Normalization by Optimization

Ralph Schiller

Technische Universität Hamburg-Harburg, 21075 Hamburg, Germany

Abstract. An approach to normalization is presented for both the affine and the projective case. The approach is based on group factorization as well as on optimizing parameter invariant integrals, in order to overcome the difficult problem of parameterization. Related work has been carried out by [6] and by [4] for affine transformations and by [5] for projective transformations. To avoid some drawbacks inherent to projective transformations it is suitable to integrate point information or explore 'thick' curves.

1 Introduction

An approach to normalization is presented mainly based on minimizing some fundamental properties of a subgroup in order to normalize objects up to this subgroup. In the projective case it is important that the normalization of the object is not a transformation to an abstract canonical frame but rather a reconstruction of the physical test pattern. This is relevant because small distortions of the object will be large distortions of the normalized object if the line which is mapped on the line at infinity is close to the object. Especially [5] had to face this problem but [2] proved the problem being inherent to projective transformations. We therefore propose centered curves which are already optimal in some way so that the normalization will be as extreme as the projective transformation of the object.

In case of affine transformations we will generalize some results of [6] and [4] in $\mathbb{R}^2$ to curves in $\mathbb{R}^n$ and surfaces in $\mathbb{R}^3$, which is necessary for stereo-graphic reconstruction. [6] were rather interested in the interpretation of 2d-images from 3d-objects and [4] emphasis was on texture so that they need not care so much for the parameterization of curves. But exactly the parameterization of curves is a specific problem for affine and projective invariant pattern recognition as the digitalization grid is rigid so that the amount of pixel coordinates of two equivalent but digitalized image contours may extremely vary. It is the very advantage of extremising low order parameter invariant integrals or sums that the parameterization problem can be neglected.

2 From Projective Transformations to Affine Transformations

The congruences and affine transformations are subgroups of the projective transformations. The first task is to normalize the non-affine part of the transformation in case we want to describe objects position invariant due to the pin-hole camera model. Therefore we remark the following factorization of projective transformations: A projective transformation can always be factorized in an affine part and a nonlinear part.

$$P : \mathbb{R}^2 \to \mathbb{R}^2 \ (x, y) \mapsto \left(\frac{x}{gx + hy + 1}, \frac{y}{gx + hy + 1} \right)$$

$$A : \mathbb{R}^2 \to \mathbb{R}^2 \ (x, y) \mapsto ((a - cg)x + (b - ch)y + c, (d - fg)x + (e - hf)y + f)$$

$$A(P(x, y)) = \left(\frac{ax + by + c}{gx + hy + 1}, \frac{ex + dy + f}{gx + hy + 1} \right)$$

Now let P be a convex polygon in $\mathbb{R}^2$. Due to our principle we want to extremize here some fundamental property of the affine group in order to normalize the object under projective transformations up to affine transformations. The first approach was to choose the ratio of two distances on a line. Using this fundamental invariant it is possible to define an affine arclength for a convex polygon based on intersections of tangents. The analog for continuous convex smooth curves would be to extremize the affine arclength $\int \sqrt[3]{| x'(s)x''(s) |} ds$. But Åström [2] has proved that projective transformations are quite powerful so that the infimum for the smooth curves would be ellipses as Blaschke [3] has proved the extremal property of ellipses with respect to affine arclength. Therefore it does not seem to be useful just to normalize a single curve. We will look at two classes of objects:

- a convex curve with a point in the interior of its bounded domain
- a convex curve with another convex curve in the interior of its bounded domain

Another fundamental property of affine transformations is that the center of mass (with respect to area) transforms consistently under affine transformations. Now it is shown that this never happens under 'pure' projective transformations and thus we can normalize curves up to affine transformations.

Theorem 1. *Let P be a convex polygon with nonempty interior. Then the only projective transforms preserving the convexity and mass center of the polygon are the affine ones (i.e. those for which the non-linear part of the projective transform is reduced to identity).*

Proof. Due to the factorization of the projective transformation in an affine and a nonlinear part, it is sufficient to verify the theorem for the nonlinear part.

