Saturation Results for the Truncated Max-Product Sampling Operators Based on Sinc and Fejér-Type Kernels

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Abstract

In this paper we obtain the saturation order, a strong localization result and local saturation results in the approximation of continuous positive functions by the truncated max-product sampling operators based on sinc (Wittaker-type) and Fejér-type kernels. The localization-type results obtained present important potential applications in signal theory.

Key words and phrases : Max-product truncated sampling operators, signal theory, sinc (Whitaker-type) kernel, Fejér-type kernel, saturation order, local inverse saturation result, localization result, local direct saturation result, Lipschitz function on subintervals.

2000 AMS Mathematics Subject Classification — Primary 94A20, 94A12, 41A36, 41A40 ; Secondary 41A27, 41A20.

1 Introduction

The *sinc*-approximation operators were first introduced and studied in [15], [5] and [20] under the name of cardinal function and of truncated cardinal function. Later on, these linear approximation operators were intensively studied in e.g. [1]–[3], [6]–[9], [13], [14], [16], [17], [18], [19] (see also the references cited there).

Based on the Open Problem 5.5.4, pp. 324-326 in [12], in a series of recent papers submitted for publication we have introduced and studied the so-called max-product operators attached to the Bernstein polynomials and to other linear Bernstein-type operators, like those of Favard-Szász-Mirakjan operators (truncated and nontruncated case), Baskakov operators (truncated and nontruncated case), Meyer-König and Zeller operators and Bleimann-Butzer-Hahn operators.

This idea applied, for example, to the linear Bernstein operators $B_n(f)(x) = \sum_{k=0}^n p_{n,k}(x) f(k/n)$, where $p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$ and f is positive, works

as follows. Writing in the equivalent form $B_n(f)(x) = \frac{\sum_{k=0}^n p_{n,k}(x)f(k/n)}{\sum_{k=0}^n p_{n,k}(x)}$ and then replacing the sum operator Σ by the maximum operator $\bigvee$, one obtains the nonlinear Bernstein operator of max-product kind

$$B_n^{(M)}(f)(x) = \frac{\bigvee_{k=0}^n p_{n,k}(x)f\left(\frac{k}{n}\right)}{\bigvee_{k=0}^n p_{n,k}(x)}, \quad x \in [0, 1], \quad n \in \mathbb{N},$$

where the notation $\bigvee_{k=0}^n p_{n,k}(x)$ means $\max\{p_{n,k}(x); k \in \{0, \dots, n\}\}$ and similarly for the numerator.

For this nonlinear max-product operator nice approximation and shape preserving properties were found in e.g. [4].

In the recent paper [10], Theorem 3.2 (iv), applying this idea to the classical Whittaker's cardinal series, we have obtained a Jackson-type estimate in terms of $\omega_1(f; 1/W)_{\mathbb{R}}$, in approximation of a continuous, positive and bounded function f on $\mathbb{R}$, by the nonlinear max-product Whittaker sampling operator given by

$$S_{W,\varphi}^{(M)}(f)(t) = \frac{\bigvee_{k=-\infty}^{\infty} \varphi(Wt - k)f\left(\frac{k}{W}\right)}{\bigvee_{k=-\infty}^{\infty} \varphi(Wt - k)}, \quad t \in \mathbb{R}, \quad W > 0,$$

where φ is the sinc kernel given by the formula $\varphi(t) = \text{sinc}(t) = \frac{\sin(\pi t)}{\pi t}$. Also, replacing above $\varphi(t) = \frac{1}{2} \cdot \text{sinc}^2(t/2)$ (the Fejér-type kernel), in [11], Theorem 2.4, we get the same Jackson-type order $\omega_1(f; 1/W)_{\mathbb{R}}$ in the approximation of a continuous, positive and bounded function f on $\mathbb{R}$. In order to be of interest, only functions f that are uniformly continuous on $\mathbb{R}$ must be considered, because they have the property that $\omega_1(f; t)_{\mathbb{R}} \rightarrow 0$ as $t \rightarrow 0$.

It is worth noting that comparing with the classical linear sampling operators to which they were attached, when for f bounded and Lipschitz of order $\alpha \in (0, 1)$ the order of approximation is $\frac{\log(n)}{n^\alpha}$ (see Butzer [6] or Butzer-Stens [9]), these two kinds of nonlinear max-product sampling operators give the approximation order $\frac{1}{n^\alpha}$, which is an essential improvement.

Going further and applying the max-product idea to the truncated Whittaker series defined by (see e.g. Borel [5], Whittaker [20])

$$W_n(f)(x) = \sum_{k=0}^n \frac{\sin(nx - k\pi)}{nx - k\pi} \cdot f\left(\frac{k\pi}{n}\right), \quad x \in [0, \pi], \quad n \in \mathbb{N},$$

and to the truncated sampling series based on the Fejér's kernel and defined by

$$T_n(f)(x) = \sum_{k=0}^n \frac{\sin^2(nx - k\pi)}{(nx - k\pi)^2} \cdot f\left(\frac{k\pi}{n}\right), \quad x \in [0, \pi], \quad n \in \mathbb{N},$$

in the very recent paper [11], we introduced the truncated max-product Whitaker operator given by

$$W_n^{(M)}(f)(x) = \frac{\bigvee_{k=0}^n \frac{\sin(nx-k\pi)}{nx-k\pi} \cdot f\left(\frac{k\pi}{n}\right)}{\bigvee_{k=0}^n \frac{\sin(nx-k\pi)}{nx-k\pi}}, \quad x \in [0, \pi], \quad n \in \mathbb{N},$$

and the truncated max-product operator based on the Fejér-type kernel, given by

$$T_n^{(M)}(f)(x) = \frac{\bigvee_{k=0}^n \frac{\sin^2(nx-k\pi)}{(nx-k\pi)^2} \cdot f\left(\frac{k\pi}{n}\right)}{\bigvee_{k=0}^n \frac{\sin^2(nx-k\pi)}{(nx-k\pi)^2}}, \quad x \in [0, \pi], \quad n \in \mathbb{N},$$

for which the Jackson-type order $\omega_1(f; 1/n)_{[0, \pi]}$ in the approximation of f was obtained. It is worth noting that the situation is completely different in the case of approximation by their linear counterparts W_n and T_n , when for the uniform convergence in compacts subintervals in $(0, \pi)$ and pointwise convergence in $[0, \pi]$, additional conditions are required like $\lim_{n \rightarrow \infty} \omega_1(f; 1/n)_{[0, \pi]} \cdot \log(n) = 0$ (see Trynin [19]), contrariwise divergence properties may appear (see Sklyarov [16]).

Also, the interpolation property is preserved by the operators $W_n^{(M)}$ and $T_n^{(M)}$, i.e. we have $W_n^{(M)}(f)(j\pi/n) = T_n^{(M)}(f)(j\pi/n) = f(j\pi/n)$, $j \in \{0, 1, \dots, n\}$ (see [11]).

