Sampling with Derivatives in the Bergman Space

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Abstract. We characterize sets of sampling for Bergman space functions and their derivatives. In particular, we show that the methods used in the setting of the Bargmann-Fock space can be adapted to our purpose.

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1. Introduction

The Bergman space A^2 consists of all functions f analytic in the unit disk $\mathbb{D}$ such that

$$\|f\|_2 = \left(\int_{\mathbb{D}} |f(z)|^2 d\sigma(z) \right)^{\frac{1}{2}} < \infty,$$

where $d\sigma(z) = \frac{dA(z)}{\pi}$ and $dA(z)$ is Lebesgue area measure. The space A^2 is a Hilbert space with inner product

$$\langle f, g \rangle = \int_{\mathbb{D}} f(z) \overline{g(z)} d\sigma(z).$$

Sampling and interpolating sequences for Bergman spaces [6] and for the Bargmann-Fock space [7],[8], which consists of entire functions that are square integrable with respect to Gaussian measure, were characterized by Seip using a density condition. Later, Brekke and Seip [1] showed that similar density conditions could be used to describe sequences that are sampling or interpolating for Bargmann-Fock space functions and their derivatives. While Luecking [5] has achieved this for interpolation in the Bergman space, we show here that the scheme of Brekke and Seip can be modified to obtain a similar characterization for sampling in the case of the Bergman space.

2. Statements of main results

For $\alpha \in \mathbb{D}$, define the Moebius translation $\varphi_\alpha(z) = \frac{\alpha - z}{1 - \bar{\alpha}z}$ and the pseudohyperbolic metric $\rho(\alpha, z) = |\varphi_\alpha(z)|$.

Let $\Gamma = \{z_j\}_{j \in \mathbb{N}}$ be a sequence of distinct points in $\mathbb{D}$. The sequence Γ is called *uniformly discrete* if there is a $\delta > 0$ such that $\rho(z_i, z_j) > \delta$ whenever $i \neq j$. We shall denote by $\delta(\Gamma) = \inf_{j \neq k} \rho(z_j, z_k)$ the separation constant of Γ .

Given a multiplicity function $\nu : \mathbb{N} \rightarrow \mathbb{N}$, we denote by Γ_ν the multiset where each point $z_j \in \Gamma$ appears $\nu(j)$ times. We will use the index set $\mathbb{J}_\nu = \{(j, k) : j \in \mathbb{N}, 0 \leq k \leq \nu(j) - 1\}$.

We next define the sampling sequences in our context.

Definition 2.1. We say Γ_ν is sampling for A^2 if there exists $0 < K_1 \leq K_2 < \infty$ such that

$$K_1 \|f\|^2 \leq \sum_{(j,k) \in \mathbb{J}_\nu} (1 - |z_j|^2)^{2k+2} |f^{(k)}(z_j)|^2 \leq K_2 \|f\|^2$$

for every $f \in A^2$. We define $K_1(\Gamma_\nu)$ to be the largest value of K_1 for which the lower inequality holds.

Let Γ be uniformly discrete and suppose ν is bounded. Let $n(\Gamma_\nu, \zeta, s)$ denote the number of points of Γ_ν in $\Delta(\zeta, s)$, the pseudohyperbolic disk of center ζ and radius s . Now define

$$E(\Gamma_\nu, \zeta, r) := \frac{\int_0^r n(\Gamma_\nu, \zeta, s) ds}{2 \int_0^r a(\Delta(\zeta, s)) ds},$$

where $a(\Omega) = \int_\Omega \frac{d\sigma(z)}{(1-|z|^2)^2}$ is the hyperbolic area of the measurable subset Ω of $\mathbb{D}$. We remark that the constant 2 in the definition of $E(\Gamma_\nu, \zeta, r)$ is present to make our results coincide with those in [6] in the case $\nu = 1$.

Definition 2.2. The lower uniform density of Γ_ν is defined to be

$$D^-(\Gamma_\nu) := \liminf_{r \rightarrow 1} \inf_{\zeta \in \mathbb{D}} E(\Gamma_\nu, \zeta, r).$$

We then obtain the following characterization of sampling with derivatives in the Bergman space.

Theorem 1. *Assume ν is bounded. Then Γ_ν is a set of sampling for A^2 if and only if Γ is a finite union of uniformly discrete sequences and there is a uniformly discrete subsequence Γ' for which $D^-(\Gamma'_\nu) > \frac{1}{2}$.*

3. Moebius invariance

An important fact about sampling sequences is that a sequence is sampling if and only if any Moebius transformation of it is sampling. This is an immediate consequence of Theorem 1, but its proof is actually required to prove Theorem 1. To examine this invariance we introduce the operators $T_\alpha : A^2 \rightarrow A^2$ given by

$$T_\alpha f(z) = f(\varphi_\alpha(z)) \varphi'_\alpha(z).$$

Note that for each $\alpha \in \mathbb{D}$, T_α is a self adjoint isometry. We will need the following alternative characterization of sampling sequences, which is analogous to the definition given in [1].

Proposition 3.1. *Let $\Gamma = \{z_j\}$ be a sequence of distinct points in $\mathbb{D}$ and $\nu : \mathbb{N} \rightarrow \mathbb{N}$ be bounded. Then Γ_ν is sampling for A^2 if and only if there exist positive constants K'_1 and K'_2 such that*

$$K'_1 \|f\|^2 \leq \sum_{(j,k) \in \mathbb{J}_\nu} |\langle f, T_{z_j} f_k \rangle|^2 \leq K'_2 \|f\|^2,$$

where $f_k(z) = \sqrt{k+1} z^k$.

Proof. First observe that

$$\langle T_\alpha f, f_k \rangle = \frac{1}{k! \sqrt{k+1}} \sum_{n=0}^k \binom{k}{n} (f \circ \varphi_\alpha)^{(n)}(0) \varphi_\alpha^{(k-n+1)}(0).$$

A simple calculation shows that

$$\varphi_\alpha^{(k)}(0) = -k! \bar{\alpha}^{k-1} (1 - |\alpha|^2). \quad (1)$$

The formula for $(f \circ \varphi_\alpha)^{(n)}(0)$ is more complicated. We have

$$(f \circ \varphi_\alpha)^{(n)}(0) = \sum \frac{n!}{m_1! m_2! \dots m_n!} f^{(m_1 + \dots + m_n)}(\varphi_\alpha(0)) \prod_{j=1}^n \left(\frac{\varphi_\alpha^{(j)}(0)}{j!} \right)^{m_j},$$

where the sum is taken over all n -tuples $(m_1, \dots, m_n)$ satisfying $\sum_{j=1}^n j m_j = n$. Let $m = m_1 + \dots + m_n$, so that

$$(f \circ \varphi_\alpha)^{(n)}(0) = \sum_{m=0}^n a_{n,m,k} \bar{\alpha}^{n-m} f^{(m)}(\alpha) (1 - |\alpha|^2)^m,$$

which implies that

$$\langle T_\alpha f, f_k \rangle = \sum_{n=0}^k a_{n,k} \bar{\alpha}^{k-n} f^{(n)}(\alpha) (1 - |\alpha|^2)^{n+1}, \quad (2)$$

where the $a_{n,k}$'s are independent of f and α . Since $|z_j| < 1$, we can conclude that for each $j \in \mathbb{N}$, there is a constant C_j , depending only on $\nu(j)$, such that

$$\sum_{k=0}^{\nu(j)-1} |\langle f, T_{z_j} f_k \rangle|^2 \leq C_j \sum_{k=0}^{\nu(j)-1} (1 - |z_j|^2)^{2k+2} |f^{(k)}(z_j)|^2$$

for every $f \in A^2$.