So we are facing the following projective transformations:

$$P(x, y) = \left(\frac{x}{gx + hy + 1}, \frac{y}{gx + hy + 1} \right) \quad gx_i + hy_i + 1 \geq 0 \qquad i = 0, \ldots, n$$

if (x_i, y_i) are the vertices of the polygon. The inequalities define the feasible region as a convex closed set, which has to be bounded. In this way the lines are described which do not meet the interior of the closed polygon. Particularly if we fix some parameter (g_1, h_1), then the corresponding line $G_1 = \{(x,y) : g_1x + h_1y + 1 = 0\}$ is mapped on the line at infinity. Let G_0 be the line through the origin parallel to G_1, then G_0 divides the convex polygon in two half-spaces, H^-, H^+, as the center of mass lies in the origin. Let H^+ be the half with points between G_0 and G_1. The line G_0 is mapped identically on itself. The polygon is convex and convex sets within the polygon are mapped on convex sets. To each point h^- on the border of the polygon in H^- corresponds a unique point h^+ on the border of the polygon in H^+ lying on a line through the origin. While the Euclidean distance of such a point h^- shrinks, the Euclidean distance of one such corresponding point h^+ expands. Therefore the origin cannot be center of mass of the transformed polygon. For that reason look without loss of generality at a line parallel to the y-axis intersecting the positive x-axis: $\{(x,y) : \; gx + 1 = 0\} \quad g < 0$. Then we find for a vector $(r\cos\phi, r\sin\phi)$ with $r \geq 0 \quad \phi \in [0, \frac{\pi}{2}] \cup [\frac{3\pi}{2}, 2\pi]$

$$\sqrt{(\frac{r\cos\phi}{gr\cos\phi+1})^2 + (\frac{r\sin\phi}{gr\cos\phi+1})^2} = \sqrt{(\frac{r^2}{gr\cos\phi+1})^2} = \frac{r}{gr\cos\phi+1}$$

As $r\cos\phi \geq 0$ and $g < 0$ and the line is admissible, the denominator lies between zero and one. If ϕ lies in the opposite angle region, the denominator will be greater than one.

If the center of mass of a convex test pattern is marked, then we are capable of computing a normalized equivalent pattern which characterizes the test pattern up to an affine transformation; for there has to be one position in which the marked point is the center of mass and so there exists one such projective transformation. If there were another transformation being not an affine equivalent one, then the above theorem would be wrong.

Frequently we know one point in the interior of a convex test pattern but it need not be the center of mass. The question is whether this point is the center of mass of a projective equivalent pattern.

Theorem 2. *Let P be a convex polygon. Then there exists at least one projective transform preserving the convexity of the polygon such that the origin will be transformed in the mass center of the transformed polygon.*

Proof. Again, only the nonlinear part of a projective transformation has to be taken into consideration, and therefore one may assume that the origin has to be mapped on itself. The single pieces of area transform in the following way:

$$\begin{vmatrix} \frac{x_k}{gx_k+hy_k+1} & \frac{x_{k+1}}{gx_{k+1}+hy_{k+1}+1} \\ \frac{y_k}{gx_k+hy_k+1} & \frac{y_{k+1}}{gx_{k+1}+hy_{k+1}+1} \end{vmatrix} = \frac{\begin{vmatrix} x_k & x_{k+1} \\ y_k & y_{k+1} \end{vmatrix}}{(gx_k+hy_k+1)(gx_{k+1}+hy_{k+1}+1)}$$