Due to the better approximation properties, it is natural to look for a complete characterization of the approximation properties of the nonlinear max-product sampling operators $T_n^{(M)}$ and $W_n^{(M)}$. In this sense, the goal of the present paper is to obtain for them the saturation order, a strong localization result and local inverse and local direct saturation results.

It is worth noting, for example, that the strong localization results in the Theorems 5.1 and 6.1 and the local shape preserving properties in the Corollaries 5.3 and 6.4, clearly show the important advantage that may have the truncated max-product operators $T_n^{(M)}(f)$ and $W_n^{(M)}(f)$ in the local representation with great accuracy of a continuous non-smooth signal f , if we compare them with the linear sampling operators $T_n(f)$ and $W_n(f)$.

The plan of the paper goes as follows : in Section 2 we deal with the saturation order for $T_n^{(M)}$, Section 3 deals with the saturation order for $W_n^{(M)}$, in Section 4 a local inverse saturation result for $T_n^{(M)}$ is presented, Section 5 contains a localization result and a local direct saturation result for $T_n^{(M)}$, while in Section 6 a localization result and local inverse and local direct saturation results for $W_n^{(M)}$ are presented.

Everywhere, by convention we take $\sin(0)/0 = 1$, which means that for every $x = k\pi/n$, $k \in \{0, 1, \dots, n\}$, we take $\frac{\sin(nx-k\pi)}{nx-k\pi} = 1$.

2 The Saturation Order for the $T_n^{(M)}$ Operator

Firstly, we need some auxiliary results.

Lemma 2.1 Denoting $s_{n,k}(x) = \frac{\sin^2(nx - \pi k)}{(nx - \pi k)^2}$, for any $j \in \{0, \dots, n\}$ we have

$$\bigvee_{k=0}^n s_{n,k}(x) = s_{n,j}(x), \quad x \in [(j - 1/2)\pi/n, (j + 1/2)\pi/n] \cap [0, \pi].$$

Proof. See the proof of Lemma 4.3 in [11]. $\square$

Lemma 2.2. For any function $f : [0, \pi] \rightarrow \mathbb{R}_+$, where $\mathbb{R}_+ = \{x \in \mathbb{R}; x \geq 0\}$, and for all $n \in \mathbb{N}$, $n \geq 1$, and $j \in \{0, 1, \dots, n\}$, $j \leq n/2$, we have:

(i)

$$T_n^{(M)}(f)(j\pi/(n+1)) \geq f(j\pi/n);$$

(ii)

$$T_{n+1}^{(M)}(f)(j\pi/n) \geq f(j\pi/(n+1)).$$

Proof. (i) Firstly, by Lemma 2.1 we observe that for $x \in \left[\frac{[(j-1)+1/2]\pi}{n}, \frac{j\pi}{n}\right]$ we have $\bigvee_{k=0}^n s_{n,k}(x) = s_{n,j}(x)$. Now, if $j \leq n/2$ then it is easy to check that $x := j\pi/(n+1) \in \left[\frac{[(j-1)+1/2]\pi}{n}, \frac{j\pi}{n}\right]$ which implies $\bigvee_{k=0}^n s_{n,k}(j\pi/(n+1)) = s_{n,j}(j\pi/(n+1))$. This implies that

$$\begin{aligned} T_n^{(M)}(f)(j\pi/(n+1)) &= \frac{\bigvee_{k=0}^n s_{n,k}(j\pi/(n+1)) f\left(\frac{k\pi}{n}\right)}{s_{n,j}(j\pi/(n+1))} \\ &\geq \frac{s_{n,j}(j\pi/(n+1)) f\left(\frac{j\pi}{n}\right)}{s_{n,j}(j\pi/(n+1))} = f\left(\frac{j\pi}{n}\right). \end{aligned}$$

(ii) Since $j \leq n/2$, one can easily prove that $j\pi/n \in \left[\frac{j\pi}{n+1}, \frac{(j+1/2)\pi}{n+1}\right]$. Therefore, by Lemma 2.1 we obtain $\bigvee_{k=0}^{n+1} s_{n+1,k}(j\pi/n) = s_{n+1,j}(j\pi/n)$. This implies that

$$\begin{aligned} T_{n+1}^{(M)}(f)(j\pi/n) &= \frac{\bigvee_{k=0}^{n+1} s_{n+1,k}(j\pi/n) f\left(\frac{k\pi}{n+1}\right)}{s_{n+1,j}(j\pi/n)} \geq \frac{s_{n+1,j}(j\pi/n) f\left(\frac{j\pi}{n+1}\right)}{s_{n+1,j}(j\pi/n)} \\ &= f\left(\frac{j\pi}{n+1}\right). \end{aligned}$$

$\square$

Since $j \geq n/2$ easily implies that $(j+1)\pi/(n+1) \in [j\pi/n, (j+1/2)\pi/n]$ and $j\pi/n \in [(j+1/2)\pi/(n+1), (j+1)\pi/(n+1)]$, by similar reasonings with those for Lemma 2.2, we also get the following.

Lemma 2.3. *For any function $f : [0, \pi] \rightarrow \mathbb{R}_+$ and for all $n \in \mathbb{N}$, $n \geq 1$, and $j \in \{0, 1, \dots, n\}$, $j \geq n/2$, we have:*

(i)

$$T_n^{(M)}(f)((j+1)\pi/(n+1)) \geq f(j\pi/n);$$

(ii)

$$T_{n+1}^{(M)}(f)(j\pi/n) \geq f((j+1)\pi/(n+1)).$$

We are now in position to determine the saturation order and the associated special class of functions for the truncated max-product operator $T_n^{(M)}$.

Theorem 2.4. *Denote $C_+[0, \pi] = \{f : [0, \pi] \rightarrow \mathbb{R}_+; f \text{ continuous on } [0, \pi]\}$ and $\|f\| = \sup\{|f(x)|; x \in [0, \pi]\}$. Then for the max-product $T_n^{(M)}$ operator, the saturation order in $C_+[0, \pi]$ is $\frac{1}{n}$, that is $\|T_n^{(M)}(f) - f\| = o(1/n)$, implies that f is a constant function on $[0, \pi]$.*

Proof. By hypothesis, there exists $a_n \in \mathbb{R}$, $n \in \mathbb{N}$ with the property $a_n \searrow 0$ as $n \rightarrow +\infty$, such that

$$\left| T_n^{(M)}(f)(x) - f(x) \right| \leq \frac{a_n}{n}, \text{ for all } x \in [0, \pi] \text{ and } n \in \mathbb{N}.$$

Let us choose arbitrary $\varepsilon > 0$. Since $a_n \searrow 0$ as $n \rightarrow +\infty$, it follows that there exists $n_0 \in \mathbb{N}$ such that $a_n < \varepsilon$ for all $n \in \mathbb{N}$, $n \geq n_0$. Noting the above relation we get

$$\left| T_n^{(M)}(f)(x) - f(x) \right| \leq \frac{\varepsilon}{n}, \text{ for all } x \in [0, \pi] \text{ and } n \in \mathbb{N}, n \geq n_0. \quad (1)$$

Then, from the uniform continuity of f it results the existence of $n_1 \in \mathbb{N}$ such that

$$|f(x) - f(y)| \leq \varepsilon \text{ for all } x, y \in [0, \pi] \text{ and } n \in \mathbb{N}, |x - y| \leq \pi/n, n \geq n_1. \quad (2)$$

We will obtain the desired conclusion by following the following steps: (A) we prove that f is constant on any interval $[a, b]$ with $0 < a < b < \pi/2$; (B) we prove that f is constant on any interval $[a, b]$ with $\pi/2 < a < b < \pi$. Indeed, if (A) holds then thanks to the continuity of f we easily obtain that f is constant on $[0, \pi/2]$. Similarly, if (B) holds then we obtain that f is constant on $[\pi/2, \pi]$. Then, from the continuity of f it easily follows that f is constant on $[0, \pi]$. So, we start by proving that (A) and (B) hold.

