Topology

Digital Lighting Functions

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Abstract. In this paper a notion of lighting function is introduced as an axiomatized formalization of the "face membership rules" suggested by Kovalevsky. These functions are defined in the context of the framework for digital topology previously developed by the authors. This enlarged framework provides the (α,β)-connectedness ($\alpha,\beta \in \{6,18,26\}$) defined on $\mathbb{Z}^3$ within the graph-based approach to digital topology. Furthermore, the Kong-Roscoe (α,β)-surfaces, with $(\alpha,\beta) \neq (6,6),(18,6)$, are also found as particular cases of a more general notion of digital surface.

Keywords: Lighting function, digital surface, pixel connectivity, digital topology.

1 Introduction

In [1,2] we introduced a framework for digital topology whose main feature is to provide a link between digital spaces and Euclidean spaces. This framework consists of a multilevel architecture made up of five levels each of them representing a different level of abstraction for a digital picture, increasing from its digital structure to the continuous perception that an observer takes on it.

The starting level is a polyhedral complex, called the device level, which represents the physical layout of the pixels in the digital space, and so the neighbouring relationship considered among them. This relationship is abstracted by means of a graph, called the logical level. Two further levels serve as a bridge towards an Euclidean polyhedron, where every digital picture is associated with a subpolyhedron called its continuous analogue.

With this framework one takes advantage of the knowledge from continuous topology to obtain results in digital topology, by translating, whenever it is possible, not only the statements but also the proofs of the corresponding continuous ones to the logical level. Indeed, this method has allowed us to introduce a general notion of digital n-manifold extending the Morgenthaler (26,6)-surfaces [1], and then to prove a generalized digital index theorem for these n-manifolds [2].

Another interesting aspect of this framework is that it gathers, at least partially, some of the various approaches to digital topology that have been appeared in literature, such as those of Kovalevsky [8], Khalimsky [5], and the graph-based spaces due to Rosenfeld and other authors [7,11,6].

Concerning the latter approach, only some of the most usual graph-based spaces, as the (8,4)- and (26,6)-connected spaces (and their generalization to arbitrary dimension) or the hexagonal one, were found as the logical level of some device level. So that, this framework was not general enough to deal with all the graph-based spaces. This is so because each device level determines a single neighbouring relationship on the pixels, and so the logical level is fixed. The goal of this paper is to present an improved version of that framework in order to avoid this restriction. This is done by adding what we call a lighting function to the architecture quoted above (see §2). This allows us, given a device level, to select the neighbouring relationship we want to work with. In this way, the ability for translating results from continuous topology is preserved, and still the (α, β)-connectedness can be defined in this setting for all pairs (α, β) with $\alpha, \beta \in \{6, 18, 26\}$ (see §4). Furthermore, the (α, β)-surfaces, for $(\alpha, \beta) \neq (6,6), (18,6)$, are also found as particular cases of a more general notion of digital surface (see §5). Finally, it is worth pointing out that this new version provides a single digital notion of connectedness which works for both the digital object and its complement (see §3).

We refer to [12] for all notions in polyhedral topology contained in this paper. For recent trends in digital topology see [4].

2 Lighting functions and digital spaces

As in [1,2], a digital space consists of a multilevel architecture which provides a bridge for transferring definitions, statements and proofs from continuous topology to digital topology.

The first level of a digital space, called the *device level*, is used to represent the spatial layout of the pixels, which are represented by the n-cells of a homogeneously n-dimensional locally finite polyhedral complex K. Namely, K is a complex of convex cells (polytopes) such that each cell is face of a finite number (non-zero) of n-cells. If σ is a face of γ we shall write $\sigma \leq \gamma$. If $\mid K \mid$ denotes the underlying polyhedron of K, a centroid-map is a map $c : K \rightarrow \mid K \mid$ such that $c(\sigma)$ belongs to the interior of σ. The set of all n-cells of K will be denoted by $\text{cell}_n(K)$. Given a device level K, a *digital object in* K is a subset of the set $\text{cell}_n(K)$ of n-cells in K.

To avoid connectivity paradoxes, Kovalevsky points out in [8] the convenience of associating to each digital object some set of lower dimensional cells in K. These cells would indicate which pairs of n-cells should be considered adjacent. For this, Kovalevsky makes two proposals. On one hand, he suggests to encode a digital image by specifying not only what pixels (n-cells) are in the object but also the faces of these pixels which are associated to it. On the other hand, to save memory space, Kovalevsky observes that some global face membership rule can be used; that is, "a rule specifying the set membership of the faces of every n-cell as a function of the membership of the n-cell itself". We have adopted this last point of view, which has been formalized through the notion of lighting function. To introduce this notion we need the following notation.

