Features

The Euler Characteristic of Discrete Object

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Abstract. We introduce the curvature indexes of the boundary of a discrete object, and using these indexes of points, we define vertex angles of discrete surfaces as an extension of the chain codes for digital curves. Next, we prove a relation between the number of points on the surface and the genus of a discrete object. This is the angular Euler characteristic of a discrete object. These relations derive a parallel algorithm for the computation of the Euler characteristic of a discrete object.

1 Introduction

In this paper we introduce the curvature indexes of points on the boundary of a discrete object using the neighborhood decomposition and the curvature indexes of planar digital curves. The decomposition of the three-dimensional neighborhood to a collection of two-dimensional neighborhoods reduces the combinatorial properties of a discrete object to the collections of combinatorial properties of planar patterns [1]. Using these indexes of points, we define the vertex angles of a discrete surface as an extension of the chain code of digital curves [1]. These indexes provide a relation between the point configurations on the boundary and the Euler characteristic of a discrete object. This is the angular Euler characteristic [5] for a 6-connected discrete object. This relation automatically derives a parallel algorithm for the computation of the Euler characteristics of a discrete object. In this paper a hole is a point set which is encircled by an object. A tunnel is a point set which connects point sets which are locally separated by an object. Denoting a wall which transform a tunnel to a pair of wells of a discrete object, we derive an equation for the computation of the Euler characteristic of a discrete object using points on the boundary [4].

Recently in computer vision, shape reconstruction from multiview images is concerned from viewpoints of the mathematical theory such as the multilinear form for corresponding points of a series of images, and practical applications such as the construction of geometric date of three-dimensional object from a series of tow-dimensional images. This relation among images provides methods for the estimation and generation of new images from observed data. The reconstruction of shape from multiview images is considered as an interpolation problem

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in three-dimensional Euclidean space from a collection of two-dimensional data on imaging planes. Since each image contains noises, errors in the reconstructed data are the collection of errors of projected images. This means the signal-to-noise ratio in a reconstructed object is usually worse than the signal-to-noise ratio of each image. These instabilities cause inaccuracies for the configuration of reconstructed points. For the reduction of these geometric instabilities of reconstructed spatial points, it is necessary to consider the topology of an object, since topological properties are expressed by Boolean values which is stable for the calculation in digital computers [6]. In digital computer, an object is expressed as a discrete object which is a collection of lattice points. Therefore, it is desired to define topological characteristics of discrete objects.

The Euler characteristic is a combinatorial relation among vertices, edges, faces, and tunnels of a polyhedron. The Euler characteristic plays important role in *differential geometry in the large* [7], and digital image processing. The numbers of holes and connected components of a planar discrete binary image derive the Euler characteristic of an image. For topological preserving transformations such as deformation and skeltonization, it is desired to check the topology characteristics of an image using the Euler characteristic [3] [8]. For digital binary images a relation among the chin codes, boundary elements, and the Euler characteristic is proven [9]. Furthermore, an algorithm for the computation of the Euler characteristic based on this property was developed. Bieri and Nef [15] proposed an algorithm for the computation of the Euler characteristic of a n-dimensional binary object as an application of the cell decomposition of a polytope. In reference [10], Toriwaki and coworkers applied this idea for the definition of the Euler characteristic of a discrete object. Furthermore, they derived a Boolean function for the computation of the Euler characteristic [11]. These results are the discrete versions of usual expression of Euler characteristic by the cell decomposition. Lee, Poston, and Rosenfeld [12] proposed a method for the definition of holes and tunnels of discrete objects using the theory of *cuts*, that is, they dealt with a discrete version of a topological property between the minimum number of cuts which eliminate holes and tunnels and the number of holes and tunnels of an object. The numbers of holes and tunnels are called the genuses of a planar pattern and a spatial object, respectively. Points on the boundary of an object also derive the topological characteristics of object.

