Partial Metrics and Co-continuous Valuations*

Michael A. Bukatin[1] and Svetlana Yu. Shorina[2]

[1] Department of Computer Science, Brandeis University, Waltham, MA 02254, USA; **bukatin@cs.brandeis.edu**; *http://www.cs.brandeis.edu/~bukatin/papers.html*
[2] Faculty of Mechanics and Mathematics, Moscow State University, Moscow, Russia; **sveta@cpm.ru**

Abstract. The existence of deep connections between partial metrics and valuations is well known in domain theory. However, the treatment of non-algebraic continuous Scott domains has been not quite satisfactory so far.

In this paper we return to the continuous normalized valuations μ on the systems of open sets and introduce notions of *co-continuity* ($\{U_i, i \in I\}$ is a filtered system of open sets $\Rightarrow \mu(\mathrm{Int}(\bigcap_{i \in I} U_i)) = \inf_{i \in I} \mu(U_i)$) and *strong non-degeneracy* ($U \subset V$ are open sets $\Rightarrow \mu(U) < \mu(V)$) for such valuations. We call the resulting class of valuations CC-valuations. The first central result of this paper is a construction of CC-valuations for Scott topologies on all continuous dcpo's with countable bases. This is a surprising result because neither co-continuous, nor strongly non-degenerate valuations are usually possible for ordinary Hausdorff topologies.

Another central result is a new construction of partial metrics. Given a continuous Scott domain A and a CC-valuation μ on the system of Scott open subsets of A, we construct a continuous partial metric on A yielding the Scott topology as $u(x, y) = \mu(A \setminus (C_x \cap C_y)) - \mu(I_x \cap I_y)$, where $C_x = \{y \in A \mid y \sqsubseteq x\}$ and $I_x = \{y \in A \mid \{x, y\} \text{ is unbounded}\}$. This construction covers important cases based on the real line and allows to obtain an induced metric on $Total(A)$ without the unpleasant restrictions known from earlier work.

1 Introduction

Recently the theory of *partial metrics* introduced by Matthews [14] undergoes active development and is used in various applications from computational description of metric spaces [9] to the analysis of parallel computation [13]. The relationship between partial metrics and *valuations* was first noticed by O'Neill in [15].

In [3] Bukatin and Scott generalized this relationship by considering valuations on powersets of bases, instead of valuations on the domains themselves, as in [15]. They also explained the computational intuition of partial metrics by generalizing them to *relaxed metrics*, which take values in the *interval numbers*.

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Partial metrics can be considered as taking values in the *upper bounds* of those interval numbers. However it is often desirable to remove the most restrictive axioms of partial metrics, like *small self-distances*, $u(x,x) \leq u(x,y)$, and strong *Vickers-Matthews triangle inequality*, $u(x,z) \leq u(x,y)+u(y,z)-u(y,y)$. Thus [3] only requires symmetry and the ordinary triangle inequality for the upper bounds of relaxed metrics.

However, it can be shown (see Section 6) that if the upper bounds $u(x,y)$ of relaxed metrics are based on the idea that common information, or more precisely, *measure of common information* about x and y, brings *negative contribution* to $u(x,y)$ — e.g. in the normalized world we can consider $u(x,y) = 1 - \mu(Info(x) \cap Info(y))$ — then all axioms of partial metrics should hold for u. In fact, it makes sense to introduce both positive and negative information, and to define $u(x,y) = 1 - \mu(Info(x) \cap Info(y)) - \mu(Neginfo(x) \cap Neginfo(y))$, then defining meaningful lower bounds $l(x,y) = \mu(Info(x) \cap Neginfo(y)) + \mu(Info(y) \cap Neginfo(x))$ and obtaining an induced metric on $Total(A)$.

This is, essentially, the approach of Section 5 of [3], where $Info(x)$ and $Neginfo(x)$ can be understood as subsets of a domain basis. However, there was a number of remaining *open problems*. In particular, while [3] builds partial metrics on all continuous Scott domains with countable bases, the reliance of [3] on finite weights of non-compact basic elements does not allow to obtain some natural partial metrics on real-line based domains, and also introduces some unpleasant restrictions on domains which should be satisfied in order to obtain an induced classical metric on $Total(A)$.

1.1 Co-continuous Valuations

This paper rectifies these particular open problems by defining partial metrics via *valuations on the systems of Scott open sets* of domains. The theory of valuations on open sets underwent a considerable development recently (see [5, 11, 18, 2] and references therein). However we have found that we need a special condition of *co-continuity* for our valuations — for a filtered system of open sets $\{U_i, i \in I\}$, $\mu(\text{Int}(\bigcap_{i \in I} U_i)) = \inf_{i \in I}(\mu(U_i))$. We need this condition to ensure Scott continuity of our partial metrics.

The paper starts as follows. In Section 2 we remind the necessary definitions of domain theory. Section 3 defines various properties of valuations and introduces the class of *CC-valuations* — continuous, normalized, strongly non-degenerate, co-continuous valuations. Section 4 builds a CC-valuation on the system of Scott open sets of every continuous dcpo with a countable basis. This is the first central result of this paper.

It seems that the notion of co-continuity of valuations and this result for the case of continuous Scott domains with countable bases are both new and belong to us. The generalization of this result to continuous dcpo's with countable bases belongs to Klaus Keimel [12]. He worked directly with *completely distributive lattices* of Scott open sets of continuous dcpo's and used the results about completely distributive lattices obtained by Raney in the fifties (see Exercise 2.30

on page 204 of [8]). Here we present a proof which can be considered a simplification of both our original proof and the proof obtained by Keimel. This proof also works for all continuous dcpo's with countable bases. A part of this proof, as predicted by Keimel, can be considered as a special case of Raney's results mentioned above. However, our construction is very simple and self-contained.