Now we observe from (2) that

$$(1 - |\alpha|^2)^{k+1} f^{(k)}(\alpha) = \langle f, T_\alpha f_k \rangle - \sum_{n=0}^{k-1} b_{n,k} \langle f, T_\alpha f_n \rangle,$$

where the $b_{n,k}$'s only involve the $a_{n,k}$'s from above and positive powers of $\bar{\alpha}$. It therefore follows that for each j , there is a constant C'_j , depending only on $\nu(j)$, such that

$$\sum_{k=0}^{\nu(j)-1} (1 - |z_j|^2)^{2k+2} |f^{(k)}(z_j)|^2 \leq C'_j \sum_{k=0}^{\nu(j)-1} |\langle f, T_{z_j} f_k \rangle|^2$$

for every $f \in A^2$.

Since ν is bounded, we can finally conclude that there are constants C_ν and C'_ν such that

$$C_j \leq C_\nu \quad \text{and} \quad C'_j \leq C'_\nu$$

for every j , which completes the proof. $\square$

We remark here that the boundedness of the multiplicity function ν is essential to the proof of the above result. In fact, it is easy to see that if ν is unbounded, there are no sampling sequences of the form presented in the introduction (since $(1 - |z|^2)^{k+1} \rightarrow 0$ as $k \rightarrow \infty$ for $z \in \mathbb{D}$); however, we do not know whether there exist sequences with an unbounded ν satisfying the condition of Proposition 3.1.

Lemma 3.2. *If Γ_ν is sampling for A^2 then so is $\varphi_\alpha(\Gamma_\nu)$ for every $\alpha \in \mathbb{D}$. In fact, $K_1(\varphi_\alpha(\Gamma_\nu)) = K_1(\Gamma_\nu)$.*

Proof. We first observe that

$$\varphi_\alpha(\varphi_{z_j}(z)) = \frac{\varphi_\alpha(z_j)}{\varphi_{z_j}(\alpha)} \varphi_{\varphi_{z_j}(\alpha)}(z),$$

so

$$\varphi'_\alpha(\varphi_{z_j}(z)) \varphi'_{z_j}(z) = \frac{\varphi_\alpha(z_j)}{\varphi_{z_j}(\alpha)} \varphi'_{\varphi_{z_j}(\alpha)}(z).$$

Therefore

$$\begin{aligned} T_{z_j} T_\alpha f(z) &= f(\varphi_\alpha(\varphi_{z_j}(z))) \varphi'_\alpha(\varphi_{z_j}(z)) \varphi'_{z_j}(z) \\ &= f\left(\frac{\varphi_\alpha(z_j)}{\varphi_{z_j}(\alpha)} \varphi_{\varphi_{z_j}(\alpha)}(z)\right) \frac{\varphi_\alpha(z_j)}{\varphi_{z_j}(\alpha)} \varphi'_{\varphi_{z_j}(\alpha)}(z). \end{aligned}$$

A calculation similar to the one in the proof of Proposition 3.1 shows

$$\langle T_{z_j} T_\alpha f, f_k \rangle = \sum_{n=0}^k a_n f^{(n)}(\varphi_\alpha(z_j)) \left(\frac{\varphi_\alpha(z_j)}{\varphi_{z_j}(\alpha)}\right)^{n+1} \overline{\varphi_{z_j}(\alpha)^{k-n}} (1 - |\varphi_{z_j}(\alpha)|^2)^{n+1},$$

since

$$f^{(n)}\left(\frac{\varphi_\alpha(z_j)}{\varphi_{z_j}(\alpha)} \varphi_{\varphi_{z_j}(\alpha)}(0)\right) = f^{(n)}(\varphi_\alpha(z_j)),$$

and

$$\left(\frac{\varphi_\alpha(z_j)}{\varphi_{z_j}(\alpha)} \varphi_{\varphi_{z_j}(\alpha)}\right)^{(n)}(0) = -\frac{\varphi_\alpha(z_j)}{\varphi_{z_j}(\alpha)} n! \overline{\varphi_{z_j}(\alpha)^{n-1}} (1 - |\varphi_{z_j}(\alpha)|^2).$$

Now we observe that $|\varphi_\alpha(z_j)| = |\varphi_{z_j}(\alpha)|$, and

$$\varphi_{z_j}(\alpha) = \frac{\varphi_{z_j}(\alpha)}{\varphi_\alpha(z_j)} \varphi_\alpha(z_j),$$

so

$$\overline{\varphi_{z_j}(\alpha)} = \overline{\frac{\varphi_{z_j}(\alpha)}{\varphi_\alpha(z_j)} \varphi_\alpha(z_j)} = \frac{\varphi_\alpha(z_j)}{\overline{\varphi_{z_j}(\alpha)}} \overline{\varphi_\alpha(z_j)}.$$

Therefore

$$\begin{aligned} \langle T_{z_j} T_\alpha f, f_k \rangle &= \left(\frac{\varphi_\alpha(z_j)}{\varphi_{z_j}(\alpha)} \right)^{k+1} \sum_{n=0}^k a_n f^{(n)}(\varphi_\alpha(z_j)) \overline{\varphi_\alpha(z_j)} (1 - |\varphi_\alpha(z_j)|^2)^{n+1} \\ &= \left(\frac{\varphi_\alpha(z_j)}{\varphi_{z_j}(\alpha)} \right)^{k+1} \langle T_{\varphi_\alpha(z_j)} f, f_k \rangle, \end{aligned}$$

which implies that

$$|\langle T_{z_j} T_\alpha f, f_k \rangle| = |\langle T_{\varphi_\alpha(z_j)} f, f_k \rangle|.$$

Therefore

$$\sum_{(j,k) \in \mathbb{J}_\nu} |\langle T_{z_j} T_\alpha f, f_k \rangle|^2 = \sum_{(j,k) \in \mathbb{J}_\nu} |\langle T_{\varphi_\alpha(z_j)} f, f_k \rangle|^2,$$

which completes the proof. $\square$

The following lemma is taken from [2].

Lemma 3.3. *For $1 < t < s$, there is a constant $C_{t,s}$ such that*

$$\int_{\mathbb{D}} \frac{(1 - |z|^2)^{t-2}}{|1 - \bar{\zeta}z|^s} d\sigma(z) \leq C_{t,s} (1 - |\zeta|^2)^{t-s}$$

for every $\zeta \in \mathbb{D}$.

4. Proof of necessity of the sampling condition in A^2

We first consider the upper sampling inequality. Lemma 2.1 from Luecking's paper [4] states that for each $0 < r < 1$ and natural number k , there is a constant C such that

$$|f^{(k)}(z)|^2 (1 - |z|^2)^{2+2k} \leq C \int_{\Delta(z,r)} |f(w)|^2 d\sigma(w) \quad (3)$$

for all $f \in A^2$ and $z \in \mathbb{D}$.

Together with (2), this yields:

Lemma 4.1. *For a fixed $0 < r < 1$ and k , there is a constant $C_{r,k}$ such that*

$$|\langle T_\alpha f, f_k \rangle|^2 \leq C_{r,k} \int_{\Delta(\alpha,r)} |f(z)|^2 d\sigma(z)$$

for every $f \in A^2$ and $\alpha \in \mathbb{D}$.

Lemma 4.2. *Let Γ be uniformly discrete and $\nu : \mathbb{N} \rightarrow \mathbb{N}$ be bounded. There is a constant C , depending only on $\delta(\Gamma)$ and ν , such that*

$$n(\Gamma_\nu, 0, r) \leq \frac{C}{1-r}$$

for all $r < 1$.