Given a cell $\alpha \in K$ and a digital object $O \subseteq \text{cell}_n(K)$, the *star of* α *in* O is the set $\text{st}_n(\alpha; O) = \{\sigma \in O : \alpha \leq \sigma\}$, and the *support of* O, $\text{supp}(O)$, is the set of all cells $\alpha \in K$ such that $\alpha = \cap\{\sigma : \sigma \in \text{st}_n(\alpha; O)\}$. Observe that if $\text{st}_n(\alpha; O)$ has only one element, then $\alpha \in \text{supp}(O)$ if and only if $\alpha \in O$, and thus $\text{st}_n(\alpha; O) = \{\alpha\}$. To ease the writing, when the digital object is the whole set $\text{cell}_n(K)$ we shall write $\text{supp}(K)$ and $\text{st}_n(\alpha; K)$ instead of $\text{supp}(\text{cell}_n(K))$ and $\text{st}_n(\alpha; \text{cell}_n(K))$, respectively. Finally, we shall write $\mathcal{P}(A)$ for the family of all subsets of a given set A.

Definition 1. Given a complex K, a function $f : \mathcal{P}(\text{cell}_n(K)) \times K \to \{0, 1\}$ is said to be a *lighting function* on K if it verifies the following properties for all $O \in \mathcal{P}(\text{cell}_n(K))$ and $\alpha \in K$.

(F1) If $\alpha \notin \text{supp}(O)$ then $f(O, \alpha) = 0$.

(F2) If $\alpha \in O$ then $f(O, \alpha) = 1$.

(F3) $f(O, \alpha) = f(\text{st}_n(\alpha; O), \alpha)$.

(F4) $f(O, \alpha) \leq f(\text{cell}_n(K), \alpha)$.

In this way, given a digital object O, the complex K is partitioned into two subsets of cells. Namely, $\{\alpha \in K : f(O, \alpha) = 1\}$ which is associated to the object, and $\{\alpha \in K : f(O, \alpha) = 0\}$ associated to its complement. In addition, these properties formalize very natural and intuitive ideas. Property (F2) expresses that in order to display a digital object its pixels must be lighted, while (F1) says that the cells which are not the intersection of pixels of the object have nothing to do with its connectivity, and so we choose to get them dark. Property (F3) states that for a given object the lighting of a cell is a local property of the object; and finally, (F4) says that a cell $\alpha \in K$ is lighted for the global object $\text{cell}_n(K)$ whenever it is lighted for some small object $O \subseteq \text{cell}_n(K)$.

Given a lighting function f on K, the *logical level of a digital object* O is an undirected graph, $\mathcal{L}_O^f$, whose vertices are the centroids of n-cells in O and two of them $c(\sigma)$, $c(\tau)$ are adjacent if there exists a common face $\alpha \leq \sigma \cap \tau$ such that $f(O, \alpha) = 1$.

The *conceptual level of* O is the digraph $\mathcal{C}_O^f$ whose vertices are the centroids $c(\alpha)$ of all cells $\alpha \in K$ with $f(O, \alpha) = 1$, and its directed edges are $(c(\alpha), c(\beta))$ with $\alpha \leq \beta$.

The *simplicial analogue of* O is the order complex $\mathcal{A}_O^f$ associated to the digraph $\mathcal{C}_O^f$. That is, $\langle x_0, x_1, \ldots, x_m \rangle$ is an m-simplex of $\mathcal{A}_O^f$ if $x_0, x_1, \ldots, x_m$ is a directed path in $\mathcal{C}_O^f$. This simplicial complex defines the simplicial level for the object O in the architecture and, finally, the continuous level is represented by the underlying polyhedron $| \mathcal{A}_O^f |$ of $\mathcal{A}_O^f$. This polyhedron is called the *continuous analogue of* O.

Example 1. Every polyhedral complex $K \neq \emptyset$ admits the following lighting functions: $f_{\max}(O, \alpha) = 1$ if and only if $\alpha \in \text{supp}(O)$; $f_{\min}(O, \alpha) = 1$ if and only if $\alpha \in O$; and, $g(O, \alpha) = 1$ if and only if $\alpha \in \text{supp}(O)$ and $\text{st}_n(\alpha; K) \subseteq O$.

Notice that both $f_{\max}$ and g are distinct from $f_{\min}$ only if there exist two n-cells in K with a common face. On the other hand, if a cell $\alpha \in K$ is the intersection of a proper subset of n-cells in $\text{st}_n(\alpha; K)$ then $g \neq f_{\max}$. Moreover, each one of these lighting functions may induce different levels for a given digital

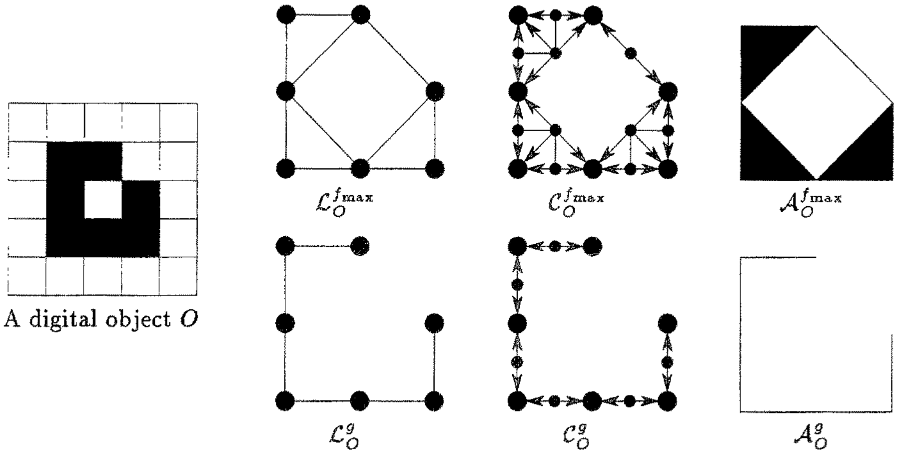

Fig. 1. Logical and conceptual levels, and simplicial analogues of a digital object for $f_{\max}$ and g in Example 1.

object, as Figure 1 shows for $f_{\max}$ and g. In this example, all levels for $f_{\min}$ are the same; namely, a set of seven discrete points. From this example it is evident that the levels for a digital object depend both on the complex K and on the lighting function f considered on it. So that, we define the notion of digital space as follows.