Keimel also pointed out in [12] that our results are quite surprising, because both co-continuity and *strong non-degeneracy*, $U \subset V$ are open sets $\Rightarrow \mu(U) < \mu(V)$, seem contradictory, as neither of them can hold for the system of open sets of the ordinary Hausdorff topology on $[0, 1]$. However, if we replace the system of open sets of this Hausdorff topology with the system of open intervals, both conditions would hold. We believe that the reason behind our results is that the Scott topology is coarse enough for its system of open sets to exhibit behaviors similar to the behaviors of typical *bases of open sets* of Hausdorff topologies.

1.2 Application to Partial Metrics

Section 5 discusses partial and relaxed metrics and their properties. Section 6 describes an approach to partial and relaxed metrics where the upper bounds $u(x, y)$ are based on the idea of common information about x and y bringing negative contribution to $u(x, y)$. We formalize this approach introducing the notion of μ*Info*-structure. However, we feel that this formalization can be further improved.

In particular, Section 6 presents the second central result of this paper — given a CC-valuation on the system of Scott open sets of any continuous Scott domain (no assumptions about the cardinality of the basis are needed here), we build a Scott continuous relaxed metric $\langle l, u\rangle : A \times A \to \mathbf{R}^I$, such that $u : A \times A \to \mathbf{R}^-$ is a partial metric, the relaxed metric topology coincides with the Scott topology, and if $x, y \in Total(A)$, $l(x, y) = u(x, y)$ and the resulting classical metric $Total(A) \times Total(A) \to \mathbf{R}$ defines a subspace topology on $Total(A)$. Here $\mathbf{R}^I$ is the domain of interval numbers, $\mathbf{R}^-$ is the domain of upper bounds, and $Total(A)$ is the set of maximal elements of A.

Section 7 discusses various examples and possibilities to weaken the strong non-degeneracy condition — to find a sufficiently general weaker condition is an open problem.

A more detailed presentation can be found in [4].

2 Continuous Scott Domains

Recall that a non-empty partially ordered set (poset), $(S, \sqsubseteq)$, is *directed* if $\forall x, y \in S.\ \exists z \in S.\ x \sqsubseteq z, y \sqsubseteq z$. A poset, $(A, \sqsubseteq)$, is a *dcpo* if it has a least element, $\perp$, and for any directed $S \subseteq A$, the least upper bound $\sqcup S$ of S exists in A. A set $U \subseteq A$ is *Scott open* if $\forall x, y \in A.\ x \in U, x \sqsubseteq y \Rightarrow y \in U$ and for any directed poset $S \subseteq A$, $\sqcup S \in U \Rightarrow \exists s \in S.\ s \in U$. The Scott open subsets of a dcpo form the *Scott topology*.

Consider dcpo's $(A, \sqsubseteq_A)$ and $(B, \sqsubseteq_B)$ with the respective Scott topologies. $f : A \to B$ is (Scott) continuous iff it is monotonic ($x \sqsubseteq_A y \Rightarrow f(x) \sqsubseteq_B f(y)$) and for any directed poset $S \subseteq A$, $f(\sqcup_A S) = \sqcup_B \{f(s) \mid s \in S\}$.

We define continuous Scott domains in the spirit of [10]. Consider a dcpo $(A, \sqsubseteq)$. We say that $x \ll y$ (*x is way below y*) if for any directed set $S \subseteq A$, $y \sqsubseteq \sqcup S \Rightarrow \exists s \in S.\ x \sqsubseteq s$. An element x, such that $x \ll x$, is called *compact*. We say that A is *bounded complete* if $\forall B \subseteq A.\ (\exists a \in A.\ \forall b \in B. b \sqsubseteq a) \Rightarrow \sqcup_A B$ exists.

Consider a set $K \subseteq A$. Notice that $\perp_A \in K$. We say that a dcpo A is a *continuous dcpo* with *basis* K, if for any $a \in A$, the set $K_a = \{k \in K \mid k \ll a\}$ is directed and $a = \sqcup K_a$. We call elements of K *basic* elements. A continuous, bounded complete dcpo is called a *continuous Scott domain*.

3 CC-valuations

Consider a topological space $(X, \mathcal{O})$, where $\mathcal{O}$ consists of all open subsets of X. The following notions of the theory of valuations can be considered standard (for the most available presentation in a regular journal see [5]; the fundamental text in the theory of valuations on Scott opens sets is [11]).

Definition 3.1. A function $\mu : \mathcal{O} \to [0, +\infty]$ is called *valuation* if

1. $\forall U, V \in \mathcal{O}.\ U \subseteq V \Rightarrow \mu(U) \leq \mu(V)$;
2. $\forall U, V \in \mathcal{O}.\ \mu(U) + \mu(V) = \mu(U \cap V) + \mu(U \cup V)$;
3. $\mu(\emptyset) = 0$.

Definition 3.2. A valuation μ is *bounded* if $\mu(X) < +\infty$. A valuation μ is *normalized* if $\mu(X) = 1$.

Remark: If a valuation μ is bounded and $\mu(X) \neq 0$, then it is always easy to replace it with a normalized valuation $\mu'(U) = \mu(U)/\mu(X)$.

Definition 3.3. Define a *directed system of open sets*, $\mathcal{U} = \{U_i, i \in I\}$, as satisfying the following condition: for any finite number of open sets $U_{i_1}, U_{i_2}, \cdots, U_{i_n} \in \mathcal{U}$ there is $U_i, i \in I$, such that $U_{i_1} \subseteq U_i, \cdots, U_{i_n} \subseteq U_i$.

Definition 3.4. A valuation μ is called *continuous* when for any directed system of open sets $\mu(\bigcup_{i \in I} U_i) = \sup_{i \in I} \mu(U_i)$.

We introduce two new properties of valuations.

Definition 3.5. A valuation $\mu : \mathcal{O} \to [0, +\infty]$ is *strongly non-degenerate* if $\forall U, V \in \mathcal{O}.\ U \subset V \Rightarrow \mu(U) < \mu(V)$.[3]

This is, obviously, a very strong requirement, and we will see later that it might be reasonable to look for weaker non-degeneracy conditions.

