Cardinal Sine Series: Convergence and Uniqueness

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Abstract. Many aspects of the cardinal sine series, particularly those associated with the mathematical theory of sampling in signal processing, are very well known due to its role in the classical sampling theorems. The objective of this article is to highlight several extensions of these classical theorems and to provide corresponding examples. Among the matter presented here are (i) necessary and sufficient conditions for convergence of the series and (ii) several new classes of entire functions that can be represented via such series. Some of these classes contain members that may be unbounded on the real axis.

Key words and phrases : cardinal series, entire functions of exponential type, sampling theorems.

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1. Introduction

The cardinal sine series with coefficients $\{c_n\}$ is the series

$$(1) \quad f(z) = \sum_{n=-\infty}^{\infty} c_n \frac{\sin \pi(z - n)}{\pi(z - n)}.$$

This series is also known as the Whittaker cardinal series in honor of the author of [20] and is sometimes called the sampling series because of the central role it plays in the well known Whittaker-Kotelnikov-Shannon (WKS) sampling theorem. There are many references that provide excellent treatments of this subject from various points of view, for example [5, 6], [7, 8], [10], [18, 19], [21, 22], and [23].

If the data sequence $\{c_n\}$ satisfies

$$(2) \quad \sum_{n=-\infty}^{\infty} \frac{|c_n|}{1+|n|} < \infty$$

then the series converges for every z in the complex plane $\mathbb{C}$. This is the case in the celebrated WKS sampling theorem by virtue of the fact that the data sequence $\{c_n\}$ is taken to be in ℓ^2 . This is also the case in the important and far reaching extension for data $\{c_n\}$ in ℓ^p , $1 < p < \infty$, attributed to Plancherel and Polya, [11, Lecture 20, Theorem 3].

If the data sequence $\{c_n\}$ fails to satisfy (2) the situation concerning the convergence and properties of (1) is not so well known. By convergence we mean the convergence, as $N \rightarrow \infty$, of the symmetric partial sums $f_N(z)$ defined by

$$(3) \quad f_N(z) = \sum_{n=-N}^N c_n \frac{\sin \pi(z-n)}{\pi(z-n)}.$$

The goal of this article is to highlight some of the results that treat cases in which $\{c_n\}$ does not necessarily satisfy (2) and to provide examples of unexpected behavior. More specifically, the material presented here includes (i) necessary and sufficient conditions on the sequence $\{c_n\}$ for the symmetric partial sums $\{f_N(z)\}$ to converge and (ii) several new classes of entire functions that enjoy representation (1). Theorem 12 is a new result and the proof of Theorem 11 is new.

We use standard notation and only alert the reader to the fact that E_σ denotes the class of entire functions of exponential type no greater than σ that have no greater than polynomial growth along the real axis. In view of the distributional variant of the Paley-Wiener Theorem, for example see [9, Theorem 1.7.7], E_σ consists of the Fourier transforms of distributions with support in the interval $[-\sigma, \sigma]$.

2. Convergence properties of generic cardinal sine series

2.1. The basic convergence criterion. Statements and proofs of the following two theorems can be found in [2, 2.1 Theorem 1 and 3.3 Proposition 2].

Theorem 1. *Suppose $\{c_n\}_{n=-\infty}^{\infty}$ is a sequence with even and odd parts $\{a_n\}$ and $\{b_n\}$, namely,*

$$a_n = \frac{c_n + c_{-n}}{2} \quad \text{and} \quad b_n = \frac{c_n - c_{-n}}{2}.$$

1) The symmetric partial sums $\{f_N(z)\}_{N=1}^{\infty}$ defined by (3) converge uniformly on compact subsets of $\mathbb{C}$ to an entire function $f(z)$ if and only if both

$$(4) \quad \sum_{n=1}^{\infty} (-1)^n \frac{a_n}{n^2} \quad \text{and} \quad \sum_{n=1}^{\infty} (-1)^n \frac{b_n}{n} \quad \text{converge.}$$

2) If $\{f_N(z)\}$ does not converge on compact subsets of $\mathbb{C}$ then $\{f_N(z)\}$ converges for at most one z that is not an integer. In other words, if $\{f_N(z)\}$ converges for two points, $z = z_1$ and $z = z_1$, that are not integers then $\{f_N(z)\}$ converges uniformly on compact subsets of $\mathbb{C}$.