2 Connectivity and Neighborhood

Setting $\mathbf{R}^2$ and $\mathbf{R}^3$ to be two- and three- dimensional Euclidean spaces, we express vectors in $\mathbf{R}^2$ and $\mathbf{R}^3$ as $\boldsymbol{x} = (x, y)^\top$ and $\boldsymbol{x} = (x, y, z)^\top$, where $\top$ is the transpose of vectors. Setting $\mathbf{Z}$ to be the set of all integers, the two- and three-dimensional discrete spaces $\mathbf{Z}^2$ and $\mathbf{Z}^3$ are sets of points such that both x and y are integers and all x, y, and z are integers, respectively.

On $\mathbf{Z}^2$ and in $\mathbf{Z}^3$

$$\mathbf{N}^4((m,n)^\top) = \{(m \pm 1, n)^\top, (m, n \pm 1)^\top\} \tag{1}$$

and

$$\mathbf{N}^6((k,m,n)^\top) = \{(k\pm 1, m, n)^\top, (k, m\pm 1, n)^\top, (k, m, n\pm 1)^\top\} \quad (2)$$

are the planar 4-neighborhood of point $(m,n)^\top$ and the spatial 6-neighborhood of point $(k,m,n)^\top$, respectively. In this paper, we assume the 4-connectivity on $\mathbf{Z}^2$ and the 6-connectivity in $\mathbf{Z}^3$.

Setting one of x, y, and z to be a fixed integer, we obtain two dimensional sets of lattice points such that

$$\mathbf{Z}_1^2((k,m,n)^\top) = \{(k,m,n)^\top | \exists k, \forall m, \forall n \in \mathbf{Z}\}, \quad (3)$$

$$\mathbf{Z}_2^2((k,m,n)^\top) = \{(k,m,n)^\top | \forall k, \exists m, \forall n \in \mathbf{Z}\}, \quad (4)$$

and

$$\mathbf{Z}_3^2((k,m,n)^\top) = \{(k,m,n)^\top | \forall k, \forall m, \exists n \in \mathbf{Z}\}. \quad (5)$$

These two dimensional discrete spaces are mutually orthogonal. Denoting

$$\mathbf{N}_1^4((k,m,n)^\top) = \{(k, m\pm 1, n)^\top, (k, m, n\pm 1)^\top\}, \quad (6)$$

$$\mathbf{N}_2^4((k,m,n)^\top) = \{(k\pm 1, m, n)^\top, (k, m, n\pm 1)^\top\}, \quad (7)$$

and

$$\mathbf{N}_3^4((k,m,n)^\top) = \{(k\pm 1, m, n)^\top, (k, m\pm 1, n)^\top\}, \quad (8)$$

the relationship

$$\mathbf{N}^6((k,m,n)^\top) = \mathbf{N}_1^4((k,m,n)^\top) \cup \mathbf{N}_2^4((k,m,n)^\top) \cup \mathbf{N}_3^4((k,m,n)^\top), \quad (9)$$

holds, since $\mathbf{N}_i^4((k,m,n)^\top)$ is the 4-neighborhood on plane $\mathbf{Z}_i^2((k,m,n)^\top)$ for $i = 1,2,3$ [1]. Equation (9) implies that the 6-neighborhood is decomposed into mutually orthogonal three 4-neighborhoods [1].

A pair of points $(k,m,n)^\top$ and $\boldsymbol{x} \in \mathbf{N}^6((k,m,n)\top)$ is a unit line segment in $\mathbf{Z}^3$, Furthermore, 6-connected four points which form a circle define a unit plane segment in $\mathbf{Z}^3$ with respect to the 6-connectivity. Therefore, we assume that our object is a complex of $2\times 2\times 2$ cubes which share at least one face each other. Thus, the surface of an object is a collection of unit squares which are parallel to planes $x=0$, $y=0$, and $z=0$.