Consider a decreasing sequence of open sets $U_1 \supseteq U_2 \supseteq \ldots$, or, more generally, a *filtered system of open sets* $\mathcal{U} = \{U_i, i \in I\}$, meaning that for any finite system of open sets $U_{i_1}, \cdots U_{i_n} \in \mathcal{U}$ there is $U_i, i \in I$, such that $U_i \subseteq$

[3] We use $U \subset V$ as an equivalent of $U \subseteq V$ & $U \neq V$.

$U_{i_1}, \cdots, U_i \subseteq U_{i_n}$. Consider the interior of the intersection of these sets. It is easy to see that for a valuation μ

$$\mu(\mathrm{Int}(\bigcap_{i \in I} U_i)) \leq \inf_{i \in I} \mu(U_i).$$

Definition 3.6. A valuation μ is called *co-continuous* if for any filtered system of open sets $\{U_i, i \in I\}$

$$\mu(\mathrm{Int}(\bigcap_{i \in I} U_i)) = \inf_{i \in I} \mu(U_i).$$

Definition 3.7. A continuous, normalized, strongly non-degenerate, co-continuous valuation μ is called a *CC-valuation*.

Informally speaking, the strong non-degeneracy provides for non-zero contributions of compact elements and reasonable "pieces of space". The co-continuity provides for single non-compact elements and borders $B \setminus \mathrm{Int}(B)$ of "reasonable" sets $B \subseteq A$ to have zero measures.

"Reasonable" sets here are Alexandrov open (i.e. upwardly closed) sets. Thus, it is possible to consider co-continuity as a method of dealing with non-discreteness of Scott topology. We follow here the remarkable definition of a discrete topology given by Alexandrov: a topology is discrete if an intersection of arbitrary family of open sets is open (e.g. see [1]). Of course, if one assumes the T_1 separation axiom, then the Alexandrov's definition implies that all sets are open — the trivial (and more standard) version of the definition. In this sense, Alexandrov topology of upwardly closed sets is discrete, but Scott topology is not.

We should also notice that since our valuations are bounded, they can be extended onto closed sets via formula $\mu(C) = \mu(A) - \mu(A \setminus C)$, and all definitions of this section can be expressed in the dual form.

A bounded valuation μ can be uniquely extended to an additive measure defined on the ring of sets generated from the open sets by operations $\cap$, $\cup$, $\setminus$ [16]. The issues of σ-additivity are not in the scope of this text (interested readers are referred to [11, 2]). We deal with the specific infinite systems of sets we need, and mainly focus on quite orthogonal conditions given to us by co-continuity of μ.

3.1 Example: Valuations Based on Weights of Basic Elements

This example essentially reproduces a construction in [3]. Consider a continuous dcpo A with a countable basis K. Assign a converging system of weights to basic elements: $w(k) > 0$, $\sum_{k \in K} w(k) = 1$. Define $\mu(U) = \sum_{k \in U} w(k)$. It is easy to see that μ is a continuous, normalized, strongly non-degenerate valuation.

However, μ is co-continuous if and only if all basic elements are compact (which is possible only if A is algebraic). This is proved in [4] using the following observations.

First, observe that arbitrary intersections of Alexandrov open (i.e. upwardly closed) sets are Alexandrov open. Also it is a well-known fact that $\{y \mid x \ll y\}$ is Scott open in a continuous dcpo.

Lemma 3.1 (Border Lemma) *Consider an Alexandrov open set* $B \subseteq A$. *Then its interior in the Scott topology,* $\mathrm{Int}(B) = \{y \in A \mid \exists x \in B.\, x \ll y\}$. *Correspondingly, the border of* B *in the Scott topology,* $B \setminus \mathrm{Int}(B) = \{y \in B \mid \neg(\exists x \in B.\, x \ll y)\}$

3.2 A Vertical Segment of Real Line

Consider the segment $[0, 1]$, $\sqsubseteq = \leq$. Define $\mu((x, 1]) = 1 - x$. Unfortunately, to ensure strong non-degeneracy we have to define $\mu([0, 1]) = 1 + \epsilon$, $\epsilon > 0$. This is the first hint that strong non-degeneracy is too strong in many cases. In order to obtain a normalized valuation we have to consider $\mu'(U) = \mu(U)/(1+\epsilon)$. The resulting μ' is a CC-valuation.

4 Constructing CC-valuations

In this section we build a CC-valuation for all continuous dcpo's with countable bases. The construction generalizes the one of Subsection 3.1. We are still going to assign weights, $w(k) > 0$, to compact elements. For non-compact basic elements we proceed as follows. We focus our attention on the pairs of non-compact basic elements, (k', k''), which do not have any compact elements between them, and call such elements *continuously connected.* We observe, that for every such pair we can construct a special kind of vertical chain, which "behaves like a vertical segment $[0, 1]$ of real line". We call such chain a *stick.* We assign weights, $v(k', k'') > 0$, to sticks as well, in such a way that the sum of all $w(k)$ and all $v(k', k'')$ is 1.

As in Subsection 3.1, compact elements k contribute $w(k)$ to $\mu(U)$, if $k \in U$. An intersection of the stick, associated with a continuously connected pair (k', k''), with an open set U "behaves as either $(q, 1]$ or $[q, 1]$", where $q \in [0, 1]$. Such stick contributes $(1 - q) \cdot v(k', k'')$ to $\mu(U)$. The resulting μ is the desired CC-valuation.

It is possible to associate a complete lattice homomorphism from the lattice of Scott open sets to $[0, 1]$ with every compact element and with every stick defined by basic continuously connected elements, k' and k''. Then, as suggested by Keimel [12], all these homomorphisms together can be thought of as an injective complete lattice homomorphism to $[0, 1]^J$. From this point of view, our construction of μ is the same as in [12].