Definition 2. A *digital space* is a pair (K, f), where K is a homogeneously n-dimensional locally finite polyhedral complex and f a lighting function on K.

From now on, when working with a digital space (K, f) and if there is no place to confusion, for a digital object O we shall write $\mathcal{L}_O$, $\mathcal{C}_O$, $\mathcal{A}_O$ and $\mid \mathcal{A}_O \mid$ instead of $\mathcal{L}_O^f$, $\mathcal{C}_O^f$, $\mathcal{A}_O^f$ and $\mid \mathcal{A}_O^f \mid$ respectively to denote the corresponding levels of O. Moreover, if the digital object is the set $\text{cell}_n(K)$ of all n-cells in K we shall write $\mathcal{L}_K, \mathcal{C}_K, \mathcal{A}_K$ and $\mid \mathcal{A}_K \mid$ for its levels, which will be called the levels of the whole digital space (K, f).

Next, we introduce some structural properties about digital spaces whose proofs are straightforward.

Proposition 3. *Let (K, f) be a digital space and $O \subseteq \text{cell}_n(K)$ a digital object. Then (1) $\mathcal{L}_O$ is a (not necessarily full) subgraph of $\mathcal{L}_K$; (2) $\mathcal{C}_O$ is a full subgraph of $\mathcal{C}_K$; (3) $\mathcal{A}_O$ is a full subcomplex of $\mathcal{A}_K$.*

Clearly, any lighting function f on K is the characteristic function of some subset of $\mathcal{P}(\text{cell}_n(K)) \times K$. However, not all the subsets of $\mathcal{P}(\text{cell}_n(K)) \times K$ define a lighting function (for instance, if $K \neq \emptyset$, the characteristic function of the empty set does not verify the property (F2)).

Theorem 4. *The set of all lighting functions on a given complex K is a distributive complete lattice, whose greatest and least elements are $f_{\max}$ and $f_{\min}$, respectively. Moreover, it is a Boolean algebra if and only if $f_{\max} = g$.*

3 Connectedness in digital spaces

In this section we introduce the notion of connectedness for subsets of n-cells in a digital space. This notion includes, as particular cases, the connectedness for both digital objects and their complements. Afterwards, we shall prove that this notion of connectedness coincides with the corresponding topological notion in the continuous analogue.

Definition 5. Let O and O' be two disjoint digital objects in a digital space (K, f). Two distinct n-cells $\sigma, \tau \in O$ are said to be *O'-adjacent in O* if there exists a common face $\alpha \leq \sigma \cap \tau$ such that $f(O', \alpha) = 0$ and $f(O \cup O', \alpha) = 1$. An *O'-path in O from σ to τ* is a finite sequence $\{\sigma_i\}_{i=0}^m \subseteq O$ such that $\sigma_0 = \sigma$, $\sigma_m = \tau$ and σ_{i-1} is O'-adjacent in O to σ_i, for $i = 1, \ldots, m$.

The digital object O will be said *O'-connected* if for any pair of n-cells $\sigma, \tau \in O$ there exists an O'-path in O from σ to τ. An object $C \subseteq O$ is an *O'-component* of O if for any pair $\sigma, \tau \in C$ there exists an O'-path in O from σ to τ and none element in C is O'-adjacent in O to some element of $O - C$. Observe that any O'-component is O'-connected.

Given a digital object O in the digital space (K, f) the previous definitions provide an entire family of notions of connectedness for O in relation to another object O', when O' is allowed to range over the set of all subsets of $\text{cell}_n(K) - O$. The extreme cases, when $O' = \emptyset$ and $O' = \text{cell}_n(K) - O$, represent the connectedness of the digital object O itself and the connectedness of O as the complement of O', respectively.

Following this line, we will call *connected* to any object which is $\emptyset$-connected, and C is a *component* of O if it is a $\emptyset$-component. Moreover, it is easy to check that two n-cells $\sigma, \tau \in O$ are $\emptyset$-adjacent in O if and only if there exists $\alpha \leq \sigma \cap \tau$ such that $f(O, \alpha) = 1$. So that, $\sigma, \tau \in O$ are $\emptyset$-adjacent in O if and only if their centroids $c(\sigma)$ and $c(\tau)$ are adjacent as vertices of the logical level $\mathcal{L}_O$ of O. This justifies to call *adjacent in O* to any pair of n-cells which are $\emptyset$-adjacent in O, and then a *path in O* is just a $\emptyset$-path in O. These observations prove the following result.

