Contact Graphs of Curves
(extended abstract)

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Abstract. Contact graphs are a special kind of intersection graphs of geometrical objects in which we do not allow the objects to cross but only to touch each other. Contact graphs of simple curves (and line segments as a special case) in the plane are considered. Several classes of contact graphs are introduced and their properties and inclusions between them are studied. Also the relation between planar and contact graphs is mentioned. Finally, it is proved that the recognition of contact graphs of curves (line segments) is *NP*-complete (*NP*-hard) even for planar graphs.

1 Introduction

The intersection graphs of geometrical objects have been extensively studied for their many practical applications. Formally the *intersection graph* of a set family $\mathcal{M}$ is defined as a graph G with the vertex set $V(G) = \mathcal{M}$ and the edge set $E(G) = \big\{\{A,B\} \subseteq \mathcal{M} \,|\, A \neq B,\ A \cap B \neq \emptyset\big\}$. Probably the first type studied were interval graphs (intersection graphs of intervals on a line), owing to their applications in biology, see [14],[1]. We may also mention other kinds of intersection graphs such as the intersection graphs of chords of a circle (circle graphs [2]), of boxes in the space [15], of curves or line segments in the plane [3],[16],[11],[12].

A special type of geometrical intersection graphs—the *contact graphs*, for which we do not allow the geometrical objects to cross but only to touch each other, are considered here. Unlike the general intersection graphs, in this field only a few results are known. There is a nice old result of Koebe [10] about representations of planar graphs as contact graphs of circles in the plane. In [5] a similar result about contact graphs of triangles is proved. The contact graphs of line segments are considered in [4],[6] and [17]. It is proved that every bipartite planar graph is a contact graph of vertical and horizontal line segments [4], and for contact graphs of line segments of any direction, with contact of 2 segments in one contact point, a characterization is given in [17].

We follow the ideas of intersection graphs of curves and of contact graphs of segments, and generalize the definition to contact graphs of simple curves in the plane. We also allow a contact of more than 2 curves in a point and for such contact points we distinguish one-sided and two-sided contacts. We define classes of contact graphs and study the inclusions among them and their properties such as the maximal clique and the chromatic number. Then we show some relations of contact graphs and planar graphs, and using them we prove the main result—that the recognition of contact graphs of curves (line segments) is *NP*-complete (*NP*-hard) even for planar graphs, while the similar question for planar triangulations is solvable in polynomial time.

For complete proofs of the presented results refer to [9].

2 Curve Contact Representations

2.1 Definitions

Simple curves of finite length (Jordan curves) in the plane are considered. Each curve has two *endpoints* and all of its other points are called interior points; they form the *interior* of the curve. We say that a curve φ *ends (passes through)* in a point X if X is an endpoint (interior point) of φ.

Definition. A finite set $\mathcal{R}$ of curves in the plane is called a *curve contact representation* of a graph G if interiors of any two curves of $\mathcal{R}$ are disjoint and G is the intersection graph of $\mathcal{R}$. The graph G is called the *contact graph* of $\mathcal{R}$ and denoted by $G(\mathcal{R})$. A curve contact representation $\mathcal{R}$ is said to be a *line segment contact representation* if each curve of $\mathcal{R}$ is a line segment.
A graph H is called a *contact graph of curves* (*contact graph of line segments*) if there exists a curve contact representation (line segment contact representation) $\mathcal{S}$ such that $H \cong G(\mathcal{S})$.

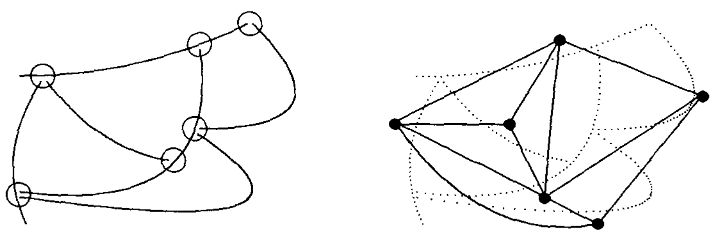

Fig. 1. An example of a curve contact representation of a graph

A curve contact representation is called simply a *representation*, a contact graph of curves simply *contact graph*. Any subset $\mathcal{S} \subseteq \mathcal{R}$ is called a *subrepresentation* of $\mathcal{R}$. A point C of the plane is said to be a *contact point* of a representation $\mathcal{R}$ if it is contained in at least two curves of $\mathcal{R}$. The *degree* of a contact point C in $\mathcal{R}$ is the number of curves of $\mathcal{R}$ containing C, a contact point of degree k is called a *k-contact point.* Note that for any k–contact point