Thus the discourse in this section yields the proof of the following:

Theorem 4.1 *For any continuous dcpo* A *with a countable basis, there is a CC-valuation* μ *on the system of its Scott open sets.*

4.1 Continuous Connectivity and Sticks

Definition 4.1. Two elements $x \ll y$ are called *continuously connected* if the set $\{k \in A | k \text{ is compact}, x \ll k \ll y\}$ is empty.

Remark: This implies that x and y are not compact.

Lemma 4.1 *If $x \ll y$ are continuously connected, then $\{z \,|\, x \ll z \ll y\}$ has cardinality of at least continuum.*

Proof. We use the well-known theorem on intermediate values that $x \ll y \Rightarrow \exists z \in A\ x \ll z \ll y$ (see [10]). Applying this theorem again and again we build a countable system of elements between x and y as follows, using rational numbers as indices for intermediate elements:

$$x \ll a_{1/2} \ll y, \quad x \ll a_{1/4} \ll a_{1/2} \ll a_{3/4} \ll y, \dots$$

All these elements are non-compact and hence non-equal. Now consider a directed set $\{a_i \,|\, i \le r\}$, where r is a real number, $0 < r < 1$. Introduce $b_r = \sqcup\{a_i \,|\, i \le r\}$. We prove that if $r < s$ then $b_r \ll b_s$, and also that $x \ll b_r \ll b_s \ll y$, thus obtaining the required cardinality. Indeed it is easy to find such n and numbers q_1, q_2, q_3, q_4, that

$$x \ll a_{q_1/2^n} \sqsubseteq b_r \sqsubseteq a_{q_2/2^n} \ll a_{q_3/2^n} \sqsubseteq b_s \ll a_{q_4/2^n} \ll y$$

□

Definition 4.2. We call the set of continuum different non-compact elements $\{a_r \,|\, r \in (0,1)\}$ between continuously connected $x \ll y$, built in the proof above, such that $x \ll a_r \ll a_q \ll z \Leftrightarrow r < q$ a (vertical) *stick*.

4.2 Proof of Theorem 4.1

Consider a continuous dcpo A with a countable basis K. As discussed earlier, with every compact $k \in K$ we associate weight $w(k) > 0$, and with every continuously connected pair (k', k''), $k', k'' \in K$, we associate weight $v(k', k'') > 0$ and a stick $\{a_r^{k',k''} \mid r \in (0,1)\}$. Since K is countable, we can require $\sum w(k) + \sum v(k', k'') = 1$.

Whenever we have an upwardly closed (i.e. Alexandrov open) set U, for any stick $\{a_r^{k',k''} \mid r \in (0,1)\}$ there is a number $q_U^{k',k''} \in [0,1]$, such that $r < q_U^{k',k''} \Rightarrow a_r^{k',k''} \notin U$ and $q_U^{k',k''} < r \Rightarrow a_r^{k',k''} \in U$. In particular, for a Scott open set U define

$$\mu(U) = \sum_{k \in U \text{is compact}} w(k) + \sum_{k',k'' \in K \text{are continuously connected}} (1 - q_U^{k',k''}) \cdot v(k', k'')$$

It is easy to show that μ is a normalized valuation. The rest follows from the following Lemmas.

Lemma 4.2 *μ is continuous.*

Lemma 4.3 *μ is strongly non-degenerate.*

Proof. Let U and V be Scott open subsets of A and $U \subset V$. Let us prove that $V \setminus U$ contains either a compact element or a stick between basic elements. Take $x \in V \setminus U$. If x is compact, then we are fine. Assume that x is not compact. We know that $x = \sqcup K_x$, $K_x = \{k \in K \mid k \ll x\}$ is directed set. Since V is open $\exists k \in K_x.\ k \in V$. Since $k \sqsubseteq x$ and $x \notin U$, $k \in V \setminus U$. If there is k' – compact, such that $k \ll k' \ll x$, we are fine, since $k' \in V \setminus U$. Otherwise, since any basis includes all compact elements, k and x are continuously connected.

Now, as in the theorem of intermediate values $x = \sqcup \widetilde{K}_x$, $\widetilde{K}_x = \{k' \in K \mid \exists k'' \in K.\ k' \ll k'' \ll x\}$ is directed set, thus $\exists k' k''.\ k \sqsubseteq k' \ll k'' \ll x$, thus (k, k'') yields the desired stick.

If $k \in V \setminus U$ and k is compact, then $\mu(V) - \mu(U) \geq w(k) > 0$. If the stick formed by (k, k') is in $V \setminus U$, then $\mu(V) - \mu(U) \geq v(k, k') > 0$.

□

Lemma 4.4 *μ is co-continuous.*

Proof. Recall the development in Subsection 3.1. Consider a filtered system of open sets $\{U_i,\ i \in I\}$. By Lemma 3.1 for $B = \bigcap_{i \in I} U_i$, $B \setminus \text{Int}(B) = \{y \in B \mid \neg(\exists x \in B.\ x \ll y)\}$. Notice that $B \setminus \text{Int}(B)$, in particular, does not contain compact elements. Another important point is that for any stick, $q_B^{k',k''} = q_{\text{Int}(B)}^{k',k''}$.

The further development is essentially dual to the omitted proof of Lemma 4.2. We need to show that for any $\epsilon > 0$, there is such $U_i, i \in I$, that $\mu(U_i) - \mu(\text{Int}(B)) < \epsilon$.