Proposition 6. *A digital object O is connected if and only if its logical level $\mathcal{L}_O$ is a connected graph.*

Furthermore, when O is considered as the complement of the digital object $O' = \text{cell}_n(K) - O$, we get that $\sigma, \tau \in O$ are O'-adjacent in O if and only if there exists $\alpha \leq \sigma \cap \tau$ such that $f(O', \alpha) = 0$ and $f(\text{cell}_n(K), \alpha) = 1$. So that, two n-cells σ, τ are O'-adjacent in the complement of O' if and only if there exists $\alpha \in K$ whose centroid $c(\alpha)$ is adjacent to both $c(\sigma)$ and $c(\tau)$ in the complement $\mathcal{C}_K \setminus \mathcal{C}_{O'}$ of the conceptual level $\mathcal{C}_{O'}$ in $\mathcal{C}_K$. In this way, the connectedness of the complement of an object can be characterized in the conceptual level, but not in the logical level. Indeed, the complement of the object O shown in Figure 1 it is not O-connected in the digital space $(K, f_{\max})$, while the complement of the logical level $\mathcal{L}_O^{f_{\max}}$ of O in $\mathcal{L}_K^{f_{\max}}$ is connected.

Theorem 7 shows how these notions of connectedness are stated at each level of our architecture. This result is an immediate consequence of Theorem 8. Below, $L_1 \setminus L_2 = \{\alpha \in L_1 : \alpha \cap \mid L_2 \mid = \emptyset\}$ will stand for the *simplicial complement of* L_2 *in* L_1, where L_1 and L_2 are subcomplexes of a simplicial complex L.

Theorem 7. *Let O be a digital object. The following properties are equivalent: (1) O is connected. (2) $\mathcal{L}_O$ is a connected graph. (3) $\mathcal{C}_O$ is a connected digraph. (4) $\mathcal{A}_O$ is a connected simplicial complex. (5) $\mid \mathcal{A}_O \mid$ is a connected space.*

Moreover, if $O' = cell_n(K) - O$, the following properties are equivalent: (1) O is O'-connected. (2) $\mathcal{C}_K \setminus \mathcal{C}_{O'}$ is a connected digraph. (3) $\mathcal{A}_K \setminus \mathcal{A}_{O'}$ is a connected simplicial complex. (4) $\mid \mathcal{A}_K \mid - \mid \mathcal{A}_{O'} \mid$ is a connected space.

Theorem 8. *Let O and O' be two disjoint digital objects in a digital space. The family $\mathcal{F}$ of O'-components of O can be described in any of the following ways*

(1) Conceptual level: $\mathcal{F} = \{O_G\}$, where $O_G = \{\sigma \in O : c(\sigma)$ is a vertex of $G\}$, and G ranges over the family of components of the digraph $\mathcal{C}_{O \cup O'} \setminus \mathcal{C}_{O'}$.

(2) Simplicial level: $\mathcal{F} = \{O_A\}$, where $O_A = \{\sigma \in O : c(\sigma) \in A\}$, and A ranges over the family of components of the simplicial complement $\mathcal{A}_{O \cup O'} \setminus \mathcal{A}_{O'}$.

(3) Continuous level: $\mathcal{F} = \{O_X\}$, where $O_X = \{\sigma \in O : c(\sigma) \in X\}$, and X ranges over the family of components of the space $\mid \mathcal{A}_{O \cup O'} \mid - \mid \mathcal{A}_{O'} \mid$.

We sketch a proof of this theorem. Firstly, the characterization in the conceptual level can be readily proved from the following proposition.

Proposition 9. (i) *Let $O_2 \subseteq O_1$ be two digital objects. If $c(\tau)$ is a vertex of the complement $\mathcal{C}_{O_1} \setminus \mathcal{C}_{O_2}$ then there exists an n-cell $\sigma \in O_1 - O_2$ such that $\tau \leq \sigma$.*

(ii) *Let O and O' be two disjoint digital objects in a digital space. Given two distinct n-cells $\sigma, \tau \in O$ there exists a O'-path in O from σ to τ if and only if their centroids $c(\sigma)$ and $c(\tau)$ are vertices of the same component of $\mathcal{C}_{O \cup O'} \setminus \mathcal{C}_{O'}$.*

Next, to obtain the characterization in the simplicial level from that in the conceptual level it is enough to observe that $\mathcal{C}_{O \cup O'} \setminus \mathcal{C}_{O'}$ can be identified with the 1-skeleton of $\mathcal{A}_{O \cup O'} \setminus \mathcal{A}_{O'}$. Finally, the characterization in the continuous level follows from the simplicial one and the next lemma from simplicial topology.

Lemma 10. *Let $L \subseteq K$ be a full subcomplex of a locally finite simplicial complex K. Then, the components of $\mid K \mid - \mid L \mid$ are in 1-1 correspondence with the components of $K \setminus L$.*

4 Lighting functions for the (α, β)-connectedness

Within the graph-based approach to digital spaces, due to Rosenfeld and other authors, many graphs on almost arbitrary grids of points have been used [7]. However, this section is only concerned with the most usual connectedness on the grid $\mathbb{Z}^3$, defined by means of the double adjacency (α, β), with $\alpha, \beta \in \{6, 18, 26\}$. The α-adjacency is considered to define the connection for digital objects and the β-adjacency for their complements. See [6] for a precise definition.

