

Typechecking of PEI Expressions

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Abstract. PEI was introduced to express and transform parallel programs. In this paper, we recall its main features and focus on the notion of *data field* in the language. We define the type of such objects and present an algorithm to infer types.

Keywords. Parallel computation, Semantics of programming languages, Type inference, Data Parallelism and automatic parallelization.

1 Introduction

PEI is a formal framework for reasoning on programs. It was introduced [14, 15] to express and transform parallel programs [3]. It defines a language used to express statements, called PEI programs, and a refinement calculus to transform such programs.

PEI was clearly born of classical methods for the synthesis of systolic arrays [12, 9]. In this approach, a statement was a set of recurrence equations which are an abstract form for single assignment loop nests. Indices are the coordinates of points over a geometrical space and scan the *computation domain*. In this context, program transformations can be considered as pure geometrical transformations: they mainly consist in remodeling the computation domain by using isometries or partitions, in order to define a computation ordering in preserving the data *dependencies* within the computation domain. So, a parallel program results of a change of basis from the geometrical space into a new *space-time domain*: in this representation all dependencies are directed along the time axis, and successive computation fronts can be defined [7, 6].

These methods and all developments on automatic parallelization of DO loops apply when drastic constraints, such as linear constraints, are satisfied. The formal approach of PEI aims at overcoming these limitations and PEI proposes program transformations in a more general point of view. However, geometry and change of basis are features of main importance in PEI: objects expressed in the language are founded on these concepts and their type depends on them. This supposes to define a precise notion of type as definition domain of partial functions on \mathbb{Z}^n .

2 The language PEI

2.1 An introduction to PEI Programming

As a general point of view, let us consider that a problem can be specified as a relation between *multisets* of value items, roughly speaking its *inputs* and *outputs*. Of course, programming may imply to put these items in a convenient organized directory, depending on the problem terms. In scientific computations for example, items such as arrays are functions on indices: the index set, that is the reference domain, is a part of some \mathbb{Z}^n . In PEI such a multiset of value items mapped on a discrete reference domain is called a *data field*.

For example the multiset of integral items $\{1, -2, 3, 1\}$ can be expressed as a data field, say **A**, each element of **A** being recognized by an index in \mathbb{Z} (e.g. from 0 to 3). Of course this multiset may be expressed as an other data field, say **M**, which places the items on points $(i, j) \in \mathbb{Z}^2$ such that $0 \leq i, j < 2$. These two data fields **A** and **M** are considered equivalent in PEI since they express the same multiset. Formally, there exists a bijection from the first arrangement onto the second one, e.g. $\sigma(i) = (i \bmod 2, i \text{ div } 2)$. This is denoted by the equation:

$$\mathbf{M} = \text{align} :: \mathbf{A}$$

where $\text{align} = \lambda(i) \mid (0 \leq i \leq 3) . (i \bmod 2, i \text{ div } 2)$

Any PEI program is composed of such unoriented equations, each of them defining an expression of a *data field*. On the example, **M** and $\text{align} :: \mathbf{A}$ have the same set of value items, placed in the same fashion in the same reference domain.

A set of p input and q output data fields and a set of equations define a program in PEI if and only if the set of equations has at most one solution, i.e. from any set of p input valued data fields there exists at most one set of q resulting output valued data fields. Here is a classical example of prefix-sum of n numbers in PEI:

Example 1. Prefix-Sum of n numbers ($n > 1$)

```

PrefixSum : A ↦ X
{
  A = dom :: A
  X = add ▷ (A /&/ (X ◁ pre))
}
dom = λ(i) | (1 ≤ i ≤ n, n > 1) . (i)
pre = λ(i) | (1 < i ≤ n) . (i-1)
add = id # λ(a;b) . (a+b)

```

- the first equation defines the mapping of the input data field **A**: its values are placed onto a line segment $[1..n]$ of \mathbb{Z} ,
- fig. 1 intuitively shows that data field **X** is a solution of the second equation: its value items are the prefix computations of the sums of the value items in **A**. The first expression $(X \triangleleft \text{pre})$ defines a data field resulting from **X** by shifting its values from left to right. They are then composed with the values of **A** in the expression $(A /\&/ (X \triangleleft \text{pre}))$ and added one to another.