Take enough (a finite number) of compact elements, $k_1, \ldots, k_n$, and continuously connected pairs of basic elements, $(k'_1, k''_1), \ldots, (k'_m, k''_m)$, so that $w(k_1) + \ldots + w(k_n) + v(k'_1, k''_1) + \ldots + v(k'_m, k''_m) > 1 - \epsilon/2$. For each $k_j \notin \text{Int}(B)$, take $U_{i_j}, i_j \in I$, such that $k_j \notin U_{i_j}$. For each (k'_j, k''_j), such that $q_{\text{Int}(B)}^{k'_j,k''_j} > 0$, take $U_{i'_j}, i'_j \in I$, such that $q_{\text{Int}(B)}^{k'_j,k''_j} - q_{U_{i'_j}}^{k'_j,k''_j} < \epsilon/(2m)$. A lower bound of these U_{i_j} and $U_{i'_j}$ is the desired U_i.

□

It should be noted that Bob Flagg suggested and Klaus Keimel showed that Lemma 5.3 of [7] can be adapted to obtain a dual proof of existence of CC-valuations (see [6] for one presentation of this). Klaus Keimel also noted that one can consider all pairs k, k' of basic elements, such that $k \ll k'$, instead of considering just continuously connected pairs and compact elements.

5 Partial and Relaxed Metrics on Domains

The motivations behind the notion of relaxed metric, its computational meaning and its relationships with partial metrics [14] were explained in [3]. Here we focus on the definitions and basic properties, revisit the issue of specific axioms of partial metrics, and list the relevant open problems.

The distance domain consists of pairs $\langle a, b\rangle$ (also denoted as $[a, b]$) of non-negative reals ($+\infty$ included), such that $a \le b$. We denote this domain as R^I. $[a, b] \sqsubseteq_{R^I} [c, d]$ iff $a \le c$ and $d \le b$.

We can also think about R^I as a subset of $R^+ \times R^-$, where $\sqsubseteq_{R^+} = \le$, $\sqsubseteq_{R^-} = \ge$, and both R^+ and R^- consist of non-negative reals and $+\infty$. We call R^+ a *domain of lower bounds*, and R^- a *domain of upper bounds*. Thus a distance function $\rho : A \times A \to R^I$ can be thought of as a pair of distance functions $\langle l, u\rangle$, $l : A \times A \to R^+$, $u : A \times A \to R^-$.

Definition 5.1. A symmetric function $u : A \times A \to R^-$ is called a *relaxed metric* when it satisfies the triangle inequality. A symmetric function $\rho : A \times A \to R^I$ is called a *relaxed metric* when its upper part u is a relaxed metric.

An *open ball* with a center $x \in A$ and a real radius ϵ is defined as $B_{x,\epsilon} = \{y \in A \mid u(x, y) < \epsilon\}$. Notice that only upper bounds are used in this definition — the ball only includes those points y, about which we are *sure* that they are not too far from x.

We should formulate the notion of a relaxed metric open set more carefully than for ordinary metrics, because it is now possible to have a ball of a non-zero positive radius, which does not contain its own center.

Definition 5.2. A subset U of A is *relaxed metric open* if for any point $x \in U$, there is an $\epsilon > u(x, x)$ such that $B_{x,\epsilon} \subseteq U$.

It is easy to show that for a continuous relaxed metric on a dcpo all relaxed metric open sets are Scott open and form a topology.

5.1 Partial Metrics

The distances p with $p(x, x) \neq 0$ were first introduced by Matthews [14, 13]. They are known as *partial metrics* and obey the following axioms:

1. $x = y$ iff $p(x, x) = p(x, y) = p(y, y)$.
2. $p(x, x) \le p(x, y)$.
3. $p(x, y) = p(y, x)$.
4. $p(x, z) \le p(x, y) + p(y, z) - p(y, y)$.

Whenever partial metrics are used to describe a partially ordered domain, a stronger form of the first two axioms is used: If $x \sqsubseteq y$ then $p(x, x) = p(x, y)$, otherwise $p(x, x) < p(x, y)$. We include the stronger form in the definition of partial metrics for the purposes of this paper.

Section 8.1 of [3] discusses the issue of whether axioms $u(x, x) \le u(x, y)$ and $u(x, z) \le u(x, y) + u(y, z) - u(y, y)$ should hold for the upper bounds of relaxed metrics. In particular, the approach in this paper is based on $u(x, y) = 1 - \mu(\text{Common information between } x \text{ and } y)$ and thus, as will be explained in details in the next section, the axioms of partial metrics hold. Further discussion of the utilitarian value of these axioms can be found in [4].

6 Partial and Relaxed Metrics via Information

6.1 μ*Info*-structures

Some of the earlier known constructions of partial metrics can be understood via the mechanism of *common information* between elements x and y bringing negative contribution to $u(x, y)$ (see [3, Section 8]). This can be further formalized as follows. Assume that there is a set $\mathcal{I}$ representing information about elements of a dcpo A. We choose a ring, $\mathcal{M}(\mathcal{I})$, of admissible subsets of $\mathcal{I}$ and introduce a measure-like structure, μ, on $\mathcal{M}(\mathcal{I})$. We associate a set, $Info(x) \in \mathcal{M}(\mathcal{I})$, with every $x \in A$, and call $Info(x)$ a set of (positive) information about x. We also would like to consider negative information about x, $Neginfo(x) \in \mathcal{M}(\mathcal{I})$, — intuitively speaking, this is information which cannot become true about x, when x is arbitrarily increased.