3 Curvature Indexes of Points

3.1 Planar Curvature Indexes

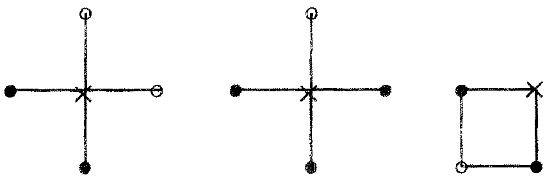
Since we are concerned with a binary discrete object, we affix 0 and 1 to points in the background and in objects, respectively. On $\mathbf{Z}^2$ three types of configurations of points which are illustrated in figure 1 exist in the neighborhood of a point $\times$ on the boundary. In figure 1, $\bullet$, and $\circ$, are points on the boundary and in the background, respectively. Setting $f_i \in \{0, 1\}$ to be the value of point $\boldsymbol{x}_i$ such that

$$\begin{array}{ccc} & \boldsymbol{x}_3 = (m, n+1)^\top & \\ \boldsymbol{x}_5 = (m-1.n)^\top & \boldsymbol{x}_0 = (m, n)^\top & \boldsymbol{x}_1 = (m+1, n)^\top \\ & \boldsymbol{x}_7 = (m, n-1)^\top & \end{array} \tag{10}$$

the curvature of point $\boldsymbol{x}_0$ is defined by

$$r(\boldsymbol{x}_0) = 1 - \frac{1}{2}\sum_{k \in N} f_k + \frac{1}{4}\sum_{k \in N} f_k f_{k+1} f_{k+2}, \tag{11}$$

where $N = \{1, 3, 5, 7\}$ and $k+8 = k$ [9]. The curvature indexes of configurations (a), (b) and (c) are positive, zero, and negative, respectively. Therefore, we call these configurations convex, flat, and concave, respectively, and affix the indexes $+$, 0, and $-$, respectively.



(a) (b) (c)

Fig. 1. Configurations of points on the boundary.

Setting n_+ and n_- to be the numbers of positive and negative points on a plane, respectively, we obtain the following theorem.

Theorem 1 . *Let g to be the number of holes of a planar discrete object. If a discrete object is 4-connected, the relation*

$$n_+ - n_- = 4(1 - g) \tag{12}$$

is held.

Proof. If an object has no hole, it is possible to deform a 4-connected object to a rectangle. A rectangle implies the relation $n_+ = 4$. Thus, an object which has no hole holds the relation. Next, a hole is also deformable to a rectangle in the background. A rectangular hole holds the relation $n_- = 4$. Therefore, an object which has a hole holds the relation. Assuming that an object which has g holes holds the relation, an object which has $g+1$ holes preserves the relation since the new hole is deformable to a rectangular hole. This concludes that n_- becomes $n_- + 4$, if a hole appears. Therefore, a 4-connected planar object holds eq. (12). □

There exist 4 congruent patterns for patterns (a) and (c) in figure 1. These 8 patterns are independent. This theorem automatically derives an algorithm for the computation of the Euler characteristic of a planar discrete object using a table with 8 entries of (3×3)-local patterns, since points the vertex angles of which are 0-s do not affect to the Euler characteristic.

Step 1 Extract the boundary of object $\mathbf{O}$.
Step 2 Count n_+ and n_- using the boundary following algorithm.
Step 3 Compute $n_+ - n_-$.

It is also possible to apply parallelly these 8 patterns to masking and matching.

3.2 Spatial Curvature Indexes

Using combinations of the planar curvature indexes on mutually orthogonal three planes which pass through a point $\boldsymbol{x}_0$, we define the curvature index of a point $\boldsymbol{x}_0$ in $\mathbf{Z}^3$ since the 6-neighborhood is decomposed into three 4-neighborhoods. Setting α_i to be the planar curvature index on plane $\mathbf{Z}_i^2(\boldsymbol{x}_0)$ for $i = 1, 2, 3$, the curvature index of a point in $\mathbf{Z}^3$ is a triplet of two-dimensional curvature indexes $(\alpha_1, \alpha_2, \alpha_3)$ such that $\alpha_i \in \{+, -, 0, \emptyset\}$. Here, if $\alpha_i = \emptyset$, the curvature index of a point on plane $\mathbf{Z}_i^2(\boldsymbol{x}_0)$ is not defined. The curvature indexes hold the following theorem.

