ON THE ACCELERATED SSOR METHOD FOR SOLVING ELLIPTIC BOUNDARY VALUE PROBLEMS

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1. INTRODUCTION

The paper is concerned with the use of the symmetric successive overrelaxation method (SSOR method) for solving large systems of linear algebraic equations arising in the solution by finite difference methods of boundary value problems involving elliptic partial differential equations. Our main result is to show that by the use of the SSOR method, combined with an acceleration procedure, one can attain a specified degree of convergence to the exact solution of the linear system in $O(h^{-\frac{1}{2}})$ iterations where h is the mesh size. The result holds for a class of problems involving the self-adjoint equation

$$\frac{\partial}{\partial x}\left(A \frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(C \frac{\partial u}{\partial y}\right) + Fu = G \tag{1.1}$$

where

$$A(x,y) > 0, \quad C(x,y) > 0, \quad F(x,y) \leq 0 \tag{1.2}$$

in the region. It is assumed that A(x,y) and C(x,y) belong to class $C^{(2)}$. It is not required that the region be a rectangle. The result represents a considerable improvement over the successive overrelaxation method (SOR method) which requires $O(h^{-1})$ iterations.

2. A MODEL PROBLEM

Let us first consider the following model problem. Given a function g(x,y) defined and continuous on the boundary S of the unit square, find a function u(x,y) continuous in R + S and satisfying Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{2.1}$$

in R. Here R is the interior of the square. The function u(x,y) is required to satisfy on S the condition

$$u(x,y) = g(x,y). \tag{2.2}$$

In order to apply the method of finite differences, we choose a positive number h (the mesh size) such that h^{-1} is an integer. At points (ph,qh), where p and q are integers, inside the square we require that the difference equation

$$\frac{u(x+h,y)+u(x-h,y)-2u(x,y)}{h^2} + \frac{u(x,y+h)+u(x,y-h)-2u(x,y)}{h^2} = 0 \tag{2.3}$$

be satisfied. If we require that (2.2) hold for mesh points on S we obtain a system of N linear algebraic equations with N unknowns, where N is the number of mesh points in R.

In the case h = 1/3 and with the points labelled as in Figure 2.1 we have, upon multiplying by $-h^2$,

$$\begin{bmatrix} 4 & -1 & -1 & 0 \\ -1 & 4 & 0 & -1 \\ -1 & 0 & 4 & -1 \\ 0 & -1 & -1 & 4 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} g_6 + g_{16} \\ g_7 + g_9 \\ g_{13} + g_{15} \\ g_{10} + g_{12} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} \tag{2.4}$$

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