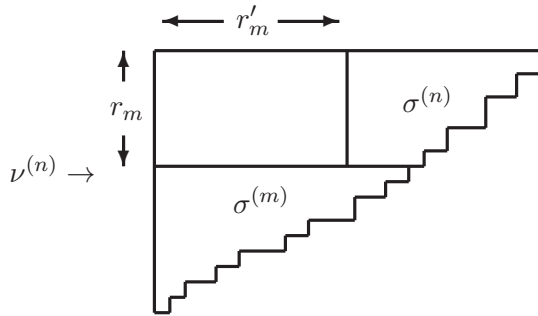
Equation (4.14) looks very similar to the equations for the formfactors of bound state (diagram $\nu^{(n)}$) in terms of formfactors of the particles (diagrams $\sigma^{(m)}$ and $\sigma^{(n)}$) which form this bound state. To underline this similarly we note that the conservation laws are fulfilled for this fusion process. For example, it is easy to check for conserved charges $q^{(3)}(\nu, x)$ defined by (3.31) that

$$q^{(3)}(\nu^{(n)}, x - x(r_m, r_m')) = q^{(3)}(\sigma^{(n)}, x - x(r_m, -r_m')) + q^{(3)}(\sigma^{(m)}, x + x(r_m, -r_m')) +$$
$$+ q^{(3)}(\Box_{r_m, r_m'}, x - x(r_m, r_m')), \quad (4.15)$$

where $\Box_{r_m, r_m'}$ is the rectangular diagram of size $r_m \times r_m'$ and $x(r, r') = -rb/2 - r'b^{-1}/2$.

To proceed further we should impose two conditions (4.2) to the vector P'. We note that if impose even $n-1$ conditions (4.2) to the vector P' the representation of W_n- algebra will not be strongly degenerate. The strongly degenerate representations are specified by the vector P' such that for all simple roots $(e_i, P) = -r_i b - r_i' b^{-1}$, where all numbers r_i, r_i' are positive. These representations contain $n! - 1$ singular vectors. The representations specified by the vector P' satisfying $n - 1$ condition (4.2) will have in general $2^{(n-1)} - 1$ singular vectors.

To make our consideration more clear and do not introduce the long notation we consider the case $n = 3$. In this case we have two simple roots e_1 e_2 and the root $\hat{e}_{12} = e_1 + e_2$. We also have $e_1 = 2\omega_1 - \omega_2, e_2 = 2\omega_2 - \omega_1$. The solution of two equations (4.2) is

$$P'_{r_1, r_1'; r_2, r_2'} = -\omega_1((r_1 - r_2)b + (r_1' - r_2')b^{-1}) - \omega_2(r_2 b + r_2' b^{-1}) \quad (4.16)$$

where $r_2, r_2' > 0$ and $r_1 - r_2 > 0, r_1' - r_2' < 0$. In this case we will have in the Verma module two singular vectors. The first at the level $r_2 r_2'$ parametrized by

$$P'_{r_1, r_1'; r_2, \hat{s}_2 r_2'} = P'_{r_1, r_1'; r_2, r_2'} + e_2 r_2'/b = -\omega_1((r_1 - r_2)b + r_1' b^{-1}) - \omega_2(r_2 b - r_2' b^{-1}) \quad (4.17)$$

and the second at the level $r_1 r_1'$ parametrized by

$$P'_{r_1, \hat{s}_1 r_1'; r_2, r_2'} = P'_{r_1, r_1'; r_2, r_2'} + \hat{e}_{12} r_1'/b = -\omega_1((r_1 - r_2)b - r_1' b^{-1}) - \omega_2(r_2 b + (r_2' - r_1')b^{-1}) \quad (4.18)$$

But the primary state parametrized by $P'_{r_1, r_1'; r_2, \hat{s}_2 r_2'}$ has a singular vector at the level $(r_1 - r_2)r_1'$ with P' equal

$$P'_{r_1, \hat{s}_2 \hat{s}_1 r_1'; r_2, \hat{s}_2 r_2'} = P'_{r_1, \hat{s}_1 r_1'; r_2, r_2'} + e_1 r_1' b^{-1} = -\omega_1((r_1 - r_2)b - r_1' b^{-1}) - \omega_2(r_2 b + (r_1' - r_2')b^{-1})$$

and primary state parametrized by $P'_{r_1, r_1'; r_2, \hat{s}_2 r_2'}$ has the a singular vector at the level $r_2(r_2' - r_1')$ with the same $P' = P'_{r_1, \hat{s}_2 \hat{s}_1 r_1'; r_2, \hat{s}_2 r_2'}$. We note that $\hat{s}_2 \hat{s}_1 = \hat{t}_1$ and $\hat{s}_2 = \hat{t}_2$, where $\hat{t}_i$ are the Weyl reflections with respect to the planes orthogonal to roots e_i. So we have in the Verma module with $P'_{r_1, r_1'; r_2, r_2'}$ three singular vectors at the levels $r_2 r_2', r_1 r_1'$ and $(r_1 - r_2)r_1' + r_2 r_2'$ with embedding diagram shown in fig 2.

We see that there are two ways to arrive to the last point corresponding to $P' = P'_{r_1, \hat{t}_1 r_1'; r_2, \hat{t}_2 r_2'}$. In the first way we derive from eq. (4.14)

$$F^{\lambda^{(1)}, \varnothing, \varnothing}_{\varnothing, \varnothing, \nu^{(3)}}(P, P'_{r_1, r_1'; r_2, r_2'} a) = R^{(3)}_{r_2, r_2'}(P, P'_{r_1, r_1'; r_2, r_2'}, a) F^{\lambda^{(1)}, \varnothing, \varnothing}_{\varnothing, \sigma^{(2)}, \sigma^{(3)}}(P, P'_{r_1, r_1'; r_2, \hat{t}_2 r_2'}, a)$$
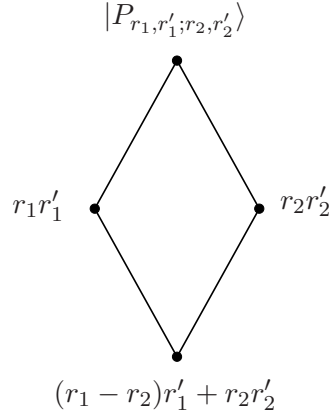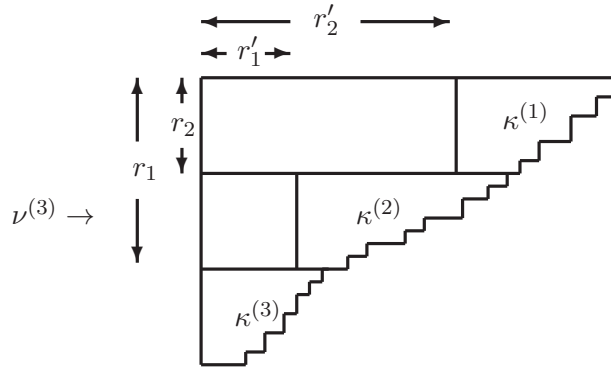
Figure 2. Embedding diagram for representation $P'_{r_1,r'_1;r_2,r'_2}$.