In order that the center of mass of the polygon and the origin coincide, two equations have to be met: (the polygon is assumed to be positively orientated)

$$0 = x_s = \sum_{k=0}^{n} \frac{\begin{vmatrix} x_k & x_{k+1} \\ y_k & y_{k+1} \end{vmatrix} (x_k(gx_{k+1} + hy_{k+1} + 1) + x_{k+1}(gx_k + hy_k + 1))}{(gx_k + hy_k + 1)^2 (gx_{k+1} + hy_{k+1} + 1)^2}$$

$$0 = y_s = \sum_{k=0}^{n} \frac{\begin{vmatrix} x_k & x_{k+1} \\ y_k & y_{k+1} \end{vmatrix} (y_k(gx_{k+1} + hy_{k+1} + 1) + y_{k+1}(gx_k + hy_k + 1))}{(gx_k + hy_k + 1)^2 (gx_{k+1} + hy_{k+1} + 1)^2}$$

and further the inequalities:

$$gx_i + hy_i + 1 \geq 0 \qquad i = 0, \ldots, n.$$

Both equalities may be regarded as conditions for the gradient of a function to be zero; for g we have

$$\frac{\partial}{\partial g} \frac{1}{(gx_k + hy_k + 1)(gx_{k+1} + hy_{k+1} + 1)} = \frac{-1(x_k(gx_{k+1} + hy_{k+1} + 1) + x_{k+1}(gx_k + hy_k + 1))}{(gx_k + hy_k + 1)^2 (gx_{k+1} + hy_{k+1} + 1)^2}$$

and alike for h. So we find

$$F(g, h) = \sum_{k=0}^{n} \frac{-1 \begin{vmatrix} x_k & x_{k+1} \\ y_k & y_{k+1} \end{vmatrix}}{(gx_k + hy_k + 1)(gx_{k+1} + hy_{k+1} + 1)}$$

This function has a minimum and a maximum on the compact region defined by the inequalities. The single summands are negative as the areas are all positive and the denominators are positive because of the constraint inequalities. The function will decrease without limit towards the border of the compact domain. Therefore a maximum will not be attained on the border. But as there has to be a maximum, the corresponding gradient must have a zero point.

Subsequently if the affine part of the decomposition is fixed the transformation is unique because of theorem 1. However, a single point is principally hard to detect. Therefore we look at objects which may be normalized in a more robust way and which are still of some practical significance. We consider two closed convex curves: $(x(t), y(t))$ lies in the bounded open domain of $(\tilde{x}(t), \tilde{y}(t))$. Particularly we will look at curves of the following type: $(\tilde{x}(t), \tilde{y}(t)) = (lx(t), ly(t)) \quad l > 1$ whose center of mass coincide. Another fundamental property of affine transformations is that area ratios are preserved. So we want to extremize the ratio of the area of the two curves in order to normalize the curves up to affine transformations.

The task is to describe the object consisting of the two convex curves projectively invariant (up to affine transformations).

$$F(g, h) = \frac{\int \frac{\tilde{x}(t)\dot{\tilde{y}}(t) - \tilde{y}(t)\dot{\tilde{x}}(t)}{(g\tilde{x}(t) + h\tilde{y}(t) + 1)^2} dt}{\int \frac{x(t)\dot{y}(t) - y(t)\dot{x}(t)}{(gx(t) + hy(t) + 1)^2} dt}$$

Consequently we face the following optimization problem with the restriction that the convexity of the objects is preserved.

$$F(g,h) \longrightarrow min \qquad g\tilde{x}(t) + h\tilde{y}(t) + 1 \geq 0$$

The feasible region is again a compact set and towards the border of the set the values of the function grow without a limit. So a minimum of the function has to lie in the interior of the set and the gradient of the function is of the following form:

$$\frac{\partial F(g,h)}{\partial g} = \frac{\int -2\tilde{x}\frac{\tilde{x}\dot{\tilde{y}}-\dot{\tilde{x}}\tilde{y}}{(g\tilde{x}+h\tilde{y}+1)^3}dt \int \frac{x\dot{y}-\dot{x}y}{(gx+hy+1)^2}dt - \int \frac{\tilde{x}\dot{\tilde{y}}-\dot{\tilde{x}}\tilde{y}}{(g\tilde{x}+h\tilde{y}+1)^2}dt \int -2x\frac{x\dot{y}-\dot{x}y}{(gx+hy+1)^3}dt}{(\int \frac{x\dot{y}-y\dot{x}}{(gx+hy+1)^2}dt)^2}$$

This implies, that the gradient of the function is zero if the centers of mass of the two curves coincide.