C of a representation $\mathcal{R}$ either all k curves of $\mathcal{R}$ containing C end in C or one curve is passing through C and the other $k-1$ curves end in C. We say an endpoint of a curve is *free* if it is not a contact point.

In Figure 1 an example of a curve contact representation and its contact graph is given. For a better view of the representation we introduce the following convention for drawing the contact representations: Each contact point is emphasized by a circle around it and the curves of this contact point are drawn into the circle but not necessarily touching each other.

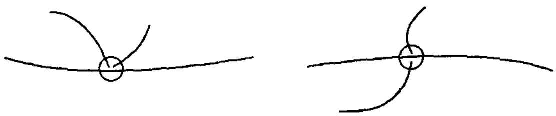

Fig. 2. The difference between one-sided and two-sided contact points

For contact representations there is an important difference in contact points with a curve passing through (see Figure 2)—whether the other curves of this contact point are only on one side of the passing curve or on both sides of it. We may formally define a *one-sided contact point* as a contact point C in which either all of its curves end or there exists a curve ϱ passing through C such that for all other curves $\sigma_1, \ldots, \sigma_{k-1}$ ending in C the cyclic order of the curves outgoing from C is $\varrho, \varrho, \sigma_1, \ldots, \sigma_{k-1}$. We say a contact point is *two-sided* if it is not one-sided. It is obvious that any 2–contact point is one-sided, but from the contact degree 3 there exist graphs that have a contact representation and cannot be represented without two-sided contact points.

A representation $\mathcal{R}$ is said to be a *k-contact representation* if each contact point of $\mathcal{R}$ has degree at most k. A representation $\mathcal{R}$ is said to be *simple* if each pair of curves from $\mathcal{R}$ has at most one common contact point. A representation $\mathcal{R}$ is said to be *one-sided* if each of its contact points is one-sided. The same definitions of a k–contact or one-sided representations are applied for line segment representations. It is clear that every line segment representation is simple. All these properties of contact representations are transferred to contact graphs, and we refer to contact graphs as k-contact, simple or one-sided in the obvious sense. Unless explicitly stated otherwise, we will consider only one-sided contact representations. Therefore by representation we will mean one-sided representation, and we will say a two-sided representation otherwise. Similarly, we will consider one-sided contact graphs by default.

2.2 Simple Results

For a description of a curve contact representation we define the following tool: The *incidence graph* of a representation $\mathcal{R}$ (denoted by $I(\mathcal{R})$) is a directed graph, whose vertices correspond to curves and contact points of $\mathcal{R}$ and each vertex of a curve is connected with all contact points that lie on it. The edge is oriented from curve to contact point iff the point is an endpoint of the curve. We understand here a directed graph as an orientation of undirected graph, i.e. strictly without multiple edges. An example of the incidence graph of a representation is presented in Figure 3.

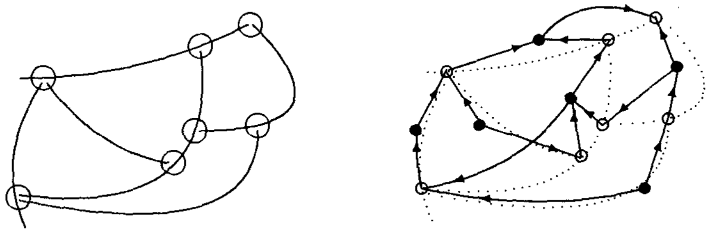

Fig. 3. An example of the incidence graph of a representation

From the definitions it follows that the contact graph of a representation is fully determined by its incidence graph, while the opposite is, of course, not true. We say that two representations are *similar* if their incidence graphs are isomorphic. It is not difficult to show the following lemma that enables us to handle a curve contact representation easier and to describe it using finite (polynomial) space.