The rate of convergence of $f_N(z)$ to $f(z)$ can be estimated in terms of the rate of convergence of the series (4). In what follows $f_e(z)$ and $f_o(z)$ denote the even and odd parts of the function $f(z)$ defined on $\mathbb{C}$, namely,

$$f_e(z) = \frac{f(z) + f(-z)}{2} \quad \text{and} \quad f_o(z) = \frac{f(z) - f(-z)}{2}.$$

Theorem 2. In the notation of Theorem 1, if both the series (4) converge and K is a compact subset of $\mathbb{C}$ then there is an $M > 0$ such that

$$\sup_{z \in K} |f_e(z) - f_{e,N}(z)| \leq M \sup_{n \geq N} \left| \sum_{j=n+1}^{\infty} \frac{(-1)^j a_j}{j^2} \right|$$

and

$$\sup_{z \in K} |f_o(z) - f_{o,N}(z)| \leq M \sup_{n \geq N} \left| \sum_{j=n+1}^{\infty} \frac{(-1)^j b_j}{j} \right|.$$

For an application of Theorem 2 see [2, Subsection 4.7].

2.2. The growth of cardinal sine series. We say that an entire function $f(z)$ is a cardinal sine series if on compact subsets of $\mathbb{C}$ it is the uniform limit of symmetric partial sum $f_N(z)$ defined by (3). Needless to say, in this case $c_n = f(n)$, $n = 0, \pm 1, \pm 2, \dots$.

Condition (4) implies that $|a_n| = o(n^2)$ and $|b_n| = o(n)$ as $n \rightarrow \infty$. These bounds on growth generalize to the resulting entire function $f(z)$, as implied by the following theorem whose proof can be found in [2, 2.2 Theorem 2].

Theorem 3. If $f(z)$ is a cardinal sine series, then

$$|f_e(z)|e^{-\pi|\operatorname{Im}(z)|} = o(|z|^2) \quad \text{and} \quad |f_o(z)|e^{-\pi|\operatorname{Im}(z)|} = o(|z|) \quad \text{as } |z| \rightarrow \infty.$$

In particular, a cardinal sine series is in E_{π} .

More restrictive bounds on the sequences of samples $\{f(n)\}$ give rise to corresponding bounds on the growth of $f(z)$ as $|z| \rightarrow \infty$.

Theorem 4. *Suppose $f(z)$ is a cardinal sine series and for some α , $0 \leq \alpha < 2$,*

$$a_n = f_e(n) = O(n^\alpha) \quad \text{as } n \rightarrow \infty.$$

then $f_e(z)$ satisfies

$$|f_e(z)|e^{-\pi|\operatorname{Im}(z)|} = O(|z|^\alpha \log |z|) \quad \text{as } |z| \rightarrow \infty.$$

If $a_n = f_e(n)$ and

$$a_{n+1} - a_n = O(n^{\alpha-1}) \quad \text{as } n \rightarrow \infty \quad \text{when } 0 < \alpha < 2$$

or

$$\sum_{n=1}^{\infty} |a_{n+1} - a_n| < \infty \quad \text{when } \alpha = 0$$

then $f_e(z)$ satisfies

$$|f_e(z)|e^{-\pi|\operatorname{Im}(z)|} = O(|z|^\alpha) \quad \text{as } |z| \rightarrow \infty.$$

Analogous bounds are valid for $f_o(z)$. Namely, if for some β , $0 \leq \beta < 1$,

$$b_n = f_o(n) = O(n^\beta) \quad \text{as } n \rightarrow \infty.$$

then $f_o(z)$ satisfies

$$|f_o(z)|e^{-\pi|\operatorname{Im}(z)|} = O(|z|^\beta \log |z|) \quad \text{as } |z| \rightarrow \infty.$$

If $b_n = f_o(n)$ and

$$b_{n+1} - b_n = O(n^{\beta-1}) \quad \text{as } n \rightarrow \infty \quad \text{when } 0 < \beta < 1$$

or

$$\sum_{n=1}^{\infty} |b_{n+1} - b_n| < \infty \quad \text{when } \beta = 0$$

then $f_o(z)$ satisfies

$$|f_o(z)|e^{-\pi|\operatorname{Im}(z)|} = O(|z|^\beta) \quad \text{as } |z| \rightarrow \infty.$$

Theorem 4 summarizes the contents of [13, Theorems 2, 3, 5, and 6] where detailed proofs can also be found.