where $\sigma^{(3)} = (\nu_1^{(3)} - r'_2, \ldots, \nu_{r_2}^{(3)} - r'_2)$, $\sigma^{(2)} = (\nu_{r_2+1}^{(3)}, \nu_{r_2+2}^{(3)}, \ldots)$ after the first step and after the second step

$$F^{\lambda^{(1)},\varnothing,\varnothing}_{\varnothing,\varnothing,\nu^{(3)}}(P, P'_{r_1,r'_1;r_2,r'_2}a) = \mathrm{R}^{(3)}_{r_2,r'_2}(P, P'_{r_1,r'_1;r_2,r'_2}, a)\mathrm{R}^{(2)}_{r_1-r_2,r'_1}(P, P'_{r_1,r'_1;r_2,\hat{t}_2 r'_2}, a) \qquad (4.19)$$

$$\times F^{\lambda^{(1)},\varnothing,\varnothing}_{\kappa^{(1)},\kappa^{(2)},\kappa^{(3)}}(P, P'_{r_1,\hat{t}_1 r'_1;r_2,\hat{t}_2 r'_2}, a) \qquad (4.20)$$

where $\kappa^{(3)} = \sigma^{(3)}$, $\kappa^{(2)} = (\nu_{r_2+1}^{(3)} - r'_2 + r'_1, \ldots, \nu_{r_1}^{(3)} - r'_2 + r'_1)$ and $\kappa^{(1)} = (\nu_{r_1+1}^{(3)}, \nu_{r_1+2}^{(3)}, \ldots)$. It is convenient to represent it by the following picture:
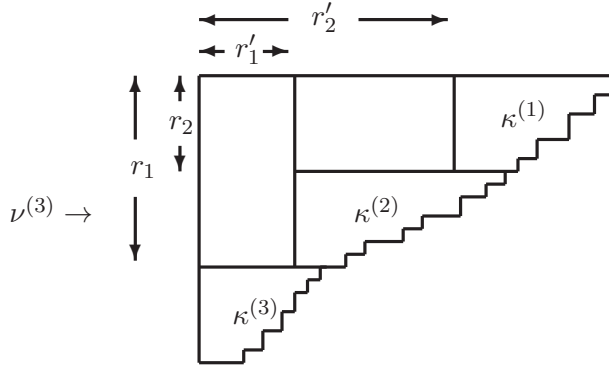

Taking the second way at the first step we have

$$F^{\lambda^{(1)},\varnothing,\varnothing}_{\varnothing,\varnothing,\nu^{(3)}}(P, P'_{r_1,r'_1;r_2,r'_2}a) = \mathrm{R}^{(3)}_{r_1,r'_1}(P, P'_{r_1,r'_1;r_2,r'_2}, a)F^{\lambda^{(1)},\varnothing,\varnothing}_{\mu^{(1)}\varnothing,\mu^{(3)}}(P, P'_{r_1,\hat{s}_1 r'_1;r_2,r'_2}, a)$$

where $\mu^{(3)} = (\nu_1^{(3)} - r'_1, \ldots, \nu_r^{(3)} - r'_1)$, $\mu^{(1)} = \kappa^{(1)}$ and after the second step

$$F^{\lambda^{(1)},\varnothing,\varnothing}_{\varnothing,\varnothing,\nu^{(3)}}(P, P'_{r_1,r'_1;r_2,r'_2}a) = \mathrm{R}^{(3)}_{r_1,r'_1}(P, P'_{r_1,r'_1;r_2,r'_2}, a)\mathrm{R}^{(3)}_{r_2,r'_2-r'_1}(P, P'_{r_1,\hat{s}_1 r'_1;r_2,r'_2}, a)\times$$

$$\times F^{\lambda^{(1)},\varnothing,\varnothing}_{\kappa^{(1)},\kappa^{(2)},\kappa^{(3)}}(P, P'_{r_1,\hat{t}_1 r'_1;r_2,\hat{t}_2 r'_2}, a), \quad (4.21)$$

which is equivalent to the picture

It follows from eqs. (4.19) and (4.21) that they are consistent if

$$\mathrm{R}^{(3)}_{r_2,r_2'}(P, P'_{r_1,r_1';r_2,r_2'}, a)\mathrm{R}^{(2)}_{r_1-r_2,r_1'}(P, P'_{r_1,r_1';r_2,\hat{t}_2 r_2'}, a) =$$
$$= \mathrm{R}^{(3)}_{r_1,r_1'}(P, P'_{r_1,r_1';r_2,r_2'}, a)\mathrm{R}^{(3)}_{r_2,r_2'-r_1'}(P, P'_{r_1,\hat{s}_1 r_1';r_2,r_2'}, a) \quad (4.22)$$

It is easy to check that this equation as well as eqs. (4.19)–(4.21) with functions $F_{\vec{\mu}}^{\vec{\lambda}}$ defined by eq. (2.23) are satisfied. The factorization equations (4.19) and (4.21) can be considered as equation for functions $F_{\kappa^{(1)},\kappa^{(2)},\kappa^{(3)}}^{\lambda^{(1)},\varnothing,\varnothing}(P, P', a)$. It can be proved that these equations have unique solution defined by eq. (2.23). Having these functions we can develop the same procedure with partition $\lambda^{(1)}$ to derive all amplitudes $F_{\vec{\mu}}^{\vec{\lambda}}$.

For arbitrary n the number of singular vectors in the Verma module corresponding to the vector P' satisfying the l equations (4.2)

$$\hat{e}_{m_i,n-1}P' = -r_{m_i}b - r'_{m_i}b^{-1}, \quad i = 1\ldots l, \quad (4.23)$$

depends on the position of points m_i (m_i form increasing sequence) on the Dynkin diagram of the Lie algebra $sl\,(n)$. We can delete all nodes on this diagram except the nodes having numbers m_i. The resulting diagram will correspond to the Lie algebra

$$sl\,(k_1) \otimes sl(k_2) \otimes \cdots \otimes sl(k_j)$$

where $\sum_{i=1}^{j} k_i = l + j$. The number of singular vectors will be in general case $N_s = \sum_{i=1}^{j}(2^{k_j-1} - 1)$. The imbedding diagram of singular vectors for each of the Lie algebra $sl\,(k)$ has $k-1$ lines outgoing from starting point which enter to $k-1$ points corresponding to the first generation of the singular vectors. From each of these points starts $k-2$ lines which enter to C_{k-1}^2 points corresponding to the second generation of the singular vectors. From each of these points starts $k-3$ lines which end at C_{k-1}^3 points and so on. To the last point enter $k-1$ lines. The imbedding diagram for the Lie algebra $sl\,(4)$ is drown in fig 3. We see that at each point corresponding to the second generation we will have one consistency condition and in the last point we have three consistency condition. For arbitrary k the number of these conditions will be $N_c = (n-1)\,(n-2)\,2^{n-4}$. It can be checked that all these conditions are satisfied.
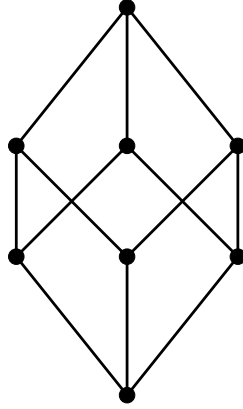
Figure 3. Embedding diagram for $sl(4)$.