Proof. The result follows from the proof of Lemma 15 in Chapter 2 of [2]. $\square$

Lemma 4.3. *Let $\nu : \mathbb{N} \rightarrow \mathbb{N}$ be bounded. Then $\Gamma = \{z_j\}$ can be written as a finite union of uniformly discrete sequence if and only if there exists a constant $C_{\delta, \nu}$ such that*

$$\sum_{(j,k) \in \mathbb{J}_\nu} |\langle T_{z_j} f, f_k \rangle|^2 \leq C_{\delta, \nu} \|f\|^2$$

for every $f \in A^2$, where $\delta = \delta(\Gamma)$.

Proof. The forward direction follows immediately from Lemma 4.1. The backward direction is a consequence of the fact that if Γ_ν satisfies the upper sampling inequality, then so does Γ . The result then follows from an application of Theorem 15 in Chapter 2 of [2]. $\square$

The remainder of this section is devoted to a discussion of the lower sampling inequality and its relationship to the density condition.

Lemma 4.4. *For a fixed $0 < r < 1$ and k , there is a constant $C_{r,k}$ such that*

$$||\langle T_\alpha f, f_k \rangle| - |\langle T_\zeta f, f_k \rangle|| \leq C_{r,k} \rho(\alpha, \zeta) \left(\int_{\Delta(\alpha, r)} |f(z)|^2 d\sigma(z) \right)^{\frac{1}{2}}$$

for every $f \in A^2$ provided $\rho(\alpha, \zeta) < \frac{r}{4}$.

Proof. The proof is similar to the proof of Lemma 2 in Chapter 7 of [2]. First note that since T_ζ is unitary (in particular, invertible), we lose no generality by replacing f with $T_\zeta f$. Since $T_\zeta T_\zeta f = f$, $|\langle T_\alpha T_\zeta f, f_k \rangle| = |\langle T_{\varphi_\zeta(\alpha)} f, f_k \rangle|$, and $\rho(\alpha, \zeta) = |\varphi_\zeta(\alpha)|$, we just need to show that there is a constant $C_{r,k}$ such that

$$||\langle T_\gamma f, f_k \rangle| - |\langle f, f_k \rangle|| \leq C_{r,k} |\gamma| \left(\int_{\Delta(\gamma, r)} |f(z)|^2 d\sigma(z) \right)^{\frac{1}{2}}$$

whenever $|\gamma| < \frac{r}{4}$. We have

$$\begin{aligned} & ||\langle T_\gamma f, f_k \rangle| - |\langle f, f_k \rangle|| \leq |\langle T_\gamma f, f_k \rangle - \langle f, f_k \rangle| \\ & = \left| \left(\sum_{n=0}^k a_n (1 - |\gamma|^2)^{n+1} \bar{\gamma}^{k-n} f^{(n)}(\gamma) \right) - f^{(k)}(0) \right| \\ & = \left| ((1 - |\gamma|^2)^{k+1} f^{(k)}(\gamma) - f^{(k)}(0)) + \sum_{n=0}^{k-1} a_n (1 - |\gamma|^2)^{n+1} \bar{\gamma}^{k-n} f^{(n)}(\gamma) \right| \end{aligned}$$

$$\begin{aligned}
&\leq |(1 - |\gamma|^2)^{k+1} f^{(k)}(\gamma) - f^{(k)}(0)| + \left| \sum_{n=0}^{k-1} a_n (1 - |\gamma|^2)^{n+1} \bar{\gamma}^{k-n} f^{(n)}(\gamma) \right| \\
&= \Psi_1 + \Psi_2.
\end{aligned}$$

Notice that

$$\begin{aligned}
\Psi_2 &= |\gamma| \left| \sum_{n=0}^{k-1} a_n (1 - |\gamma|^2)^{n+1} \bar{\gamma}^{k-1-n} f^{(n)}(\gamma) \right| \\
&\leq |\gamma| C_{r,k} \left(\int_{\Delta(\gamma,r)} |f(z)|^2 d\sigma(z) \right)^{\frac{1}{2}}
\end{aligned}$$

following a similar argument to the proof of Lemma 4.1. Now we turn to Ψ_1 .

$$\begin{aligned}
\Psi_1 &= |(1 - |\gamma|^2)^{k+1} f^{(k)}(\gamma) - f^{(k)}(0)| \\
&\leq |(1 - |\gamma|^2)^{k+1} f^{(k)}(\gamma) - f^{(k)}(\gamma)| + |f^{(k)}(\gamma) - f^{(k)}(0)| \\
&= (1 - (1 - |\gamma|^2)^{k+1}) |f^{(k)}(\gamma)| + |f^{(k)}(\gamma) - f^{(k)}(0)| \\
&= \Phi_1 + \Phi_2.
\end{aligned}$$

Now apply (3) to infer that

$$\begin{aligned}
\Phi_1 &\leq \frac{1 - (1 - |\gamma|^2)^{k+1}}{(1 - |\gamma|^2)^{k+1}} C_{r,k} \left(\int_{\Delta(\gamma,r)} |f(z)|^2 d\sigma(z) \right)^{\frac{1}{2}} \\
&\leq |\gamma| C_{r,k} \left(\int_{\Delta(\gamma,r)} |f(z)|^2 d\sigma(z) \right)^{\frac{1}{2}}
\end{aligned}$$

for $|\gamma| < \frac{r}{4}$. Now it only remains to estimate Φ_2 . For this purpose apply Cauchy's integral formula to obtain

$$\begin{aligned}
\Phi_2 &= |f^{(k)}(\gamma) - f^{(k)}(0)| = \left| \int_0^\gamma f^{(k+1)}(z) d\sigma(z) \right| \\
&\leq |\gamma| \max_{|z| \leq |\gamma|} |f^{(k+1)}(z)| \\
&= |\gamma| \max_{|z| \leq |\gamma|} \frac{1}{2\pi} \left| \int_{|w-z|=\frac{r}{4}} \frac{f(w)}{(w-z)^{k+2}} dw \right| \\
&\leq C_{r,k} |\gamma| \max_{|w| \leq |\gamma| + \frac{r}{4}} |f(w)|.
\end{aligned}$$

Note that $|w| \leq |\gamma| + \frac{r}{4} < \frac{r}{2} < \frac{1}{2}$ since $|\gamma| < \frac{r}{4}$, which means that $\Delta(w, \frac{r}{4}) \subseteq \Delta(\gamma, r)$. Applying (3) (with $k = 0$), we see

$$\begin{aligned}
(1 - |w|^2)^2 |f(w)|^2 &\leq C_r \int_{\Delta(w, \frac{r}{4})} |f(z)|^2 d\sigma(z) \\
&\leq C_r \int_{\Delta(\gamma, r)} |f(z)|^2 d\sigma(z).
\end{aligned}$$

Thus

$$\max_{|w| \leq |\gamma| + \frac{r}{4}} |f(w)| \leq C_r \left(\int_{\Delta(\gamma, r)} |f(z)|^2 d\sigma(z) \right)^{\frac{1}{2}},$$

which implies

$$\Phi_2 \leq C_{r,k} |\gamma| \left(\int_{\Delta(\gamma, r)} |f(z)|^2 d\sigma(z) \right)^{\frac{1}{2}},$$

completing the proof. $\square$

A natural notion of distance between subsets of the unit disk is the Hausdorff distance with respect to the pseudohyperbolic metric. For a relatively closed set $A \subset \mathbb{D}$, let

$$A_t = \{z \in \mathbb{D} : \rho(z, A) < t\}, \quad 0 < t < 1,$$

where $\rho(z, A) = \inf_{\zeta \in A} \rho(z, \zeta)$. The *pseudohyperbolic Hausdorff distance* between relatively closed sets A and B in $\mathbb{D}$ is defined by

$$[A, B] = \inf \{t : A \subset B_t \text{ and } B \subset A_t\}.$$

The next two lemmas follow immediately.

