Symmetry Reductions in Model Checking *

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Abstract. The use of symmetry to alleviate state-explosion problems during model-checking has become a important research topic. This paper investigates several problems which are important to techniques exploiting symmetry. The most important of these problems is the *orbit problem*. We prove that the orbit problem is equivalent to an important problem in computational group theory which is at least as hard as the graph isomorphism but not known to be *NP*-complete. This paper also shows classes of commonly occurring groups for which the orbit problem is easy. Some methods of deriving symmetry for a shared variable model of concurrent programs are also investigated. Experimental results providing evidence of reduction in state space by using symmetry are also provided.

1 Introduction

Temporal Logic Model Checking is a technique for determining whether a temporal logic formula is valid in a finite state system $M = (S, R, L)$, where S is the state space, R is the state transition relation, and L is a function that labels states with sets of atomic propositions [4]. Such a structure is usually called a *Kripke structure* and may have an enormous number of states because of the *state explosion* problem. A system with n boolean state variables can have a state space which is exponential in n. An efficient Model Checking procedure tries to reduce the number of states that are actually explored [2, 4]. Recently, techniques which exploit the inherent symmetry of the system while performing model checking have become quite popular [3, 6, 11].

Basically, the idea of exploiting symmetry is the following: given a Kripke Structure $M = (S, R, L)$, a symmetry group G is a group acting on the state set S that *preserves* the transition relation R, G partitions the state set S into equivalence classes called *orbits*. A quotient model M_G is constructed that contains one or more representative from each orbit. The state space S_G of the

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quotient model will, in general, be much smaller than the original state space S. This makes it possible to verify much larger structures.

This paper investigates several problems associated with exploiting symmetry. For example, given a group G and two states s and s', the *orbit problem* asks whether s and s' are in the same orbit. Determining whether two states are in the same orbit is at the core of any model checking procedure exploiting symmetry. We prove that the orbit problem is equivalent to an important problem in computational group theory. We explore ways of deriving symmetries of shared variable concurrent programs. Since the orbit problem in its full generality is quite hard, this paper also shows that the orbit problem is easy for certain commonly occurring groups and provides alternative techniques which do not require solving the orbit problem. We have also built a model-checker called SYMM which allows the user to specify the symmetries of the model and then uses the symmetries to perform efficient model-checking. Few of these problems were considered in [3, 6]. The results provided in this paper represent significant advances over those presented in [3, 6]. Although the paper only presents results for asynchronous composition of processes, all the results do extend to synchronous composition of processes but are not presented here because of lack of space. Complete details will be provided in the full version of the paper.

Our paper is organized as follows: In Section 2 we introduce a shared variable model. Section 3 gives a technique to derive symmetries of shared variable programs. Section 4 investigates the complexity of the orbit problem. Section 5 investigates some special classes of the orbit problem. Experimental results on an arbiter example are shown in section 6. Section 7 concludes with some interesting future directions. Due to space limitations we have not presented the background material here. For background on group theory the reader is referred to [9]. For general background on symmetry groups and model-checking see [3, 6].

2 A Shared Variable Model of Computation

We adopt the model of computation from [6]. A *shared variable program* is defined with the state sets and the transition relation as follows:

- $S = Loc^I \times D^V$ is the finite set of *states*, with Loc a finite set of individual process *locations*, I the set of process indices, and V is a finite set of shared *variables* over a finite *data domain* D.
- $R \subseteq S \times S$ which represents the transitions of the system.

For convenience, each state $s = (s', s'') \in S$ can be written in the form $(\ell_1, \ldots, \ell_n, v = d, \ldots, v' = d')$ indicating that processes $1, \ldots, n$ are in locations $\ell_1, \ldots, \ell_n$, respectively and the shared variables $v, \ldots, v'$ are assigned data values $d, \ldots, d'$, respectively.