Definition 6.1. Given a dcpo A, the tuple of $(A, \mathcal{I}, \mathcal{M}(\mathcal{I}), \mu, Info, Neginfo)$ is called a *μInfo-structure* on A, if $\mathcal{M}(\mathcal{I}) \subseteq \mathcal{P}(\mathcal{I})$ — a ring of subsets closed with respect to $\cap$, $\cup$, $\setminus$ and including $\emptyset$ and $\mathcal{I}$, $\mu : \mathcal{M}(\mathcal{I}) \to [0, 1]$, $Info : A \to \mathcal{M}(\mathcal{I})$, and $Neginfo : A \to \mathcal{M}(\mathcal{I})$, and the following axioms are satisfied:

1. **(VALUATION AXIOMS)**
   (a) $\mu(\mathcal{I}) = 1$, $\mu(\emptyset) = 0$;
   (b) $U \subseteq V \Rightarrow \mu(U) \leq \mu(V)$;
   (c) $\mu(U) + \mu(V) = \mu(U \cap V) + \mu(U \cup V)$;
2. **(*Info* AXIOMS)**
   (a) $x \sqsubseteq y \Leftrightarrow Info(x) \subseteq Info(y)$;
   (b) $x \sqsubset y \Rightarrow Info(x) \subset Info(y)$;
3. **(*Neginfo* AXIOMS)**
   (a) $Info(x) \cap Neginfo(x) = \emptyset$;
   (b) $x \sqsubseteq y \Rightarrow Neginfo(x) \subseteq Neginfo(y)$;
4. **(STRONG RESPECT FOR TOTALITY)**
   $x \in Total(A) \Rightarrow Info(x) \cup Neginfo(x) = \mathcal{I}$;
5. **(CONTINUITY OF INDUCED RELAXED METRIC)**
   if B is a directed subset of A and $y \in A$, then
   (a) $\mu(Info(\sqcup B) \cap Info(y)) = sup_{x \in B}(\mu(Info(x) \cap Info(y))$,
   (b) $\mu(Info(\sqcup B) \cap Neginfo(y)) = sup_{x \in B}(\mu(Info(x) \cap Neginfo(y))$,
   (c) $\mu(Neginfo(\sqcup B) \cap Info(y)) = sup_{x \in B}(\mu(Neginfo(x) \cap Info(y))$,
   (d) $\mu(Neginfo(\sqcup B) \cap Neginfo(y)) = sup_{x \in B}(\mu(Neginfo(x) \cap Neginfo(y))$;
6. **(SCOTT OPEN SETS ARE RELAXED METRIC OPEN)**
   for any (basic) Scott open set $U \subseteq A$ and $x \in U$, there is an $\epsilon > 0$, such that $\forall y \in A.\ \mu(Info(x)) - \mu(Info(x) \cap Info(y)) < \epsilon \Rightarrow y \in U$.

In terms of lattice theory, μ is a (normalized) valuation on a lattice $\mathcal{M}(\mathcal{I})$. The consideration of unbounded measures is beyond the scope of this paper, and $\mu(\mathcal{I}) = 1$ is assumed for convenience. Axioms relating $\sqsubseteq$ and *Info* are in the spirit of information systems [17], although we are not considering any inference structure over $\mathcal{I}$ in this paper.

The requirements for negative information are relatively weak, because it is quite natural to have $\forall x \in A.\ Neginfo(x) = \emptyset$ if A has a top element.

The axiom that for $x \in Total(A)$, $Info(x) \cup Neginfo(x) = \mathcal{I}$, is desirable because indeed, if some $i \in \mathcal{I}$ does not belong to $Info(x)$ and x can not be further increased, then by our intuition behind $Neginfo(x)$, i should belong to $Neginfo(x)$. However, this axiom might be too strong and will be further discussed later.

The last two axioms are not quite satisfactory — they almost immediately imply the properties, after which they are named, but they are complicated and might be difficult to establish. We hope, that these axioms will be replaced by something more tractable in the future. One of the obstacles seems to be the fact in some valuable approaches (in particular, in this paper) it is not correct that $x_1 \sqsubseteq x_2 \sqsubseteq \cdots$ implies that $Info(\sqcup_{i \in \mathbf{N}} x_i) = \bigcup_{i \in \mathbf{N}} Info(x_i)$.

The nature of these set-theoretical representations, $\mathcal{I}$, of domains may vary: one can consider sets of tokens of information systems, powersets of domain bases, or powersets of domains themselves, custom-made sets for specific domains, etc. The approach via powersets of domain bases (see [3]) can be thought of as a partial case of the approach via powersets of domains themselves adopted in the present paper.

6.2 Partial and Relaxed Metrics via $\mu Info$-structures

Define the (upper estimate of the) distance between x and y from A as $u : A \times A \to \mathbf{R}^-$:

$$u(x, y) = 1 - \mu(Info(x) \cap Info(y)) - \mu(Neginfo(x) \cap Neginfo(y)).$$

I.e. the more information x and y have in common the smaller is the distance between them. However a partially defined element might not have too much information at all, so its self-distance $u(x, x) = 1 - \mu(Info(x)) - \mu(Neginfo(x))$ might be large.

It is possible to find information which will never belong to $Info(x) \cap Info(y)$ or $Neginfo(x) \cap Neginfo(y)$ even when x and y are arbitrarily increased. In particular, $Info(x) \cap Neginfo(y)$ and $Info(y) \cap Neginfo(x)$ represent such information. Then we can introduce the lower estimate of the distance $l : A \times A \to \mathbf{R}^+$:

$$l(x, y) = \mu(Info(x) \cap Neginfo(y)) + \mu(Info(y) \cap Neginfo(x)).$$

The proof of Lemma 9 of [3] is directly applicable and yields $l(x, y) \leq u(x, y)$. Thus we can form an **induced relaxed metric**, $\rho : A \times A \to R^I$, $\rho = \langle l, u \rangle$, with a meaningful lower bound.

The following theorem is proved in [4] without using the **strong respect for totality** axiom.

Theorem 6.1 *Function u is a partial metric. Function ρ is a continuous relaxed metric. The relaxed metric topology coincides with the Scott topology.*

Due to the axiom $\forall x \in Total(A).\ Info(x) \cup Neginfo(x) = \mathcal{I}$, the proof of Lemma 10 of [3] would go through, yielding

$$x, y \in Total(A) \Rightarrow l(x, y) = u(x, y)$$

and allowing to obtain the following theorem (cf. Theorem 8 of [3]).

