§29. Computation of low-degree cases

To illustrate (28.2) we consider the low degree cases. Assume $\phi(t)$ is primitive and let Δ denote its discriminant.

1) Degree $\phi(t) = 2$: $Z[\alpha, \alpha^{-1}]$ is Dedekind unless, for some prime p, $p^2 | \Delta$, but at most one coefficient of $\phi(t)$ is divisible by p, and $\Delta \equiv 0$ or $4 \bmod 16$ if $p = 2$.

In this case, the criterion of (28.2) reduces to the existence of a nontrivial double root a of $\phi(t) \pmod p$ such that $\phi(a)$ is divisible by p^2. It is not hard to see this is equivalent to the stated criterion.

2) Degree $\phi(t) = 3$: $Z[\alpha, \alpha^{-1}]$ is Dedekind unless for some prime p there exists an integer a such that

$$\phi'(a) \equiv 0 \mod p$$

$$\phi(a) \equiv 0 \mod p^2$$

(thus $p | \Delta$).

In order that $Z[\alpha, \alpha^{-1}]$ be not Dedekind, (28.2) implies we may write $\phi(t) = c(t - a)^2(t - b) + p\gamma(t)$ for some prime p, where $c, a \not\equiv 0 \mod p$, and $\gamma(a) \equiv 0 \mod p$. This is easily seen to be equivalent to the stated criterion.

3) Degree $\phi(t) = 4$; $\phi(t) = \phi(t^{-1})$: (This is satisfied when A supports a nondegenerate ϵ-pairing--see (19.1).) $Z[\alpha, \alpha^{-1}]$ is Dedekind unless $\phi(1)$ or $\phi(-1)$ is divisible by p^2, for some prime p, or, writing $\phi(t) = at^2 + bt + c + bt^{-1} + at^{-2}$, and, setting $\Delta = b^2 - 4a(c - 2a)$, $p^2 | \Delta$ for some prime p such that $p \nmid a$ and, if $p = 2$, $\Delta \equiv 0$ or $4 \mod 16$.

We omit the proof.