2.3. Behavior under shifts. It would be desirable for convergence of the symmetric partial sums $\{f_N\}_N$ to be invariant under shifts of the data. In general this is not the case, as a shift in $\{c_n\}_{n=-\infty}^{\infty}$ can rearrange its odd and even parts so that condition (4) of Theorem 1 may fail, see Example 4.1 in Section 4. Theorem 5 below presents conditions under which $\{f_N(z)\}$ behaves nicely given integer-shifted data, see [2, 2.3 Theorem 3] for complete details.

Theorem 5. Suppose $\{c_n\}_{n=-\infty}^{\infty}$ is a sequence for which the corresponding symmetric partial sums $\{f_N(z)\}$ converge to $f(z)$.

1) The symmetric partial sums of the cardinal sine series corresponding to the sequence $\{c_{n+m}\}_{n=-\infty}^{\infty}$,

$$f_N(m; z) = \sum_{n=-N}^N c_{n+m} \frac{\sin \pi(z - n)}{\pi(z - n)},$$

converge for every integer m if and only if

$$(5) \quad \lim_{n \rightarrow \infty} \frac{c_n - c_{n+1}}{n} = 0.$$

3) More generally, the symmetric partial sums, $\{f_N(m; z)\}$, of the cardinal sine series corresponding to the sequence $\{c_{n+m}\}_{n=-\infty}^{\infty}$ converges for the given integer m if and only if

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{2m-1} (-1)^k c_{n+k} = 0.$$

If $\{f_N(m; z)\}$ converges, then so does $\{f_N(km; z)\}$ for every integer k .

3) If the symmetric partial sums of the cardinal sine series corresponding to the sequence $\{c_{n+m}\}_{n=-\infty}^{\infty}$ converge, then the corresponding limit is $f(z + m)$.

Note that if $m \neq 0$ then $f_N(m; z)$ is not necessarily equal to $f_N(z + m)$.

3. Sampling and uniqueness theorems

In spite of Theorem 1, given an entire function $u(z)$ in E_π , with the exception of the cases covered by the Plancherel-Polya Theorem, it is not always easy to determine whether the corresponding sequences of samples $c_n = u(n)$, $n = 0, \pm 1, \pm 2, \dots$, gives rise to a convergent cardinal series. Furthermore, in view of the functions $P(z) \sin \pi z$, where $P(z)$ is a polynomial, that are in E_π and vanish on the integers $\mathbb{Z}$, even if the cardinal series does converge, the limiting function $f(z)$ need not necessarily be the same as $u(z)$.

In what follows we describe several classes of entire functions $f(z)$ whose samples $c_n = f(n)$, $n = 0, \pm 1, \pm 2, \dots$, give rise to convergent cardinal sine series. In most of the cases the functions are uniquely defined by the corresponding series.

3.1. A growth condition on samples. We begin with the following theorem that is a routine consequence of Theorems 1 and 4 above and Liouville's Theorem [1, page 122]. A detailed proof can be found in [13, Theorem 2].

Theorem 6. *Suppose the data sequence $\{c_n\}$ is even and for some α , $0 \leq \alpha < 1$, satisfies $c_n = O(n^\alpha)$ as $n \rightarrow \infty$. Then the partial sums $f_N(z)$ converge to an entire function $f(z)$ that satisfies*

$$(6) \quad |f(z)|e^{-\pi|\operatorname{Im}(z)|} = O(|z|^\alpha \log |z|) \quad \text{as } |z| \rightarrow \infty.$$

$f(z)$ is the unique even function in E_π that satisfies (6) and interpolates the data, $f(n) = c_n$, $n = 0, \pm 1, \pm 2, \dots$.