Theorem 6.2 *For all x and y from $Total(A)$, $l(x, y) = u(x, y)$. Consider $d : Total(A) \times Total(A) \to \mathbf{R}$, $d(x, y) = l(x, y) = u(x, y)$. Then $(Total(A), d)$ is a metric space, and its metric topology is the subspace topology induced by the Scott topology on A.*

However, in [3] $x \in Total(A) \Rightarrow Info(x) \cup Neginfo(x) = \mathcal{I}$ holds under an awkward condition, the regularity of the basis. While bases of algebraic Scott domains and of continuous lattices can be made regular, there are important continuous Scott domains, which cannot be given regular bases. In particular, in $\mathbf{R}^I$ no element, except for $\perp$, satisfies the condition of regularity, hence a regular basis cannot be provided for $\mathbf{R}^I$.

The achievement of the construction to be described in Section 6.4 is that by removing the reliance on the weights of non-compact basic elements, it eliminates the regularity requirement and implies $x \in Total(A) \Rightarrow Info(x) \cup Neginfo(x) = \mathcal{I}$ for all continuous Scott domains equipped with a CC-valuation (which is built above for all continuous Scott domains with countable bases) where $Info(x)$ and $Neginfo(x)$ are as described below in the Subsection 6.4.

However, it still might be fruitful to consider replacing the axiom $\forall x \in Total(A).\ Info(x) \cup Neginfo(x) = \mathcal{I}$ by something like $\forall x \in Total(A).\ \mu(\mathcal{I} \setminus (Info(x) \cup Neginfo(x))) = 0$.

6.3 A Previously Known Construction

Here we recall a construction from [3] based on a generally non-co-continuous valuation of Subsection 3.1. We will reformulate it in our terms of $\mu Info$-structures. In [3] it was natural to think that $\mathcal{I} = K$. Here we reformulate that construction in terms of $\mathcal{I} = A$, thus abandoning the condition $x \in Total(A) \Rightarrow Info(x) \cup Neginfo(x) = \mathcal{I}$ altogether.

Define $I_x = \{y \in A \mid \{x, y\} \text{ is unbounded}\}$, $P_x = \{y \in A \mid y \ll x\}$ (cf. $I_x = \{k \in K \mid \{k, x\} \text{ is unbounded}\}$, $K_x = \{k \in K \mid k \ll x\}$ in [3]).

Define $Info(x) = P_x$, $Neginfo(x) = I_x$. Consider a valuation μ of Subsection 3.1: for any $S \subseteq \mathcal{I} = A$, $\mu(S) = \sum_{k \in S \cap K} w(k)$. μ is a continuous strongly non-degenerate valuation, but it is not co-continuous unless K consists only of compact elements.

Because of this we cannot replace the inconvenient definition of $Info(x) = P_x$ by $Info(x) = C_x = \{y \in A \mid y \sqsubseteq x\}$ (which would restore the condition $x \in Total(A) \Rightarrow Info(x) \cup Neginfo(x) = A$) as $\mu(C_k)$ would not be equal to $\sup_{k' \ll k} \mu(C_{k'})$ if k is a non-compact basic element, leading to the non-continuity of the partial metric $u(x, y)$.

Also the reliance on countable systems of finite weights excludes such natural partial metrics as metric $u : \mathbf{R}^-_{[0,1]} \times \mathbf{R}^-_{[0,1]} \rightarrow \mathbf{R}^-$, where $\mathbf{R}^-_{[0,1]}$ is the set $[0, 1]$ equipped with the dual partial order $\sqsubseteq = \geq$, and $u(x, y) = max(x, y)$. We rectify all these problems in the next Subsection.

6.4 Partial and Relaxed Metrics via CC-valuations

Assume that there is a CC-valuation $\mu(U)$ on Scott open sets of a domain A. Then it uniquely extends to an additive measure μ on the ring of sets generated by the system of open sets. Define $\mathcal{I} = A$, $Info(x) = C_x$, $Neginfo(x) = I_x$. It is easy to see that valuation, *Info*, and *Neginfo* axioms of $\mu Info$-structure hold. We have $x \in Total(A) \Rightarrow C_x \cup I_x = A$. Thus we only need to establish the axioms of **continuity of induced relaxed metrics** and **Scott open sets are relaxed metric open** in order to prove theorems 6.1 and 6.2 for our induced relaxed metric $(u(x, y) = 1 - \mu(C_x \cap C_y) - \mu(I_x \cap I_y), l(x, y) = \mu(C_x \cap I_y) + \mu(C_y \cap I_x))$. These axioms are established by the Lemmas below.

You will also see that for such bare-bones partial metrics, as $u(x, y) = 1 - \mu(C_x \cap C_y)$, which are nevertheless quite sufficient for topological purposes and for domains with $\top$, only *co-continuity* matters, continuity is not important.

Observe also that since the construction in Section 3.1 does form a CC-valuation for algebraic Scott domains with bases of compact elements, the construction in [3] can be considered as a partial case of our current construction if the basis does not contain non-compact elements.

Lemma 6.1 *Assume that μ is a co-continuous valuation and B is a directed subset of A. Then $\mu(C_{\sqcup B} \cap Q) = \sup_{x \in B}(\mu(C_x \cap Q))$, where Q is a closed or open subset of A.*

Remark: Note that continuity of μ is not required here.

Lemma 6.2 *Assume that μ is a continuous valuation and B is a directed subset of A. Then $\mu(I_{\sqcup B} \cap Q) = \sup_{x \in B}(\mu(I_x \cap Q))$, where Q is an open or closed subset of A.*

Remark: Co-continuity is not needed here.

Lemma 6.3 *Assume that μ is a strongly non-degenerate valuation. Then the $\mu Info$-structure axiom* **Scott open sets are relaxed metric open** *holds.*

Remark: Neither continuity, nor co-continuity required, and even the strong non-degeneracy condition can probably be made weaker (see the next Section).

7 Examples and Non-degeneracy Issues

In this section we show some examples of "nice" partial metrics, based on valuations for vertical and interval domains of real numbers. Some of these valuations are strongly non-degenerate, while others are not, yet all examples are quite natural.