Lemma 4.5. *Let $\Gamma = \{z_j\}$ be uniformly discrete with separation constant δ and $\nu : \mathbb{N} \rightarrow \mathbb{N}$ be bounded. Then there is a constant $C_{\delta, \nu}$ such that*

$$|K_1(\Gamma_\nu)^{\frac{1}{2}} - K_1(\Gamma'_\nu)^{\frac{1}{2}}| \leq C_{\delta, \nu} [\Gamma, \Gamma']$$

provided $[\Gamma, \Gamma'] < \frac{\delta}{8}$.

Lemma 4.6. *Let Γ_ν be sampling for A^2 . Then Γ contains a uniformly discrete subset Γ' such that Γ'_ν is still sampling for A^2 .*

Before we begin with the proof of the forward direction of Theorem 1, we shall present another form of the density that will be more amenable to calculations. To this end, we let

$$D(\Gamma, \alpha, r) = \left(\log \frac{1}{1-r} \right)^{-1} \sum_{z_j \in \Delta(\alpha, \frac{1}{2}, r)} \nu(j) \log \frac{1}{|\varphi_\alpha(z_j)|},$$

where $\Delta(\alpha, \frac{1}{2}, r) = \{z : \frac{1}{2} < \rho(\alpha, z) < r\}$, and $\frac{1}{2} < r < 1$.

The proof of the next lemma is almost identical to the proof of the analogous result for the case $\nu \equiv 1$, given in Section 6.6 of [2].

Lemma 4.7.

$$D^-(\Gamma_\nu) = \liminf_{r \rightarrow 1} \inf_{\zeta \in \mathbb{D}} D(\Gamma_\nu, \zeta, r).$$

The proof of Theorem 1 requires that we determine the effect a radial outward shift has on the density of a sequence. Given $0 < \eta < 1$ let

$$g_\eta(z) = \frac{z}{|z|} \left(\frac{|z| + \eta}{1 + \eta|z|} \right), \quad z \neq 0$$

Lemma 4.8. *If Γ is uniformly discrete with separation constant $\delta = \delta(\Gamma)$, and $\nu : \mathbb{N} \rightarrow \mathbb{N}$ is bounded, then there is a constant C_δ such that*

$$D(g_\eta(\Gamma)_\nu, 0, r) \leq (1 - \eta)D(\Gamma_\nu, 0, r) + C_\delta \left(\log \frac{1}{1 - r} \right)^{-1}$$

for $0 < \eta \leq \frac{1}{4}$, and $\frac{1}{2} < r < 1$.

Proof. The proof is similar to the proof of Lemma 4 in Chapter 7 of [2] and will be omitted. $\square$

We can now prove the necessity direction of Theorem 1. The proof is a modified version of the argument from [7].

Let Γ_ν be sampling for A^2 , where $\Gamma = \{z_j\}$, and assume $\nu : \mathbb{N} \rightarrow \mathbb{N}$ is bounded. Lemma 4.6 allows us to assume that Γ is uniformly discrete.

Choose $0 < \epsilon_\ell < \frac{1}{2}$ so that $\epsilon_\ell \rightarrow 0$. Then for each ℓ , there is a point $\zeta_\ell \in \mathbb{D}$ and a radius r_ℓ , such that $1 - \epsilon_\ell < r_\ell < 1$, and

$$D(\Gamma_\nu, \zeta_\ell, r_\ell) < D^-(\Gamma_\nu) + \epsilon_\ell.$$

Since $D(\varphi_\zeta(\Gamma_\nu), 0, r) = D(\Gamma_\nu, \zeta, r)$ for every $\zeta \in \mathbb{D}$, the above inequality becomes

$$D(\Gamma_{\ell, \nu}, 0, r_\ell) < D^-(\Gamma_\nu) + \epsilon_\ell, \quad (4)$$

where $\Gamma_{\ell, \nu} = \varphi_{\zeta_\ell}(\Gamma_\nu)$. Next, observe that each $\Gamma_{\ell, \nu}$ is uniformly discrete and that $\delta(\Gamma_{\ell, \nu}) = \delta(\Gamma)$. Also, Lemma 3.2 implies that each $\Gamma_{\ell, \nu}$ is sampling for A^2 and that $K_1(\Gamma_{\ell, \nu}) = K_1(\Gamma_\nu)$. Now choose $\eta < \frac{\delta}{8}$ and apply Lemma 4.5 to see

$$K_1(g_\eta(\Gamma_{\ell, \nu}))^{\frac{1}{2}} \geq K_1(\Gamma_{\ell, \nu})^{\frac{1}{2}} - C_{\delta, \nu}[\Gamma_{\ell, \nu}, g_\eta(\Gamma_{\ell, \nu})] \geq K_1(\Gamma_\nu)^{\frac{1}{2}} - C_{\delta, \nu}\eta.$$

Therefore,

$$K_1(g_\eta(\Gamma_{\ell, \nu})) \geq (K_1(\Gamma_\nu)^{\frac{1}{2}} - C_{\delta, \nu}\eta)^2 > 0,$$

which implies that $g_\eta(\Gamma_{\ell, \nu})$ is sampling for A^2 for η sufficiently small. Moreover, $K_1(g_\eta(\Gamma_{\ell, \nu}))$ has a positive lower bound independent of ℓ . It can also be shown that each $g_\eta(\Gamma_{\ell, \nu})$ is uniformly discrete and that $\delta(g_\eta(\Gamma_{\ell, \nu})) \geq \frac{3\delta}{4}$, where $\delta = \delta(\Gamma)$.

Let $g_\eta(\Gamma_{\ell, \nu}) = \{w_{\ell, j}\}$, and for each ℓ define the function

$$h_\ell(w) = \prod_{|w_{\ell, m}| < r_\ell} \left(\frac{1}{|w_{\ell, m}|} \frac{w_{\ell, m} - w}{1 - \overline{w_{\ell, m}}w} \right)^{\nu(m)}.$$

Observe that $\langle h_\ell, T_{w_{\ell,j}} f_k \rangle = 0$ whenever $|w_{\ell,j}| < r_\ell$, so

$$\begin{aligned}
\sum_{(j,k) \in \mathbb{J}_\nu} |\langle h_\ell, T_{w_{\ell,j}} f_k \rangle|^2 &= \sum_{|w_{\ell,j}| \geq r_\ell} \sum_{k=0}^{\nu(j)-1} |\langle h_\ell, T_{w_{\ell,j}} f_k \rangle|^2 \\
&\leq \sum_{|w_{\ell,j}| \geq r_\ell} C_{\delta,\nu} \int_{\Delta(w_{\ell,j}, \frac{\delta}{4})} |h_\ell(w)|^2 d\sigma(w) \\
&\leq \left(\prod_{|w_{\ell,m}| < r_\ell} \frac{1}{|w_{\ell,m}|^{2\nu(m)}} \right) \left(\sum_{|w_{\ell,j}| \geq r_\ell} C_{\delta,\nu} \int_{\Delta(w_{\ell,j}, \frac{\delta}{4})} d\sigma(w) \right) \\
&\leq \left(\prod_{\frac{1}{2} < |w_{\ell,m}| < r_\ell} \frac{1}{|w_{\ell,m}|^{2\nu(m)}} \right) \left(\sum_{|w_{\ell,j}| \geq r_\ell} C_{\delta,\nu} \int_{\Delta(w_{\ell,j}, \frac{\delta}{4})} d\sigma(w) \right),
\end{aligned}$$