Theorem 2 . *For the boundary points, seven configurations*

$$\begin{array}{c} (+,+,+),\ (+,+,-),\ (+,0,0), \\ (0,0,\emptyset), \\ (-,-,-),\ (+,-,-),\ (-,0,0) \end{array} \tag{13}$$

and their permutations are possible.

Proof. From the definition of the curvature indexes for points on $\mathbf{Z}^2$ and the connectivity of points in $\mathbf{Z}^3$, obviously the configurations with the indexes $+$ and $-$ exist. Therefore, we prove that the configurations such that $(*, *, 0)$, and $(0, 0, \emptyset)$, where $* \in \{+, -, 0\}$, and their permutations exist. From the configurations of points in a $3 \times 3 \times 3$ cube, configurations with 0-curvature on a plane exist. This geometric property automatically concludes that one of $*$ is zero. Thus, the curvature index is $(*, 0, 0)$ for $* \in \{+, -, \emptyset\}$. Furthermore, a congruence transform in $\mathbf{Z}^3$ yields a permutation of a code. □

4 Curvature and Spatial Vertex Angle

4.1 Local Configurations on the Boundary

Since a triplet of mutually orthogonal planes separates a space into eight parts, we call a one eighth of space an octspace. The number of octspaces determines

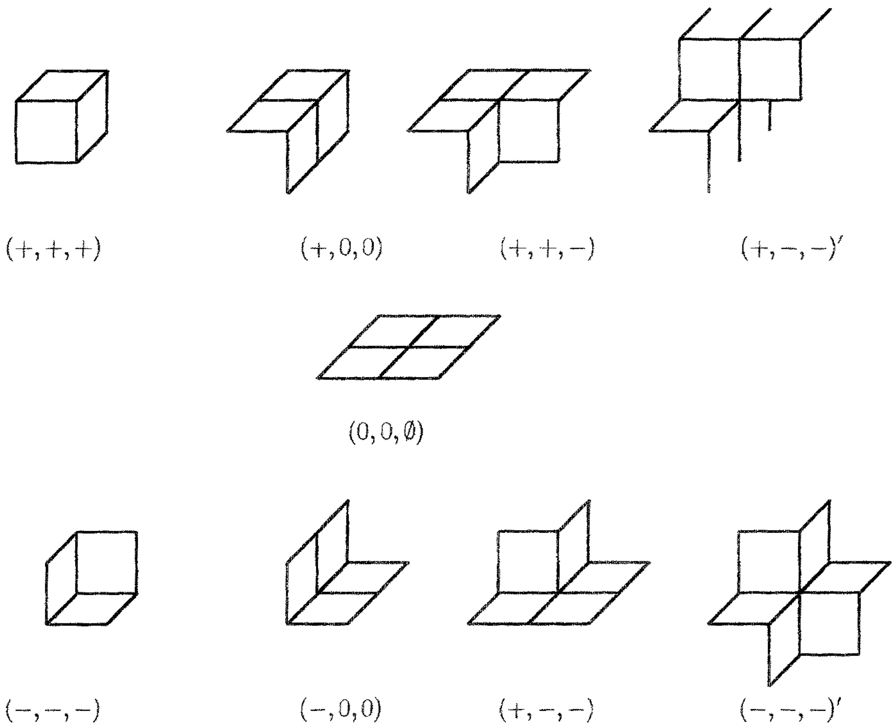


Fig. 2. Angles and configurations on the boundary.

configurations of points in a $3 \times 3 \times 3$ cube. There exist nine configurations in the $3 \times 3 \times 3$ neighborhood of a point on the boundary since these configurations separate $\mathbf{Z}^3$ to two parts which do not share any common points. These configurations are the same things introduced by Françon [13] for the analysis of discrete planes. The curvature analysis of discrete surfaces also derives these configurations.