With vectors P' satisfying eq. (4.23) we can reproduce using factorization property the functions $F_{\vec{\sigma}}^{\lambda^{(1)}, \varnothing, \ldots, \varnothing}(P, P', a)$ starting from known function $F_{\varnothing, \ldots, \varnothing, \nu^{(n)}}^{\lambda^{(1)}, \varnothing, \ldots, \varnothing}(P, P', a)$. The "vector" $\vec{\sigma}$ is specified by non-empty partitions $\sigma^{(m_i)}$ where numbers m_i correspond the position of point m_i on the Dynkin diagram of $sl(n)$ and the partition $\sigma^{(n)}$. Then if we impose $n-1$ condition (4.23) we can reproduce the functions $F_{\vec{\sigma}}^{\lambda^{(1)}, \varnothing, \ldots, \varnothing}(P, P', a)$ with $\vec{\sigma} = (\sigma^{(1)}, \ldots, \sigma^{(n)})$. Proceeding in the same way with the vector P and partitions $\vec{\lambda}$ we can reproduce from these functions all functions $F_{\vec{\sigma}}^{\vec{\lambda}}(P, P', a)$ defined by eq. (2.24).

Thus we constructed the basis of states $|\Psi_{\vec{\nu}, P}\rangle$ and primary operators (2.17) such that the matrix elements (2.20) are given by factorized expression (2.24). The norms of these states $N(\vec{\nu}, P) = \langle \Psi_{\vec{\nu}, P} | \Psi_{\vec{\nu}, P} \rangle$ can be obtained from (2.24) by setting $a = 0$, $\vec{\nu} = \vec{\nu}'$ and $P = -P'$ i.e.

$$N(\vec{\nu}, P) = F_{\vec{\nu}}^{\vec{\nu}}(0, -P, P). \tag{4.24}$$

Then the combinatorial expansion of the function (1.3) takes the form

$$\mathbb{Z}_{\vec{j}} = \sum_{\vec{\nu}_1, \ldots, \vec{\nu}_{k-3}} N^{-1}(\vec{\nu}_1, P_1) \ldots N^{-1}(\vec{\nu}_{k-3}, P_{k-3}) \times \tag{4.25}$$

$$\times F_{\varnothing}^{\vec{\nu}_1}(a_2, P_1, P) F_{\vec{\nu}_1}^{\vec{\nu}_2}(a_3, P_2, P_1) \ldots F_{\vec{\nu}_{k-4}}^{\vec{\nu}_{k-3}}(a_{k-2}, P_{k-3}, P_{k-4}) F_{\vec{\nu}_{k-3}}^{\varnothing}(a_{k-1}, \hat{P}, P_{k-3}),$$

where $P = \alpha_1 - \mathcal{Q}$ and $\hat{P} = \alpha_k - \mathcal{Q}$.

5 Classical Integrability

Here we consider the classical reduction of our integrable system. In the classical limit we define the fields

$$\mathsf{v} \to Q\phi', \quad \mathsf{u} \to Q^2 T, \quad \mathsf{u}_k \to Q^k \hat{\mathsf{W}}_k, \quad [,] \to \frac{2\pi i}{Q^2} \{,\}, \tag{5.1}$$

and the operator D: $\mathrm{D}F = i(F_+ - F_-)$, where F_+ and F_- are the positive and negative frequency part of F. Then the densities of the first three integrals can be written as $I^{(cl)}_{n,(k)} = \frac{1}{2\pi}\int_0^{2\pi} G_k^{(n)}dx$

$$G_2^{(n)} = \mathsf{u} - \mathsf{v}^2/2,$$

$$G_3^{(n)} = \frac{n}{2i}\mathsf{v}\mathrm{D}\mathsf{v} + \frac{2}{\sqrt{n}}\mathsf{uv} - \frac{\mathsf{v}^3}{3\sqrt{n}} + i\mathsf{u}_3$$

$$G_4^{(n)} = -\frac{(1+n^2)}{2}(\partial\mathsf{v})^2 + \frac{(6-n)\mathsf{u}^2}{n} + \frac{3}{2i}\sqrt{n}(2\mathsf{u} - \mathsf{v}^2)\mathrm{D}\mathsf{v} - \frac{1}{4n}\mathsf{v}^4 + \frac{3i}{\sqrt{n}}\mathsf{u}_3\mathsf{v} + \mathsf{u}_4, \qquad (5.2)$$

where the terms u_3 and u_4 appear only for $n > 2$ and $n > 3$ respectively. The Poisson brackets for the fields $\mathsf{u}, \mathsf{u}_3, \mathsf{u}_4 \ldots$ are the standard Gel'fand-Dikii brackets and can be found in refs [34–36]. The Poisson bracket for the field v is: $\{\mathsf{v}(\mathsf{x}), \mathsf{v}(\mathsf{y})\} = -\delta'(x - y)$.

The field u_3 transforms as the primary field of spin 3. It means that its Poisson bracket with u has canonical form

$$\{\mathsf{u}(\mathsf{x}), \mathsf{u}_3(y)\} = (\delta'(x - y)(\mathsf{u}_3(y) + 2\mathsf{u}_3(x)).$$

The field u_4 is not "primary". In quantum case we can represent the field $\hat{\mathrm{W}}_4$ in the form

$$\hat{\mathrm{W}}_4 = \widetilde{\mathrm{W}}_4 + \frac{(n-2)(n-3)}{2}\left(\frac{(\frac{5}{n} + 3(5 - 3n)Q^2)}{17 + 5n(n^2 - 1)Q^2}\Lambda + \frac{3nQ^2}{10}\partial^2 T\right), \qquad (5.3)$$

where $\Lambda = {}_\times^\times T\,T_\times^\times - \frac{3}{10}\partial^2 T$ (here symbol ${}_\times^\times \ldots {}_\times^\times$ denotes regularized product) and $\widetilde{\mathrm{W}}_4$ is a primary field of spin four. In the semiclassical limit $Q \to \infty$ we find that

$$\mathsf{u}_4 = \widetilde{\mathsf{u}}_4 + \frac{(n-2)(n-3)}{2}\left(\frac{3(5-3n)}{5n(n^2-1)}\mathsf{u}^2 + \frac{3n}{10}\partial^2\mathsf{u}\right).$$