(A) Let us choose arbitrary $a, b \in \mathbb{R}$ such that $0 < a < b < \pi/2$. Further one, let x_0 and y_0 be the points where f attains its minimum and maximum respectively on the interval $[a, b]$. Without any loss of generality we may suppose

that $x_0 \neq y_0$ (contrariwise it follows that f is constant on $[a, b]$ and there is nothing to prove). We have two subcases: A₁) $x_0 < y_0$ and A₂) $x_0 > y_0$.

Subcase A₁) Let us choose arbitrary $n \in \mathbb{N}$ with $n > \max\{n_0, n_1, 2\pi/(y_0 - x_0)\}$. By relation (1) it follows that

$$T_n^{(M)}(f)(j\pi/(n+1)) - f(j\pi/(n+1)) \leq \frac{\varepsilon}{n} \text{ for all } j \in \{0, 1, \dots, n\}.$$

Moreover, combining the inequality in Lemma 2.2 (i) with the above inequality, we get

$$f(j\pi/n) - f(j\pi/(n+1)) \leq \frac{\varepsilon}{n} \text{ for all } j \in \{0, 1, \dots, n\}, j \leq n/2. \quad (3)$$

Further one, let us choose $j_1 \in \{0, 1, \dots, n-1\}$ such that $j_1\pi/n \leq y_0 \leq (j_1+1)\pi/n$ and $x_0 \leq j_1\pi/n$. Note that there exists such an index j_1 , because the previous inequalities are equivalent to $\frac{ny_0}{\pi} - 1 \leq j_1 \leq \frac{ny_0}{\pi}$, $\frac{nx_0}{\pi} \leq j_1 \leq \frac{ny_0}{\pi}$, while the condition $n > 2\pi/(y_0 - x_0)$ is equivalent to the condition $\frac{ny_0}{\pi} - \frac{nx_0}{\pi} > 2$.

Also, from $j_1\pi/n \leq y_0 \leq b < \pi/2$ it easily follows that $j_1 \leq n/2$.

As a first consequence, from the relation (2) we obtain

$$|f(j_1\pi/n) - f(y_0)| < \varepsilon. \quad (4)$$

Then, since $\lim_{l \rightarrow \infty} \frac{j_1\pi}{n+l} = 0$, by $x_0 > 0$ and $x_0 \leq j_1\pi/n$ it follows that there exists $l_0 \in \mathbb{N}$ such that $\frac{j_1\pi}{n+l_0+1} \leq x_0 \leq \frac{j_1\pi}{n+l_0}$.

It is worth noting here that indeed, the above l_0 cannot be equal to 0, because if we would have $l_0 = 0$, then we would obtain $j_1\pi/(n+1) \leq x_0 < y_0 \leq (j_1+1)\pi/n \leq (j_1+2)\pi/(n+1)$, which would imply $y_0 - x_0 \leq 2\pi/(n+1) < 2\pi/n$, in contradiction with the supposition that $n > 2\pi/(y_0 - x_0)$.

The inequality $\frac{j_1\pi}{n+l_0+1} \leq x_0 \leq \frac{j_1\pi}{n+l_0}$ and (2) also implies that

$$|f((j_1\pi/(n+l_0))) - f(x_0)| < \varepsilon. \quad (5)$$

Since $j_1 \leq n/2$, applying successively relation (3) we obtain

$$\begin{aligned} f(j_1\pi/n) - f(j_1\pi/(n+1)) &\leq \frac{\varepsilon}{n}, \\ f(j_1\pi/(n+1)) - f(j_1\pi/(n+2)) &\leq \frac{\varepsilon}{n+1}, \\ &\vdots \\ f(j_1\pi/(n+l_0-1)) - f(j_1\pi/(n+l_0)) &\leq \frac{\varepsilon}{n+l_0-1}. \end{aligned}$$

Taking the sum of all these inequalities we get

$$\begin{aligned} f(j_1\pi/n) - f(j_1\pi/(n+l_0)) &\leq \frac{\varepsilon}{n} + \frac{\varepsilon}{n+1} + \dots + \frac{\varepsilon}{n+l_0-1} \\ &\leq \frac{l_0\varepsilon}{n}. \end{aligned}$$

Then, by relations (4)-(5) we obtain

$$\begin{aligned}
& f(y_0) - f(x_0) \\
&= (f(y_0) - f(j_1\pi/n)) + (f(j_1\pi/n) - f(j_1\pi/(n+l_0))) \\
&\quad + (f(j_1\pi/(n+l_0)) - f(x_0)) \\
&\leq |f(y_0) - f(j_1\pi/n)| + |f(j_1\pi/n) - f(j_1\pi/(n+l_0))| \\
&\quad + |f(j_1\pi/(n+l_0)) - f(x_0)| \\
&\leq 2\varepsilon + \frac{l_0\varepsilon}{n}
\end{aligned}$$

and since $0 \leq f(y_0) - f(x_0)$, we obtain

$$0 \leq f(y_0) - f(x_0) \leq 2\varepsilon + \frac{l_0\varepsilon}{n}. \quad (6)$$

On the other hand, since $0 < x_0 \leq j_1\pi/(n+l_0)$, after some simple calculations we get (note that $j_1 \leq n/2$)

$$l_0 \leq j_1\pi/x_0 - n \leq n(\pi/(2x_0) - 1)$$

Using this information in relation (6) we obtain

$$0 \leq f(y_0) - f(x_0) \leq \varepsilon(2 + \pi/(2x_0) - 1)$$

where $\varepsilon > 0$ was chosen arbitrary. Therefore, passing to the limit for $\varepsilon \searrow 0$ in the previous inequality, we obtain $f(x_0) = f(y_0)$ (here, it is important that $x_0 > 0$). Since on the interval $[a, b]$ the maximum value and the minimum value of the function f coincide, we obtain that f is a constant function on the interval $[a, b]$ and hence we obtained the desired conclusion for this case.

The proof of Subcase A_2) is similar to that in the case of Subcase A_1), which proves the Case (A).

(B) Let us choose arbitrary $a, b \in \mathbb{R}$ such that $\pi/2 < a < b < \pi$ and further one, let x_0 and y_0 be the points where f attains its minimum and maximum respectively on the interval $[a, b]$. Again we have two subcases B_1) $x_0 < y_0$ and B_2) $x_0 > y_0$, which by similar reasonings with those in the Case (A) implies that (B) holds.