In this section we show how all types of (α, β)-connectedness on the grid $\mathbb{Z}^3$ can be recovered in our framework by selecting suitable lighting functions. These functions will be defined on the device level R^3, called the *standard cubical decomposition* of the 3-dimensional Euclidean space $\mathbb{R}^3$. That is, R^3 is the complex determined by the collection of unit 3-cubes in $\mathbb{R}^3$ whose edges are parallel to the coordinate axes and whose centres are in the set $\mathbb{Z}^3$. The centroid-map we will consider in R^3 associates to each cube σ its barycentre $c(\sigma)$. In particular, if $\dim \sigma = 3$ then $c(\sigma) \in \mathbb{Z}^3$, where $\dim \sigma$ stands for the dimension of σ. So that, every digital object O in R^3 can be identified with a subset of points in $\mathbb{Z}^3$. Henceforth we shall use this identification without further comment.

Definition 11. We say that a lighting function $f_{\alpha,\beta}$ on R^3 *provides the* (α, β)-*connectedness* if the two following properties hold for any digital object O: (1) O is connected if and only if it is α-connected; and, (2) whenever O is considered as the complement of the object $O' = \text{cell}_3(R^3) - O$, then O is O'-connected if and only if it is β-connected.

In Example 2 below we give some lighting functions providing the (α, β)-connectedness for each pair (α, β). For this we consider a new polyhedral decomposition of $\mathbb{R}^3$, denoted $R^3(\mathbb{Z}^3)$, consisting of unit cubes with vertices in $\mathbb{Z}^3$. To avoid misunderstandings, we keep the terminology "cube" for the 3-cells in R^3 and we call $\mathbb{Z}^3$-cell to the closed cubes in $R^3(\mathbb{Z}^3)$. Given a digital object O in R^3 and a $\mathbb{Z}^3$-cell C, the *configuration of* O *in* C is the set $C(O) = \{c(\sigma) \in \mathbb{Z}^3 : \sigma \in O\} \cap C$ of vertices of C which are the centroids of cells in O. Observe that the centre of C coincides with some 0-cell $\rho \in R^3$. So that, $C(O)$ is the set of centroids of cubes in $\text{st}_3(\rho; O)$. In Figure 2 are shown all the possible configurations of a given object after a suitable rotation or reflection.

Example 2. The lighting functions $f^n_{\alpha,\beta}$ listed below are providing the corresponding (α, β)-connectedness, for all pairs (α, β) with $\alpha, \beta \in \{6, 18, 26\}$. To prove this fact is a tedious but mechanical task, which involves to check properties (F1)-(F4) in Definition 1 and to prove that the components and the $(\text{cell}_3(R^3) - O)$-components of a given digital object O coincide with the α-components and the β-components, respectively.

a) $f^0_{6,6}(O, \alpha) = 1$ iff $\alpha \in O$ for $\dim \alpha = 3$, and $\alpha \in \text{supp}(O)$ for $\dim \alpha = 2$.

b) $f^0_{6,18}(O, \alpha) = 1$ iff $\alpha \in O$ for $\dim \alpha = 3$, $\alpha \in \text{supp}(O)$ for $\dim \alpha = 2$, $\text{st}_3(\alpha; R^3) \subseteq O$ for $\dim \alpha = 1$, and $\text{st}_3(\alpha; O)$ contains the configuration (6c) in Figure 2 for $\dim \alpha = 0$.

c) $f^1_{6,18}(O, \alpha) = f^0_{6,18}(O, \alpha)$ for $\dim \alpha = 3, 2, 0$, and, for $\dim \alpha = 1$, $f^1_{6,18}(O, \alpha) = 1$ iff $|\text{st}_3(\alpha; O)| \geq 3$.

d) $f^0_{6,26}(O, \alpha) = 1$ iff $\text{st}_3(\alpha; R^3) \subseteq O$ for any cell $\alpha \in R^3$.

e) $f^0_{18,6}(O, \alpha) = 1$ iff $\alpha \in O$ for $\dim \alpha = 3$, $\alpha \in \text{supp}(O)$ for $\dim \alpha = 1, 2$, and $\text{st}_3(\alpha; O)$ contains either the configuration (3c) or (4e) for $\dim \alpha = 0$.

f) $f^1_{18,6}(O, \alpha) = 1$ iff $\alpha \in O$ for $\dim \alpha = 3$, $\alpha \in \text{supp}(O)$ for $\dim \alpha = 1, 2$, and $|\text{st}_3(\alpha; O)| \geq 3$ and $\alpha \in \text{supp}(O)$ for $\dim \alpha = 0$.