With the Gel'fand-Dikii brackets we can write the equation of motion for the functions $\mathsf{v}, \mathsf{u}, \mathsf{u}_i$. It is however easier to use the bosonization procedure described above and express the integrals $I^{(cl)}_{n,(k)}$ in terms of the field $\phi_i = Q\Phi_i$ which have simple Poisson brackets $\{\phi_i(x), \phi_j(y)\} = -\delta_j^i\delta'(x - y)$. The densities $\mathrm{G}_k^{(n)}$ of the first three integrals in these variables have a form (we put for simplicity $x_i = 0$)

$$\mathrm{G}_2^{(n)} = -\sum_{j=1}^n \phi_j^2,$$

$$\mathrm{G}_3^{(n)} = \frac{1}{2i}\hat{\phi}\mathrm{D}\hat{\phi} + \sum_{j>k}^n \phi_j\partial\phi_k - \frac{1}{3}\sum_{j=1}^n\phi_j^3,$$

$$\mathrm{G}_4^{(n)} = -\frac{n\partial\hat{\phi}\partial\hat{\phi}}{2} + \sum_{j=1}^n\frac{(n-1)}{2}\partial\phi_j\partial\phi_j - \frac{\phi_j^4}{4} - \frac{1}{i}(\phi_j^2 - (n+1-2j)\partial\phi_j)\mathrm{D}\hat{\phi},$$

$$+ \sum_{k=1}^{n-1}(n-2k)\sum_{l=1}^{n-k-1}\partial\phi_l\partial\phi_{l+k} + \sum_{j=1}^n\sum_{k=1}^n\phi_k^2\sigma[j-k]\partial\phi_j, \qquad (5.4)$$

where $\hat{\phi} = \sum_i^n \phi_i$ and $\sigma[i-j] = 1$ if $i > j$, 0 if $i = j$ and -1 if $i < j$.

The equations of motion corresponding to the dynamics with Hamiltonian $\mathcal{H}_3 = i\, I_{n,(3)}^{(cl)}$ ($\partial_t \phi_j = \{\phi_j, \mathcal{H}_3\}$) have a simple form

$$\frac{1}{i}\partial_t \phi_j + \frac{1}{i}\mathrm{D}\hat{\phi} + \sum_{k=1}^n \sigma[j-k]\partial_x^2 \phi_k - 2\phi_j \partial_x \phi_j = 0. \tag{5.5}$$

They admit the "reality" condition $\phi_j^* = -\phi_{n+1-j}$. To study these equations we consider first the case $n = 2$. In this case we have

$$\begin{aligned}
\frac{1}{i}\partial_t \phi_1 + \frac{1}{i}\mathrm{D}(\phi_1 + \phi_2)_x - \partial_x^2 \phi_2 - 2\phi_1 \partial_x \phi_1 &= 0 \\
\frac{1}{i}\partial_t \phi_2 + \frac{1}{i}\mathrm{D}(\phi_1 + \phi_2)_x + \partial_x^2 \phi_1 - 2\phi_2 \partial_x \phi_2 &= 0.
\end{aligned} \tag{5.6}$$

The reality condition can be written as $\phi_1 = i\psi^*, \phi_2 = i\psi$. In terms of function ψ these equations can be written as

$$\partial_t \psi + 2\,\mathrm{Re}\,\mathrm{D}\psi_x + i\partial_x^2 \psi^* + 2\psi\partial_x \psi = 0. \tag{5.7}$$

This equation (up to the substitution $t \to 2t$) coincides with the equation derived in [12]. For the functions ψ analytical in the upper half plane this equation due to relation $\mathrm{D}(\psi + \psi^*) = i(\psi_x - \psi_x^*)$ reduces to the complex Burgers equation

$$\partial_t \psi + i\partial_x^2 \psi + 2\psi\partial_x \psi = 0, \tag{5.8}$$

which can be linearized by Cole-Hopf substitution $\psi = i(\log \theta)_x$

$$\partial_t \theta + i\theta_{xx} = 0. \tag{5.9}$$

We note that for all n eqs. (5.5) can be reduced to the complex Burgers equation if we put $\phi_2 = \phi_3 = \cdots = \phi_{n-1} = 0$ and functions $\phi_1 = i\psi^*$, $\phi_n = i\psi$ with ψ analytical in the upper half plane.

If we take equation with $j = 1$ and the difference between equations $j+1$ and j in (5.5) we obtain

$$\frac{1}{i}\partial_t \phi_1 + \frac{1}{i}\mathrm{D}\hat{\phi}_x - \sum_{j=2}^n \partial^2 \phi_j - 2\phi_1 \partial_x \phi_1 = 0, \tag{5.10}$$

$$\frac{1}{i}\partial_t(\phi_{j+1} - \phi_j) + \partial^2(\phi_j + \phi_{j+1}) - \partial_x(\phi_{j+1}^2 - \phi_j^2) = 0$$

In the limit $n \to \infty$ we can define variable $y = j/n$, denote $\phi_j = \chi_{j/n}$ and introduce the field $\rho(x, y, t) = n\chi_{j/n}(x, nt)$, then this system of equations can be written as

$$\frac{1}{i}\partial_y\partial_t \rho(x, y, t) + 2\partial_x^2 \rho(x, y, t) - \partial_x\partial_y \rho^2(x, y, t) = 0, \tag{5.11}$$

$$\frac{1}{i}\partial_t \rho(x, 0, t) + \int_0^1 (\frac{1}{i}\mathrm{D}\rho(x, y, t)_x - \rho(x, y, t)_{xx})dy - \partial_x \rho^2(x, 0, t) = 0.$$

These equations have a simple stationary solution

$$\rho(x, y) = (1/2 - y)\cot(x/2 + i\,\mathrm{sign}(y - 1/2)\eta)$$

where $\eta > 0$. It is easy to check that for all n the functions

$$\phi_j = \left(\frac{n+1}{2} - j\right) \cot\left(x/2 + i\,\mathrm{sign}\left(j - \frac{n+1}{2}\right)\eta\right)$$

give the stationary solution of eqs. (5.5).

We can consider dynamics generated by the next Hamiltonian $\mathcal{H}_4 = I^{(cl)}_{n,(4)}$ ($\partial_{t_2}\phi_i = \{\phi_i, \mathcal{H}_4\}$). The corresponding equation looks rather complicated. However, if we take the same reduction: $\phi_1 = i\psi^*$, $\phi_n = i\psi$ an all other $\phi_j = 0$, the equation for ψ analytical in upper half plane has a form

$$\partial_{t_2}\psi + \partial_x^3\psi - 3i\partial_x(\psi\partial_x\psi) - \partial_x(\psi)^3 = 0 \tag{5.12}$$

This equation also can be linearized by Cole-Hopf substitution (5.9)

$$\partial_{t_2}\theta + \theta_{xxx} = 0$$

It looks reasonable to think that for higher Hamiltonians $\mathcal{H}_k$ the equations of motion for the function ψ analytical in upper half plane can be linearized by the same substitution.

We suppose to consider more general solutions of eqs. (5.5) as well as their relations with semiclassical limit of matrix elements of the fields Φ_j in the separate publication.