Now, by the discussion just before the beginning of the case (A), we conclude that f is constant on the whole interval $[0, \pi]$. $\square$

Remark. Because it is easy to check that $T_n^{(M)}$ reproduces the constant functions in $C_+[0, \pi]$, it follows that the saturation class in $C_+[0, \pi]$ for $T_n^{(M)}$ is exactly the class of constant functions.

3 The Saturation Order for the $W_n^{(M)}$ Operator

Firstly, we need some auxiliary results.

Lemma 3.1 Denoting now $s_{n,k}(x) = \frac{\sin(nx - \pi k)}{nx - \pi k}$, for any $j \in \{0, \dots, n\}$ we have

$$\bigvee_{k=0}^n s_{n,k}(x) = s_{n,j}(x), \quad x \in [(j - 1/2)\pi/n, (j + 1/2)\pi/n] \cap [0, \pi].$$

Proof. See the proof of Lemma 3.3 in [11].

Lemma 3.2. For any function $f : [0, \pi] \rightarrow \mathbb{R}_+$ and for all $n \in \mathbb{N}$, $n \geq 1$, and $j \in \{0, 1, \dots, n\}$, $j \leq n/2$, we have:

(i)

$$W_n^{(M)}(f)(j\pi/(n+1)) \geq f(j\pi/n);$$

(ii)

$$W_{n+1}^{(M)}(f)(j\pi/n) \geq f(j\pi/(n+1)).$$

Proof. The proof is identical with the proof of Lemma 2.2. $\square$

Lemma 3.3. For any function $f : [0, \pi] \rightarrow \mathbb{R}_+$ and for all $n \in \mathbb{N}$, $n \geq 1$, and $j \in \{0, 1, \dots, n\}$, $j \geq n/2$, we have:

(i)

$$W_n^{(M)}(f)((j+1)\pi/(n+1)) \geq f(j\pi/n);$$

(ii)

$$W_{n+1}^{(M)}(f)(j\pi/n) \geq f((j+1)\pi/(n+1)).$$

Proof. The proof is identical with the proof of Lemma 2.3. $\square$

Since Theorem 2.4 is a direct consequence of the Lemmas 2.1-2.3 and in its proof does not intervene at all the form of the kernel $s_{n,k}(x)$, as a consequence of the Lemmas 3.1-3.3, analogously we can prove the following saturation result for the truncated max-product operator $W_n^{(M)}$.

Theorem 3.4. For the max-product $W_n^{(M)}$ operator, the saturation order in $C_+[0, \pi]$ is $\frac{1}{n}$, that is $\|W_n^{(M)}(f) - f\| = o(1/n)$, implies that f is a constant function on $[0, \pi]$.

Remark. Because it is easy to see that $W_n^{(M)}$ reproduces the constant functions in $C_+[0, \pi]$, it follows that the saturation class for $W_n^{(M)}$ in $C_+[0, \pi]$ is exactly the class of constant functions.

4 Local Inverse Saturation Result for the $T_n^{(M)}$ Operator

In this section we present the following local inverse saturation result for the operator $T_n^{(M)}$.

Theorem 4.1. *Let $f : [0, \pi] \rightarrow \mathbb{R}_+$ and $0 < \alpha < \beta < \pi$ be such that f is continuous on $[\alpha, \beta]$. If there exists a constant $M > 0$ (independent of n but depending on f, α and β) such that*

$$\|T_n^{(M)}(f) - f\|_{[\alpha, \beta]} \leq M/n, \text{ for all } n \in \mathbb{N},$$

then $f|_{[\alpha, \beta]} \in \text{Lip}[\alpha, \beta]$, that is f is a Lipschitz function on $[\alpha, \beta]$. Here $\|f\|_{[\alpha, \beta]} = \sup\{|f(x)|; x \in [\alpha, \beta]\}$ and

$$\text{Lip}[\alpha, \beta] = \{g : [\alpha, \beta] \rightarrow \mathbb{R}; |g(x) - g(y)| \leq C|x - y|, \text{ for all } x, y \in [\alpha, \beta]\}.$$

The proof of Theorem 4.1 requires the following three lemmas.

Lemma 4.2. *Let $f : [0, \pi] \rightarrow \mathbb{R}$, $n \in \mathbb{N}$ and $0 < \alpha < \beta \leq \pi/2$ be fixed, such that f is continuous on $[\alpha, \beta]$. Also, denote*

$$M_n(\alpha, \beta) = \max \left\{ \left| f\left(\frac{k\pi}{n}\right) - f\left(\frac{k\pi}{n+1}\right) \right| : k \in \{0, \dots, n\}, \alpha \leq \frac{k\pi}{n+1} \leq \frac{k\pi}{n} \leq \beta \right\}.$$

Then

$$\limsup_{n \rightarrow \infty} n \cdot \omega_1(f, \pi/n)_{[\alpha, \beta]} = \infty \text{ if and only if } \limsup_{n \rightarrow \infty} n \cdot M_n(\alpha, \beta) = \infty.$$

where

$$\omega_1(f, \delta)_{[\alpha, \beta]} = \sup\{|f(x) - f(y)|; x, y \in [\alpha, \beta], |x - y| \leq \delta\}.$$

Proof. We prove only the direct implication since the converse one is immediate. Since f is continuous on the interval $[\alpha, \beta]$, it easily follows that for each $n \in \mathbb{N}$, $n \geq 2$, $\pi/n \leq \beta - \alpha$, there exist $x_n, y_n \in [\alpha, \beta]$ satisfying $|x_n - y_n| \leq \pi/n$ and $\omega_1(f, \pi/n)_{[\alpha, \beta]} = |f(x_n) - f(y_n)|$. Without any loss of generality, we may suppose that $x_n \neq y_n$ and $x_n < y_n$, for all $n \in \mathbb{N}$.

Let us consider the sequences $(a_n)_{n \geq 1}$ and $(b_n)_{n \geq 1}$, $a_n = n\omega_1(f, \pi/n)_{[\alpha, \beta]} = n|f(x_n) - f(y_n)|$ and $b_n = n \cdot M_n(\alpha, \beta)$.

Let us fix $n \in \mathbb{N}$. Since f is uniformly continuous on $[\alpha, \beta]$, it follows that there exists $m \in \mathbb{N}$ such that for all $x, y \in [\alpha, \beta]$ satisfying $|x - y| \leq \pi/m$ we have $|f(x) - f(y)| \leq 1/n$. In addition, we may choose sufficiently large $m \in \mathbb{N}$ such that $y_n - x_n > 2\pi/m$, that is $m > 2\pi/(y_n - x_n)$.

Since $0 < \alpha < y_n \leq \beta < \pi/2$, clearly there exists $j \in \{1, \dots, m-1\}$ (depending on m and n) such that $j\pi/m \leq y_n \leq (j+1)\pi/m$.