Lemma 2.1. *For each two-sided representation $\mathcal{R}$ there exists a two-sided representation $\mathcal{S}$ similar to $\mathcal{R}$, so that each curve from $\mathcal{S}$ is a piecewise linear curve, consisting of linear number of segments. Additionally, if $\mathcal{R}$ is one-sided, then $\mathcal{S}$ can be also chosen one-sided.*

For one-sided contact representations another description, which is using the incidence graph of a representation, is proposed next.

Proposition 2.2 *For a graph G there exists a contact representation $\mathcal{R}$ such that $G \cong I(\mathcal{R})$ iff G is a planar directed graph and its vertices can be divided into two independent set $V(G) = A \cup B$ so that the outdegrees in A are at most 2, the outdegrees in B are at most 1 and the total degrees in B are at least 2.*

We omit proofs of this technical results, they may be found in [9]. To show the difference between one-sided and two-sided contact representations, we present the representation in Figure 4. It is a two-sided contact representation with non-planar incidence graph, and its contact graph has no one-sided contact representation (for example by Proposition 2.4).

The following characterization of the 2–contact graphs of line segments is given in [17]: Graph G is a 2–contact graph of line segments iff G is planar and

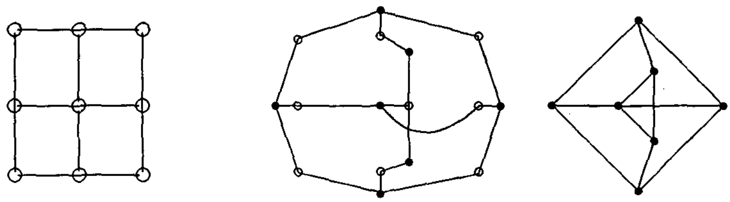

Fig. 4. A two-sided contact representation with non-planar incidence graph

for each subgraph $\boldsymbol{H} \subseteq \boldsymbol{G}$, $|E(\boldsymbol{H})| \leq 2 \cdot |V(\boldsymbol{H})| - 3$. A similar characterization of 2–contact graphs of curves can be easily derived from Proposition 2.2.

Proposition 2.3 *Graph $\boldsymbol{G}$ is a 2–contact graph iff $\boldsymbol{G}$ is planar and for each subgraph $\boldsymbol{H} \subseteq \boldsymbol{G}$, $|E(\boldsymbol{H})| \leq 2 \cdot |V(\boldsymbol{H})|$.*

It also follows from Proposition 2.2 that one-sided 3–contact graphs of curves are planar too. However, there is probably no such nice characterization as the previous ones, due to the results presented in Section 5.

Proposition 2.4 *If $\boldsymbol{G}$ is a contact graph of a one-sided 3–contact representation $\mathcal{R}$, then $\boldsymbol{G}$ is planar. Moreover, there exists a planar drawing of $\boldsymbol{G}$ such that for each 3–contact point X of curves $u, v, w \in \mathcal{R}$ the triangle u, v, w forms a face.*

3 Classes of Contact Graphs

Various classes of contact graphs of curves or line segments, with bounds on contact degrees and simplicity, are defined. Remember that only one-sided contact representations are considered here. The inclusions among the classes are described in Theorem 1, see the diagram in Figure 5.

Definition. For an integer $k \geq 2$, we denote by *CONCUR* (k-*CONCUR*) the class of all contact (k–contact) graphs of curves, by *SCONCUR* (k-*SCONCUR*) the class of all simple contact (simple k–contact) graphs of curves, and by *CONSEG* (k-*CONSEG*) the class of all contact (k–contact) graphs of line segments.

Theorem 1. *All the inclusions among contact graph classes described in Figure 5 are strict and no other inclusion holds.*

Sketch of proof. All inclusions shown in Figure 5 are obvious from definitions. The equalities 2-*SCONCUR* = 2-*CONCUR* and 3-*SCONCUR* = 3-*CONCUR* follow from the fact (not proved here) that any 3–contact representation may be rearranged to be simple.