Consider the example from Subsection 3.2. The partial metric, based on the strongly non-degenerate valuation μ' of that example would be $u'(x,y) = (1 - \min(x,y))/(1+\epsilon)$, if $x, y > 0$, and $u'(x,y) = 1$, if x or y equals to 0. However, another nice valuation, μ'', can be defined on the basis of μ of Subsection 3.2: $\mu''((x,1]) = \mu((x,1]) = 1 - x$, $\mu''([0,1]) = 1$. μ'' is not strongly non-degenerate, however it yields the nice partial metric $u''(x,y) = 1 - \min(x,y)$, yielding the Scott topology.

Now we consider several valuations and distances on the domain of interval numbers located within the segment $[0,1]$. This domain can be thought of as a triangle of pairs $\langle x, y \rangle$, $0 \leq x \leq y \leq 1$. Various valuations can either be concentrated on $0 < x \leq y < 1$, or on $x = 0, 0 \leq y \leq 1$ and $y = 1, 0 \leq x \leq 1$, or, to insure non-degeneracy, on both of these areas with an extra weight at $\langle 0, 1 \rangle$.

Among all these measures, the classical partial metric $u([x,y],[x',y']) = \max(y,y') - \min(x,x')$ results from the valuation accumulated at $x = 0, 0 \leq y \leq 1$, and $y = 1, 0 \leq x \leq 1$, namely $\mu(U) = (Length(\{x = 0, 0 \leq y \leq 1\} \cap U) + Length(\{y = 1, 0 \leq x \leq 1\} \cap U))/2$. Partial metrics generated by strongly non-degenerate valuations contain quadratic expressions.

It is our current feeling, that instead of trying to formalize weaker non-degeneracy conditions, it is fruitful to build a $\mu Info$-structure based on $\mathcal{I} = [0,1] + [0,1]$ in situations like this.

8 Conclusion

We introduced notions of co-continuous valuations and CC-valuations, and built CC-valuations for all continuous dcpo's with countable bases. Given such a valuation, we presented a new construction of partial and relaxed metrics for all continuous Scott domains, improving a construction known before.

The key open problem is to learn to construct not just topologically correct, but canonical measures and relaxed metrics for higher-order functional domains and reflexive domains, and also to learn how to compute these measures and metrics quickly.

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References

1. Aleksandrov P.S. *Combinatory Topology*, vol.1, Graylock Press, Rochester, NY, 1956. p.28.
2. Alvarez M., Edalat A., Saheb-Djahromi N. *An extension result for continuous valuations*, 1997, available via URL
`http://theory.doc.ic.ac.uk/people/Edalat/extensionofvaluations.ps.Z`
3. Bukatin M.A., Scott J.S. Towards computing distances between programs via Scott domains. In S. Adian, A. Nerode, eds., Logical Foundations of Computer Science, *Lecture Notes in Computer Science*, **1234**, 33–43, Springer, 1997.
4. Bukatin M.A., Shorina S.Yu. *Partial Metrics and Co-continuous Valuations (Extended Version)*, Unpublished notes, 1997, available via one of the URLs
`http://www.cs.brandeis.edu/~bukatin/ccval_draft.{dvi,ps.gz}`
5. Edalat A. Domain theory and integration. *Theoretical Computer Science*, **151** (1995), 163–193.
6. Flagg R. *Constructing CC-Valuations*, Unpublished notes, 1997. Available via URL `http://macweb.acs.usm.maine.edu/math/archive/flagg/biCts.ps`
7. Flagg R., Kopperman R. Continuity spaces: Reconciling domains and metric spaces. *Theoretical Computer Science*, **177** (1997), 111–138.
8. Gierz G., Hofmann K., Keimel K., Lawson J., Mislove M., Scott D. *A Compendium of Continuous Lattices*, Springer, 1980.
9. Heckmann R. Approximation of metric spaces by partial metric spaces. To appear in *Applied Categorical Structures*, 1997.
10. Hoofman R. Continuous information systems. *Information and Computation*, **105** (1993), 42–71.
11. Jones C. *Probabilistic Non-determinism*, PhD Thesis, University of Edinburgh, 1989. Available via URL
`http://www.dcs.ed.ac.uk/lfcsreps/EXPORT/90/ECS-LFCS-90-105/index.html`
12. Keimel K. *Bi-continuous Valuations*, to appear in the Proceedings of the Third Workshop on Computation and Approximation, University of Birmingham, Sept. 1997. Available via URL
`http://theory.doc.ic.ac.uk/forum/comprox/data/talk.3.1.6.ps.gz`
13. Matthews S.G. An extensional treatment of lazy data flow deadlock. *Theoretical Computer Science*, **151** (1995), 195–205.
14. Matthews S.G. Partial metric topology. In S. Andima et al., eds., Proc. 8th Summer Conference on General Topology and Applications, *Annals of the New York Academy of Sciences*, **728**, 183–197, New York, 1994.
15. O'Neill S.J. Partial metrics, valuations and domain theory. In S. Andima et al., eds., Proc. 11th Summer Conference on General Topology and Applications, *Annals of the New York Academy of Sciences*, **806**, 304–315, New York, 1997.
16. Pettis B.J. On the extension of measures. *Annals of Mathematics*, **54** (1951), 186-197.
17. Scott D.S. Domains for denotational semantics. In M. Nielsen, E. M. Schmidt, eds., Automata, Languages, and Programming, *Lecture Notes in Computer Science*, **140**, 577–613, Springer, 1982.
18. Tix R. *Stetige Bewertungen auf topologischen Räumen*, (Continuous Valuations on Topological Spaces, in German), Diploma Thesis, Darmstadt Institute of Technology, 1995. Available via URL
`http://www.mathematik.th-darmstadt.de/ags/ag14/papers/papers.html`