$$\frac{\int \tilde{x}\frac{\tilde{x}\dot{\tilde{y}}-\tilde{y}\dot{\tilde{x}}}{(g\tilde{x}+h\tilde{y}+1)^3}dt}{\int \frac{\tilde{x}\dot{\tilde{y}}-\tilde{y}\dot{\tilde{x}}}{(g\tilde{x}+h\tilde{y}+1)^2}dt} = \frac{\int x\frac{x\dot{y}-y\dot{x}}{(gx+hy+1)^3}dt}{\int \frac{x\dot{y}-y\dot{x}}{(gx+hy+1)^2}dt}$$

So if our test object consists of two curves which are centered, then the test pattern will already be an extremal object. In order to demonstrate uniqueness of the optimization problem we restrict to the particular class of objects already mentioned above: $(\tilde{x}(t), \tilde{y}(t)) = (lx(t), ly(t)) \quad l > 1$ where the center of mass of $(x(t), y(t))$ lies in the origin. If we parameterize our curve $(x(t), y(t))$ with the area-parameterization [1] we will derive the following simplified function which is to be optimized:

$$F(g,h) = \frac{\int \frac{l^2}{(glx(t)+hly(t)+1)^2}dt}{\int \frac{1}{(gx(t)+hy(t)+1)^2}dt} \longrightarrow min \quad glx(t) + hly(t) + 1 \geq 0$$

We know that the parameter $(g,h) = (0,0)$ represents a local minimum as the centers of mass of our curves coincide due to their construction. Further our two curves are apparently projective equivalent. Due to theorem 1 we know that the center of mass of our curve cannot remain in the origin under the above 'pure' projective transformations unless $(g,h) = (0,0)$. The relation of our objects is $(x(t), y(t)) \sim (lx(t), ly(t))$ and the relation of the projective transformed objects is

$$\left(\frac{x(t)}{gx(t)+hy(t)+1}, \frac{y(t)}{gx(t)+hy(t)+1}\right) \sim \left(\frac{lx(t)}{glx(t)+hly(t)+1}, \frac{ly(t)}{glx(t)+hly(t)+1}\right)$$

and if there were a second minimum then the centers of mass would have to coincide. But as $l > 1$ the line $\{(x,y) : glx + hly + 1 = 0\}$ is parallel to $\{(x,y) : gx + hy + 1 = 0\}$ and closer to the object. Thus similarly as in theorem 1 the centers of mass of the transformed objects cannot coincide. So there has to be a unique minimum.

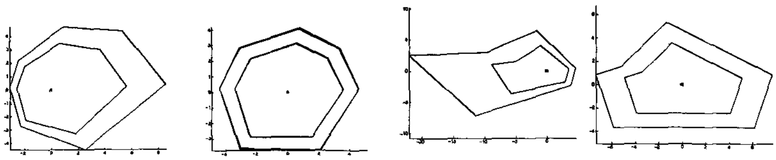

Fig. 1. Projective transformed object, normalized object up to affine transformations after some iterations. (The normalization of the non-convex curves is based on the corresponding convex curves)

3 From Affine Transformations to Congruences

Due to our principle we want to extremize some fundamental property of the congruences in order to normalize our objects under equiaffine transformations up to congruences. We therefore chose the Euclidean distance and found later on that there had been performed some research in this respect.

Already Brady and Yuille [6] have pointed out for a large class of curves $\gamma(t) = (x(t), y(t))$ that the following minimization problem has a unique solution in $\mathbb{R}^2$ up to congruences:

$$\int \| A\gamma(\dot{t}) + b \| \; dt \longrightarrow \quad min$$

under the restriction that the determinant of A is equal one. The particular problem is in this case that it is not possible to determine the parameters explicitly in general so that we are forced to use iterative algorithms. The advantage is that we need not care for the parameterization problem as the integral is parameter independent. We also used the squared distances and derived the same explicit formula as [4]. But [4] were not that interested in contours and therefore could neglect the parameterization problem which raises in this case as the integral is no longer parameter independent. Nevertheless a complete explicit formula was derived in $\mathbb{R}^2$ [9] by using Arbter's [1] area parameterization. The whole algorithm was tested with real images [9] and the object recognition results were very satisfying.