Since $\lim_{l \rightarrow \infty} j\pi/(m+l) = 0$ and since $x_n \geq \alpha > 0$, it results the existence of $l_0 \in \mathbb{N}$ (depending on j and m) such that $j\pi/(m+l_0+1) \leq x_n \leq j\pi/(m+l_0)$.

By the inequalities $x_n \leq j\pi/(m+l_0) < j\pi/m \leq y_n$, we get

$$|f(x_n) - f(y_n)|$$

$$\begin{aligned}
&\leq |f(x_n) - f(j\pi/(m+l_0))| + |f(j\pi/(m+l_0)) - f(j\pi/(m+l_0-1))| \\
&\quad + \dots + |f(j\pi/(m+1)) - f(j\pi/m)| + |f(j\pi/m) - f(y_n)| \\
&\leq |f(x_n) - f(j\pi/(m+l_0))| + |f(j\pi/m) - f(y_n)| \\
&\quad + l_0 |f(j\pi/(m+p)) - f(j\pi/(m+p+1))|
\end{aligned}$$

where $p \in \{0, 1, \dots, l_0 - 1\}$ is such that

$$\begin{aligned}
&|f(j\pi/(m+p)) - f(j\pi/(m+p+1))| \\
&= \max \{|f(j\pi/(m+k)) - f(j\pi/(m+k+1))| : k \in \{0, 1, \dots, l_0 - 1\}\}.
\end{aligned}$$

On the other hand, we observe that $\max\{|j\pi/(m+l_0) - x_n|, |j\pi/m - y_n|\} \leq \pi/m$, which implies $|f(x_n) - f(j\pi/(m+l_0))| \leq 1/n$ and $|f(j\pi/m) - f(y_n)| \leq 1/n$. We thus obtain that

$$|f(x_n) - f(y_n)| \leq 2/n + l_0 |f(j\pi/(m+p)) - f(j\pi/(m+p+1))|. \quad (7)$$

By the inequalities $x_n \leq j\pi/(m+l_0) \leq j\pi/m \leq y_n$ we get $j\pi/m - j\pi/(m+l_0) \leq y_n - x_n \leq \pi/n$ and this implies $j\pi l_0/(m(m+l_0)) \leq \pi/n$ and then $l_0 \leq m/j \cdot (m+l_0)/n \leq \pi/\alpha \cdot (m+l_0)/n$. (Here we used that $\alpha \leq x_n < j\pi/m$).

Then, by the inequalities $0 < \alpha \leq x_n \leq A := j\pi/(m+l_0) \leq B := j\pi/m \leq y_n \leq \beta$ we easily get $B/A \leq \beta/\alpha$, which immediately implies $j\pi/(m+l_0) \geq j\pi/m \cdot \alpha/\beta$. From here we get $m+l_0 \leq m\beta/\alpha$, that is $l_0 \leq m(\beta/\alpha - 1)$. Replacing this last inequality in the inequality $l_0 \leq \pi/\alpha \cdot (m+l_0)/n$ just proved above, we conclude that $l_0 \leq \pi\beta/\alpha^2 \cdot m/n$.

Replacing now in relation (7) and then multiplying with n , we get

$$\begin{aligned}
n \cdot |f(x_n) - f(y_n)| &\leq 2 + \pi\beta/\alpha^2 \cdot m \cdot |f(j\pi/(m+p)) - f(j\pi/(m+p+1))| \\
&\leq 2 + \pi\beta/\alpha^2 \cdot (m+p) \cdot |f(j\pi/(m+p)) - f(j\pi/(m+p+1))|
\end{aligned}$$

and clearly this implies that $a_n \leq 2 + \pi\beta/\alpha^2 \cdot M_{m+p}(\alpha, \beta)$. Summarizing, for any $n \in \mathbb{N}$ there exist $m+p \in \mathbb{N}$ such that $a_n \leq \pi\beta/\alpha^2 \cdot b_{m+p} + 2$. Since $m > 2\pi/(y_n - x_n)$ and $y_n - x_n < \pi/n$, we get $m > 2n$. Therefore, by $\limsup_{n \rightarrow \infty} a_n = \infty$, it easily follows that $\limsup_{n \rightarrow \infty} b_n = \infty$ and the lemma is proved. $\square$

By similar reasoning with those for the Lemma 4.2, we also get the following

Lemma 4.3. *Let $f : [0, \pi] \rightarrow \mathbb{R}$, $n \in \mathbb{N}$ and $\pi/2 \leq \alpha < \beta < \pi$ be fixed, such that f is continuous on $[\alpha, \beta]$. Also, denote*

$$\begin{aligned}
&P_n(\alpha, \beta) \\
&= \max \left\{ \left| f\left(\frac{k\pi}{n}\right) - f\left(\frac{(k+1)\pi}{n+1}\right) \right| : k \in \{0, \dots, n\}, \alpha \leq \frac{k\pi}{n} \leq \frac{(k+1)\pi}{n+1} \leq \beta \right\}.
\end{aligned}$$

Then

$$\limsup_{n \rightarrow \infty} n \cdot \omega_1(f, \pi/n)_{[\alpha, \beta]} = \infty \quad \text{if and only if} \quad \limsup_{n \rightarrow \infty} n \cdot P_n(\alpha, \beta) = \infty.$$

where

$$\omega_1(f, \delta)_{[\alpha, \beta]} = \sup\{|f(x) - f(y)|; x, y \in [\alpha, \beta], |x - y| \leq \delta\}.$$

Lemma 4.4. *Let $f : [0, \pi] \rightarrow \mathbb{R}_+$ and $0 < \alpha < \beta < \pi$ be such that f continuous on $[\alpha, \beta]$. If*

$$\limsup_{n \rightarrow \infty} n \cdot \omega_1(f, \pi/n)_{[\alpha, \beta]} = \infty,$$

then

$$\limsup_{n \rightarrow \infty} n \cdot \left\| T_n^{(M)}(f) - f \right\|_{[\alpha, \beta]} = \infty.$$

Here $\|f\|_{[\alpha, \beta]} = \sup\{|f(x)|; x \in [\alpha, \beta]\}$.

Proof. If $\alpha < \pi/2 < \beta$ then by the hypothesis it is elementary to prove that either $\limsup_{n \rightarrow \infty} n \cdot \omega_1(f, \pi/n)_{[\alpha, \pi/2]} = \infty$ or $\limsup_{n \rightarrow \infty} n \cdot \omega_1(f, \pi/n)_{[\pi/2, \beta]} = \infty$. Therefore, without any loss of generality we may suppose that we have only two cases: (i) $0 < \alpha < \beta \leq \pi/2$ and (ii) $\pi/2 \leq \alpha < \beta < \pi$.

Case (i) For fixed $n \in \mathbb{N}$ with $n \geq (\pi + \alpha)/(\beta - \alpha)$, let us choose $k(n) \in \{1, \dots, n\}$ such that $\alpha \leq \frac{k(n)\pi}{n+1} \leq \frac{k(n)\pi}{n} \leq \beta$ and

$$M_n(\alpha, \beta) = \left| f\left(\frac{k(n)\pi}{n}\right) - f\left(\frac{k(n)\pi}{n+1}\right) \right|.$$

Note that such an index $k(n)$ exists, because the inequalities $\alpha \leq k(n)\pi/(n+1) \leq k(n)\pi/n \leq \beta$ imply $\alpha(n+1)/\pi \leq k(n) \leq \beta n/\pi$, where $\beta n/\pi - \alpha(n+1)/\pi \geq 1$.