Recalling that an entire function $f(z)$ is a cardinal sine series if on compact subsets of $\mathbb{C}$ it is the uniform limit of symmetric partial sum $f_N(z)$ defined by (3), it follows that if $f(z)$ is an even entire function in E_π that for some $\alpha < 1$ satisfies $f(x) = O(x^\alpha)$ as $x \rightarrow \infty$ on the real axis then $f(z)$ is a cardinal sine series.

On the other hand, if $f(z)$ is an entire function in E_π such that $f(x) = o(1)$ as $x \rightarrow \pm\infty$ that is not even then it is not readily apparent that the samples $f(n) = c_n$, $n = 0, \pm 1, \pm 2, \dots$, satisfy (4) of Theorem 1.

It turns out, somewhat unexpectedly in our view, that such data sequences do indeed satisfy (4) of Theorem 1. This is a consequence of the following Theorem and is explicitly recorded in Theorem 9 below.

3.2. Asymptotic conditions on $f(z)$. As a consequence of Theorem 1 above, a necessary condition for $f \in E_\pi$ to be a cardinal sine series is $f_o(n) = o(n)$ and $f_e(n) = o(n^2)$ as $n \rightarrow \infty$. If, in addition, we impose a mild asymptotic oscillation condition on f , we can in fact conclude that f is a cardinal sine series. This is precisely stated in the next theorem.

Theorem 7. *Suppose $f(z)$ is in E_π .*

1) If $f(z)$ is an odd function such that on the real axis

$$\lim_{x \rightarrow \infty} f^{(k)}(x) = 0 \quad \text{for some integer } k, k = 0, 1, 2, \dots,$$

and

$$f(n) = o(n) \quad \text{as } n \rightarrow \infty \text{ over the integers,}$$

then f is a cardinal sine series.

2) If f is an even function such that on the real axis

$$\lim_{x \rightarrow \infty} \frac{f^{(k)}(x)}{x} = 0 \quad \text{for some integer } k, k = 0, 1, 2, \dots,$$

and

$$f(n) = o(n^2) \quad \text{as } n \rightarrow \infty \text{ over the integers,}$$

then f is a cardinal sine series.

The proof can be found in [3, Theorem 4]. As a useful corollary we state the following.

Theorem 8. *Suppose $f(z)$ is in E_π . If $f_e(x) = o(x)$ and $f_o(x) = o(1)$ as $x \rightarrow \infty$, then f is a cardinal sine series.*

Note that the functions $z \sin(\pi z)$ and $\sin(\pi z)$, that are even and odd respectively, are both zero at the integers and are not cardinal sine series. So the rates in Theorem 8 cannot be significantly improved.

The next theorem, also a corollary of Theorem 7, addresses an issue raised earlier. It implies that if $f(z)$ is an entire function in E_π such that $f(x) = o(1)$ as $x \rightarrow \pm\infty$ then the samples $f(n) = c_n$, $n = 0, \pm 1, \pm 2, \dots$, satisfy (4) of Theorem 1.

In what follows, we use the notation $C_0(\mathbb{R})$ to denote the class of those functions $f(x)$ that are continuous on the real axis $\mathbb{R}$ with $\lim_{x \rightarrow \pm\infty} f(x) = 0$. $E_\pi \cap C_0(\mathbb{R})$ consists of those entire functions in E_π whose restriction to the real axis is in $C_0(\mathbb{R})$. $C_0(\mathbb{Z})$ is the class of those sequences $\{c_n\}$ such that $\lim_{n \rightarrow \pm\infty} c_n = 0$.