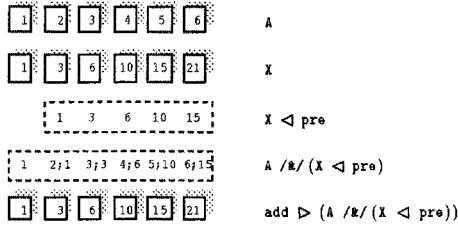
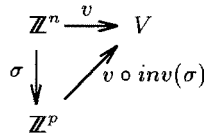


Fig. 1. X is a solution of the second equation

2.2 Formal definitions

The previous example points out that data fields are the central concept in PEI. A data field represents a multiset of values. It is characterized by a *drawing* of the multiset: a drawing associates a geometrical point on \mathbb{Z}^n with each value of a multiset. Formally, assuming the values of the multiset are in V , a drawing of the multiset is a function $v : \mathbb{Z}^n \mapsto V$.

As it has been observed in section 2.1, many data fields can represent the same multiset of values and a bijection links any two of them. It is the reason why, besides its drawing, a bijection characterizes also a data field: it links the data field with a virtual reference domain and can be changed by a change of basis. In fact, the bijection of a data field is not explicit in PEI expressions and it only expresses the *conformity* of objects in such a way that two objects can be combined if and only if one of them conforms to the other. Formally, the bijection is denoted as σ , and it defines an other drawing ($v \circ inv(\sigma)$) if $dom(v) \subseteq dom(\sigma)$.



Definition 1. A data field is a pair, denoted as $(v : \sigma)$, composed of a drawing v and of a bijection σ such that $dom(v) \subseteq dom(\sigma)$.

The *superimposition* combines the data fields in conformity. More precisely, we say that a data field conforms with an other one if its bijection is a restriction of the other's bijection. The drawing of the result is the union of the drawings and the operation builds sequences of values on the intersection. As a consequence, we consider all values are sequences and we use two operators on sequences: an associative constructor denoted as “,” and the function *id* which is the identity on sequences of one element.

Definition 2. Let X_1 and X_2 be two data fields in conformity *i.e.* $\sigma_1 = \sigma_2 \setminus_{dom(\sigma_1)}$. The *superimposition* defines the data field $X_1/\&/X_2$ as $(w : \sigma_2)$, where $w(z) = v_1(z); v_2(z)$.

The other operations apply a function on a data field $\mathbf{X} = (v : \sigma)$ and form a new data field. The PEI notation for partial functions is derived from the lambda-calculus: any function \mathbf{f} of domain $dom(\mathbf{f}) = \{x \mid P(x)\}$ and whose image is $img(\mathbf{f}) = \{\mathbf{f}(x) \mid P(x)\}$ is denoted as $\lambda x \mid P(x) . \mathbf{f}(x)$. Moreover, a function \mathbf{f} defined on disjunctive sub-domains is denoted as a partition $\mathbf{f}_1 \# \mathbf{f}_2$ of functions, and the domain of a composed function $\mathbf{f} \circ \mathbf{g}$ is $\{x \in dom(\mathbf{g}) \mid \mathbf{g}(x) \in dom(\mathbf{f})\}$. Last, $inv(\mathbf{h})$ denotes the inverse of a bijection \mathbf{h} .

Definition 3. Let \mathbf{f} be a partial function from V to W such that $img(v) \subseteq dom(\mathbf{f})$. Let \mathbf{g} be a partial function from $dom(\sigma)$ to $dom(v)$. Let \mathbf{h} be a bijection from $dom(\sigma)$ to \mathbb{Z}^p such that $dom(v) \subseteq dom(\mathbf{h})$.

- The *functional operation* defines the data field $\mathbf{f} \triangleright \mathbf{X}$ as $(\mathbf{f} \circ v : \sigma)$.
- The *geometrical operation* defines the data field $\mathbf{X} \triangleleft \mathbf{g}$ as $(v \circ \mathbf{g} : \sigma)$.
- The *change of basis* defines the data field $\mathbf{h} :: \mathbf{X}$ as $(v \circ inv(\mathbf{h}) : \sigma \circ inv(\mathbf{h}))$.

PEI was originally defined to describe and reason on parallel programs and their implementation. It includes a refinement calculus [14, 16] that makes possible to transform statements by associating algebraic laws and symbolic evaluation of functions with the classical geometrical foundations in parallel programming or parallel compiling. In that context, typechecking is a particularly relevant issue.

2.3 Relevance of typechecking in PEI

As seen in the previous section, PEI operations are not allowed on any data fields: this means that some phrases are forbidden according to some constraints. In other words, if the constraints do not hold, then we say that no semantics is associated with such phrases. Note that this is not absolutely necessary and we could decide to associate a specific meaning with such phrases. But this addresses the following crucial question: is a given PEI specification, feasible or not? It is important to be able to check for feasibility at any step of the refinement process [10]: in PEI, feasibility checking is just typechecking.