In $\mathbb{R}^3$ we prefer the ordinary Euclidean distance, as it is not so easy to parameterize our curve in this case. Therefore it will be shown, that the problem - to find among all equiaffine equivalent curves that one, with minimal Euclidean arclength - has a unique solution for a large class of point sets in $\mathbb{R}^n$ unless the object is degenerate. We want to mention that it is not so difficult to demonstrate the existence of a solution. Consider three points in $\mathbb{R}^2$ which are not collinear: $(x_1, y_1), \cdots, (x_3, y_3)$. The linear transformation with determinant one minimizing the following term $\sum_{k=1}^{n} \sqrt{(ax_k + by_k)^2 + (cx_k + dy_k)^2}$ will transform the three points in such a way that they represent the vertices of an equilateral triangle [9]. So if there are more points then there will always be a lower bound build up by an equilateral triangle. In $\mathbb{R}^3$ the same is true for a regular tetrahedron and so on.

Theorem 3. *The optimization problem has a unique solution up to congruences, unless the polygon can be embedded in a hyperplane.*

Proof. Let P_1 be a polygon: $(x_1, y_1), \cdots, (x_n, y_n)$ in $\mathbb{R}^n$ and $P_2 = AP + z$ an affine equivalent polygon. Suppose that both these polygons are optimal due to our minimization problem. According to the matrix decomposition theorem [7] we can factorize A in the following way: $A = O_1 D O_2$ where O_1 and O_2 are orthogonal matrices and D is a diagonal matrix with positive diagonal elements. Thus we redefine our polygons in the following way: $P_1 := O_2 P_1$ and $P_2 := DP_1$. P_1 and P_2 are again optimal as an orthogonal matrix does not change the Euclidean distances. So the transformation matrix between the two optimal polygons is a diagonal matrix. Now we can prove uniqueness by proving that the following restricted optimization problem has a unique solution.

$$\sum_{k=1}^{n} \sqrt{\sum_{i=1}^{m} \lambda_i^2 x_{ik}^2} \longrightarrow \quad min \qquad \prod_{i=1}^{m} \lambda_i = 1 \qquad \lambda_i > 0$$

- The objective function:
  In order to verify that there is a unique solution we consider one single summand as a function: $\sqrt{\sum_{i=1}^{m} \lambda_i^2 x_i^2}$. This function is convex, because, if λ and μ are two vectors and $0 \leq \alpha \leq 1$ the convexity condition can be expressed in the following relation:

  $$\sqrt{\sum_{i=1}^{m} (\alpha\lambda_i + (1-\alpha)\mu_i)^2 x_i^2} \leq \alpha \sqrt{\sum_{i=1}^{m} \lambda_i^2 x_i^2} + (1-\alpha) \sqrt{\sum_{i=1}^{n} \mu_i^2 x_i^2}$$

  The relation is true due to the Minkowski inequality. As each single summand is a convex function and as a sum of convex functions leads to a convex function, our function has to be convex too.
  If F is a convex function than the level-sets $N_c := \{x : F(x) \leq c\}$ are convex closed sets, and our special level-sets are restricted too in case none of the components k of the vectors x are identically zero; for if you assume that the level-sets are not restricted, then we could find a $\lambda \neq O$, because of the convexity of the level sets, such that for all $\alpha \geq 0$ we find $\alpha\lambda \in N_c$, i.e.

  $$\sum_{k=1}^{n} \sqrt{\sum_{i=1}^{m} (\alpha\lambda_i)^2 x_{ik}^2} \leq c \Longleftrightarrow \alpha \sum_{k=1}^{n} \sqrt{\sum_{i=1}^{m} \lambda_i^2 x_{ik}^2} \leq c$$