Theorem 9. *If $f(z)$ is in $E_\pi \cap C_0(\mathbb{R})$, then f is a cardinal sine series. Furthermore, the sequence of samples $\{f(n)\}_{n=-\infty}^\infty$ is in $C_0(\mathbb{Z})$ and*

$$\|\{f(n)\}\|_{\ell^\infty(\mathbb{Z})} \leq \|f\|_{L^\infty(\mathbb{R})}.$$

Given the hypothesis, the statements involving the samples, of course, are completely evident properties of $f(z)$. The reason for this particular formulation is to highlight the fact that Theorem 9 may be viewed as a partial extension of the Plancherel-Polya Theorem since $\ell^p(\mathbb{Z}) \subset C_0(\mathbb{Z})$ and, by virtue of [11, page 150, item 2], $E_\pi \cap L^p(\mathbb{R}) \subset E_\pi \cap C_0(\mathbb{R})$ when p satisfies $1 < p < \infty$. This extension is at best only partial because Theorem 9 fails to extend other important aspects, e. g. see Example 4.2. A more complete examination of the issues that fail to generalize can be found in [3].

Using the fact that there is a member of E_π that is a cardinal sine series and that satisfies both $\lim_{x \rightarrow \infty} f(x) = 1$ and $\lim_{x \rightarrow -\infty} f(x) = -1$, see Example 4.3 below, Theorem 9 can be extended to include functions $f \in E_\pi$ that have nice limiting behavior along the real axis while not necessarily decaying to zero at infinity.

Theorem 10. *Suppose $f(z)$ is in E_π .*

1) *If there are complex numbers a and b such that*

$$\lim_{x \rightarrow \infty} f(x) = a \quad \text{and} \quad \lim_{x \rightarrow -\infty} f(x) = b$$

then $f(z)$ is a cardinal sine series.

2) *If $f(z)$ is an even function and there is a complex number a such that*

$$\lim_{x \rightarrow \infty} \frac{f(x)}{x} = a$$

then $f(z)$ is a cardinal sine series.

For complete details see [3, Theorem 3 and its proof].

3.3. Conditions on finite differences of sample data. The theorems in subsection 3.2 involve conditions on asymptotic behavior of $f(z)$ in order to conclude that its sequence of samples $c_n = f(n)$, $n = 0, \pm 1, \pm 2, \dots$, gives rise to a convergent cardinal sine series. In what follows we present results involving conditions on the finite differences of the sampled values only.

First recall that the Plancherel-Polya Theorem can be extended in a direction different from that suggested by Theorem 9 by considering the classes of functions $PW^{p,k}$ and sequences $\ell^{p,k}$, $k = 0, 1, 2, \dots$

If $k = 0$ then $PW^{p,k}$ is just the usual Paley-Wiener space $PW^p = E_\pi \cap L^p(\mathbb{R})$. If $k > 0$ then $PW^{p,k}$ denotes the class of those entire functions u in E_π whose derivative of order k , $u^{(k)}$, is in PW^p . The class $PW^{p,k}$ is endowed with the natural semi-norm $\|u\|_{PW^{p,k}} = \|u^{(k)}\|_{L^p(\mathbb{R})}$. The space $\ell^{p,k}$ is the set of all bi-infinite sequences whose k^{th} order forward differences are in ℓ^p .

In [12], it was shown that if $1 < p < \infty$ then the mapping $u \mapsto \{u(n)\}$ is a continuous isomorphism between $PW^{p,k}$ and $\ell^{p,k}$ thus providing an extension of the classical Plancherel-Polya Theorem. However, unlike the classical Plancherel-Polya Theorem, when $k \geq 1$ the inverse mapping $\{u(n)\} \mapsto u(z)$ is not provided by the cardinal sine series. See Comment 4.4 below.

We mention that $PW^{p,k} \subset PW^{p,k+1}$ where the containment is proper. If $k \geq 1$, members u of $PW^{p,k}$ need not decay or even be bounded on the real axis but, when restricted to the real axis, satisfy

$$|u(x)| = o\left(|x|^{k-1/p}\right), \quad |x| \rightarrow \infty.$$

The sequence spaces $\ell^{p,k}$ and their members satisfy analogous properties. Thus, for example, if $\alpha > 0$ then the sequence $\{c_n\}$ defined by $c_n = |n|^\alpha$, for $n \neq 0$, is in $\ell^{p,k}$ if $\alpha < k - 1/p$.

If the sequence $\{c_n\}$ is in $\ell^{p,k}$ then a simple restriction on its growth suffices to imply that conditions (4) of Theorem 1 are satisfied.