Since $\beta \leq \pi/2$, it results that $k(n) \leq n/2$ and hence we can use the conclusion of Lemma 2.2. This means that we have

$$T_n^{(M)}(f)(k(n)\pi/(n+1)) \geq f(k(n)\pi/n)$$

and

$$T_{n+1}^{(M)}(f)(k(n)\pi/n) \geq f(k(n)\pi/(n+1)).$$

If $f(k(n)\pi/n) \geq f(k(n)\pi/(n+1))$ then

$$\begin{aligned} & n \cdot \left(T_n^{(M)}(f)(k(n)\pi/(n+1)) - f(k(n)\pi/(n+1)) \right) \\ & \geq n \cdot (f(k(n)\pi/n) - f(k(n)\pi/(n+1))) = n \cdot M_n(\alpha, \beta) \end{aligned}$$

and this implies

$$n \cdot M_n(\alpha, \beta) \leq n \cdot \left\| T_n^{(M)}(f) - f \right\|_{[\alpha, \beta]}.$$

If $f(k(n)\pi/n) < f(k(n)\pi/(n+1))$ then

$$\begin{aligned} & (n+1) \cdot \left(T_{n+1}^{(M)}(f)(k(n)\pi/n) - f(k(n)\pi/n) \right) \\ & \geq (n+1) \cdot (f(k(n)\pi/(n+1)) - f(k(n)\pi/n)) \geq n \cdot M_n(\alpha, \beta). \end{aligned}$$

and this implies

$$n \cdot M_n(\alpha, \beta) \leq (n+1) \cdot \left\| T_{n+1}^{(M)}(f) - f \right\|_{[\alpha, \beta]}.$$

In conclusion, for any $n \in \mathbb{N}$ with $n \geq \pi/(\beta - \alpha)$, we have

$$n \cdot M_n(\alpha, \beta) \leq \max \left\{ n \cdot \left\| T_n^{(M)}(f) - f \right\|_{[\alpha, \beta]}, (n+1) \cdot \left\| T_{n+1}^{(M)}(f) - f \right\|_{[\alpha, \beta]} \right\}.$$

Since by Lemma 4.3 we have $\limsup_{n \rightarrow \infty} n \cdot M_n(\alpha, \beta) = \infty$, it easily follows now that

$$\limsup_{n \rightarrow \infty} n \cdot \left\| T_n^{(M)}(f) - f \right\|_{[\alpha, \beta]} = \infty.$$

The proof of the Case (ii) is absolutely similar, which proves the lemma. $\square$

Now, we are in position to prove Theorem 4.1.

Proof of Theorem 4.1. Firstly, it is immediate that f is a Lipschitz function on $[\alpha, \beta]$ if and only if $\limsup_{n \rightarrow \infty} n \cdot \omega_1(f, 1/n)_{[\alpha, \beta]} < \infty$.

Now, by the hypothesis it follows $n \cdot \left\| T_n^{(M)}(f) - f \right\|_{[\alpha, \beta]} \leq M$, for all $n \in \mathbb{N}$. Supposing that f is not a Lipschitz function on $[\alpha, \beta]$, by the above considerations it follows that $\limsup_{n \rightarrow \infty} n \cdot \omega_1(f, 1/n)_{[\alpha, \beta]} = \infty$. But then, by Lemma 4.4 we get

$$\limsup_{n \rightarrow \infty} n \cdot \left\| T_n^{(M)}(f) - f \right\|_{[\alpha, \beta]} = \infty,$$

which is a contradiction. The theorem is proved. $\square$

5 Localization and Local Direct Saturation Result for the $T_n^{(M)}$ Operator

In this section we obtain a local direct saturation result as a consequence of the following strong localization theorem, of independent importance.

Theorem 5.1. *Let $f, g : [0, \pi] \rightarrow \mathbb{R}_+$ be both bounded on $[0, \pi]$ with strictly positive lower bounds and suppose that there exists $a, b \in [0, \pi]$, $0 < a < b < \pi$ such that $f(x) = g(x)$ for all $x \in [a, b]$. Then for all $c, d \in [a, b]$ satisfying $a < c < d < b$ there exists $\tilde{n} \in \mathbb{N}$ which depends only on f, g, a, b, c, d such that $T_n^{(M)}(f)(x) = T_n^{(M)}(g)(x)$ for all $x \in [c, d]$ and $n \in \mathbb{N}$, $n \geq \tilde{n}$.*

Proof. Let us choose arbitrary $x \in [c, d]$ and for each $n \in \mathbb{N}$ let $j_x \in \{0, 1, \dots, n\}$ (j_x depends on n too, but there is no need at all to complicate on the notations) be such that $x \in [j_x \pi/n, (j_x + 1)\pi/n]$. Then we know that

$$T_n^{(M)}(f)(x) = \frac{\bigvee_{k=0}^n s_{n,k}(x) f\left(\frac{k\pi}{n}\right)}{\bigvee_{k=0}^n s_{n,k}(x)},$$

where for $k \in \{0, 1, \dots, n\}$ we have $s_{n,k}(x) = \frac{\sin^2(nx - k\pi)}{(nx - k\pi)^2}$. Since $x \in [c, d] \cap [j_x\pi/n, (j_x + 1)\pi/n]$ and since $a < c < d < b$ it is immediate that for $n \geq n_0$ where n_0 is chosen such that $\pi/n_0 < \min\{c - a, d - b\}$, we obtain $a < j_x\pi/n < b$ which gives $na/\pi < j_x < nb/\pi$ for all $n \geq n_0$ (indeed, if we would suppose that there exists $n > n_0$ which does not satisfy the previous double inequalities, then we would easily get a contradiction).

It is important to notice here that n_0 does not depend on x . From the previous inequality it follows that if $n \geq n_0$ then for any $x \in [c, d]$ there exists $\alpha_x \in [a, b]$ such that $j_x = n\alpha_x/\pi$.

In what follows, it will serve to our purpose to use the sequence $(a_n)_{n \geq 1}$, $a_n = \sqrt{n}$. For this sequence there exists $n_1 \in \mathbb{N}$ such that $na/\pi - a_n > 0$ for all $n \geq n_1$.

Our intention is to prove as an intermediate result, that there exists an absolute constant $N_0 \in \mathbb{N}$ independent of $x \in [c, d]$ such that for any $n \geq N_0$ and $x \in [c, d]$ we have $\bigvee_{k=0}^n s_{n,k}(x)f(\frac{k\pi}{n}) = \bigvee_{k \in I_{n,x}} s_{n,k}(x)f(\frac{k\pi}{n})$ where $I_{n,x} = \{k \in \{0, 1, \dots, n\} : j_x - a_n \leq k \leq j_x + a_n\}$. In order to obtain this conclusion, for $n \geq \max\{n_0, n_1\}$ let us choose $k \in \{0, 1, \dots, n\} \setminus I_{n,x}$. We have two cases: i) $k + a_n < j_x$ and ii) $j_x + a_n < k$.