For a spatial curvature index α, setting $n(\alpha)$ to be the number of octspaces in the $3 \times 3 \times 3$ neighborhood of a point, the relationships

$$\begin{array}{c} n((+,+,+)) = 1, \quad n((+,0,0)) = 2, n((+,+,-)) = 3, \\ n((0,0,\emptyset)) = 4, \\ n((+,-,-)) = 4,5, n((-,0,0)) = 6, n((-,-,-)) = 4,7, \end{array} \tag{14}$$

are held. From the decomposition property of the neighborhood and configurations of points in a 3×3 neighborhood on a plane, there exist two configurations for codes $(-,-,-)$ and $(+,-,-)$. Therefore, we set

$$n((-,-,-)) = 7, n((-,-,-)') = 4, \\ n((+,-,-)) = 5, n((+,-,-)') = 4. \tag{15}$$

4.2 The Vertex Angles

The curvature indexes of eq. (13) correspond to nine configurations of points which are illustrated in figure 2. Since $\frac{i}{8} = 1 - \frac{8-i}{8}$, we define the vertex angles of configurations as

$$\begin{aligned} &\gamma((+,+,+)) = \frac{1}{8}, \quad \gamma((+,0,0)) = \frac{2}{8}, \gamma((+,+,-)) = \frac{3}{8}, \\ &\gamma((+,-,-)') = \frac{4}{8}, \quad \gamma((0,0,\emptyset)) = 0, \quad \gamma((-,-,-)') = \frac{-4}{8}, \\ &\gamma((+,-,-)) = \frac{-3}{8}, \gamma(-,0,0) = \frac{-2}{8}, \gamma((-,-,-)) = \frac{-1}{8}. \end{aligned} \tag{16}$$

Considering congruent transformations in $\mathbf{Z}^3$, since a code uniquely corresponds to a configuration, the vertex angle for a point on a surface is an extension of the chain code for a point on a curve.

Next, we clarify the geometric properties of nine configurations of points which are illustrated in figure 2. A configuration the vertex angle of which is 0 is flat in a $3 \times 3 \times 3$ local region.

A point the vertex angle of which is $\frac{1}{8}$ touches a plane the normal vector of which is one of $(\pm1, \pm1, \pm1)^\top$, and a point the vertex angle of which is $\frac{-1}{8}$ also touches a plane the normal vector of which is one of $(\pm1, \pm1, \pm1)^\top$. Therefore, these points are elliptic points in $\mathbf{Z}^3$.

A point the vertex angle of which is $\frac{2}{8}$ includes a line segment which is parallel to one of three axis of the coordinate system on a plane the normal vector of which is one of $(\pm1, \pm1, 0)^\top$, $(\pm1, 0, \pm1)^\top$, and $(0, \pm1, \pm1)^\top$. Furthermore, a point the vertex angle of which is $\frac{-2}{8}$ also includes a line segment which is parallel to one of three axis of the coordinate system on a plane the normal vector of which is one of $(\pm1, \pm1, 0)^\top$, $(\pm1, 0, \pm1)^\top$ and $(0, \pm1, \pm1)^\top$. Therefore, these points are parabolic points in $\mathbf{Z}^3$.

Points in the $3 \times 3 \times 3$ neighborhood of a point the vertex angle of which is $\frac{3}{8}$ exists in a half space separated by a plane the normal vector of which is one of $(\pm1, 0, 0)^\top$, $(0, \pm1, 0)^\top$, and $(0, 0, \pm1)^\top$. Furthermore, points in the $3 \times 3 \times 3$ neighborhood of a point the vertex angle of which is $-\frac{3}{8}$ exists in a half space separated by a plane the normal vector of which is one of $(\pm1, 0, 0)^\top$, $(0, \pm1, 0)^\top$, and $(0, 0, \pm1)^\top$. Therefore, these points are discrete hyperbolic-parabolic points in $\mathbf{Z}^3$.