Theorem 11. *Suppose $\{c_n\}$ is in $\ell^{p,k}$ for some p , $1 < p < \infty$, and some k , $k = 1, 2, \dots$*

- 1) *If $c_n = o(n)$ as $n \rightarrow \pm\infty$ then the unique interpolant of $\{c_n\}$ in $PW^{p,k}$ is the cardinal sine series $f(z)$ with coefficients $\{c_n\}$.*
- 2) *If c_n is even, $c_{-n} = c_n$, $n = 1, 2, \dots$, and $c_n = o(n^2)$ as $n \rightarrow \infty$ then the unique interpolant of $\{c_n\}$ in $PW^{p,k}$ is the cardinal sine series $f(z)$ with coefficients $\{c_n\}$.*

A direct but technically tedious proof of this theorem can be found in [2]. Below we provide a more efficient proof based on Theorem 8.

Proof. If $\{c_n\}$ is in $\ell^{p,k}$ then there is a unique $f(z)$ in $PW^{p,k}$ such that $f(n) = c_n$, $n = 0, \pm 1, \pm 2, \dots$. Since $f^{(k)}(x)$ is in $L^p(\mathbb{R})$ it follows that $f^{(k)}(x) = o(1)$ as $x \rightarrow \pm\infty$. Since $f(n) = c_n = o(n)$ as $n \rightarrow \pm\infty$ the hypothesis of Theorem 8 are satisfied and we may conclude that $f(z)$ is a cardinal sine series completing the proof of item 1).

Item 2) follows via a completely analogous argument, *mutatis mutandis*. $\square$

In view of the bound on the growth of functions in $PW^{p,k}$ when restricted to the real axis, Theorem 11 has the following corollary.

Corollary 1. *If $u(z)$ is in $PW^{p,1}$ or if $u(z)$ is an even function in $PW^{p,2}$, where $1 < p < \infty$, then the cardinal sine series of its samples, $c_n = u(n)$, $n = 0, \pm 1, \pm 2, \dots$, converges to $u(z)$.*

For another corollary, see Example 4.5 below.

3.4. Conditions on the Fourier transform of $f(z)$.

Lemma 1. *Suppose $f(z)$ is an odd function in E_π such that $f(z)/z$ is the inverse Fourier transform of a function $g(\xi)$ in L^1 . Namely,*

$$f(z) = \frac{z}{2\pi} \int_{-\pi}^{\pi} g(\xi) e^{iz\xi} d\xi \quad \text{where} \quad \int_{-\pi}^{\pi} |g(\xi)| d\xi < \infty.$$

Then a necessary and sufficient condition for the convergence of the cardinal sine series with coefficients $c_n = f(n)$, $n = 0, \pm 1, \pm 2, \dots$, is that the Fourier series of $g(\xi)$ converge when $\xi = \pi$.

Proof. The Fourier series of $g(\xi)$ is

$$(7) \quad f'(0) + \sum_{n=1}^{\infty} \frac{f(n)}{n} \cos n\xi.$$

The convergence of this series at $\xi = \pi$ is exactly the necessary and sufficient condition (4) of Theorem 1 for the convergence of the cardinal sine series with coefficients $c_n = f(n)$, $n = 0, \pm 1, \pm 2, \dots$. $\square$

If the function $g(\xi)$ in Lemma 1 vanishes when $|\xi| > \sigma$ for some positive σ such that $\sigma < \pi$ then by the principle of localization, [24, Vol. 1, page 52, item (6.3)], it follows that the series (7) converges when $\xi = \pi$ and hence so does the corresponding cardinal sine series. We summarize this observation as follows.