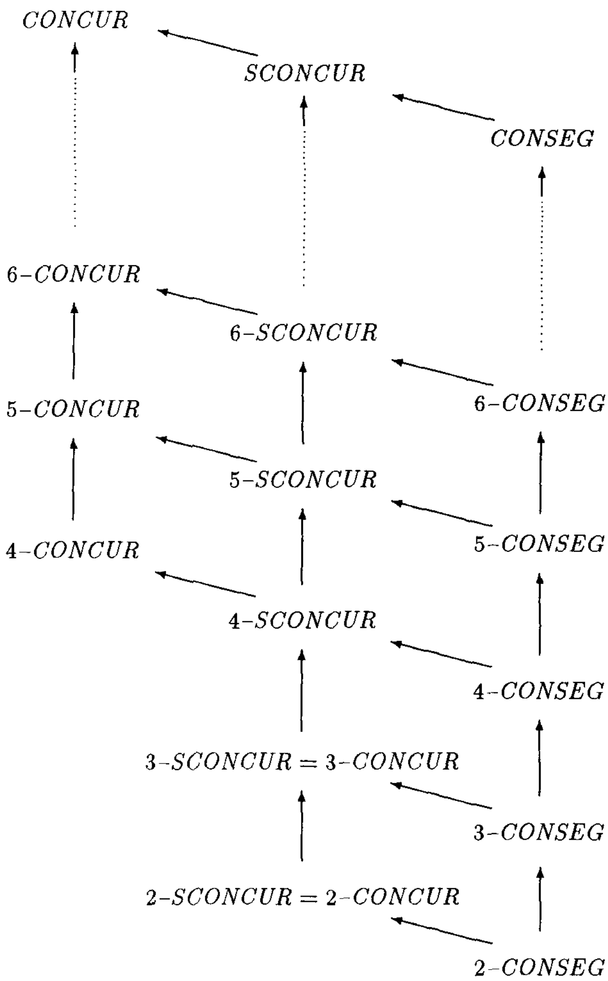

Fig. 5. The inclusions between classes of contact graphs

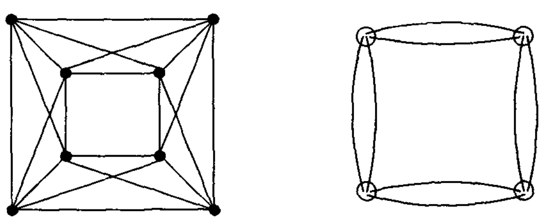

Fig. 6. A 4-contact graph that has no simple contact representation

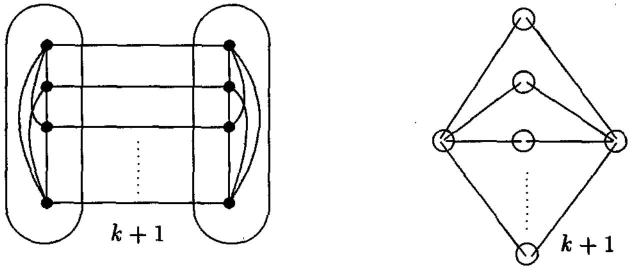

Fig. 7. A $(k+1)$–contact graph of segments that has no k–contact representation

The differences between distinct classes are proved by constructing special graphs. For example, in Figure 6 there is a graph that has a 4–contact representation but no simple contact representation. In Figure 7 a scheme of a $(k + 1)$–contact graph of segments that has no k–contact representation, is shown for $k \geq 3$. The detailed proofs of all cases may be found in [9].

4 Maximal Clique and Other Problems in Contact Graphs

We study the contact representations of complete graphs. Several examples of them are shown in Figure 8. It may be proved that these representations are, in some sense, the only possibilities to represent cliques. We present here only the weaker version of the result, bounding the clique size of contact graphs.