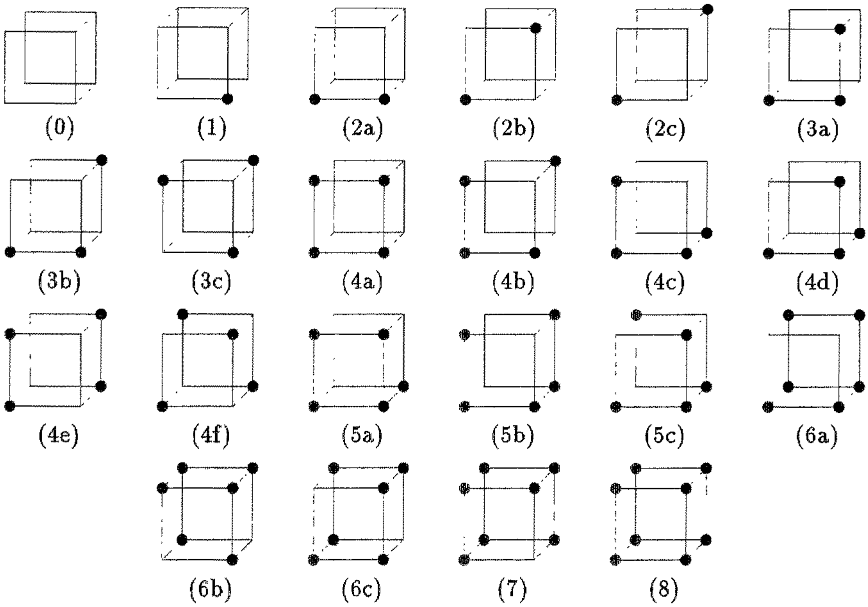

Fig. 2. Possible configurations of a digital object.

g) $f^2_{18,6}(O,\alpha) = 1$ iff $\alpha \in O$ for $\dim\alpha = 3$, $\alpha \in \text{supp}(O)$ for $\dim\alpha = 2$, $\alpha \in \text{supp}(O)$ and $|\text{st}_3(\alpha;O)| = 2,4$ for $\dim\alpha = 1$, and $\text{st}_3(\alpha;O)$ is one of the configurations (3c), (4b), (4d), (4e), (4f) or $|\text{st}_3(\alpha;O)| \geq 5$ for $\dim\alpha = 0$.

h) $f^0_{18,18}(O,\alpha) = 1$ iff $\alpha \in O$ for $\dim\alpha = 3$, $\alpha \in \text{supp}(O)$ for $\dim\alpha = 1,2$, and $\text{st}_3(\alpha;O)$ contains the configuration (6c) in Figure 2 for $\dim\alpha = 0$.

i) $f^1_{18,18}(O,\alpha) = 1$ iff $\alpha \in O$ for $\dim\alpha = 3$, $\alpha \in \text{supp}(O)$ for $\dim\alpha = 2$, $\alpha \in \text{supp}(O)$ and $|\text{st}_3(\alpha;O)| = 2,4$ for $\dim\alpha = 1$, and $\text{st}_3(\alpha;O)$ is one of the configurations (6c), (7) or (8) for $\dim\alpha = 0$.

j) $f^0_{18,26}(O,\alpha) = 1$ iff $\alpha \in O$ for $\dim\alpha = 3$, $\alpha \in \text{supp}(O)$ for $\dim\alpha = 1,2$, and $|\text{st}_3(\alpha;O)| \geq 7$ for $\dim\alpha = 0$.

k) $f^0_{26,6}(O,\alpha) = 1$ iff $\alpha \in \text{supp}(O)$ for any cell $\alpha \in R^3$; that is, $f^0_{26,6} = f_{\max}$.

l) $f^0_{26,18}(O,\alpha) = 1$ iff $\alpha \in O$ for $\dim\alpha = 3$, $\alpha \in \text{supp}(O)$ for $\dim\alpha = 2$, $\alpha \in \text{supp}(O)$ and $|\text{st}_3(\alpha;O)| = 2,4$ for $\dim\alpha = 1$, and $\text{st}_3(\alpha;O)$ is one of the configurations (2c), (4c), (5a), (6a), (6c), (7) or (8) for $\dim\alpha = 0$.

m) $f^0_{26,26}(O,\alpha) = 1$ iff $\alpha \in O$ for $\dim\alpha = 3$, $\alpha \in \text{supp}(O)$ for $\dim\alpha = 2$, $\alpha \in \text{supp}(O)$ and $|\text{st}_3(\alpha;O)| = 2,4$ for $\dim\alpha = 1$, and $\text{st}_3(\alpha;O)$ is one of the configurations (2c), (6a), (7) or (8) for $\dim\alpha = 0$.

Observe that, in general, the (α,β)-connectedness can be provided by several lighting functions. However, it is not difficult to prove that $f^0_{6,6}$ is the only function providing the (6,6)-connectedness.

It can be readily checked that the simplicial analogue of the whole space $(R^3, f^n_{\alpha,\beta})$ is the barycentric subdivision of R^3 for all the lighting functions given in Example 2, except for the case $f^0_{6,6}$. Thus, their continuous analogues are always the 3-dimensional Euclidean space $\mathbb{R}^3$. The continuous analogue of the special case $(R^3, f^0_{6,6})$ is the subset $(\mathbb{Z}\times\mathbb{Z}\times\mathbb{R})\cup(\mathbb{Z}\times\mathbb{R}\times\mathbb{Z})\cup(\mathbb{R}\times\mathbb{Z}\times\mathbb{Z}) \subseteq \mathbb{R}^3$.