Points in the $3 \times 3 \times 3$ neighborhood of a point the vertex angle of which is $\frac{4}{8}$ never exists in a half space separated by a plane the normal vector of which is one of $(\pm1, 0, 0)^\top$, $(0, \pm1, 0)^\top$ $(0, 0, \pm1)^\top$, $(\pm1, \pm1, 0)^\top$, $(\pm1, 0, \pm1)^\top$, $(0, \pm1, \pm1)^\top$, and $(\pm1, \pm1, \pm1)^\top$. Furthermore, points in the $3 \times 3 \times 3$ neighborhood of a point the vertex angle of which is $\frac{-4}{8}$ never exists in a half space separated by a plane the normal vector of which is one of $(\pm1,0,0)^\top$, $(0,\pm1,0)^\top$ $(0,0,\pm1)^\top$, $(\pm1,\pm1,0)^\top$, $(\pm1,0,\pm1)^\top$ $(0,\pm1,\pm1)^\top$, and $(\pm1,\pm1,\pm1)^\top$. These points are hyperbolic points in $\mathbf{Z}^3$.

5 The Euler Characteristic of Discrete Object

In this section, we prove the relation between the number of points n_i the vertex angles of which are $\frac{i}{8}$ and the number of tunnels of a discrete object.

5.1 The Euler Equation

Setting

$$\boldsymbol{n} = (n_{-4}, n_{-3}, n_{-2}, n_{-1}, n_0, n_1, n_2, n_3, n_4)^\top \tag{17}$$

and

$$\boldsymbol{a} = (-2,-1,0,1,0,1,0,-1,-2)^\top, \tag{18}$$

we obtain the following theorems.

Theorem 3 . *For an object which has no tunnel, an object holds the relationship*

$$\boldsymbol{a}^\top \boldsymbol{n} = 8. \tag{19}$$

Theorem 4 . *For an object with g tunnels, the object holds the relationship*

$$\boldsymbol{a}^\top \boldsymbol{n} = 8(1-g). \tag{20}$$

Theorem 5 . *Setting g, c, and p to be the number of tunnels, holes, and poles which are tunnels in holes, respectively, an object holds the relationship*

$$\boldsymbol{a}^\top \boldsymbol{n} = 8(1-g+c-p). \tag{21}$$

There exist 8 congruent patterns for 6 patterns which affect to the Euler characteristic. These 48 patterns are independent. These theorems automatically derive an algorithm for the computation of the Euler characteristic of a discrete object using a table with 48 entries of $(3\times3\times3)$-local patterns, since points the vertex angles of which are 0-s and $\frac{\pm2}{8}$-s do not affect to the Euler characteristic. The properties of indexes derive the following simple framework for the computation of the Euler characteristic.

Step 1 Decompose object $\mathbf{O}$ to collections of slices of planar patterns

$$\{\mathbf{O}_{1\alpha}\}_{\alpha=1}^{k}, \{\mathbf{O}_{2\beta}\}_{\beta=1}^{m}, \{\mathbf{O}_{3\gamma}\}_{\gamma=1}^{n},$$

which are perpendicular to $(1,0,0)^\top$, $(0,1,0)^\top$, and $(0,0,1)^\top$, respectively.
Step 2 Compute $r(\boldsymbol{x})$ for points on the boundaries of $\mathbf{O}_{1\alpha}$, $\mathbf{O}_{2\beta}$, and $\mathbf{O}_{3\gamma}$.
Step 3 Compute γ for points $r(\boldsymbol{x}) \neq 0$, according to the configurations.
Step 4 Count n_i for $i = \pm1, \pm3, \pm4$.
Step 5 Compute $\boldsymbol{a}^\top \boldsymbol{n}$.