Next, we define a labeling function for a shared variable program. The set of *terms* are expressions of the form l_i $(i \in I)$ and $v = d$ $(v \in V$ and $d \in D)$. The set of atomic propositions AP are constructed from the set of terms by the

logical connectives $\wedge$ and $\neg$. Given an atomic proposition $p \in AP$ and a state $s \in S$, the satisfaction relation $s \models p$ is defined in the following way:
$s \models l_i$ iff the i-th process in the state s is in location l_i.
$s \models (v = d)$ iff the shared variable v has the value d in the state s.
$s \models f \wedge g$ iff $s \models f$ and $s \models g$.
$s \models \neg f$ iff $s \not\models f$.

Given a shared variable program, we can construct a corresponding Kripke Structure $M = (S, R, L)$ (S and R were defined before) by constructing the following labeling function $L : S \to 2^{AP}$: $p \in L(s) \Leftrightarrow s \models p$.

In practice, for ordinary model checking, M is the Kripke Structure corresponding to a finite state *concurrent program* $\mathcal{P}$ of the form $\|_{i=1}^{n} K_i$ consisting of processes $K_1, \ldots, K_n$ running in parallel. Each K_i may be viewed as a finite state transition graph with node set *Loc*. An arc from node ℓ to node ℓ' may be labeled by a guarded command $B \to A$. The guard B is an atomic proposition that can inspect shared variables and local states of "accessible" processes. A is a set of *simultaneous assignments* to shared variables $v := d \parallel \cdots \parallel v' := d'$. When process K_i is in local state ℓ and the guard B evaluates to *true* in the current global state, the program $\mathcal{P}$ can nondeterministically choose to advance by firing this transition of K_i which changes the local state of K_i to be ℓ' and the shared variables in V according to A. Thus the arc from ℓ to ℓ' in K_i represents a *local transition* of K_i denoted by $\ell : B \to A : \ell'$.

The Kripke structure $M = (S, R, L)$ corresponding to $\mathcal{P}$ is thus defined using the obvious formal operational semantics. First, the set of (all possible) states S is determined from $\mathcal{P}$ because it provides us with the set of local (i.e., individual process) locations *Loc*, process indices I, variables V, and data domain D. For states $s, t \in S$, we define $s \to t \in R$ iff $\exists i \in I$ such that the process K_i can cause s to move to t, denoted by $s \to_i t$ iff $\exists i \in I$ $\exists$ local transition $\tau_i = \ell_i : B_i \to A_i : m_i$ of K_i which *drives* $s = (s', s'')$ to $t = (t', t'')$; i-th component of s' equals ℓ_i, the i-th component of t' equals m_i, all other components of s' equal the corresponding component of t', predicate $B_i(s) = true$, and $t'' = A_i(s'')$. $A_i(s'')$ is constructed from s'' by replacing the values of the shared variables according to the simultaneous assignment statement A_i. The labeling function L is defined as before. Notice that we use asynchronous composition in the definition given here. Analogous definition can be given for synchronous composition. All the results given in the paper hold for synchronous composition, but are not stated because of lack of space.

3 Deriving Symmetry

This section analyzes how one can derive symmetry for shared variable programs introduced in the previous section. Intuitively, if one has a graph G whose nodes corresponds to processes and the processes communicate over the edges of G, an automorphism of the graph G should manifest itself into a symmetry of the underlying structure [3, 6]. Succinctly speaking, *structural symmetry introduces*

symmetry in the model [3](cf. [5]). This section proves that for certain cases one can derive the symmetry of the model from the topology of the system. Given a concurrent program $\mathcal{P} = \|_{i=1}^{n} K_i$, we build a hypergraph $HG(\mathcal{P})$. Under certain restrictions, we prove that each automorphism of $HG(\mathcal{P})$ is also a symmetry of the underlying Kripke Structure M. A restricted version of the theorem already appeared in [6]. There the authors assumed that all processes are isomorphic and the variables are only shared between two processes. The new and more general version of the theorem, presented here, can handle a broader class of systems. For example, an arbiter which maintains a global shared variable (indicating who has the resource) can now be accomodated.