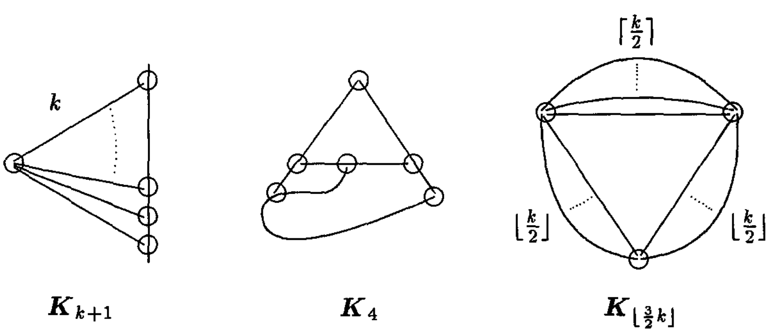

Fig. 8. Contact representations of complete graphs

Theorem 2. *The complete graphs contained in the contact graph classes are*

1. *for every* $k \geq 2$, $K_m \in k$*–CONSEG iff* $m \leq k+1$,

2. for every $k \geq 3$, $\boldsymbol{K}_m \in k$-SCONCUR iff $m \leq k+1$,
3. for every $k \geq 3$, $\boldsymbol{K}_m \in k$-CONCUR iff $m \leq \lfloor \frac{3}{2}k \rfloor$,
4. specially $\boldsymbol{K}_m \in 2$-CONCUR = 2-SCONCUR iff $m \leq 4$.

Corollary 4.1. *For $k \geq 3$, the maximal clique size of a k-contact graph, simple k-contact graph, is bounded by $\lfloor \frac{3}{2}k \rfloor$, $k+1$.*

Sketch of proof. We include here only the proof of the easy case of simple contact representations. The whole proof is in [9].

Let us suppose that there exists a simple k-contact representation $\mathcal{R}$ of the graph $\boldsymbol{K}_{k+2}$, $k \geq 3$. If there is no contact point of degree at least 3 in $\mathcal{R}$, we get a 2-contact representation of $\boldsymbol{K}_5$, a contradiction to Proposition 2.4. Otherwise we take a contact point X of degree at least 3 and curves $\varrho_1, \varrho_2, \varrho_3$ containing X. Because the contact degree of X is at most k, there exist two curves σ_1, σ_2 not containing X. Then $\varrho_1, \varrho_2, \varrho_3, \sigma_1, \sigma_2$ form a 3-contact subrepresentation of $\boldsymbol{K}_5$ (in contact points distinct from X there may be only one of the curves $\varrho_1, \varrho_2, \varrho_3$), again a contradiction to Proposition 2.4.

Many graph problems that are hard in the general case, can be solved quickly for special intersection graphs. For example, it is easy to find the chromatic number, maximal clique or independent set of an interval graph, using the simplicial decomposition of it. We show, based on the previous result, that the maximal clique of a contact graph can be found in polynomial time if the contact representation is given.

Proposition 4.2 *There exists a polynomial algorithm that for given contact representation of a graph $\boldsymbol{G}$ finds the maximal clique of $\boldsymbol{G}$, while the INDEPENDENT SET and the 3-COLOURABILITY problems remain NP-complete for contact graphs (2-contact graphs) even when the contact representation is given.*

Further we show that the contact graphs are "almost perfect", i.e. their chromatic number is bounded by a linear function of the maximal clique size. However, an infinite sequence of contact graphs, for which the chromatic number grows faster than the maximal clique, may be constructed.

Proposition 4.3 *For any contact graph $\boldsymbol{G}$, $\chi(\boldsymbol{G}) \leq 2 \cdot \omega(\boldsymbol{G})$. There exists a contact graph $\boldsymbol{H}_m$ with $\omega(\boldsymbol{H}_m) = m$ and $\chi(\boldsymbol{H}_m) \geq m + \lceil \frac{m-1}{4} \rceil$, for any integer m.*

5 Recognition of Contact Graphs

The problem to decide, whether a given graph can be represented as an intersection graph of specified objects, is important in studying the intersection graphs. The decision version of the problem is called the *recognition* of intersection graphs (of a special kind). For the interval graphs a simple characterization is given in [14], and a more efficient algorithm for their recognition is in [1]. Circle graphs (intersection graphs of chords of a circle) may be mentioned [2] as other kind of intersection graphs that can be quickly recognized, but the algorithm is not easy. On the other hand, the recognition of intersection graphs of curves in the plane is proved to be *NP*–hard [11], moreover graphs with at least exponential complex representations [13] are known in that case.