  But this can only happen if the whole expression is equal zero.
- The feasible region:
  The set $\prod_{i=1}^{m} \lambda_i \geq 1 \qquad \lambda_i > 0$ is a convex set; for if

  $$\prod_{i=1}^{m} \lambda_i \geq 1 \quad \prod_{i=1}^{m} \mu_i \geq 1 \qquad \lambda_i > 0 \ \mu_i > 0$$

we calculate due to the concavity of the logarithm for $1 \geq \alpha \geq 0$

$$\log\left(\prod_{i=1}^{m}(\alpha\lambda_i + (1-\alpha)\mu_i)\right) = \sum_i^m \log(\alpha\lambda_i + (1-\alpha)\mu_i)$$
$$\geq \alpha\sum_i^m \log\lambda_i + (1-\alpha)\sum_i^m \log\mu_i$$

Now taking the exponential of the two sides, one gets:

$$\prod_{i=1}^{m}(\alpha\lambda_i + (1-\alpha)\mu_i) \geq (\prod_{i=1}^{m}\lambda_i)^{\alpha}(\prod_{i=1}^{m}\mu_i)^{(1-\alpha)} \geq 1$$

if the two products are greater than one. And especially due to the last inequality we find that the constraint set $\prod_{i=1}^{m}\lambda_i = 1 \quad \lambda_i > 0$ is the border of a strictly convex set.

There exists a neighbourhood of the origin which does not contain a point of the feasible region as the Euclidean norm of an admissible point is always greater than one. Therefore one can find a factor such that the unit-level set, stretched by this factor, will intersect the constraint set in a unique point due to the strict convexity of the constraint set.

3.1 Line objects in $\mathbb{R}^3$

The algorithm was tested on some objects in $\mathbb{R}^3$ and each time it led to a 'unique' solution, see figure 2. The algorithm may be as well applied on objects which are build up by several curves.

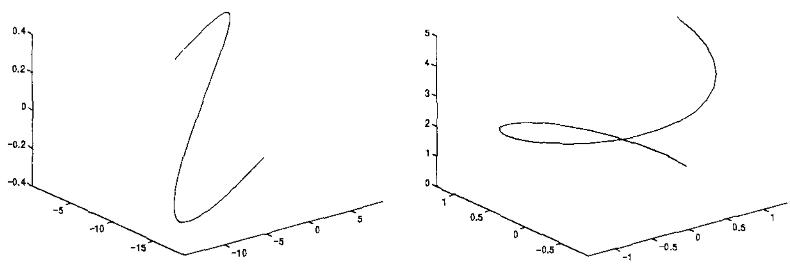

Fig. 2. Affine transformed and normalized helix

3.2 Normalization of Surfaces in $\mathbb{R}^3$

Closely related to the problem of minimizing the arclength under equiaffine transformations is the problem of minimizing the surface of an object in $\mathbb{R}^3$ under equiaffine transformations:

Let $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ be two vectors in $\mathbb{R}^3$. Then the crossproduct $Z = x \times y$ is the vector whose components consists of the minors of the matrix which rows are x and y. There is an important relation among a vector z, the crossproduct Z and the affine transformed vectors z^*, Z^*. For example Blaschke [3] has proved that: $z^* Z^* = zZ$. So we can easily prove that the crossproduct acts as a contravariant tensor under affine transformations: $Z^* = A^{t^{-1}} Z$. Let $\Phi(u,v) := X(u,v)i + Y(u,v)j + Z(u,v)k$ be a surface with a compact parameter domain K. The area $I(\Phi)$ of the surface Φ on K is defined by

$$\int_K |N(u,v)| d(u,v) = \int_K |\frac{\partial \Phi}{\partial u} \times \frac{\partial \Phi}{\partial v}| d(u,v).$$

Particularly we find

$$I(\Phi) = \int_K \sqrt{(\frac{\partial(Y,Z)}{\partial(u,v)})^2 + (\frac{\partial(Z,X)}{\partial(u,v)})^2 + (\frac{\partial(X,Y)}{\partial(u,v)})^2} d(u,v) \text{ with } \frac{\partial(Y,Z)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial Y}{\partial u} & \frac{\partial Y}{\partial v} \\ \frac{\partial Z}{\partial u} & \frac{\partial Z}{\partial v} \end{vmatrix}$$