6 Concluding remarks

In this paper we have shown that for spherical geometry the AGT conjecture is valid for the $N = 2$ supersymmetric quiver theories with gauge group $U(n)$. To prove this conjecture we considered the algebra $A = W_n \otimes H$, which is a tensor product of commuting W_n and Heisenberg algebras. We constructed the commuting integrals of motion in the space A which have simple eigenvectors an eigenvalues. The remarkable property of the orthogonal basis of eigenvectors is that all matrix elements of vertex operators in this basis have the factorized form and determine the combinatorial expansion for multipoint conformal blocks.

It is natural to think that the similar construction can give the combinatorial expansion for all CFT where conformal blocks are completely defined by the symmetry algebra together with three point functions of primary fields. For example, in $N = 1$ and $N = 2$ superconformal Liouville theories, parafermionic Liouville CFT (see for example [37]), sine-Liouville model and closely related with it $sl(2)$ WZN model. An evidence of this statement is confirmed by the results of the paper [38].

We note that corresponding integrable systems can have the own interest. They can be related with integrable many body systems. For example, the integral of motion constructed in this paper describes n Calogero-Sutherland systems coupled in the special way which preserve the integrability.

As the simplest example of integrable system in the extended space A we consider $A = G * \Psi$, where G is $N = 1$ supersymmetry algebra generated by super-current $G(z)$ and stress energy tensor $T(z)$ with the standard commutation relations and Ψ is the algebra

generated by the free Majorana fermion $\psi(z)$. In the space A we have two integrals at the level 2.

$$
\begin{aligned}
I_2 &= L_0 + l_0 \\
\hat{I}_2 &= 2iQ \sum_{k>0}^{\infty} k\psi_{-k}\psi_k + \sum_{-\infty}^{\infty} G_{-k}\psi_k
\end{aligned}
\tag{6.1}
$$

here L_n and l_n are the Fourier components of stress energy tensor T and stress energy tensor of free fermion theory respectively, ψ_k and G_k with half-integer (integer) are the modes of the of fields ψ and G in the NS (R) sectors. The spectrum of I_2 is trivial and $\hat{I}_2$ has spectrum linear in momentum P.

To show that this system is integrable we give here one next integral commuting with integrals $I_2, \hat{I}_2$, which has a form:

$$
\hat{I}_4 = \sum_{n+k+l=0} : L_n G_k : \psi_l + iQ \sum_{k>0} k G_{-k}G_k + 4iQ \sum_{-\infty}^{\infty} L_{-n}l_n + \tag{6.2}
$$
$$
\left(1 - \frac{3Q^2}{2}\right) \sum_{-\infty}^{\infty} k^2 G_{-k}\psi_k + iQ\left(\frac{5}{2} - 2Q^2\right) \sum_{k>0}^{\infty} k^3 \psi_{-k}\psi_k + \frac{iQ(1+2Q^2)}{8} \sum_{k>0}^{\infty} k\psi_{-k}\psi_k.
$$

We think that the integrable systems of this type could play rôle in supersymmetric version of AGT relation proposed recently in [39] and discussed in [40–42]. We propose to discuss this question in a future publication.

Acknowledgments

A   Calculation of the integral (3.35)

Here we calculate the ratio (3.35). In our case

$$
\mathcal{O} = \mathrm{J}_\mu^{(1/g)}[p_k(t^{(1)}) + \eta]\mathrm{J}_\lambda^{(1/g)}[p_{-k}(t^{(n-1)})].
$$

Following ref [12] we change the normalization tof Jack polynomials i.e. introduce the polynomials

$$
\mathrm{P}_\lambda^{(1/g)}(t_1, \ldots, t_k) = \frac{1}{\mathrm{c}_\lambda(g)}\mathrm{J}_\lambda^{(1/g)}(t_1, \ldots, t_k)
\tag{A.1}
$$

with

$$
\mathrm{c}_\lambda(g) = \prod_{s\in\lambda}(1 + l_\lambda(s) + g^{-1}a_\lambda(s))
$$

where $a_\lambda(s)$ and $l_\lambda(s)$ are the arm length and the leg length of the square s in the partition λ.

The polynomials $P_\lambda^{(1/g)}(t_1, \ldots, t_k)$ satisfy the Kadell identity [43] which we will use further

$$P_\lambda^{(1/g)}[p_{-m}] = \prod_{j=1}^k t_j^{-\lambda_1} P_{\hat\lambda}^{(1/g)}[p_m] \tag{A.2}$$

where for given partition $\lambda = \{\lambda_1 \geq \lambda_2, \ldots\}$ the hatted partition $\hat\lambda = \left\{\hat\lambda_1 \geq \hat\lambda_2, \ldots\right\}$ is defined by

$$\hat\lambda_j = \lambda_1 - \lambda_{k-j+1} \tag{A.3}$$

We introduce also the generalized Pochammer symbol

$$[z]_\lambda = \prod_{j\geq 1}(z + (1-j)g)_{\lambda_j} = \prod_{j\geq 1} \frac{\Gamma(z + \lambda_j + (1-j)g)}{\Gamma(z + (1-j)g)} \tag{A.4}$$

and the new notations, which are usual for calculation of $sl(n+1)$ type Selberg Integrals. Namely,

$$g \to \gamma, \quad A_r \to \alpha_r - 1, \quad r = 1, \ldots, n-1, \quad A_n - \lambda_1 = \alpha_n - 1, \quad B = \beta - 1 \tag{A.5}$$

It is more convenient to consider the integral not over the collection of the contours $\mathcal{C}(n+1)$ shown on figure 1 but the integral over the special domain $C_\gamma^{k_1, \ldots, k_n}[0, 1]$ which will be described bellow. These two integral are related by overall factor independent on the single-valued function $\mathcal{O}$ and cancels in the ratio (3.35). Namely,

$$\int\limits_{\mathcal{C}(n+1)} \mathcal{O}(t^{(1)}|\ldots|t^{(n)})G_{n+1}(t^{(1)}|\ldots|t^{(n)})d^{k_1}t^{(1)}\ldots d^{k_n}t^{(n)}$$

$$= \mathcal{N} \int\limits_{C_\gamma^{k_1, \ldots, k_n}[0,1]} \mathcal{O}(t^{(1)}|\ldots|t^{(n)})|G_{n+1}(t^{(1)}|\ldots|t^{(n)})|d^{k_1}t^{(1)}\ldots d^{k_n}t^{(n)} \tag{A.6}$$

where $\mathcal{N} = N_{k_1}(\beta, \gamma)N_{k_2}(-k_1\gamma, \gamma)\ldots N_{k_{n-1}}(-k_{n-2}\gamma, \gamma)$. Here $N_k(\beta, \gamma)$ is [30, 31]

$$N_k(\beta, \gamma) = \frac{2ie^{i\pi\beta}\sin(\pi(\beta + (j-1)\gamma)\sin(\pi j\gamma)}{\sin(\gamma)}$$

and $G_{n+1}(t^{(1)}|\ldots|t^{(n)})$ is defined by eq. (3.37).