where the first inequality follows from Lemma 4.1. The disks $\Delta(w_{\ell,j}, \frac{\delta}{4})$ are disjoint, and since

$$\rho(z, \alpha) \leq \frac{\rho(z, w) + \rho(w, \alpha)}{1 + \rho(z, w)\rho(w, \alpha)}$$

for all $z, w, \alpha \in \mathbb{D}$, they all lie in the annulus $R_\ell < |w| < 1$ where

$$R_\ell = \frac{r_\ell - \frac{\delta}{4}}{1 - r_\ell \frac{\delta}{4}}.$$

Therefore,

$$\begin{aligned}
\sum_{|w_{\ell,j}| \geq r_\ell} C_{\delta,\nu} \int_{\Delta(w_{\ell,j}, \frac{\delta}{4})} d\sigma(w) &\leq C_{\delta,\nu} \int_{|w| \geq R_\ell} d\sigma(w) \\
&\leq C_{\delta,\nu} (1 - R_\ell^2) \\
&\leq C_{\delta,\nu} \left(\frac{1 + \frac{\delta}{4}}{1 - \frac{\delta}{4}} \right) (1 - r_\ell^2) \\
&\leq C_{\delta,\nu} (1 - r_\ell).
\end{aligned}$$

But Lemma 4.8 shows

$$\begin{aligned}
\prod_{\frac{1}{2} < |w_{\ell,m}| < r_\ell} \frac{1}{|w_{\ell,m}|^{2\nu(m)}} &= \exp \left(2 \sum_{\frac{1}{2} < |w_{\ell,m}| < r_\ell} \nu(m) \log \frac{1}{|w_{\ell,m}|} \right) \\
&= \exp \left(2 \left(\log \frac{1}{1 - r_\ell} \right) D(g_\eta(\Gamma_{\ell,\nu}), 0, r_\ell) \right) \\
&\leq \exp \left(2(1 - \eta) \left(\log \frac{1}{1 - r_{\ell,\nu}} \right) D(\Gamma_\ell, 0, r_\ell) + C \right).
\end{aligned}$$

Thus, we have

$$\sum_{(j,k) \in \mathbb{J}_\nu} |\langle h_\ell, T_{w_{\ell,j}} f_k \rangle|^2 \leq C_{\delta,\nu} (1 - r_\ell)^{1-2(1-\eta)(D^-(\Gamma_\nu) + \epsilon_\ell)}.$$

On the other hand, we know that $\|h_\ell\| \geq |h_\ell(0)| = 1$, so the lower sampling inequality tells us

$$K_1(g_\eta(\Gamma_{\ell,\nu})) \leq \sum_{(j,k) \in \mathbb{J}_\nu} |\langle h_\ell, T_{w_{\ell,j}} f_k \rangle|^2.$$

Combining the last two inequalities and letting $\epsilon_\ell \rightarrow 0$, so that $r_\ell \rightarrow 1$, we see that we must have

$$1 - 2(1 - \eta)D^-(\Gamma_\nu) \leq 0,$$

so

$$D^-(\Gamma_\nu) \geq \frac{1}{2(1 - \eta)} > \frac{1}{2}.$$

□

5. Sampling in $A^{-\beta}$

We will need to consider the sampling problem in the *growth space* $A^{-\beta}$, which is the space of analytic functions f for which

$$\|f\|_{-\beta} = \sup_{z \in \mathbb{D}} (1 - |z|^2)^\beta |f(z)| < \infty, \quad \beta > 0.$$

Proposition 5.1. *There exists a constant $C_{\beta,k}$ such that*

$$(1 - |z|^2)^{\beta+k} |f^{(k)}(z)| \leq C_{\beta,k} \|f\|_{-\beta}$$

for every $f \in A^{-\beta}$ and every $z \in \mathbb{D}$.

Proof. Let $z \in \mathbb{D}$ and let C_z denote the circle of radius $\frac{1-|z|}{2}$ centered at z . By Cauchy's integral formula we have

$$\begin{aligned} (1 - |z|^2)^{\beta+k} |f^{(k)}(z)| &\leq (1 - |z|^2)^{\beta+k} \int_{C_z} \left| \frac{f(w)}{(w - z)^{k+1}} \right| dw \\ &= (1 - |z|^2)^{\beta+k} \int_{C_z} \left| \frac{f(w)}{(w - z)^{k+1}} \right| \frac{(1 - |w|^2)^\beta}{(1 - |w|^2)^\beta} dw \\ &\leq (1 - |z|^2)^{\beta+k} \|f\|_{-\beta} \int_{C_z} \frac{dw}{|w - z|^{k+1} (1 - |w|^2)^\beta} \\ &= \frac{2^k (1 - |z|^2)^{\beta+k} \|f\|_{-\beta}}{(1 - |z|)^{k+1}} \int_{C_z} \frac{dw}{(1 - |w|^2)^\beta} \\ &\leq C_{\beta,k} \left(\frac{(1 - |z|^2)^{\beta+k} \|f\|_{-\beta}}{(1 - |z|)^{k+1}} \right) \left(\frac{1 - |z|}{(1 - |z|^2)^\beta} \right) \\ &\leq C_{\beta,k} \|f\|_{-\beta}. \end{aligned}$$

□

Definition 5.2. We say that Γ_ν is sampling for $A^{-\beta}$ if there is an $L > 0$ such that

$$L\|f\|_{-\beta} \leq \sup_{(j,k) \in \mathbb{J}_\nu} |(1 - |z_j|^2)^{\beta+k} f^{(k)}(z_j)|$$

for every $f \in A^{-\beta}$. We call the largest L for which this condition holds the sampling constant of Γ_ν and denote it by $L(\Gamma_\nu)$.

Theorem 2. Let Γ be a sequence of distinct points in $\mathbb{D}$, $\nu : \mathbb{N} \rightarrow \mathbb{N}$ bounded. Then Γ_ν is sampling for $A^{-\beta}$ if and only if Γ has a uniformly discrete subsequence Γ' for which $D^-(\Gamma'_\nu) > \beta$.

The proof of necessity of Theorem 2 is similar to that of Theorem 1 and will be omitted to avoid repetition.

For the proof of sufficiency of Theorem 2 we will need to use the notion of weak convergence of sequences of sequences of points in $\mathbb{D}$ used by Seip [7]. The following definitions are taken almost verbatim from [2].

We say a sequence $\Gamma = \{z_j\}_{j \in \mathbb{N}}$ is *naturally ordered* if $|z_1| \leq |z_2| \leq \dots$. We say a sequence $\Omega = \{\omega_j\}_{j \in \mathbb{N}}$ is a *rearrangement* of Γ if there is a bijection $\phi : \mathbb{N} \rightarrow \mathbb{N}$ such that $\omega_j = z_{\phi(j)}$ for every j . A sequence $\{\Gamma^\ell\}_\ell$ of sequences $\Gamma^\ell = \{z_j^\ell\}$ is called *equidiscrete* if there is a positive lower bound on the separation constants of the sequences $\{\Gamma^\ell\}_\ell$, that is $\inf_\ell \delta(\Gamma^\ell) > 0$.