5 About digital surfaces

In Section 3 we have used our multilevel architecture to show that a merely combinatorial definition, as the notion of O'-connectedness, is a correct counterpart of the topological notion we have at the continuous level. But, in order to define a digital notion, one may proceed along the inverse way. That is, given a topological property P, we can say that a digital object O satisfies the digital counterpart of P by requiring that the continuous analogue $\mathcal{A}_O$ satisfies P. However, doing that, it arises the problem of characterizing property P at a level as close to the logical one as possible.

In this section we will present a case of this method, by defining the notion of digital surface throughout the continuous analogue of objects and then, finding characterizations in the logical level for those digital spaces $(R^3, f^n_{\alpha,\beta})$ in Example 2.

Definition 12. A digital object S in a digital space (K, f) is said to be a *digital surface* if its continuous analogue $\mid \mathcal{A}_S \mid$ is a surface without boundary. We will call S a *f-surface* in case the digital space is (R^3, f).

Kong-Roscoe [6], generalizing the Morgenthaler-Rosenfeld surfaces [11], define in $\mathbb{Z}^3$ the notion of (α,β)-surface for all pairs $\alpha,\beta \in \{6,18,26\}$. Next theorem states the characterization in the logical level of the $f^n_{\alpha,\beta}$-surfaces throughout their relation with the corresponding (α,β)-surfaces.

Theorem 13. *The following properties are verified for the lighting functions given in Example 2.*

(1) The family of $f^n_{\alpha,\beta}$-surfaces coincides with the corresponding family of (α,β)-surfaces for the lighting functions: $f^0_{6,26}$, $f^0_{6,18}$, $f^0_{18,26}$, $f^0_{26,6}$, $f^0_{26,18}$ and $f^0_{26,26}$.

(2) The family of $f^n_{\alpha,\beta}$-surfaces is strictly contained in the family of (α,β)-surfaces in the cases: $f^1_{6,18}$, $f^0_{18,6}$, $f^1_{18,6}$ and $f^0_{18,18}$. In addition, the families of $f^0_{18,6}$-surfaces and $f^1_{18,6}$-surfaces coincide.

(3) The families of (18,6)-surfaces and (18,18)-surfaces are strictly contained in the families of $f^2_{18,6}$-surfaces and $f^1_{18,18}$-surfaces, respectively.

The equality between the families of $f^0_{26,6}$-surfaces and (26,6)-surfaces was originally proved in [1]. The same technique can be adapted for each one of the remaining cases in Theorem 13. Notice that we only get an inclusion in (2) because not all the configurations permitted in an (α,β)-surface can appear in an $f^n_{\alpha,\beta}$-surface. Also we only get an inclusion in (3) since an $f^2_{18,6}$-surface may contain the configuration (5b) in Figure 2 which is not permitted in a (18,6)-surface.

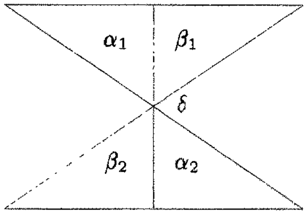

Fig. 3. A polyhedral complex K_0.

Observe that Theorem 13 shows that the family of (α,β)-surfaces, for $(\alpha,\beta) \neq (6,6),(18,6)$, coincides with the family of $f^n_{\alpha,\beta}$-surfaces for some lighting function $f^n_{\alpha,\beta}$. As it was pointed out by Kong-Roscoe in [6], there exists (6,6)-surfaces whose complement is 6-connected. So that, these are not truly digital surfaces. This fact agrees with our point of view because the continuous analogue of the only digital space $(R^3, f^0_{6,6})$ providing the (6,6)-connectedness is 1-dimensional. Concerning the pair (18,6), it can be proved that there exists no lighting function such that the corresponding family of digital surfaces exactly coincides with the (18,6)-surfaces. These surfaces have, in some sense, an anomalous behaviour with respect the others. In a future work we will intend a more detailed analysis of them; and moreover we also plan to find lighting functions gathering other definitions of digital surface, as those due to Malgouyres [9,10].

6 Final discussions

In this last section we want to discuss some interesting points of our framework as it is compared with other approaches to digital topology.

Firstly, we focus our attention on our notion of connectedness (Definition 5), which is slightly different from that normally used in abstract cell complexes. To explain this difference, let us consider the polyhedral complex K_0 in Fig. 3, where we will distinguish two digital objects $A = \{\alpha_1, \alpha_2\}$ and $B = \{\beta_1, \beta_2\}$, and the vertex δ. Let us also consider the lighting function f ("membership rule" in Kovalevsky's terminology) defined on K_0 by $f(O,\gamma) = 1$ for any digital object $O \subseteq \text{cell}_n(K_0)$ and any cell $\gamma \in \text{supp}(O)$ with $\gamma \neq \delta$, and $f(O,\gamma) = 0$ otherwise. Observe that the "membership rule" f associates the 0-cell δ only to complements of objects; see the comment after Definition 1.