Let $\mathcal{P} = \|_{i=1}^{n} K_i$ be a concurrent program. In this section the index set is $I = [n]$. Each shared variable v is subscripted by the set of indices of the processes which access that shared variable. For example, if x is accessed by processes 1, 4, and 5, we write x as $x_{\{1,4,5\}}$. Notice that each shared variable is uniquely determined by its name and subscript, but we allow shared variables to have the same name as long as their subscripts are different. For example, $x_{\{1,2\}}$ and $x_{\{3,4\}}$ are allowed. A permutation $\pi \in S_n$ acts on the variables in a natural manner, i.e., $\pi(x_w) = x_{\pi(w)}$. A permutation π acting on $[n]$ is called *consistent* if and only if for every shared variable x_w, $x_{\pi(w)}$ is a variable as well. This means that we only allow permutations which map shared variables to shared variables.

We define how a consistent permutation π acts on states, atomic propositions, and processes. Let π be a consistent permutation.

- Given a state $s = (l_1, \cdots, l_n, v_{w_1} = d_1, \cdots, v_{w_k} = d_k)$, the state $\pi(s)$ is defined as follows: i-th process is in location $l_{\pi(i)}$ in the state $\pi(s)$, and the shared variable $v_{\pi(w)}$ in the state $\pi(s)$ has the same value as the variable v_w in the state s.
- Let $p \in AP$ be an atomic proposition. $\pi(p)$ is recursively defined as follows: $\pi(f \wedge g) = \pi(f) \wedge \pi(g)$, $\pi(\neg f) = \neg\pi(f)$, $\pi(l_i) = l_{\pi(i)}$, and $\pi(v_w = d) = (v_{\pi(w)} = d)$.
- Given a simultaneous assignment $A = (v_{w_1} = d_1 \| \cdots \| v_{w_k} = d_k)$, define $\pi(A)$ as the following simultaneous assignment: $v_{\pi(w_1)} = d_1 \| \cdots \| v_{\pi(w_k)} = d_k$
- Given a process K_i, the process $\pi(K_i)$ is constructed in the following manner: $l : B \rightarrow A : l'$ is a transition in K_i iff $l : \pi(B) \rightarrow \pi(A) : l'$ is a transition in $\pi(K_i)$.

Definition 1. A *colored hypergraph* with n vertices and k colors is a 3-tuple $H = ([n], E, L)$ such that $E \subseteq 2^{[n]}$ is the *edge set*, and $L : [n] \rightarrow [k]$ is the *coloring function* which colors each node with one of the k colors. A permutation π acting on $[n]$ is called an *automorphism* of the hypergraph H iff the following two conditions hold:
For all $1 \leq i \leq n$, $L(i) = L(\pi(i))$.
$w \in E$ iff $\pi(w) \in E$.
The group of automorphisms of the hypergraph H is denoted by $Aut(H)$.

Given a process K_i, let $\tau(K_i)$ be the *type* of that process. For example, if process K_i is an instance of `MODULE` m [14], then $\tau(K_i) = m$. Next, we define the

concept of isomoprhism between two processes. Let $\Gamma(K_i)$ be the set of indices r such that there exists a shared variable x_w such that $\{i, r\} \subseteq w$. Intuitively, if $r \in \Gamma(K_i)$, then K_i and K_r share some variables. $\Gamma(K_i)$ is called the *neighborhood* of K_i. We require that $i \in \Gamma(K_i)$, i.e., a process is in its own neighborhood. We say that $K_i \cong K_j$ if and only if for every consistent permutation π such that $\pi(i) = j$ and such that for all $r \in \Gamma(K_i)$, $\tau(K_r) = \tau(K_{\pi(r)})$ it is the case that $\pi(K_i) = K_j$ and vice versa. Many times the condition $K_i \cong K_j$ can be checked by checking that $\pi(K_i) = K_j$ with respect to all permutations π that map i to j and that map elements in $\Gamma(K_i)$ to elements in $\Gamma(K_j)$ and that satisfy local consistency conditions (i.e. x_w is a variable iff $x_{\pi(w)}$ is a variable for cases where $i \in w$); this property can be checked efficiently.

Definition 2. Given a concurrent program $\mathcal{P} = \|_{i=1}^{n} K_i$, define the *corresponding* colored hypergraph $HG(\mathcal{P}) = ([n], E, L)$ in the following manner:

- $w \in E$ iff there exists a shared variable with subscript w.
- Partition the processes $K_1, \cdots, K_n$ into equivalence classes induced by the relation $\cong$. Let $c_1, \cdots, c_k$ be the k equivalence classes. The coloring function L is defined as follows: $L(i) = r$ iff the process K_i is in the equivalence class c_r.