Moreover, refinement rules are founded on algebraic properties of operations. More precisely, a refined statement is obtained by replacing one occurrence of a PEI expression by an other one in such a way that the conditions required for the new expression to be well-formed are stronger than the ones required for the old expression. In fact, refinement calculus in PEI is based on types: it preserves feasibility. Let us explain this last point with an example.

Example 2. Routing composition law

$$\mathbf{X} \triangleleft \mathbf{f1} \circ \mathbf{f2} \longrightarrow (\mathbf{X} \triangleleft \mathbf{f1}) \triangleleft \mathbf{f2}$$

This is a classical law of refinement in PEI which says that the expression on the left can be replaced by the expression on the right without condition. Proof of this law directly lies on types: it shows that if the expression on the right is well-formed, then the expression on the left will be well-formed and equal.

This shows that typing objects in PEI is an important issue for reasoning on PEI statements in a practical manner.

3 Typing objects in PEI

As seen before, we are particularly interested in checking the type of the domains of a data field, but not the type of values in the data field. For sake of simplicity, the following definition does not take the type of values into account.

Definition 4. Let $(v : \sigma)$ be a data field, the pair $(dom(v), dom(\sigma))$ is called the *type* of this data field. Domains $dom(v)$ and $dom(\sigma)$ are called respectively the *value domain* and the *reference domain* of the data field.

3.1 Structural definition of types

Writing inference rules is an elegant way to define the type of expressions in a language [1]. Classically, formulae have the form $\vdash \mathbf{E} : \tau$ where \mathbf{E} is an expression and τ its type. We use this notation to define types in PEI. Let D , D_1 and D_2 be value domains and Δ , Δ_1 and Δ_2 be reference domains.

$$\text{Superimposition} \quad \frac{\vdash \mathbf{E}_1 : D_1 \times \Delta_1 \quad \vdash \mathbf{E}_2 : D_2 \times \Delta_2}{\vdash \mathbf{E}_1/\&/\mathbf{E}_2 : (D_1 \cup D_2) \times \Delta_2} \quad \text{if } \Delta_1 \subseteq \Delta_2$$

$$\text{Functional operation} \quad \frac{\vdash \mathbf{E} : \tau}{\vdash \mathbf{f} \triangleright \mathbf{E} : \tau}$$

$$\text{Geometrical operation} \quad \frac{\vdash \mathbf{E} : D \times \Delta}{\vdash \mathbf{E} \triangleleft \mathbf{g} : dom(\mathbf{g}) \times \Delta} \quad \text{if } \begin{cases} img(\mathbf{g}) \subseteq D \\ dom(\mathbf{g}) \subseteq \Delta \end{cases}$$

$$\text{Change of basis} \quad \frac{\vdash \mathbf{E} : D \times \Delta}{\vdash \mathbf{h} :: \mathbf{E} : \mathbf{h}(D) \times img(\mathbf{h})} \quad \text{if } \begin{cases} D \subseteq dom(\mathbf{h}) \\ dom(\mathbf{h}) \subseteq \Delta \end{cases}$$

$$\text{Equation} \quad \frac{\vdash \mathbf{E}_1 : \tau \quad \vdash \mathbf{E}_2 : \tau}{\vdash \mathbf{E}_1 = \mathbf{E}_2}$$

where $\vdash \mathbf{E}_1 = \mathbf{E}_2$ means that the equation $\mathbf{E}_1 = \mathbf{E}_2$ is well-formed.

These rules express the conditions we have associated with the definition of PEI operations and define well-formed phrases in PEI. As said in 2.1, PEI equations are unoriented: they cannot be considered as definitions as a general rule. For that reason, the inference system associated with these rules cannot be carried out: these rules only enable to formally define the type of a data field and to build a “type system” associated with a PEI statement.

3.2 Type system

Typechecking consists in solving a *type system*. The system contains all conditions on domains which ensure that every operation on data fields occurring in the program can be applied. Unknowns of the system denote either a value domain, or a reference domain of a data field \mathbf{X} . We write them $\mathbf{val} \mathbf{X}$ and $\mathbf{ref} \mathbf{X}$ respectively. A type system is a set of equalities and inclusions. Each of them connects two domain expressions. A domain expression is either a variable, or the union of two domains, or the image of a domain by a bijection, or a constant domain. Moreover, note that for any data field \mathbf{X} , the assertion $\mathbf{val} \mathbf{X} \subseteq \mathbf{ref} \mathbf{X}$, deduced from data field definition, is implicitly added to this system.