Lemma 2. *Suppose $f(z)$ is an odd function in E_σ where σ satisfies $0 < \sigma < \pi$. If, in addition, $f(z)/z$ is the inverse Fourier transform of a function $g(\xi)$ in L^1 then $f(z)$ is a cardinal sine series.*

While both these lemmas seem rather obvious they have corollaries that are not quite so obvious without them, especially when combined with results concerning even cardinal sine series. For example:

Theorem 12. *Suppose $f(z)$ is an entire function in E_σ where σ satisfies $0 < \sigma < \pi$. If $f(n) = O(|n|^\alpha)$ as $n \rightarrow \pm\infty$ for some α , $\alpha < 1/2$, then $f(z)$ is a cardinal sine series.*

Proof. Express $f(z)$ as the sum of its even and odd parts, $f(z) = f_e(z) + f_o(z)$, and note that $f_e(n)$ enjoys the same asymptotic growth bound as $f(n)$. So Theorem 6 implies that $f_e(z)$ is a cardinal sine series.

The argument used to prove Cartwright's theorem, [11, Theorem 4, Lecture 21], shows that the above hypothesis implies that $f(x) = O(|x|^\alpha)$ as $x \rightarrow \pm\infty$. Since $f_o(x)$ enjoys the same asymptotic growth bound and $\alpha < 1/2$ it follows that $f_o(x)/x$ is in $L^2(\mathbb{R})$. Thus $f_o(z)/z$ is in $E_\sigma \cap L^2(\mathbb{R})$ and hence is the inverse Fourier transform of a function in $L^2([-\pi, \pi])$. Since $L^2([-\pi, \pi]) \subset L^1([-\pi, \pi])$, Lemma 2 now implies that $f_o(z)$ is a cardinal sine series.

This completes the proof. $\square$

The case $\alpha = 0$ reproduces a particular example of a theorem due to Seip [17, Theorem 2] that can be re-articulated as follows:

Corollary 2. *Suppose $f(z)$ is in $E_\sigma \cap L^\infty(\mathbb{R})$ for some σ , $0 < \sigma < \pi$. Then $f(z)$ is a cardinal sine series.*

The cardinal series whose coefficients are samples, $c_n = u(n)$, $n = 0, \pm 1, \pm 2, \dots$, of functions u in E_π whose Fourier transforms are bounded measures μ , namely

$$u(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i\xi z} \mu(d\xi),$$

also converge and essentially represent u .

Theorem 13. *Suppose the Fourier transform of u is a bounded complex Borel measure μ supported on $[-\pi, \pi]$ and $\{f_N\}_{N=1}^\infty$ are the symmetric partial sums (3) of the cardinal sine series whose coefficients are samples of u , namely $c_n = u(n)$, $n = 0, \pm 1, \pm 2, \dots$. Then*

$$\lim_{N \rightarrow \infty} f_N(z) = u(z) + i[\mu(\{-\pi\}) - \mu(\{\pi\})] \frac{\sin(\pi z)}{2\pi}$$

uniformly on compact subsets of $\mathbb{C}$.

See [3, Subsection 2.4, Theorem 5] for details. Note that if $\mu(\{-\pi\}) = \mu(\{\pi\})$ the cardinal series converges to $u(z)$. This would be the case, for example, if the Fourier transform $\hat{u}$ were integrable with $\mu(d\xi) = \hat{u}(\xi)d\xi$. See Comment 4.6.

4. Comments and examples

4.1. **Example.** The even sequence

$$c_n = c_{-n} = \begin{cases} n & \text{if } n = 4k \text{ or } n = 4k + 1, k = 0, 1, 2, \dots \\ -n & \text{if } n = 4k + 2 \text{ or } n = 4k + 3, k = 0, 1, 2, \dots \end{cases}$$

satisfies condition (4) of Theorem 1 and hence it is the coefficient sequence of a convergent cardinal sine series.

On the other hand, the shifted sequence $c'_n = c_{n+1}$ fails to satisfy condition (4) of Theorem 1. This can be seen directly by expressing the odd part b'_n as $2b'_n = c'_n - c'_{-n} = c_{1+n} - c_{1-n} = c_{n+1} - c_{n-1}$ so that when $n = 4k$ we have $b'_n = n$ and it follows that the second series in (4) fails to converge. Hence $\{c'_n\}$ is not the coefficient sequence of a convergent cardinal sine series.