Theorem 3. Let $HG(\mathcal{P})$ be the hypergraph corresponding to the program $\mathcal{P} = \|_{i=1}^{n} K_i$. Let M be the Kripke Structure corresponding to $\mathcal{P}$. Given these conditions, $Aut(HG(\mathcal{P})) \leq Aut(M)$.

4 Complexity of the Orbit Problem

In this section we assume that the state space of our system is given by assignments to n boolean state variables $x_1, \cdots, x_n$. Therefore, the state space is isomorphic to B^n (where $B = \{0, 1\}$). We assume that the symmetry group $G \leq S_n$ acts on B^n in the natural way: a permutation σ maps a vector $(z_1, \cdots, z_n)$ to $(z_{\sigma(1)}, \cdots, z_{\sigma(n)})$. The orbit problem is at the core of any method exploiting symmetry [3, 6, 12, 11]. The orbit problem asks whether two states s and s' (which in this case are two 0-1 vectors of size n) are in the same orbit, i.e., there exists a permutation $\sigma \in G$ such that $s' = \sigma(s)$. In [3] it was proved that the graph isomorphism problem can be reduced to the orbit problem. Therefore, the orbit problem is atleast as hard as the graph isomorphism problem. Here we show that the orbit problem is equivalent to the problem of finding a set stabilizer of a set Y in a coset (we call this problem SSC). Since the graph isomorphism problem can be reduced to SSC [8], this result subsumes the result which appeared in [3]. Moreover, SSC (and hence the orbit problem) is equivalent to several important problems in computational group theory, which are harder than graph isomorphism, but not known to be NP-complete. Proofs of most the theorems are based on techniques introduced in [13].

The Orbit Problem (OP): Given two 0-1 vectors x and y of size n and a

group $G \leq S_n$, does there exist a permutation $\sigma \in G$ which maps x to y, i.e., $y = \sigma(x)$.

Set Stabilizer in a coset (SSC): Given a set $Y \subseteq [n]$, let $G \leq S_n$ be a group and $\gamma \in S_n$ be a permutation. The problem is to find whether there exists $\sigma \in G\gamma$ which *stabilizes* the set Y, i.e., $\sigma(Y) = Y$.

Constructive Set Stabilizer in a coset ($CSSC$): Given a set $Y \subseteq [n]$, let $G \leq S_n$ be a group and $\gamma \in S_n$ be a permutation. The problem is to find whether there exists $\sigma \in G\gamma$ which stabilizes the set Y, i.e., $\sigma(Y) = Y$ and if yes, to exhibit such a σ.

Lemma 4. The problems SSC and $CSSC$ are polynomially equivalent.

Theorem 5. The problems OP and SSC are polynomially equivalent.

In general, the SSC problem is harder than *graph isomorphism* [8]. Conditions under which SSC can be solved in polynomial time are discussed in [8, 13]. In [8] it is proved that the *Coset Intersection Emptiness* problem stated below is polynomially equivalent to several important problems in computational group theory.

Definition 6. Coset Intersection Emptiness (CIE) Given the groups $A, B \leq S_n$ by generating sets and given a permutation $\pi \in S_n$, test whether $A\pi \cap B$ is empty.

Theorem 7. The set stabilizer in a coset problem (SSC) is polynomially equivalent to **CIE**.

Therefore, using the previous theorems one can deduce that OP is polynomially equivalent to **CIE** and hence several problems in computational group theory.

4.1 The Constructive Orbit Problem

Modeling states by boolean variables, in some cases, is too cumbersome and detailed. For example, consider the shared variable program introduced in Section 2. Let $\mathcal{P} = \|_{i=1}^{n} K_i$ be a concurrent program which does not have shared variables. Let the size of the set of locations Loc be k. In this case, a typical state in $\mathcal{P}$ is given by a vector of size n whose elements are integers between 1 and k, i.e., the space $[k]^n$. Permuting the processes K_i amounts to permuting the corresponding integers in that state. A symmetry group $G \leq S_n$ acts on the space $[k]^n$ in the following way: a permutation $\sigma \in G$ maps $(x_1, \cdots, x_n)$ to $(x_{\sigma(1)}, \cdots, x_{\sigma(n)})$.