In [8] it is proved that the surface area as defined above is invariant under 'admissible' parameter transformations. $N(u,v)$ is the normal vector. So if we consider the problem to find one surface among all equiaffine equivalent surfaces with minimal surface area, we find due to the contravariance of the crossproduct the following problem:

$$\int_K \| A^{t^{-1}} ((\frac{\partial(Y,Z)}{\partial(u,v)}), (\frac{\partial(Z,X)}{\partial(u,v)}), (\frac{\partial(X,Y)}{\partial(u,v)})) d(u,v) \| \quad \rightarrow \quad min$$

with the restriction that the determinant of the matrix A is one. But as the matrices form a group we may replace the term $A^{t^{-1}}$ with A. The structure of the problem, to minimize Euclidean distances under equiaffine transformations is the same as above. Therefore this new optimization problem for surfaces will again have a unique solution.

4 Conclusion

The emphasis in our paper is on parameter invariant integrals - area in the projective case and perimeter in the affine case - in order to overcome the difficult problem of parameterization. To normalize the rotation due to our principle we used such a normalization scheme as: $\sum_{k=1}^{n} (ax_k + by_k)^2 \quad \rightarrow \quad min \quad a^2 + b^2 = 1$. But we omit this passage here as the problem of parameterization is not so difficult in the case of congruences and finish with some normalized objects:

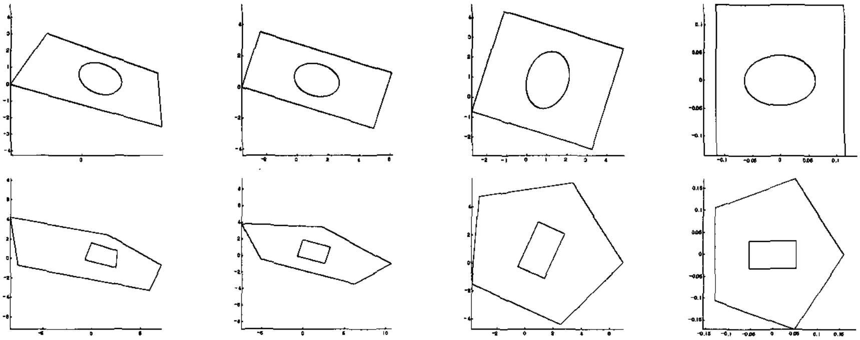

Fig. 3. a) Projective transformed object b) projective normalized object c) affine normalized object d) congruent normalized object

5 Acknowledgments

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References

1. Arbter, K., Snyder, W., Burkhardt, H. and Hirzinger, G.: Application of Affine-Invariant Fourier Descriptors to 3-D Objects. IEEE Trans. on Pattern Analysis and Machine Intelligence, PAMI-12(7):640-647, Juli 1990
2. Åström, K.: Fundamental difficulties with projective normalization of planar curves. Second Joint European - US Workshop Ponta Delgada, Azores, Portugal, October 1993 Proceedings
3. Blaschke, W.: Vorlesungen über Differentialgeometrie und goemetrische Grundlagen von Einsteins Relativitätstheorie. Berlin Verlag von Julius Springer 1923
4. Blake, A., Marinos, C.: Shape form texture: Estimation, isotropy and moments. Artificial Intell., vol. 45, pp. 323-380, 1990
5. Sinclair, D., Blake, A.: Isoperimetric Normalization of Planar Curves. IEEE Transactions on Pattern Analysis and Machine Intelligence, Vol. 16, No. 8, August 1994
6. Brady, M., Yuille, A.: An extremum principle for shape from contour. IEEE Transactions on Pattern Analysis and Machine Intelligence, vol PAMI-6, 1984
7. Fischer, G.: Analytische Geometrie. Wiesbaden: Vieweg, 1985
8. Heuser, H.: Lehrbuch der Analysis, Teil 2, Teubner
9. Schiller, R.: Distance geometric approach to recognize a set of finitely many points. Interner Bericht 10/94, Technische Informatik I, TU-HH, Juli 1994.