Case i) We have two subcases: i_a) $x \in [j_x\pi/n, (j_x + 1/2)\pi/n]$ and i_b) $x \in [(j_x + 1/2)\pi/n, (j_x + 1)\pi/n]$.

Subcase i_a) Since $x \in [j_x\pi/n, (j_x + 1/2)\pi/n]$, we observe that $(nx - j_x\pi) \in [0, \pi/2]$ and by the well-known property $\sin(x) \geq 2x/\pi$, $x \in [0, \pi/2]$, it results that $s_{n,j_x}(x) \geq 4/\pi^2$. This implies

$$\begin{aligned} \frac{s_{n,j_x}(x)f(\frac{j_x\pi}{n})}{s_{n,k}(x)f(\frac{k\pi}{n})} &\geq \frac{4/\pi^2 \cdot f(\frac{j_x\pi}{n})}{s_{n,k}(x)f(\frac{k\pi}{n})} = \frac{(nx - k\pi)^2}{\sin^2(nx - k\pi)} \cdot \frac{4}{\pi^2} \cdot \frac{f(\frac{j_x\pi}{n})}{f(\frac{k\pi}{n})} \\ &\geq (nx - k\pi)^2 \cdot \frac{4}{\pi^2} \cdot \frac{f(\frac{j_x\pi}{n})}{f(\frac{k\pi}{n})}. \end{aligned}$$

Since $nx \geq j_x\pi \geq k\pi$ it follows that

$$\frac{s_{n,j_x}(x)f(\frac{j_x\pi}{n})}{s_{n,k}(x)f(\frac{k\pi}{n})} \geq (j_x\pi - k\pi)^2 \cdot \frac{4}{\pi^2} \cdot \frac{f(\frac{j_x\pi}{n})}{f(\frac{k\pi}{n})} \geq 4a_n \cdot \frac{f(\frac{j_x\pi}{n})}{f(\frac{k\pi}{n})}.$$

Then, denoting the infimum and the supremum of f on $[0, 1]$ with m_f and M_f , respectively (according to the hypotheses they are strictly positive real numbers), we get that

$$\frac{s_{n,j_x}(x)f(\frac{j_x\pi}{n})}{s_{n,k}(x)f(\frac{k\pi}{n})} \geq 4a_n \cdot \frac{m_f}{M_f}$$

Because $\lim_{n \rightarrow \infty} \frac{4a_n m_f}{\pi M_f} = \infty$, it follows that there exists $n_2 \in \mathbb{N}$, $n_2 \geq \max\{n_0, n_1\}$ such that $\frac{s_{n,j_x}(x)f(\frac{j_x\pi}{n})}{s_{n,k}(x)f(\frac{k\pi}{n})} > 1$ for all $x \in [c, d]$, $n \geq n_2$ and $k \in \{0, 1, \dots, n\}$, $k < j_x - a_n$ (as $k \notin I_{n,x}$). In addition, it is important to notice that n_2 does not depend on $x \in [c, d]$, but of course it depends on f .

Subcase i_b) It is easy to check that $x \in [(j_x + 1/2)\pi]/n, (j_x + 1)\pi/n]$ implies $(nx - (j_x + 1)\pi) \in [-\pi/2, 0]$. Therefore, reasoning as in case i_a) (because of $\sin^2$) we obtain that for sufficiently large n we have

$$\frac{s_{n,j_x+1}(x)f\left(\frac{(j_x+1)\pi}{n}\right)}{s_{n,k}(x)f\left(\frac{k\pi}{n}\right)} > 1, x \in [c, d].$$

Summarizing, we conclude that there exists an absolute constant $N_1 \in \mathbb{N}$ which depends only on a, b, c, d, f such that

$$\frac{\max\{s_{n,j_x}(x)f(j_x\pi/n), s_{n,j_x+1}(x)f((j_x+1)\pi/n)\}}{s_{n,k}(x)f(k\pi/n)} > 1$$

for all $x \in [c, d]$, $n \geq N_1$ and $k \in \{0, 1, \dots, n\}$, $k < j_x - a_n$.

Case ii) The proof is identical with the proof of the above Case i) and therefore we conclude that there exists an absolute constant $N_2 \in \mathbb{N}$ which depends only on a, b, c, d, f such that

$$\frac{\max\{s_{n,j_x}(x)f(j_x\pi/n), s_{n,j_x+1}(x)f((j_x+1)\pi/n)\}}{s_{n,k}(x)f(k\pi/n)} > 1$$

for all $x \in [c, d]$, $n \geq N_2$ and $k \in \{0, 1, \dots, n\}$, $k > j_x + a_n$.

Analyzing the results obtained in cases i)-ii), it results that for all $x \in [c, d]$, $n \geq N_0$, $N_0 = \max\{N_1, N_2\}$ and $k \in \{0, 1, \dots, n\}$, with $k < j_x - a_n$ or $k > j_x + a_n$, we have

$$\frac{\max\{s_{n,j_x}(x)f(j_x\pi/n), s_{n,j_x+1}(x)f((j_x+1)\pi/n)\}}{s_{n,k}(x)f(k\pi/n)} > 1.$$

In conclusion, we obtain our preliminary result, that is

$$\bigvee_{k=0}^n s_{n,k}(x)f\left(\frac{k\pi}{n}\right) = \bigvee_{k \in I_{n,x}} s_{n,k}(x)f\left(\frac{k\pi}{n}\right),$$

where $I_{n,x} = \{k \in \{0, 1, \dots, n\} : j_x - a_n \leq k \leq j_x + a_n\}$.

Next, let us choose arbitrary $x \in [c, d]$ and $n \in \mathbb{N}$ so that $n \geq N_0$. If there exists $k \in I_{n,x}$ such that $k\pi/n \notin [c, d]$ then we distinguish two cases. Either $k\pi/n < c$ or $k\pi/n > d$. In the first case we observe that

$$0 < c - \frac{k\pi}{n} \leq x - \frac{k\pi}{n} \leq \frac{(j_x+1)\pi}{n+1} - \frac{k\pi}{n} \leq \frac{(j_x+1)\pi}{n} - \frac{k\pi}{n} \leq \frac{(a_n+1)\pi}{n}.$$

Since $\lim_{n \rightarrow \infty} \frac{(a_n+1)\pi}{n} = 0$, it results that for sufficiently large n we necessarily have $\frac{(a_n+1)\pi}{n} < c - a$ which clearly implies that $k\pi/n \in [a, c]$. In the same manner, when $k\pi/n > d$, for sufficiently large n we necessarily have $k\pi/n \in [d, b]$.