4.2. **Example.** The even sequence defined by $c_0 = 0$ and

$$c_n = \frac{(-1)^n}{\log(e|n|)}, \quad \text{for } n = \pm 1, \pm 2, \dots$$

that is in $C_0(\mathbb{R})$ satisfies condition (4) of Theorem 1. The corresponding cardinal sine series is the function

$$(8) \quad f(z) = \frac{2z \sin(\pi z)}{\pi} \sum_{n=1}^{\infty} \frac{1}{\log(en)} \frac{1}{z^2 - n^2}.$$

In [3, Section 4] it is shown that $f(z)$ is in $C_0(\mathbb{R}) \cap E_\pi$ but, in contrast to the conclusions of the Plancherel-Polya Theorem, the symmetric partial sums $\{f_N(z)\}$ of its cardinal sine series do not converge uniformly on $\mathbb{R}$.

For an examination of the contrast between Theorem 9 and the classical Plancherel-Polya Theorem that includes further examples see [3].

4.3. **Example.** The function $\text{Sgn}(z)$ defined by

$$\text{Sgn}(z) = \frac{2 \sin \pi z}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n n}{z^2 - n^2}$$

is a cardinal sine series in E_π that satisfies both $\lim_{x \rightarrow \infty} \text{Sgn}(x) = 1$ and $\lim_{x \rightarrow -\infty} \text{Sgn}(x) = -1$.

For a verification of these limits see [3, Lemma 4]. For other properties of $\text{Sgn}(z)$, including its relationship to the digamma function, see [2, Subsection 2.5]. For

yet other properties, including plots, see [13, Subsection 4.2 and Figures 1 and 2].

4.4. Comment. The fact that the mapping $u(z) \mapsto \{u(n)\}$ is a continuous isomorphism between $PW^{p,k}$ and $\ell^{p,k}$ in the case $p = 2$ is shown in [16], where the spline summability method is used to reconstruct $u(x)$ on the real axis from the samples $\{u(n)\}$. The general case, $1 < p < \infty$, is considered in [12], where the samples are taken on a general complete interpolating sequence and the inverse mapping is defined in terms of a series of entire functions that, roughly speaking, interpolates the divided differences.

4.5. Example. Consider the sequence

$$c_n = |n|^\alpha e^{i\lambda|n|^\beta} \quad n = 0, \pm 1, \pm 2, \dots,$$

where $\alpha \geq 0$, λ is real, and $\beta < 1$. If $k > \frac{1}{1-\beta} \left(\alpha + \frac{1}{p} \right)$ then $\{c_n\}_{n=-\infty}^\infty$ is in $\ell^{p,k}$ and therefore its terms are the samples, $u(n) = c_n$, of a function $u(z)$ in $PW^{p,k}$. Theorem 11 implies that

$$(9) \quad \sum_{n=1}^{\infty} (-1)^n |n|^{\alpha-2} e^{i\lambda|n|^\beta} \quad \text{converges if} \quad \alpha < 2 \quad \text{and} \quad \beta < 1.$$

and if $k > \frac{1}{1-\beta} \left(\alpha + \frac{1}{p} \right)$

$$u(z) = \lim_{N \rightarrow \infty} \sum_{n=-N}^N |n|^\alpha e^{i\lambda|n|^\beta} \frac{\sin \pi(z-n)}{\pi(z-n)}$$

is the function in $PW^{p,k}$ that satisfies $u(n) = |n|^\alpha e^{i\lambda|n|^\beta}$.

It is not evident how to obtain conclusion (9) without recourse to Theorems 1 and 11.

4.6. Comment. Corollary 2 and Theorem 13 suggest the possibility that any function in $E_\pi \cap L^\infty(\mathbb{R})$ might be essentially reconstructible from its samples via a cardinal sine series. That is, up to the possible addition of a constant multiple of $\sin \pi x$. It was also suggested that such functions might be essentially reconstructible on the real axis via the spline summability method in [15]. To our knowledge neither of these issues have yet been resolved.

We mention that Theorem 13 may be regarded as an extension of the result that is known when $\hat{u}$ is integrable, in other words, when the measure μ is absolutely continuous with respect to Lebesgue measure. For example see [4, page 124], [7, Theorem 3 on page 70], [8], [13, Corollary 6], or [14, page 499].

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