Example 3. Data replication

```
Broadcast : B  $\mapsto$  A
{
  project :: T = B
  A = T  $\triangleleft$  column
}
```

```
project =  $\lambda(i, j) \mid (i=1, 1 \leq j \leq n) . (j)$ 
column =  $\lambda(i, j) \mid (1 \leq i, j \leq n) . (1, j)$ 
```

The associated type system is:

$\text{val } A \subseteq \text{ref } A$ $\text{val } B \subseteq \text{ref } B$ $\text{val } T \subseteq \text{ref } T$ $\text{val } T \subseteq \text{dom}(\text{project})$ $\text{dom}(\text{project}) \subseteq \text{ref } T$ $\text{img}(\text{column}) \subseteq \text{val } T$ $\text{dom}(\text{column}) \subseteq \text{ref } T$ $\text{project}(\text{val } T) = \text{val } B$ $\text{project}(\text{ref } T) = \text{ref } B$ $\text{val } A = \text{dom}(\text{column})$ $\text{ref } A = \text{ref } T$	<div style="display: flex; flex-direction: column; align-items: center;"> <div style="margin-bottom: 10px;">conditions to define data fields</div> <div>equations</div> </div>
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4 Solving the type system: a type algorithm

The correctness of this algorithm lies on the equivalence of successive type systems obtained step by step from the initial one.

Example 4. Data replication (continued)

From the previous type system, the algorithm results in the following definition:

$$\left\{ \begin{array}{l} \text{val } A = \{(i, j) \mid 1 \leq i, j \leq n\} \\ \text{ref } A = \text{ref } T \\ \text{val } B = \{j \mid 1 \leq j \leq n\} \\ \text{ref } B = \{j \mid 1 \leq j \leq n\} \\ \text{val } T = \{(1, j) \mid 1 \leq j \leq n\} \\ \{(1, j) \mid 1 \leq j \leq n\} \subseteq \text{ref } T \end{array} \right.$$

The typechecking algorithm works in two steps: it first normalizes the system and then removes its unknowns one after the others.

4.1 System normalization

A normalized system is a set of inclusion: $\langle L \rangle \subseteq \langle R \rangle$ where $\langle L \rangle$ is a variable or a constant domain and $\langle R \rangle$ is a domain expression in which any variable occurs only once and is different from $\langle L \rangle$.

Example 5. Data replication (continued)

By definition, only the equational part has to be normalized. The last two equations are easy to normalize, by replacing each of them with two inclusions. For example, $\text{val } A = \text{dom}(\text{column})$ leads to $\text{val } A \subseteq \text{dom}(\text{column})$ and $\text{dom}(\text{column}) \subseteq \text{val } A$. The same idea applied to the former two equations leads to two new inclusions which must be normalized: $\text{project}(\text{val } T) \subseteq \text{val } B$ (1) and $\text{project}(\text{ref } T) \subseteq \text{ref } B$ (2).

- Since $\text{val } T$ is a value domain and by definition of the change of basis operation, $\text{val } T \subseteq \text{dom}(\text{project})$ is already checked. So the inclusion (1) is equivalent to $\text{val } T \subseteq \text{inv}(\text{project})(\text{val } B)$ which has a normal form.

- Since ref T is a reference domain and by definition of the change of basis operation, $\text{dom}(\text{project}) \subseteq \text{ref T}$. So the inclusion (2) is equivalent to $\text{img}(\text{project}) \subseteq \text{ref B}$.

The normal form of a type system can be obtained by applying rules **R1** to **R7**:

R1	$\langle D \rangle_{.1} = \langle D \rangle_{.2}$	$\rightarrow \langle D \rangle_{.1} \subseteq \langle D \rangle_{.2}, \langle D \rangle_{.2} \subseteq \langle D \rangle_{.1}$	
R2	$\langle D \rangle_{.1} \cup \langle D \rangle_{.2} \subseteq \langle D \rangle$	$\rightarrow \langle D \rangle_{.1} \subseteq \langle D \rangle, \langle D \rangle_{.2} \subseteq \langle D \rangle$	
R3	$h(\langle D \rangle_{.1} \cup \langle D \rangle_{.2})$	$\rightarrow h(\langle D \rangle_{.1}) \cup h(\langle D \rangle_{.2})$	
R4	$h_1(h_2(\langle D \rangle))$	$\rightarrow (h_1 \circ h_2)(\langle D \rangle)$	
R5	$h(\text{val } X) \subseteq \langle D \rangle$	$\rightarrow \text{val } X \subseteq \text{inv}(h)(\langle D \rangle)$	
R6	$h_1(\text{val } X) \cup \dots \cup h_n(\text{val } X)$	$\rightarrow \text{max}\{h_1 \dots h_n\}(\text{val } X)$	$(n > 1)$
R7	$\text{val } X \subseteq \langle D \rangle \cup h(\text{val } X)$	$\rightarrow \text{true}$	

where the relation “is a restriction of” is the order between bijections.