The complexity of recognizing the contact graphs of curves or line segments is considered here, especially for planar graphs.

5.1 Contact Representations of Planar Triangulations

Firstly we present an important lemma that is used to disprove existence of certain contact representations of planar graphs.

Lemma 5.1. *Let $\mathcal{R}$ be a two-sided 3–contact representation of a graph $\boldsymbol{G}$ containing f free endpoints of curves. Then the representation $\mathcal{R}$ must contain at least $(|E(\boldsymbol{G})| - 2 \cdot |V(\boldsymbol{G})| + f)$ 3–contact points forming non-neighbouring triangles in $\boldsymbol{G}$ (two triangles are said to be neighbouring if they have a common edge).*

We already know by Proposition 2.3 that the recognition of 2–contact graphs is polynomial—the edge number condition may be checked using the polynomial algorithm for network flows, and the planarity is also known to be polynomial. There are also other results on representing planar graphs as contact graphs that follow.

From [4] it is known that every bipartite planar graph is a 2–contact graph of segments. A planar triangulation is a planar graph that has all faces, including the outer face, triangles. In [6], representations of planar triangulations by contacts of segments are considered: A 4–connected planar triangulation is a 3–contact graph of segments iff it is 3–colourable (and this condition can be checked in linear time).

We study the curve contact representations of planar triangulations. From Lemma 5.1 the following statement is derived:

Theorem 3. *There exists a polynomial algorithm that for a given planar triangulation decides whether it is a 3–contact graph, and finds the representation if exists.*

In further constructions we need a special graph that has a simple 3–contact representation, but no contact representation in which some curve has a free endpoint (an endpoint of a curve is called free if it is not a contact point). This graph is presented in Figure 9, we denote it by $\boldsymbol{\mathcal{E}}$. The property of having no free endpoint in any contact representation follows from Lemma 5.1— $\boldsymbol{\mathcal{E}}$ has only 16 non-neighbouring triangles. It is used to "eat" an endpoint of a curve in a contact representation—if any vertex v of it is adjacent to some other vertex w (of another graph), the only way to represent the edge $\{v, w\}$ is to use one endpoint of the curve w in the contact point of v, w.

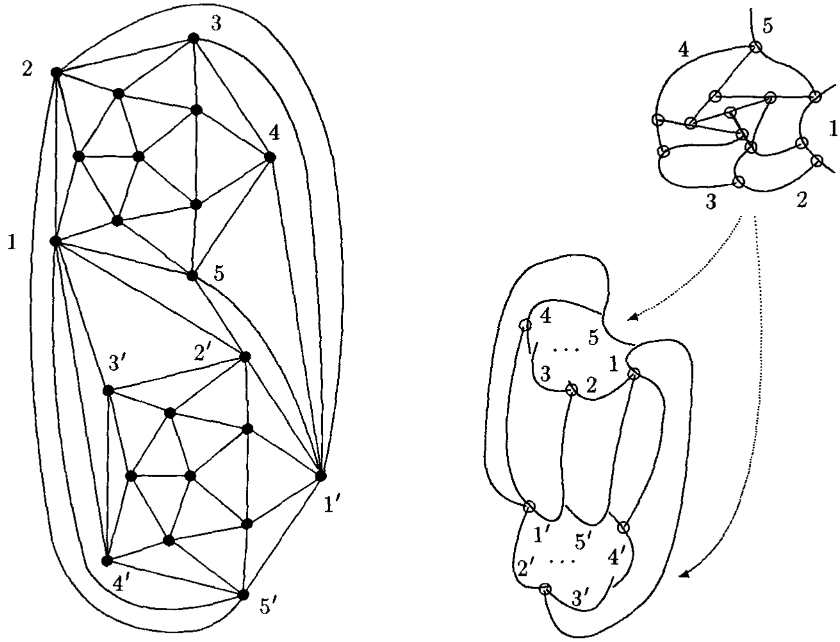

Fig. 9. The end-eating graph $\mathcal{E}$ and a scheme of its contact representation

5.2 *NP*–completeness of Recognizing 3–contact Graphs

Unlike the planar triangulations, the problem of deciding whether a given general planar graph is a 3–contact graph, is *NP*–complete. The recognition of (two-sided) contact graphs clearly belongs to *NP* from Lemma 2.1. For the *NP*–completeness reduction of our problem we use the PLANAR 3–SAT problem, that is defined as a special case of the SAT problem (a formula Φ with a set variables V and a set of clauses C) for which the bipartite graph $\boldsymbol{F}$, $V(\boldsymbol{F}) = C \cup V$, $E(\boldsymbol{F}) = \{xc : x \in c \text{ or } \neg x \in c\}$, is planar with degrees of all vertices bounded by 3.