Let now consider the digital space (K_0, f); i.e., the set $\text{cell}_n(K_0)$ of all pixels in K_0 together with the set of lower dimensional cells associated to it

According to the arcwise connectedness usually defined on cell complexes (see Definition 4 in [8]), the digital space (K_0, f) has two connected components whose sets of pixels are $\{\alpha_1, \beta_1\}$ and $\{\alpha_2, \beta_2\}$ respectively. Moreover, if we consider B as an object, its complement A is connected although A meets both components $\{\alpha_1, \beta_1\}$ and $\{\alpha_2, \beta_2\}$. This situation is avoided with our notion of connectedness.

The crucial point is that, in the usual definition, each lower dimensional cell γ associated to the complement $\text{cell}_n(K) - O$ of an object O is connecting the pixels

(n-cells) in the star of γ in $\text{cell}_n(K) - O$. In contrast our Definition 5 expresses only that γ is not a cut-point for $\text{st}_n(\gamma; \text{cell}_n(K) - O)$. In order to ensure that γ connects $\text{st}_n(\gamma; \text{cell}_n(K) - O)$ it is required in addition that γ is lighted in the global object $\text{cell}_n(K)$; i.e., γ connects the pixels in $\text{st}_n(\gamma; \text{cell}_n(K))$.

In relation to the notion of connectedness used in the graph-based approach to digital topology, it may seem puzzling the existence of lighting functions providing the (α, β)-connectedness, for $\alpha, \beta \in \{18, 26\}$. These pairs are usually discarded on the ground that their restrictions on the grid $\mathbb{Z}^2 \times \{0\} \subseteq \mathbb{Z}^3$ produce the paradoxical $(8,8)$-connectedness. However, this is not the case of our lighting functions $f_{\alpha,\beta}$.

For instance, consider the lighting function $f^0_{18,18}$ given in Example 2(h) and the subcomplex $R^3_0 = \{\alpha \in R^3 : \alpha \leq \sigma, c(\sigma) \in \mathbb{Z}^2 \times \{0\}\}$ of R^3. It is easy to show that the restriction of $f^0_{18,18}$ to the plane R^3_0, given by $f^0_{18,18} \mid_{R^3_0} (O, \alpha) = 1$ if and only if $\alpha \in \text{supp}(O)$ for any object $O \subseteq \text{cell}_3(R^3_0)$ and any cell $\alpha \in R^3_0$, is a lighting function providing the $(8,4)$-connectedness.

Next, we will illustrate how is this possible through an example. Let consider the digital object $O = \{\sigma_n \in \text{cell}_3(R^3) : c(\sigma_n) = (n, n, 0), n \in \mathbb{Z}\}$ in the digital space $(R^3, f^0_{18,18})$ consisting of a diagonal line in the digital plane R^3_0, and let $\tau_1, \tau_2 \in \text{cell}_3(R^3) - O$ be the two only 3-cells sharing the edge $\gamma = \sigma_1 \cap \sigma_2$. According to the $(18,18)$-adjacency both pairs of 3-cells, σ_1, σ_2 and τ_1, τ_2, should be adjacent. Indeed, σ_1 and σ_2 are adjacent (i.e., $\emptyset$-adjacent) in O through their common edge γ, while τ_1 and τ_2 are O-adjacent in $\text{cell}_3(R^3) - O$ through any one of the extremes of γ. On one hand, O and $\text{cell}_3(R^3) - O$ are connected and O-connected objects respectively, as well as they are 18-connected. On the other hand, consider the same object O and cells τ_1, τ_2 in the digital subspace $(R^3_0, f^0_{18,18} \mid_{R^3_0})$. In this space, σ_1 and σ_2 are again adjacent in O through γ. But now, τ_1 and τ_2 are not O-adjacent in $\text{cell}_3(R^3_0) - O$ because the extremes of γ do not belong to $\text{supp}(R^3_0)$. Thus, O is connected as an object in the digital subspace $(R^3_0, f^0_{18,18} \mid_{R^3_0})$, but its complement $\text{cell}_3(R^3_0) - O$ is not O-connected.

Finally, we are going to justify briefly why a multilevel architecture seems to us very suitable for the development of digital topology.

The goal of digital topology is to analize and to study topological properties on digital objects, under the assumption that, although these objects have strictly a discrete nature, they are perceived as continuous objects. Because of this, any framework for digital topology should contain at least these two levels: a digital one, in which digital images can be easily processed, and a continuous level, where digital methods and results can be justified in accordance with the continuous perception of objects. Obviously these levels are of very different nature and we have considered convenient to introduce some other levels which, in conjunction with suitable transformations, make easier the translation of notions and results between the digital and continuous levels.

This architecture has allowed us to define, in a very natural way, a continuous analogue close enough to the perception of objects. And, what has greater importance, it makes possible to reuse knowledges and experiences of continuous topology, by translating them to the digital level throughout the whole architecture. In addition, it is worth pointing out that our architecture gathers in its levels the somehow scattered proposals by other authors. Actually, the device level corresponds to Kovalevsky's approach, the logical level falls within the graph-based models due to Rosenfeld and others, and finally, Khalimsky's spaces are a particular case of the conceptual level.

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