The domain of integration $C_g^{k_1, \ldots, k_n}[0, 1]$ for $sl(n+1)$ Selberg Integrals is defined in the following way [30, 31]. Let all points $t_i^{(r)}$ are ordered

$$0 \leq t_i^{(r)} \leq t_2^{(r)} \leq \cdots \leq t_k^{(r)} \leq 1, r = 1, \ldots, n. \tag{A.7}$$

We introduce $n-1$ non-decreasing maps M_r

$$M_r : \{1, \ldots k_r\} \to \{1, \ldots, k_{r-1}\}, \quad M_r(i) \leq M_r(i+1), \quad r = 2, \ldots n. \tag{A.8}$$

such that $1 \leq M_r(i) \leq k_{r-1} - k_r + i$ and

$$t_{M_r(i)-1}^{(r-1)} \leq t_i^{(r)} \leq t_{M_r(i)}^{(r-1)}, \quad r = 2, \ldots n. \tag{A.9}$$

where $t_0^{(r-1)} = 0$. Given admissible maps $M_r(i)$ satisfying eq. (A.9) we define

$$D_{M_1...M_{n-1}}^{k_1,...,k_n}[0,1] \subseteq D^{k_1,...,k_n}[0,1],$$

where $D^{k_1,...,k_n}[0,1]$ is a domain (A.7). As chain we have

$$D^{k_1,...,k_n}[0,1] = \sum_{M_2,...,M_n} D_{M_2...M_n}^{k_1,...,k_n}[0,1]$$

The domain of integration $C_\gamma^{k_1,...,k_n}[0,1]$ is defined as

$$C_\gamma^{k_1,...,k_n}[0,1] = \sum_{M_2,...,M_n} F_{M_2...M_n}^{k_1,...,k_n}(\gamma) D_{M_2...M_n}^{k_1,...,k_n}[0,1]$$

where coefficients $F_{M_2...M_n}^{k_1,...,k_n}(\gamma)$

$$\prod_{r=2}^{n} \prod_{i=1}^{k_r} \frac{\sin(\pi(k_{r-1} - k_r - M(i) + i + 1)\gamma)}{\sin((\pi(k_{r-1} - k_r + i)\gamma)}$$

coincide up to overall factor $\mathcal{N}$ with the factors which appear in contour integral in l.h.s. of eq. (A.6) (if we tend all contours to the real axis) in the corresponding domain of integration due to the avoiding of branching points of the integrand.

To make our transformations more explicit we consider the $sl(3)$ Selberg integral. The calculation in the case of general n follows the same steps. In the $sl(3)$ we have only one non-decreasing map $M_2 = M : \{1, \ldots k_2\} \to \{1, \ldots, k_1\}$ such that $t_{M(i)-1}^{(1)} \le t_i^{(2)} \le t_{M(i)}^{(1)}$. The $sl(3)$ Selberg integral has a form

$$\mathbf{J}_{\mu,\lambda}^{(k_1,k_2)}(\alpha_1, \alpha_2, \beta, \gamma) =$$
$$= \int_{C_\gamma^{k_1,k_2}[0,1]} P_\mu^{(1/\gamma)}[p_m(t^{(1)}) + (\beta - \gamma)/\gamma]P_\lambda^{(1/\gamma)}[p_m(t^{(2)})]D^{2\gamma}(t^{(1)})D^{2\gamma}(t^{(2)})$$

$$|D(t^{(1)}, t^{(2)})|^{-\gamma} \prod_{i=1}^{k_1} (t_i^{(1)})^{\alpha_1 - 1}(1 - t_i^{(1)})^{\beta - 1} \prod_{j=1}^{k_2} (t_j^{(2)})^{\alpha_1 - 1} d^{(k_1)}t^{(1)} d^{(k_2)}t^{(2)} \quad \text{(A.10)}$$

It follows from eqs. (A.2) and (A.5) that

$$\frac{\left\langle \mathrm{J}_\mu^{(1/g)}[p_k + \eta] \mathrm{J}_\lambda^{(1/g)}[p_{-k}] \right\rangle_{\mathrm{Sel}}^{\mathrm{sl}(n)}}{\langle 1 \rangle_{\mathrm{Sel}}^{\mathrm{sl}(n)}} = \mathrm{c}_\lambda(g)\mathrm{c}_\mu(g) \frac{\mathbf{J}_{\mu,\hat{\lambda}}^{(k_1,k_2)}(1 + A_1, 1 + A_2 - \lambda_1, 1 + B, g)}{\mathbf{J}_{\varnothing,\varnothing}^{(k_1,k_2)}(1 + A_1, 1 + A_2 - \lambda_1, 1 + B, g)} \quad \text{(A.11)}$$

where the partition $\hat{\lambda}$ is defined by eq. (A.3).

The calculation of the integral (A.10) follows almost exactly the same steps that in the $sl(2)$ case [12]. It is based on two identities (A.1 and A.2 in ([12]) which we reproduce here for completeness. Namely [44] identity A.1:

Identity A.1. Let $\tau^{(2)} = (\tau_1^{(2)}, \ldots, \tau_{k_2-1}^{(2)})$ and $(t_1^{(2)}, \ldots, t_{k_2}^{(2)})$ satisfy the interlacing property

$$t_1^{(2)} < \tau_1^{(2)} < t_2^{(2)} < \tau_2^{(2)} < \ldots < t_{k_2-1}^{(2)} < \tau_{k_2-1}^{(2)} < t_{k_2}^{(2)}$$

denoted by $\tau^{(2)} \prec t^{(2)}$. Then for $\nu = (\nu_1 \geq \nu_2 \ldots)$ a partition of length at most $k_2 - 1$

$$D(t^{(2)})^{2\gamma-1} P_\nu^{(1/\gamma)}(t^{(2)}) = \Lambda_\nu(\gamma) \int\limits_{\tau^{(2)} \prec t^{(2)}} P_\nu^{(1/\gamma)}(\tau^{(2)}) D(\tau^{(2)}) |D(\tau^{(2)}, t^{(2)})|^{\gamma-1} d^{(k_2-1)}\tau^{(2)} \tag{A.12}$$

where

$$\Lambda_\nu(\gamma) = \frac{\Gamma(k_2\gamma)}{\Gamma(\gamma)^{k_2}} \frac{[k_2\gamma]}{[(k_2-1)\gamma]}$$

The second identity states:

Identity A.2. Let $t^{(1)} = (t_1^{(1)} \ldots t_{k_1}^{(1)})$ and $\tau^{(1)} = (\tau_1^{(1)}, \ldots, \tau_{k_1-1}^{(1)})$ satisfy the interlacing property

$$0 < t_1^{(1)} < \tau_1^{(1)} < t_2^{(1)} < \tau_2^{(1)} < \ldots < t_{k_1-1}^{(1)} < \tau_{k_1-1}^{(1)} < t_{k1}^{(1)} < 1$$

denoted by $t^{(1)} \prec \tau^{(1)}$. Then for partition $\mu = (\mu_1 \geq \mu_2 \ldots)$ a partition

$$\int\limits_{t^{(1)} \prec \tau^{(1)}} P_\mu^{(1/\gamma)} [p_m(t^{(1)}) + (\beta-\gamma)/\gamma] D(t^{(1)}) |D(t^{(1)}, \tau^{(1)})|^{\gamma-1} \prod_{i=1}^{k_1} (t_i^{(1)})^{\alpha_1-1} (1 - t_i^{(1)})^{\beta-1} d^{k_1} t^{(1)}$$

$$= \Xi_\mu(\alpha_1, \beta, \gamma) \prod_{i=1}^{k_1-1} (\tau_i^{(1)})^{\alpha_1-1} (1 - \tau_i^{(1)})^{\beta-1} D(\tau^1)^{2\gamma-1} P_\mu^{(1/\gamma)}[p_m(\tau^{(1)}) + \beta/\gamma] \tag{A.13}$$

where

$$\Xi_\mu(\alpha_1, \beta, \gamma) = \frac{\Gamma(\alpha_1)\Gamma(\beta)\Gamma^{k_1-1}(\gamma)}{\Gamma(\alpha_1 + \beta + (k_1-1)\gamma)} \frac{(\alpha_1 + \beta + (k_1-2)\gamma)_\mu}{(\alpha_1 + \beta + (k_1-1)\gamma)_\mu}$$

For our purposes we need also third identity which is the analog of identity derived in [45] for the complex integrals (i.e. for integrals over all plane for each of variables). For our case this identity can be formulated as

$$\int\limits_{\tau^{(2)} \prec t^{(2)}} \int\limits_{C_\gamma^{k_1,k_2}[0,1]} \prod_{i=1}^{k_2} (t_i^{(2)})^{\alpha_2-1} D(t^{(2)}) |D(\tau^{(2)}, t^{(2)})|^{\gamma-1} |D(t^{(1)}, t^{(2)})|^{-\gamma} d^{k_2} t^{(2)} =$$

$$= \Omega_{k_1,k_2}(\alpha_2, \gamma) D(\tau^{(2)})^{2\gamma-1} D(t^{(1)})^{1-2\gamma} \prod_{i=1}^{k_2-1} (\tau_i^{(2)})^{\alpha_2+\gamma-1} \prod_{j=1}^{k_1} (t_j^{(1)})^{\alpha_2-\gamma}$$

$$\int\limits_{t^{(1)} \prec \tau^{(1)}} \int\limits_{C_\gamma^{k_1-1,k_2-1}[0,1]} \prod_{i=1}^{k_1-1} (\tau_i^{(1)})^{-\alpha_2} D(\tau^{(1)}) |D(\tau^{(1)}, \tau^{(2)})|^{-\gamma} |D(t^{(1)}, \tau^{(1)})|^{\gamma-1} d^{(k_1-1)}\tau^{(1)}$$

$$\tag{A.14}$$

where

$$\Omega_{k_1,k_2}(\alpha_2,\gamma) = \frac{\pi \Gamma(\alpha_2)\Gamma^{k_2-k_1-1}(\gamma)}{\sin(\pi k_1 \gamma)\Gamma(1+\alpha_2-(k_2-k_1+1)\gamma)}$$

Now we can reduce the number of integrations in (A.10) performing the following steps

1. For $\lambda = (\lambda_1,\ldots,\lambda_{k_2})$ we use the relation

$$P_\nu^{(1/\gamma)}(t^{(2)}) = P_\lambda^{(1/\gamma)}[p_m(t^{(2)})] = (t_1^{(2)}\ldots t_{k_2}^{(2)})^{\lambda_{k_2}} P_\nu^{(1/\gamma)}[p_m(t^{(2)})]$$

   where $\nu = (\lambda_1 - \lambda_{k_2},\ldots,\lambda_{k_2-1}-\lambda_{k_2},0)$

2. We represent $D(t^{(2)})^{2\gamma-1} P_\nu^{(1/\gamma)}(t^{(2)})$ in eq. (A.10) using (A.12)

3. We use the integral relation (A.14)

4. We compute the integral over variables $t^{(1)}$ using (A.13).

As a result we reduce our integral (A.10) to the integral of the same form but with lower number of integrations $(k_1, k_2) \to (k_1-1, k_2-1)$

$$\mathbf{J}_{\mu,\lambda}^{(k_1,k_2)}(\alpha_1,\alpha_2,\beta,\gamma) =$$
$$= \frac{[k_2\gamma]\,\Gamma(k_2\gamma)\Gamma(\alpha_2+\lambda_{k_2})\Gamma(\hat\alpha_{12}+\lambda_{k_2}-\gamma)\Gamma(\beta)\,\Gamma(1-k_1\gamma)\Gamma(k_1\gamma)}{[(k_2-1)\gamma]\,\Gamma(\gamma)^2\Gamma(1+\alpha_2+\lambda_{k_2}-(k_1-k_2+1)\gamma)\Gamma(\hat\alpha_{12}+\beta+\lambda_{k_2}+(k_1-2)\gamma)}$$
$$\frac{[\hat\alpha_{12}+\beta+\lambda_{k_2}+(k_!-3)\gamma]_\mu}{[\hat\alpha_{12}+\beta+\lambda_{k_2}+(k_1-2)\gamma]_\mu}\,\mathbf{J}_{\mu,\nu}^{(k_1-1,k_2-1)}(\alpha_1,\alpha_2+\gamma,\beta+\gamma,\gamma) \quad \text{(A.15)}$$

where $\nu = (\lambda_1 - \lambda_{k_2},\ldots\lambda_{k_2-1}-\lambda_{k_2},0)$. Now we can proceed by induction, We should be careful, because after k_2 steps our integral transforms to the $sl(2)$ Selberg integral $\mathbf{J}_\mu^{(k_1-k_2)}(\alpha_1,\beta+k_2\gamma,\gamma)$ and further steps of induction give

$$\mathbf{J}_\mu^{(k)}(\alpha,\beta',\gamma) = \frac{\Gamma(\alpha)\Gamma(\beta')\,\Gamma(k\gamma)[\alpha+\beta+(k-2)\gamma]_\mu}{\Gamma(\alpha+\beta'+(k-1)\gamma)\Gamma(\gamma)[\alpha+\beta+(k-1)\gamma]_\mu}\mathbf{J}_\mu^{(k-1)}(\alpha+\gamma,\beta'+\gamma,\gamma)$$