Given a formula Φ (of the PLANAR 3–SAT problem), we construct a graph $\boldsymbol{R}(\Phi)$ that has a contact representation iff the formula Φ is satisfiable. In the construction each variable and each clause of Φ are replaced by special graphs, then clauses are connected with their variables by connectors. The positive and negated occurrences of a variable are distinguished by connecting to clauses using different terminals of the variable graph. For clauses with less than 3 variables special false terminators are used. The variable and clause graphs are constructed using the end-eating graph, and from Lemma 5.1 it is derived that the only possible representations of the variable graph reflects the values 0 and 1 of the variable, and the clause graph is representable iff at least one of its literals is true.

For the complete construction including proofs refer to [9]. An example of the graph $\boldsymbol{R}(\Phi)$ for a simple formula Φ is presented in Figure 10, the end-eating graphs added to some vertices are schematically drawn by double circles.

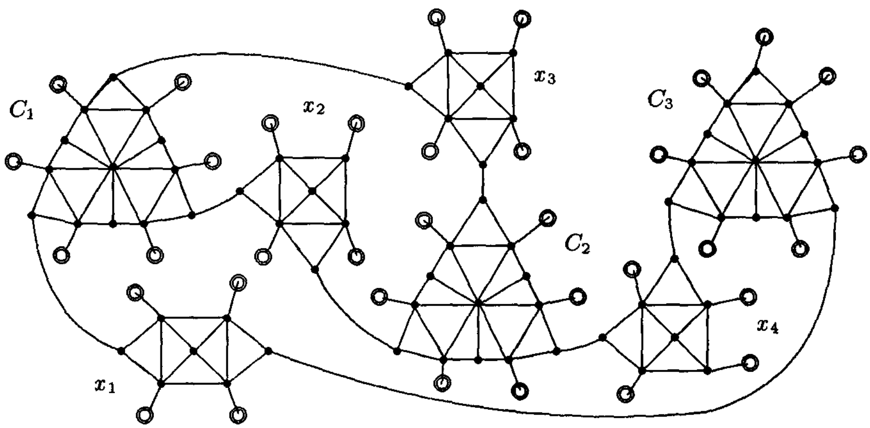

Fig. 10. An example of the graph $R(\Phi)$ for $\Phi = C_1 \wedge C_2 \wedge C_3$, $C_1 = (\neg x_1 \vee \neg x_2 \vee \neg x_3)$, $C_2 = (x_2 \vee x_3 \vee \neg x_4)$, $C_3 = (\neg x_1 \vee x_4)$

Lemma 5.2. *For a formula Φ of a given instance of the PLANAR 3-SAT problem the graph $\boldsymbol{R}(\Phi)$ is planar, and is a one-sided 3-contact graph if Φ is satisfiable but has no two-sided contact representation if Φ is not satisfiable.*

Theorem 4. *The recognition of contact graphs (simple contact graphs, k-contact graphs for $k \geq 3$) is NP-complete.*
The recognition of two-sided contact graphs (two-sided k-contact graphs for $k \geq 3$) is NP-complete.
The recognition of contact graphs is NP-complete even within the class of planar graphs.

Using more involved methods, a similar reduction can be constructed for contact graphs of line segments. However, we do not know whether the recognition of contact graphs of segments belongs to *NP*. Thus it is proved:

Theorem 5. *The recognition of contact graphs (k-contact graphs for $k \geq 3$) of segments is NP-hard.*
The recognition of two-sided contact graphs (two-sided k-contact graphs for $k \geq 3$) of segments is NP-hard.
The recognition of contact graphs of segments is NP-hard even within the class of planar graphs.

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