Property 5. *The rule system **R1** to **R7** forms a terminating system.*

4.2 Unknown removing

Assuming a type system is under normal form, any unknown X is removed by applying this algorithm: let S_X be the set of inclusions where X occurs. The system can be reduced to two inclusions which bound X : $e \subseteq X \subseteq E$ where e and E are set expressions which do not use X and a set of inclusions in the normal form which do not contain X . The details of this algorithm follows:

- Let us consider the inclusions of S_X of the form $X \subseteq \langle D \rangle_{.i}$. Since S_X has a normal form, $\langle D \rangle_{.i}$ does not contain X . These inclusions can be grouped into a single one of the form $X \subseteq E$ with $E = \bigcap_i \langle D \rangle_{.i}$.
- Let us consider the other inclusions of S_X . Since S_X has a normal form, the unknown X occurs in the right side. We obtain an inclusion of the form $\langle D \rangle_{.i} \subseteq X$ by isolating X . The new inclusions can be grouped into a single one of the form $e \subseteq X$ with $e = \bigcup_i \langle D \rangle_{.i}$.

To show how to isolate X on the right side of an inclusion of S_X , let us consider the most general form: $\langle L \rangle \subseteq \langle D \rangle \cup h(X)$. This inclusion is equivalent to: $\text{inv}(h)(\langle L \rangle) - \text{inv}(h)(\langle D \rangle) \subseteq X$ (3) and $\langle L \rangle \subseteq \langle D \rangle \cup \text{img}(h)$ (4) where (3) has the required form and (4) does not use X . Moreover, we note that (4) has a normal form. So, we obtain bounds for X and a system in the normal form where X does not appear anymore and from which another unknown can be removed.

4.3 Algorithm termination

The unknowns of the type system are removed one after the other. At the end the resulting system does not contain any unknown: the inclusions can then be evaluated by computing constant domain expressions. If all inclusions are

evaluated to true, then PEI expression is well-typed. In this case, the algorithm returns a type range for all data fields in the program. In the other case, there exists a data field without a type. Some informations about the type error source can be obtained, but this point is out of the scope of the paper.

Since the algorithm uses a structural scanning of the abstract syntax tree, its complexity depends on the number of operations. But hopefully, rules of normalization can be applied while building the type system. Moreover, the complexity also depends on the number of identifiers in the statement: the unknowns removing is clearly linear relatively to their number.

5 Implementation

Based on this algorithm, a typechecker for PEI programs has been written in CAML [17]. It uses the OMEGA library [5, 11] for evaluating set expressions: the OMEGA library allows to handle subsets of $\mathbb{Z}^n \times \mathbb{Z}^m$. If $m \neq 0$, the subsets define *relations* which connect n -tuples to m -tuples. If $m=0$, they define subsets of \mathbb{Z}^n . Tuples relations and sets are described by using *Presburger formulae* a class of logical formulas built from affine constraints over integer variables, the logical connectives \neg , \wedge and \vee , and the quantifiers \forall and \exists .

To link these tools, we developed a syntactic analyzer of PEI in CAML and we extended an interface for using the functions of the OMEGA library in CAML. A representation function translates a PEI function into an OMEGA relation. Note that this representation is not always possible because PEI expressions are not necessary limited to Presburger formulae.

A PEI statement can be typed if the geometrical operations or change of basis inside can be coded into an OMEGA relation.

6 Conclusion

This paper defined the notion of type associated with the objects used in the language PEI and presented an algorithm that can infer the type of PEI expressions. Our algorithm presents a weak limitation: all data fields resulting from a change of basis applied on the same data field must be in conformity inside an equation (cf. rule R6). In order to be eventually implemented, the algorithm also requires some basic operations on domains of \mathbb{Z}^n to be available.

Beyond this approach, this allows to determine well-formed phrases of the language [13] and further research may consist in associating classical semantics with programs. This notion of type is of great importance to prove correctness of parallel programs as well as to determine allowable program transformations.

In other works in the area of program transformations, such as ALPHA [8] or CRYSTAL [2], types are not inferred: they are declared and defined as polyhedral domains in \mathbb{Z}^n , in ALPHA for example. Typing variables consists in determining their exact domain and important restrictions on domains are made to insure operations can be applied. PEI is an attempt to overcome these limitations.

In the area of functional languages and data parallel programming, [4] defines *data fields* (not to be confused with data fields defined here), which can be seen as the first part v of data fields in our sense. An extent analysis proposes to find, for a data field f , an approximation to the index set of f in order to perform a “parallel evaluation”.

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