Given a symmetry group G, one frequently needs a *representative function* $\xi : S \to S$ [3, 6, 11] (S is the state space of the system) which has the following properties:

- s and $\xi(s)$ are in the same orbit.
- If s and s' are in the same orbit, then $\xi(s) = \xi(s')$.

Such a representative function is used during state exploration in [3, 6, 11]. The need to find such a representative function motivates the following problem.

Definition 8. The Constructive Orbit Problem (*COP*): Given a group acting on $[n]$ and vector $x = (x_1, \cdots, x_n)$ find the lexicographically least element (or lex-least element for short) in the orbit of x (the group G permutes the indices of x)

Notice that if one can solve *COP* in polynomial time, one can construct the representative function ξ. Given a state x, $\xi(x)$ is simply the lex-least element in its orbit. In [1] it is proved that the problem is NP-hard. The paper also shows that for certain special groups *COP* can be solved in polynomial time. Actually, for our purposes it is enough to find a canonical element from each orbit.

5 Working Around the Orbit Problem

Results of section 4 prove that the Orbit problem and the Constructive Orbit Problem are quite hard in its full generality. In this section we discuss two possible techniques which will help circumvent the complexity of the orbit problem.

1. We prove that for a large class of groups, which occur commonly in practice, the orbit problem can be easily solved. This means that if the symmetries are restricted to this class of groups, the orbit problem can be easily solved.
2. We also describe an approach that uses multiple representatives from each orbit rather than just one. This approach is partcularly useful for symbolic model checking [3].

The ensuing subsections outline these approaches.

5.1 Easy Groups

Notice that if a group $G \leq S_n$ has polynomial size, *COP* for G can be solved in polynomial time by exhaustive enumeration. For example, a rotation group acting on set of size $[n]$ has order n. Therefore, for the rotation group one can solve *COP* in linear time. The lemma given below states that if *COP* can be solved in polynomial time for two disjoint groups J and K, then *COP* can be solved in polynomial time for their direct product.

Lemma 9. Let G be a disjoint product of J and K. If *COP* for J and K can be solved in polynomial time, then *COP* for G can be solved in polynomial time.

The next lemma is similar to the previous one but refers to wreath products.

Lemma 10. Let $G = J \wr K$ be the wreath product of J and K. The group J, K, G act on the sets $[n], [m], [nm]$ respectively. If *COP* for J, K can be solved in polynomial time, then *COP* for G can be solved in polynomial time.

Lemma 11. Let S_n be the full symmetric group acting on the set $[n]$. The *COP* problem for S_n can be solved in polynomial time.

Proof: Given a vector $x = (x_1, \cdots, x_n)$, the lex-least element of x under the group S_n can be obtained by sorting the elements x_i. □

In practice, symmetries are given as a set of transpositions. For example, in a system which has the star topology, the two outer processes can be switched. The lemma given below states that if the group is only generated by transpositions, then *COP* for it can be solved in polynomial time.

Lemma 12. Let G be a permutation group acting on the set $[n]$. Assume that G is generated by a set of transpositions S. The *COP* problem for G can be solved in polynomial time.

This means that during model-checking if one restricts to the class of groups mentioned above, exploiting symmetry is relatively easy. Hence while exploiting symmetry one should try to work with these easy groups.

5.2 Multiple Representatives

Multiple Representatives were discussed in [3]. The account given here is much cleaner and general. Assume that we are given a Kripke Structure $M = (S, R, L)$ and a symmetry group G of M. Let $C \subseteq G$ be an arbitrary set of permutations which is *inverse closed*, i.e., $\pi \in C$ implies that $\pi^{-1} \in C$. We also assume that the identity permutation e is in C. Let $Rep \subseteq S$ be a set of *representatives* which satisfies the following requirements:

1. Every orbit of S under the action of G has a non-empty intersection with *Rep*.
2. Given an $s \in S$, there exists a $\sigma \in C$ such that $\sigma(s) \in Rep$.