Using that

$$\frac{[k_2\gamma]_\nu}{[(k_2-1)\gamma]_\nu} = \frac{P_\lambda^{(1/\gamma)}[k_2]}{P_\nu^{(1/\gamma)}[k_2-1]}$$

we find

$$\mathbf{J}_{\mu,\lambda}^{(k_1,k_2)} = P_\lambda^{(1/\gamma)}[k_2]\,P_\mu^{(1/\gamma)}[k_1+(\beta-\gamma)/\gamma]\prod_{j=1}^{k_1}\frac{\Gamma(\beta+(j-1)\gamma)\Gamma(j\gamma)}{\Gamma(\gamma)}$$

$$\prod_{j=1}^{k_1-k_2}\frac{\Gamma(\alpha_1+(j-1)\gamma)}{\Gamma(\alpha_1+\beta+(k_1+j-2)\gamma)}\frac{[\alpha_1+\beta+(k_1+j-3)\gamma]_\mu}{[\alpha_1+\beta+(k_1+j-2)\gamma]_\mu}$$

$$\prod_{j=1}^{k_2}\frac{\Gamma(\alpha_2+(k_2-j)\gamma+\lambda_j)}{\Gamma(1+\alpha_2+(2k_2-k_1-j-1)\gamma+\lambda_j)\Gamma(\gamma)}$$

$$\frac{\Gamma(\hat\alpha_{12}+(k_2-j-1)\gamma+\lambda_j)\Gamma(j\gamma)\,\Gamma(1-(k_1-k_2+j)\gamma)}{\Gamma(\hat\alpha_{12}+\beta+(k_1+k_2-j-2)\gamma+\lambda_j)}$$

$$\frac{(\hat\alpha_{12}+\beta+(k_1+k_2-j-3)\gamma+\lambda_j)_\mu}{(\hat\alpha_{12}+\beta+(k_1+k_2-j-2)\gamma+\lambda_j)_\mu} \quad \text{(A.16)}$$

here $\hat{\alpha}_{12} = \alpha_1 + \alpha_2$.

Now we can calculate the ratio (A.11). Using evaluation formula [46]

$$P_\lambda^{(1/\gamma)}[N] = \gamma^{-|\lambda|}\frac{[N\gamma]}{c_\lambda(g)}$$

and eqs. (A.11), (A.3) and (A.16) we find that the ratio $\dfrac{\left\langle J_\mu^{(1/g)}[p_k+\rho]J_\lambda^{(1/g)}[p_{-k}]\right\rangle_{\mathrm{Sel}}^{\mathrm{sl}(3)}}{\langle 1\rangle_{\mathrm{Sel}}^{\mathrm{sl}(3)}}$ is

$$g^{-|\lambda|-|\mu|}\,[k_2 g]_\lambda\,[1+B+(k_1-1)g]_\mu$$

$$\prod_{j=1}^{k_2}\frac{\Gamma(1+A_2+(j-1)\,g-\lambda_j)\Gamma(2+\hat{A}_{12}+(j-2)g-\lambda_j)}{\Gamma(1+A_2+(j-1)\,g)\Gamma(2+\hat{A}_{12}+(j-2)g)}$$

$$\frac{\Gamma(2+A_2+(k_2-k_1+j-2)g)\Gamma(3+\hat{A}_{12}+B+(k_1+j-3)g)}{\Gamma(2+A_2+(k_2-k_1+j-2)g-\lambda_j)\Gamma(3+\hat{A}_{12}+B+(k_1+j-3)g-\lambda_j)}$$

$$\frac{[3+\hat{A}_{12}+B+(k_1+j-4)g-\lambda_j]_\mu}{[3+\hat{A}_{12}+B+(k_1+j-3)g-\lambda_j]_\mu}\frac{[2+A_1+B+(k_1-2)g]_\mu}{[2+A_1+B+(2k_1-k_2-2)g]_\mu} \quad \text{(A.17)}$$

After simple transformations the r.h.s. of this equality can be rewritten as

$$\frac{g^{-|\lambda|-|\mu|}\,[1+B+(k_1-1)g]_\mu\,[2+A_1+B+(k_1-2)g]_\mu\,[k_2 g]_\lambda\,[-1-A_2+(k_1-k_2+1)g]_\lambda}{[2+A_1+B+(2k_1-k_2-2)g]_\mu[3+\hat{A}_{12}+B+(k_1+k_2-3)g]_\mu\,[-A_2]_\lambda\left[-1-\hat{A}_{12}+g\right]_\lambda}$$

$$\prod_{s\in\lambda}(2+\hat{A}_{12}+B+(k_1-3)g-a_\mu(s)g-l_\lambda(s))$$

$$\prod_{t\in\mu}(3+\hat{A}_{12}+B+(k_1-2)g+a_\lambda(t)g+l_\mu(s)) \quad \text{(A.18)}$$

We note that screening condition (3.34) can be written in $sl(3)$ case as

$$-x_1+x_1'+2a/3+k_1 b=0, \quad -x_2+x_2'-a/3+(k_2-k_1)b=0, \quad -x_3+x_3'-a/3-k_2 b=0.$$

If we take any two from these three equations to express k_1, k_2 in terms of x, x' and a and take into account eq. (3.38) we derive desirable result (3.35).

The calculation of $sl(n+1)$ -Selberg integral $\mathbf{J}_{\mu,\lambda}^{(k_1,\ldots,k_n)}(\vec{\alpha},\beta,\gamma)$ (here $\vec{\alpha}=(\alpha_1,\ldots,\alpha_n)$) follows exactly the same steps. We should only repeat the integral relation (A.14) $n-1$ times. In this way we derive the relation

$$\mathbf{J}_{\mu,\lambda}^{(k_1,\ldots,k_n)}(\vec{\alpha},\beta,\gamma)=$$

$$=\frac{[k_n\gamma]\,\Gamma(k_2\gamma)\Gamma(\alpha_n+\lambda_{k_n})\Gamma(\beta)\,[\hat{\alpha}_{1n}+\beta+(k_1-n-1)\gamma+\lambda_{k_n}]_\mu}{[(k_n-1)\gamma]\,\Gamma(\gamma)^n\Gamma(\hat{\alpha}_{1n}+\beta+(k_1-n)\,\gamma+\lambda_{k_n})[\hat{\alpha}_{1n}+\beta+(k_1-n\gamma+\lambda_{k_n}]_\mu}$$

$$\prod_{j=1}^{n-1}\frac{\Gamma(1-k_j\gamma)\Gamma(k_j\gamma)\Gamma(\hat{\alpha}_{n-j,n}-j\gamma+\lambda_{k_n})}{\Gamma(1+\hat{\alpha}_{n-j+1,n}-(k_{n-j}-k_{n-j+1}+1)\gamma+\lambda_{k_n})}\mathbf{J}_{\mu,\nu}^{(k_1-1,\ldots,k_n-1)}(\vec{\alpha}',\beta+\gamma,\gamma)$$

where $\nu=(\lambda_1-\lambda_{k_n},\ldots,\lambda_{k_n-1}-\lambda_{k_n},0)$, $\hat{\alpha}_{r,n}=\alpha_r+\cdots+\alpha_n$, $\hat{\alpha}_{n,n}=\alpha_n$ and $\vec{\alpha}'=(\alpha_1,\ldots,\alpha_{n-1},\alpha_n+\gamma)$.

If we solve this recurrence relation and use the screening conditions to express $k_1,\ldots,k_n$ in terms of terms of x, x' and a we derive the expected result for matrix element (3.35).

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