Definition 5.3. A sequence $\{\Gamma^\ell\}_\ell$ of sequences is said to be weakly convergent to a sequence $\Gamma = \{z_j\}$, written $\Gamma^\ell \rightharpoonup \Gamma$, if there are naturally ordered rearrangements $\{z_j^\ell\}$ of the sequences Γ^ℓ such that

$$\lim_{\ell \rightarrow \infty} z_j^\ell = z_j$$

for every $j = 1, 2, \dots$. If, in addition, $\nu_\ell : \mathbb{N} \rightarrow \mathbb{N}$ satisfy $\nu_\ell(j) \rightarrow \nu(j)$ as $\ell \rightarrow \infty$ (where $\nu : \mathbb{N} \rightarrow \mathbb{N}$ is bounded), we say $\Gamma_\nu^\ell \rightharpoonup \Gamma_\nu$.

A sequence Γ^ℓ converges weakly to a finite sequence $\Gamma = \{z_1, \dots, z_N\}$ if each Γ^ℓ has a naturally ordered rearrangement such that

$$\begin{aligned} \lim_{\ell \rightarrow \infty} z_j^\ell &= z_j & j = 1, 2, \dots, N & \quad \text{and} \\ \lim_{\ell \rightarrow \infty} |z_j^\ell| &= 1 & j > N. \end{aligned}$$

The number N is called the *length* of Γ , where we could have $N = 0$ or ∞ . ($N = 0$ means $\Gamma^\ell \rightharpoonup \emptyset$.) Let

$W(\Gamma_\nu) := \{\Omega_\nu : \varphi_{\zeta_\ell}(\Gamma)_\nu \rightharpoonup \Omega_\nu \text{ as } \ell \rightarrow \infty \text{ for some sequence of points } \{\zeta_\ell\} \subseteq \mathbb{D}\}$.

Lemma 5.4. Let Γ be uniformly discrete, $\nu : \mathbb{N} \rightarrow \mathbb{N}$ bounded, $\Omega_\nu \in W(\Gamma_\nu)$. Then

$$D^-(\Omega_\nu) \geq D^-(\Gamma_\nu).$$

Lemma 5.5. Suppose $\Gamma_\nu^\ell \rightharpoonup \Gamma_\nu$, where $\Gamma^\ell = \{z_{\ell j}\}$, $\Gamma = \{z_j\}$, and $z_{\ell j} \rightarrow z_j$. Let $\{f_\ell\}$ be a sequence of analytic functions in $\mathbb{D}$ which converges locally uniformly to a function f . Then, for every j and for each $0 \leq k \leq \nu(j)$, $f_\ell^{(k)}(z_{\ell j}) \rightarrow f^{(k)}(z_j)$ as $\ell \rightarrow \infty$.

In particular, if all of the f_ℓ 's vanish on Γ^ℓ with multiplicities $\nu(j)$, then so does f . A set Γ_ν is said to be a *set of uniqueness* for $A^{-\beta}$ if $\{f : f^{(k)}(z_j) = 0 \forall (j, k) \in \mathbb{J}_\nu\} = \{0\}$.

Lemma 5.6. *If Γ is uniformly discrete, $\nu : \mathbb{N} \rightarrow \mathbb{N}$ bounded, and every element of $W(\Gamma_\nu)$ is a set of uniqueness for $A^{-\beta}$, then Γ_ν is sampling for $A^{-\beta}$.*

We can now prove the sufficiency of Theorem 2. Again, the argument is adapted from [7].

Proof. Suppose $\Gamma = \{z_j\}$ is uniformly discrete, $\nu : \mathbb{N} \rightarrow \mathbb{N}$ is bounded, and $D^-(\Gamma_\nu) > \beta$. If Γ_ν is not sampling for $A^{-\beta}$, then Lemma 5.6 implies that there is some $\Omega_\nu \in W(\Gamma_\nu)$ admitting a nontrivial $f \in A^{-\beta}$ that vanishes on Ω with multiplicity at least $\nu(j)$. Lemma 5.4 says that $D^-(\Omega_\nu) > \beta$, so we can choose some γ with $\beta < \gamma < D^-(\Omega_\nu)$. Then

$$\gamma < D^-(\Omega_\nu) = \liminf_{r \rightarrow 1} \inf_{\zeta \in \mathbb{D}} D(\Omega_\nu, \zeta, r) \leq \liminf_{r \rightarrow 1} D(\Omega_\nu, 0, r),$$

so $D(\Omega_\nu, 0, r) > \gamma$ for every r in some interval $R < r < 1$ (say $R > \frac{1}{2}$).

If $0 \in \Omega$, then f can be modified (by dividing by an appropriate monomial) so that $f(0) \neq 0$ but f vanishes at every other point of $\Omega = \{\lambda_j\}$ with multiplicity $\nu(j)$. So we lose no generality by assuming that $0 \notin \Omega$ and $f(0) = 1$. Now for $R < r < 1$, Jensen's formula gives

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\theta})| d\theta &= \sum_{|\lambda_j| < r} \nu(j) \log \left(\frac{r}{|\lambda_j|} \right) \\ &\geq \sum_{\frac{1}{2} < |\lambda_j| < r} \nu(j) \log \left(\frac{1}{|\lambda_j|} \right) + n(\Omega_\nu, 0, r) \log r \\ &\geq \sum_{\frac{1}{2} < |\lambda_j| < r} \nu(j) \log \left(\frac{1}{|\lambda_j|} \right) + C_{\delta(\Omega)} \frac{\log r}{1-r} \\ &= (\log \frac{1}{1-r}) D(\Omega_\nu, 0, r) + C_{\delta(\Omega)} \frac{\log r}{1-r} \\ &> \gamma \log \frac{1}{1-r} + C_{\delta(\Omega)} \frac{\log r}{1-r}, \end{aligned}$$

where the inequality in the third line follows from Lemma 4.2.

Since $f \in A^{-\beta}$, we know

$$|f(z)| \leq \|f\|_{-\beta} (1 - |z|)^{-\beta},$$

so

$$\frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\theta})| d\theta \leq \log \|f\|_{-\beta} + \beta \log \frac{1}{1-r}.$$

Combining these two estimates gives

$$\gamma \log \frac{1}{1-r} + C_{\delta(\Omega)} \frac{\log r}{1-r} < \log \|f\|_{-\beta} + \beta \log \frac{1}{1-r}$$

for $r > R$. Now let $r \rightarrow 1$ to conclude that $\gamma \leq \beta$, which is a contradiction. Therefore $W(\Gamma_\nu)$ contains only sets of uniqueness for $A^{-\beta}$, so Lemma 5.6 says Γ_ν is sampling for $A^{-\beta}$. $\square$

6. Proof of sufficiency of sampling condition in A^2

The proof of the sufficiency direction of Theorem 1 will require consideration of sampling sequences for a certain subspace of $A^{-\beta}$. The arguments below are adapted from [7].

The subspace $A_0^{-\beta}$ consists of analytic functions f with the property

$$\lim_{|z| \rightarrow 1} (1 - |z|^2)^\beta |f(z)| = 0.$$

If Γ is a sequence of distinct points in $\mathbb{D}$, and $\nu : \mathbb{N} \rightarrow \mathbb{N}$ is bounded, then we say that Γ_ν is sampling for $A_0^{-\beta}$ if there is an $L > 0$ such that

$$L \|f\|_{-\beta} \leq \sup_{(j,k) \in \mathbb{J}_\nu} |(1 - |z_j|^2)^{\beta+k} f^{(k)}(z_j)|$$

for every $f \in A_0^{-\beta}$. Denote by $L_0(\Gamma_\nu)$ the largest L for which this condition holds.