Let $M_{Rep} = (Rep, R_{Rep}, L_{Rep})$, where R_{Rep} and L_{Rep} are defined as follows:

- We have that $(r_1, r_2) \in R_{Rep}$ iff there exists $s \in S$ and $\sigma \in C$ such that $(s, r_2) \in R$ and $\sigma(s) = r_1$.
- $L_{Rep}(r) = L(r)$.

The *representative relation* $\xi \subseteq S \times Rep$ is defined as follows: $(s, r) \in \xi$ iff there exists a $\sigma \in C$ such that $\sigma(s) = r$. Notice that $R_{Rep} = \eta^{-1} \circ R$.

Theorem 13. let $M = (S, R, L)$ be a Kripke structure, G be a *symmetry group* of M, and h be a $CTL^\star$ formula. If G is an invariance group for all the atomic propositions p occurring in h, then for all r such that $(s, r) \in \xi$

$$M, s \models h \Leftrightarrow M_{Rep}, r \models h$$

A natural question to ask is: how does one choose *Rep* given a set of polynomial size C? One natural choice of *Rep* is given below:

$$r \in Rep \Leftrightarrow \forall(\sigma \in C)(r \geq \sigma(r))$$

where $\leq$ represents the lexicographical ordering on the 0-1 vectors. It is easy to check that *Rep* has the required properties. Also notice that given an $s \in S$ and $r \in Rep$ one can decide the following questions in polynomial time?

1. Is s a representative, i.e., $s \in Rep$?
2. Is $(s, r) \in \xi$?

First we argue that the choice of C is very important on how much reduction is achieved. For example, let us consider the case when $C = G^{[1,k]}$. In this case we will only permute the first indices k indices of the states. Therefore if there exists an orbit $\Theta \subseteq S$ which does not involve these k indices, then $\Theta \subseteq Rep$, i.e., the whole orbit has to be represented. We need a way of choosing C such that we have uniform savings across all orbits of S in G. One possibility is to take a subgroup $H \leq G$ such that $[G : H]$ is polynomial in n. Let B be the full right traversal of H in G. Let C be the inverse closure of B. The subgroup H can be chosen in many ways and its choice will depend upon the structure of the system being verified. One choice is as follows: Fix a set $I = \{i_1, \cdots, i_k\} \subseteq [n]$ of size k. Let H be the pointwize stabilizer of I in G. The *pointwise stabilizer* of I in G is the following group:

$$\{\sigma \in G \mid \forall(i \in I)(\sigma(i) = i)\}$$

Notice that $[G : H] \leq n^k$. The subgroup H and its full traversal in G can be found in polynomial time [7].

6 Empirical Results

Using the multiple representatives theory presented in section 5.2 we have built a symbolic model checker called SYMM. The tool SYMM has a language based on the shared variable model of computation described earlier. It allows the user to give *CTL* specifications. The tool also provides the facility for the user to give symmetries of the system. We have verified several examples using our tool. Here we describe our results for an arbiter example. In this arbiter, each module has a priority number. While competing for the bus, modules with higher priority number are given preference. Among competing modules with the same priority number, the winner is decided non-deterministically. Winner is the module which wins the competing phase. The arbitration cycle (which results in a winner) has six phases. We give a brief overview of these phases. The interested reader is referred to the IEEE Futurebus standard for a more detailed account [10].

Phase 0: In this phase modules decide to compete for the bus.
Phase 1: Noticing that a competition phase is about to begin, other modules might decide to compete.

Phase 2: In this phase a winner (called the *master elect*) is selected from the set of competing modules. Winner is selected according to the rules described earlier.
Phase 3: Other modules check that the master elect had the highest priority number among the competing modules. If this is not the case, modules assert an error. If an error has not occurred, this phase continues until the master of the bus relinquishes its control. In this phase a module with higher priority than the master elect might start a new arbitration cycle. This is called *deposing* the master elect.
Phase 4: In this phase the current master of the bus might inhibit transfer of control of the bus to the master elect. If the master relinquishes its control over the bus, the arbitration cycle moves to the last phase.
Phase 5: In this phase the master elect gets control of the bus. This phase is called the *transfer of tenure* phase.