It is also possible to apply parallelly these 48 patterns to masking and matching.

5.2 Proofs of Theorems

Proof of Theorem 3 Since an object the vertex angles of which are 0-s, $\frac{1}{8}$-s, and $\frac{2}{8}$-s is a parallelopipedon, it holds eq. (19). Furthermore, since an object the vertex angles of which are 0-s $\frac{1}{8}$-s, $\frac{\pm 2}{8}$-s, and $\frac{\pm 3}{8}$-s is a collection of an appropriate number of parallelopipedons which contain points the vertex angles of which are $\frac{1}{8}$-s and $\frac{2}{8}$-s. Although the decomposition of an object into parallelopipedons is not uniquely determined, the decomposition is always possible. In the construction of an object from parallelopipedons, if k_1 points the vertex angles of which are $\frac{1}{8}$-s change k_3 points the vertex angles of which are $\frac{3}{8}$-s and k_{-3} points the vertex angles of which are $\frac{-3}{8}$-s, configurations of points in a $3 \times 3 \times 3$ cube imply the relationship $k_1 = k_3 + k_{-3}$. These geometric properties conclude that eq. (19) holds for all objects the vertex angles of which are 0-s $\frac{1}{8}$-s, $\frac{\pm 2}{8}$-s, and $\frac{\pm 3}{8}$-s.

Next for an object the vertex angles of which are 0-s, $\frac{\pm 1}{8}$-s, $\frac{\pm 2}{8}$-s, and $\frac{\pm 3}{8}$-s, it is possible to decompose an object into a collection of objects which do not contain points the vertex angles of which are $\frac{-1}{8}$-s. Furthermore, in the construction process, a point the vertex angle of which is $\frac{1}{8}$ and a point the vertex angle of which is $\frac{-2}{8}$ forms a point the vertex angle of which is $\frac{-1}{8}$. These geometric properties also conclude that eq. (19) holds for an object the vertex angles of which are 0-s $\frac{\pm 1}{8}$-s, $\frac{\pm 2}{8}$-s, and $\frac{\pm 3}{8}$-s.

Finally, it is possible to decompose an object which contains points the vertex angles of which are $\frac{4}{8}$-s and $\frac{-4}{8}$-s into a collection of objects which do not contain points the vertex angles of which are $\frac{\pm 4}{8}$-s. Conversely, combining a pair of objects the vertex angles of which are $\frac{1}{8}$-s and $\frac{2}{8}$-s, it is possible to construct an object which contains points the vertex angles of which are $\frac{4}{8}$-s and $\frac{-4}{8}$-s. These geometric properties also conclude that eq. (19) holds for an object the vertex angles of which are 0-s, $\frac{\pm 1}{8}$-s, $\frac{\pm 2}{8}$-s, $\frac{\pm 3}{8}$-s, and $\frac{\pm 4}{8}$-s. Thus, eq. (19) holds for objects without tunnels.

Proof of Theorem 4 From theorem 3 all discrete objects without tunnels are topologically equivalent to an object which has four corners the vertex angles of which are $\frac{1}{8}$-s. These objects are parallelopipedons, which are topologically equivalent to a cube which is a sphere in the 6-connected discrete space.

Considering an object with a tunnel, it is possible to eliminate a tunnel using a wall which is parallel to one of planes $\mathbf{Z}_i^2(\boldsymbol{x}_0)$ for $i = 1, 2, 3$ [4]. Thus, a tunnel is transformed to a pair of wells. Since an object with a pair of wells is an object without tunnels, it is possible to deform this object to an object with four vertices the vertex angles of which are $\frac{1}{8}$-s, eight vertices the vertex angles of which are $\frac{-1}{8}$-s, eight vertices the vertex angles of which are $\frac{-3}{8}$-s. If we eliminate a wall which separates a pair of wells, we obtain a tunnel, and eight vertices the vertex angles of which are $\frac{-1}{8}$-s disappear. Thus, for an object with a tunnel, eq. (20) holds.