Lemma 6.1. *If $f \in A_0^{-\beta}$, then $\lim_{|z| \rightarrow 1} (1 - |z|^2)^{\beta+k} |f^{(k)}(z)| = 0$ for every k .*

Proof. Let $f \in A_0^{-\beta}$, $\epsilon > 0$. The polynomials are dense in $A_0^{-\beta}$, so we can choose a sequence of polynomials p_n which converges to f in $A_0^{-\beta}$. Proposition 5.1 now implies that $(1 - |z|^2)^{\beta+k} p_n^{(k)}(z)$ converges to $(1 - |z|^2)^{\beta+k} f^{(k)}(z)$ uniformly on $\mathbb{D}$. Therefore, we can choose an N so that $n \geq N$ implies

$$(1 - |z|^2)^{\beta+k} |f^{(k)}(z) - p_n^{(k)}(z)| < \frac{\epsilon}{2}.$$

Now observe that the statement is clearly true for polynomials since the derivative of a polynomial is another polynomial and all polynomials are bounded on $\mathbb{D}$. Therefore, we can find a $t < 1$ so that $|z| > t$ implies

$$(1 - |z|^2)^{\beta+k} |p_N^{(k)}(z)| < \frac{\epsilon}{2}.$$

So for each such z we have

$$\begin{aligned} (1 - |z|^2)^{\beta+k} |f^{(k)}(z)| &= (1 - |z|^2)^{\beta+k} |f^{(k)}(z) - p_N^{(k)}(z) + p_N^{(k)}(z)| \\ &\leq (1 - |z|^2)^{\beta+k} |f^{(k)}(z) - p_N^{(k)}(z)| + (1 - |z|^2)^{\beta+k} |p_N^{(k)}(z)| \\ &= \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

$\square$

Theorem 3. *Let $\nu : \mathbb{N} \rightarrow \mathbb{N}$ be bounded, and suppose that $\Gamma = \{z_j\}$ is a uniformly discrete sequence such that Γ_ν is sampling for $A_0^{-\beta}$. Then there is a sequence*

$\{g_{j,k}\}$ of functions such that for each $\zeta \in \mathbb{D}$,

$$f(\zeta) = (1 - |\zeta|^2)^{-\beta} \sum_{(j,k) \in \mathbb{J}_\nu} (1 - |z_j|^2)^{k+\beta} f^{(k)}(z_j) g_{j,k}(\zeta)$$

for every $f \in A_0^{-\beta}$, where

$$\sum_{(j,k) \in \mathbb{J}_\nu} |g_{j,k}(\zeta)| \leq C < \infty,$$

and

$$|g_{j,k}(\zeta)| \leq C \frac{(1 - |\zeta|^2)(1 - |z_j|^2)}{|1 - \bar{z}_j \zeta|^2}$$

for some constant C that depends only on $L_0(\Gamma_\nu)$.

Proof. First observe that $(1 - |z|^2)^{\beta+k} |f^{(k)}(z_j)| \rightarrow 0$ as $j \rightarrow \infty$ for every $f \in A_0^{-\beta}$ by Lemma 6.1. Define the operator $S_{\Gamma_\nu}^{-\beta} : A^{-\beta} \rightarrow \ell^\infty(\mathbb{J}_\nu)$ by

$$S_{\Gamma_\nu}^{-\beta} f = \{(1 - |z_j|^2)^{\beta+k} f^{(k)}(z_j) : (j, k) \in \mathbb{J}_\nu\}.$$

It follows that $S_{\Gamma_\nu}^{-\beta}(A_0^{-\beta})$ is a subspace of $c_0(\mathbb{J}_\nu)$, where $c_0(\mathbb{J}_\nu)$ is the subspace of $\ell^\infty(\mathbb{J}_\nu)$ consisting of sequences that converge to 0. Since Γ_ν is sampling, we know $S_{\Gamma_\nu}^{-\beta}$ is bounded below, so its range is closed and $(S_{\Gamma_\nu}^{-\beta})^{-1}$ maps $S_{\Gamma_\nu}^{-\beta}(A_0^{-\beta})$ boundedly onto $A_0^{-\beta}$. Also, $\|(S_{\Gamma_\nu}^{-\beta})^{-1}\| = L_0(\Gamma_\nu)$ (the sampling constant of Γ_ν with respect to $A_0^{-\beta}$.)

For a fixed $\zeta \in \mathbb{D}$, let ψ_ζ be the bounded linear functional defined by

$$\psi_\zeta(f) = (1 - |\zeta|^2)^\beta f(\zeta).$$

It induces the linear functional

$$\tilde{\psi}_\zeta(\{w_{j,k}\}) = \psi_\zeta((S_{\Gamma_\nu}^{-\beta})^{-1}\{w_{j,k}\}), \quad \{w_{j,k}\} \in S_{\Gamma_\nu}^{-\beta}(A_0^{-\beta})$$

with norm $\|\tilde{\psi}_\zeta\| \leq L_0(\Gamma_\nu)$ since $\|\psi_\zeta\| \leq 1$. By the Hahn-Banach Theorem we can extend $\tilde{\psi}_\zeta$ to a linear functional $\tilde{\Psi}_\zeta$ on $c_0(\mathbb{J}_\nu)$ of norm $\|\tilde{\Psi}_\zeta\| = \|\tilde{\psi}_\zeta\|$. But the dual space $c_0(\mathbb{J}_\nu)^*$ can be identified with $\ell^1(\mathbb{J}_\nu)$, and $\tilde{\Psi}_\zeta$ has a representation of the form

$$\tilde{\Psi}_\zeta(\{w_{j,k}\}) = \sum_{(j,k) \in \mathbb{J}_\nu} w_{j,k} b_{j,k}(\zeta), \quad \{w_{j,k}\} \in c_0(\mathbb{J}),$$

where

$$\sum_{(j,k) \in \mathbb{J}_\nu} |b_{j,k}(\zeta)| = \|\tilde{\Psi}_\zeta\| \leq \|(S_{\Gamma_\nu}^{-\beta})^{-1}\|.$$

Restrict the representation to $S_{\Gamma_\nu}^{-\beta}(A_0^{-\beta})$ to see that

$$(1 - |\zeta|^2)^\beta f(\zeta) = \sum_{(j,k) \in \mathbb{J}_\nu} (1 - |z_j|^2)^{k+\beta} f^{(k)}(z_j) b_{j,k}(\zeta) \quad (5)$$

for every $f \in A_0^{-\beta}$.

For each $j \in \mathbb{N}$, let $n_j = |\{k : |b_{j,k}(\zeta)| > 1 - |\varphi_\zeta(z_j)|^2\}|$, and let

$$\Lambda_\zeta = \{z_j : |\varphi_\zeta(z_j)| > \frac{1}{2}, n_j \geq 1\}.$$

In view of the identity

$$1 - |\varphi_\zeta(z)|^2 = \frac{(1 - |\zeta|^2)(1 - |z|^2)}{|1 - \bar{\zeta}z|^2},$$

and the fact that $\sum_{(j,k) \in \mathbb{J}_\nu} |b_{j,k}(\zeta)| \leq L_0(\Gamma_\nu)$, we see that Λ_ζ is a Blaschke sequence with corresponding Blaschke product

$$B_\zeta(z) = \prod_{z_j \in \Lambda_\zeta} \left(\frac{|z_j|}{z_j} \frac{z_j - z}{1 - \bar{z}_j z} \right)^{\nu(j)}.$$