There are three boolean variables in each module which ensures the proper sequencing of the phases in an arbitration cycle. The internal state of the module depends on the outcome of the arbitration cycle and is shown below.

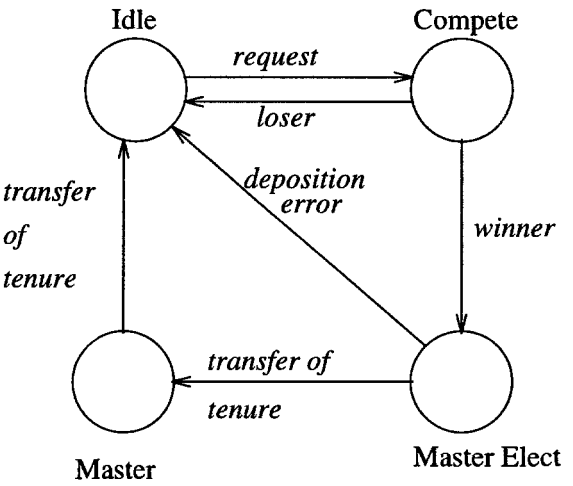

Fig. 1. States of the Arbiter

Notice that two modules with the same priority number can be permuted without changing the behavior of the system. Formally, the permutation corresponding to exchanging two modules with the same priority number is a symmetry of the system. However, two modules with different priority numbers cannot be exchanged. In the table given below the first column shows the configuration of the system. Assume that we have m sets of n modules (denoted by $\{M_{1,1}, \cdots, M_{1,n}\}, \cdots, \{M_{m,1}, \cdots, M_{m,n}\}$). Moreover, the priority numbers of two modules $M_{j,l}$ and $M_{r,t}$ are the same iff $j = r$, or they belong to the same

set. In this case the set of representatives *Rep* is the set of states where the first module from a particular set is always chosen master elect. The set C contains all the transpositions of the form $\sigma_{i,l} = ((i,1),(i,l))$ $(1 \leq i \leq m$ and $1 \leq l \leq n)$ and their products. The transposition $\sigma_{i,l}$ corresponds to exchanging modules $M_{i,1}$ and $M_{i,l}$. It is easy to see that C is inverse closed. Moreover, given an arbitrary state s there exists a representative state $r \in Rep$ and $\sigma \in C$ such that $\sigma(s) = r \in Rep$. Consider a state s where $M_{j,l}$ is the master elect. In this case $\sigma_{i,l}(s)$ is the state where $M_{j,1}$ is the master elect. Hence all conditions for applying the multiple representative theory are satisfied. The table of experimental results is shown below.

System Config	Time	BDD size	Time (Symm)	BDD size (Symm)
10m	163.41	369,705	37.04	27,671
12m	487.56	921,034	43.12	36,135
10m10m	1171.85	932,429	126.17	73,514
12m12m	-	-	198.195	93,094

The property that was checked was that only one module is picked master elect. The first column shows the system configuration. For example, 10m is the configuration with 10 modules having the same priority number. 10m10m denotes the configuration where the first 10 modules have a higher priority number than the last 10 modules (there are 20 modules in all). Columns 2 and 3 shows the time (in seconds) and the maximum BDD size encountered during checking the temporal property without using symmetry. The experiments were run on an Sun ULTRA SPARC. The last column shows the same statistic with using symmetry. The symbol - means that we were unable to check the property in the time allowed. As can be seen from the table we get considerable savings by using symmetry.

7 Conclusion

In this paper we investigated various problems associated with exploiting symmetry in model checking. We also provided ways of deriving symmetry for shared variable concurrent programs. An important research problem will be to take some existing hardware description languages and derive symmetry information statically from the system descriptions written in that language. Ideas presented in section 3 are applicable in this context. This paper also makes connection between exploiting symmetry in model checking and computational group theory. An important research direction will be to use some of the powerful techniques available in the computational group theory literature in the model checking domain. It will be also very useful to apply some of the symmetry ideas on industrial size examples.

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