Assuming eq. (20) for an object with $(p-1)$ tunnels, for an object with p tunnels, it is possible to operate the same procedure in order to transform a tunnel to a pair of wells. Furthermore, it is also possible to deform a pair of wells to a pair of wells which consist form eight vertices the vertex angle of which are $\frac{-1}{8}$-s, and eight vertices the vertex angles of which are $\frac{-3}{8}$-s. If we eliminate a tunnel which separates a pair of wells, we obtain a tunnel, and eight vertices the vertex angles of which are $\frac{-1}{8}$-s disappear. Thus, for an object with p tunnels, eq. (20) holds.

Proof of Theorem 5 The vertexes of the inner surface holds eq. (19), because the inner surface has the same vertex angles with the vertex angles of the outer surfaces. Since the Euler characteristic of an object with holes is the sum of the Euler characteristics of the outer boundary and the inner boundaries, we have

$$\boldsymbol{a}^{\top}\boldsymbol{n} = 8(1-g) + \sum_{k=1}^{c} 8(1-p_k), \tag{22}$$

where c is the number of holes and p_i is the number of poles in the i-th hole.

6 Conclusions

We introduced the curvature indexes of the boundary of a discrete object, and using these indexes of points, we defined the vertex angles of discrete surfaces as an extension of the chain codes of digital curves. Furthermore, we proved the relation between point configurations and the genus of a discrete object. This is the angular Euler characteristic of a discrete object. This relation derives a parallel algorithm for the computation of the Euler characteristic of a discrete object. In this paper we assumed the 6-connectivity for points in $\mathbf{Z}^3$. However, extensions of the angular Euler characteristics to objects which defined using 18- and 26- connectivities are possible if we define the curvature indexes for these connectivities.

In a series of papers [16] [17] [18] Bieri and Nef proposed sweeping algorithms for the computation of geometric properties of a polytope. The main idea of them is the sweeping of a space by a hyperplane. This idea is equivalent to the decomposition of a space to collections of lower dimensional spaces. This idea goes back to the Radon transform, which is now the theoretical background of computerized tomography [19] [20]. Our method is considered as a discrete version of the space-sweeping algorithm. Our method, however, requires many predetermined planes.

Regular 3-graphs enjoy nice properties for proving the combinatorial characteristics of polyhedrons [14]. The surface of a 6-connected object defines a graph the degrees of nodes of which run from 3 to 6, if we consider points and bonds as nodes and edges, respectively. In the proof of theorem 3, we dealt with an object the vertex angles of which are 0-s, $\pm\frac{1}{8}$-s, $\pm\frac{2}{8}$-s, and $\pm\frac{3}{8}$-s. For these graphs, if we first eliminate all bonds connecting points the vertex angles of which are 0-s, second, remove all isolated nodes, and third change all nodes the vertex angle of which are $\pm\frac{2}{8}$-s to edges, we obtain a 3-graph. In the proof of theorem 4, we changed an object which contains points the vertex angles of which are $\pm\frac{4}{8}$-s to an object without them. Furthermore, in the proof of theorem 5, we used this methodology. Therefore, the proofs provide us rewriting rules from a graph defined by the connectivity of points on the boundary of a discrete object to a 3-graph.

The decomposition of an object into complexes derives the definition of the computation method for the Euler characteristic of a digital object, that is, the total number of the all possible simplexes in an object defines the Euler characteristic. In reference [10], Toriwaki and coworkers defined the topological properties of discrete set using all points in the region of interest [11], and applied this idea for the computation of the Euler characteristic of a discrete binary object. Furthermore, they derived a logic function for the computation of the Euler characteristic. Their method requires all points in the region of interest. Our method, however, requires points on the surface which is extracted using an appropriate method [1] [2] [3]. It is possible to define topological characteristics of objects using only points on the boundary [5]. Our method is a discrete version of this idea. From computational view points, the later method requires less amounts of the memories.

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