Note that $B_\zeta(\zeta) \neq 0$ since $\zeta \notin \Lambda_\zeta$. Now apply the formula (5) to the function $B_\zeta f$ to see

$$\begin{aligned} (1 - |\zeta|^2)^\beta f(\zeta) &= B_\zeta(\zeta)^{-1} \sum_{(j,k) \in \mathbb{J}_\nu} (1 - |z_j|^2)^{k+\beta} \left(\sum_{n=0}^k \binom{k}{n} f^{(n)}(z_j) B_\zeta^{(k-n)}(z_j) \right) b_{j,k}(\zeta) \\ &= \sum_{(j,k) \in \mathbb{J}_\nu} (1 - |z_j|^2)^{k+\beta} f^{(k)}(z_j) g_{j,k}(\zeta), \end{aligned}$$

where

$$g_{j,k}(\zeta) = B_\zeta(\zeta)^{-1} \sum_{n=k}^{\nu(j)-1} \binom{n}{k} (1 - |z_j|^2)^n B_\zeta^{(n-k)}(z_j) b_{j,n}(\zeta).$$

Applying the inequality $x^n \geq e^{-2n(1-x)}$ for $\frac{1}{4} \leq x \leq 1$, we find

$$\begin{aligned} |B_\zeta(\zeta)|^2 &= \prod_{z_j \in \Lambda_\zeta} |\varphi_\zeta(z_j)|^{2\nu(j)} \geq \prod_{z_j \in \Lambda_\zeta} \exp(-2\nu(j)(1 - |\varphi_\zeta(z_j)|^2)) \\ &\geq \prod_{z_j \in \Lambda_\zeta} \exp(-2\nu(j)|b_{j,k_j}(\zeta)|) = \exp\left(-C_\nu \sum_{z_j \in \Lambda_\zeta} |b_{j,k_j}(\zeta)|\right) \\ &\geq \exp(-C_\nu L_0(\Gamma_\nu)), \end{aligned}$$

where for each j , k_j satisfies $|b_{j,k_j}(\zeta)| > 1 - |\varphi_\zeta(z_j)|^2$. (The existence of such a k_j is guaranteed by the definition of Λ_ζ .)

Since $|B_\zeta(z)| \leq 1$ for every $z \in \mathbb{D}$, Proposition 5.1 shows that for each n there is a constant C_n such that $(1 - |z|^2)^n |B_\zeta^{(n)}(z)| \leq C_n$ for every $z \in \mathbb{D}$. Combining these last two estimates yields $|g_{j,k}(\zeta)| \leq C_\nu e^{C_\nu L_0(\Gamma_\nu)} \sum_{n=k}^{\nu(j)-1} |b_{j,n}(\zeta)|$, so

$$\sum_{(j,k) \in \mathbb{J}_\nu} |g_{j,k}(\zeta)| \leq C_\nu e^{C_\nu L_0(\Gamma_\nu)} \sum_{(j,k) \in \mathbb{J}_\nu} |b_{j,k}(\zeta)| \leq C_\nu L_0(\Gamma_\nu) e^{C_\nu L_0(\Gamma_\nu)}.$$

If $z_j \in \Lambda_\zeta$, then $g_{j,k}(\zeta) = 0$ for every $0 \leq k \leq \nu(j) - 1$, since $B_\zeta^{(n)}(z_j) = 0$ for every $n \in [\nu(j)]$. If $z_j \notin \Lambda_\zeta$, then either $|b_{j,k}(\zeta)| \leq 1 - |\varphi_\zeta(z_j)|^2$ for every

$0 \leq k \leq \nu(j) - 1$ or $|\varphi_\zeta(z_j)| \leq \frac{1}{2}$. In the first case we have

$$|b_{j,k}(\zeta)| \leq 1 - |\varphi_\zeta(z_j)|^2 = \frac{(1 - |\zeta|^2)(1 - |z_j|^2)}{|1 - \bar{z}_j\zeta|^2}.$$

If $|\varphi_\zeta(z_j)| \leq \frac{1}{2}$, then

$$\frac{(1 - |\zeta|^2)(1 - |z_j|^2)}{|1 - \bar{z}_j\zeta|^2} \geq \frac{3}{4},$$

so

$$|b_{j,k}(\zeta)| \leq \sum_{(n,m) \in \mathbb{J}_\nu} |b_{n,m}(\zeta)| \leq L_0(\Gamma_\nu) \leq \frac{4}{3} L_0(\Gamma_\nu) \frac{(1 - |\zeta|^2)(1 - |z_j|^2)}{|1 - \bar{z}_j\zeta|^2}.$$

In either case, the desired inequality follows from the fact that

$$|g_{j,k}(\zeta)| \leq C_\nu e^{C_\nu L_0(\Gamma_\nu)} \sum_{n=k}^{\nu(j)-1} |b_{j,n}(\zeta)|. \quad \square$$

We now complete the proof of Theorem 1.

Proof. Let $\Gamma = \{z_j\} \subseteq \mathbb{D}$ be uniformly discrete, and $\nu : \Gamma \rightarrow \mathbb{N}$ be bounded so that $D^-(\Gamma_\nu) > \frac{1}{2}$. Note that our assumptions on Γ_ν guarantee that the upper sampling inequality holds by Corollary 4.3. Thus, we are to show that there exists $K_1 > 0$ such that

$$K_1 \|f\|^2 \leq \sum_{(j,k) \in \mathbb{J}_\nu} (1 - |z_j|^2)^{(2k+2)} |f^{(k)}(z_j)|^2$$

for all $f \in A^2$.

It is enough to establish the inequality for all polynomials, since polynomials are dense in A^2 and the upper sampling inequality holds. Choose β so that $\frac{1}{2} < \beta < 1$ and $\beta < D^-(\Gamma_\nu)$. Then, by Theorem 2, Γ_ν is sampling for $A_0^{-\beta}$. Let f be a polynomial. Then since $f \in A_0^{-\beta}$, we can apply Theorem 3 and Hölder's inequality to see

$$\begin{aligned} |f(\zeta)|^2 &= (1 - |\zeta|^2)^{-2\beta} \left| \sum_{(j,k) \in \mathbb{J}_\nu} (1 - |z_j|^2)^{k+\beta} f^{(k)}(z_j) g_{j,k}(\zeta) \right|^2 \\ &\leq (1 - |\zeta|^2)^{-2\beta} \left(\sum_{(j,k) \in \mathbb{J}_\nu} (1 - |z_j|^2)^{2\beta+2k} |f^{(k)}(z_j)|^2 |g_{j,k}(\zeta)| \right) \left(\sum_{(j,k) \in \mathbb{J}_\nu} |g_{j,k}(\zeta)| \right) \\ &\leq C(1 - |\zeta|^2)^{1-2\beta} \sum_{(j,k) \in \mathbb{J}_\nu} \frac{(1 - |z_j|^2)^{2k+1+2\beta}}{|1 - \bar{z}_j\zeta|^2} |f^{(k)}(z_j)|^2. \end{aligned}$$

By Lemma 3.3, we see

$$\begin{aligned} \int_{\mathbb{D}} |f(\zeta)|^2 d\sigma(\zeta) &\leq C \sum_{(j,k) \in \mathbb{J}_\nu} (1 - |z_j|^2)^{2k+1+2\beta} |f^{(k)}(z_j)|^2 \int_{\mathbb{D}} \frac{(1 - |\zeta|^2)^{1-2\beta}}{|1 - \bar{z}_j\zeta|^2} d\sigma(\zeta) \\ &\leq C \sum_{(j,k) \in \mathbb{J}_\nu} (1 - |z_j|^2)^{2k+2} |f^{(k)}(z_j)|^2, \end{aligned}$$

since $\frac{1}{2} < \beta < 1$. □

Remark 6.2. It recently came to our attention that Newton Foster proved a generalization of our result in his PhD thesis [3] after our work was completed. It appears that he was unaware of our results.

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