Summarizing, there exists $\tilde{N}_1 \in \mathbb{N}$ independent of any $x \in [c, d]$, such that

$$\bigvee_{k=0}^n s_{n,k}(x) f\left(\frac{k\pi}{n}\right) = \bigvee_{k \in I_{n,x}} s_{n,k}(x) f\left(\frac{k\pi}{n}\right), \quad n \geq \tilde{N}_1$$

and in addition for any $x \in [c, d]$, $n \geq \tilde{N}_1$ and $k \in I_{n,x}$, we have $k\pi/n \in [a, b]$. Also, it is easy to check that $\tilde{N}_1$ depends only on a, b, c, d, f . We thus obtain that

$$T_n^{(M)}(f)(x) = \frac{\bigvee_{k \in I_{n,x}} s_{n,k}(x) f\left(\frac{k\pi}{n}\right)}{\bigvee_{k=0}^n s_{n,k}(x)}, \quad n \geq \tilde{N}_1, x \in [c, d]$$

and in addition for any $x \in [c, d]$, $n \geq \tilde{N}_1$ and $k \in I_{n,x}$, we have $k\pi/n \in [a, b]$.

Reasoning for the function g exactly as in the case of the function f , it follows that there exists $\tilde{N}_2 \in \mathbb{N}$ which depends only on a, b, c, d, g such that

$$T_n^{(M)}(g)(x) = \frac{\bigvee_{k \in I_{n,x}} s_{n,k}(x) g\left(\frac{k\pi}{n}\right)}{\bigvee_{k=0}^n s_{n,k}(x)}, \quad n \geq \tilde{N}_2, x \in [c, d]$$

and in addition for any $x \in [c, d]$, $n \geq \tilde{N}_2$ and $k \in I_{n,x}$, we have $k\pi/n \in [a, b]$. Taking $\tilde{n} = \max\{\tilde{N}_1, \tilde{N}_2\}$, we easily obtain the desired conclusion. $\square$

As a consequence of the localization result in Theorem 5.1 we present a local shape preserving property.

Corollary 5.2. *Let $f : [0, \pi] \rightarrow \mathbb{R}_+$ be bounded on $[0, \pi]$ with strictly positive lower bound and suppose that there exists $a, b \in [0, \pi]$, $0 < a < b < \pi$ such that f is constant on $[a, b]$ with the constant value $\alpha > 0$. Then for any $c, d \in [a, b]$ with $a < c < d < b$, there exists $\tilde{n} \in \mathbb{N}$ which depends only on a, b, c, d and f such that $T_n^{(M)}(f)(x) = \alpha$ for all $x \in [c, d]$ and $n \in \mathbb{N}$, $n \geq \tilde{n}$.*

Proof. Let $g : [0, 1] \rightarrow \mathbb{R}_+$ be given by $g(x) = \alpha > 0$ for all $x \in [0, \pi]$. Since $f(x) = g(x)$ for all $x \in [a, b]$ and since obviously $T_n^{(M)}(g)(x) = \alpha$ for all $x \in [0, \pi]$, by Theorem 5.1 we easily obtain the desired conclusion. $\square$

At the end of this section, as a consequence of the localization result in Theorem 5.1 we present a direct result, as follows.

Corollary 5.3. *Let $f : [0, \pi] \rightarrow \mathbb{R}_+$ be bounded on $[0, \pi]$ with the lower bound strictly positive and suppose that there exists $a, b \in [0, \pi]$, $0 < a < b < \pi$ and the constant C_0 which depends only on a and b , such that*

$$|f(x) - f(y)| \leq C_0 |x - y| \quad \text{for all } x \in [a, b], \quad (8)$$

that is $f|_{[a,b]} \in Lip[a,b]$. Then, for any $c, d \in [0, 1]$ satisfying $a < c < d < b$, we have

$$\left| T_n^{(M)}(f)(x) - f(x) \right| \leq \frac{C}{n} \text{ for all } n \in \mathbb{N} \text{ and } x \in [c, d],$$

where the constant C depends only on f and the values a, b, c, d .

Proof. Let us define the function $F : [0, \pi] \rightarrow \mathbb{R}$,

$$F(x) = \begin{cases} f(x) & \text{if } x \in [a, b], \\ f(a) & \text{if } x \in [0, a], \\ f(b) & \text{if } x \in [b, \pi]. \end{cases}$$

The hypothesis imply that F is continuous strictly positive and, according to Theorem 4.4 in [11] it results that

$$\left| T_n^{(M)}(F)(x) - F(x) \right| \leq 4\omega_1 \left(F, \frac{1}{n} \right), x \in [0, 1], n \in \mathbb{N},$$

Since by the definition of F we have $\omega_1(F, \frac{1}{n})_{[0,\pi]} \leq \omega_1(f, \frac{1}{n})_{[a,b]}$ and since by the relation (8) it easily follows $\omega_1(f, \frac{1}{n})_{[a,b]} \leq C_0/n$, we get

$$\left| T_n^{(M)}(F)(x) - F(x) \right| \leq 4C_0/n, x \in [0, \pi], n \in \mathbb{N},$$

Now, let us choose arbitrary $c, d \in [a, b]$ such that $a < c < d < b$. Then, by Theorem 5.1 (applicable to F and f) it results the existence of $\tilde{n} \in \mathbb{N}$ which depends only on a, b, c, d, f, F such that $T_n^{(M)}(F)(x) = T_n^{(M)}(f)(x)$ for all $x \in [c, d]$. But since actually the function F depends on the function f , it is clear that in fact $\tilde{n}$ depends only on a, b, c, d and f .

Therefore, for arbitrary $x \in [c, d]$ and $n \in \mathbb{N}$ with $n \geq \tilde{n}$ we obtain

$$\left| T_n^{(M)}(f)(x) - f(x) \right| = \left| T_n^{(M)}(F)(x) - F(x) \right| \leq \frac{4C_0}{n},$$

where C_0 and $\tilde{n}$ depend only on a, b, c, d and f .

Now, denoting

$$C_1 = \max_{1 \leq n < \tilde{n}} \{n \cdot \|T_n^{(M)}(f) - f\|_{[c,d]}\},$$

we finally obtain

$$\left| T_n^{(M)}(f)(x) - f(x) \right| \leq \frac{C}{n}, \text{ for all } n \in \mathbb{N}, x \in [c, d],$$

with $C = \max\{C_0, C_1\}$ depending only on a, b, c, d and f . This proves the corollary. $\square$

Remark. The results in Theorem 5.1 and Corollary 5.2 show the nice property of the truncated max-product operator $T_n^{(M)}$ to reproduce locally with great

accuracy the graph of non-smooth strictly positive, bounded and locally continuous function f , with important implications in the signals' representation. For example, if on a subinterval $[c, d]$ the signal f is zero, we may consider there to be constant equal to an arbitrary small $\varepsilon > 0$ and then, the above Corollary 5.2 proves that for sufficiently large n , $T_n^{(M)}(f)(x) = \varepsilon$ for all $x \in [c, d]$. Note that the truncated linear sampling operator T_n does not have these properties.

6 Localization, Local Inverse and Local Direct Saturation Results for the $W_n^{(M)}$ Operator

In this section we present a local inverse saturation result, a strong localization result and a local direct saturation result for the $W_n^{(M)}$ operator.

In this sense, let us make the following remarks. Firstly, note that the proof of Theorem 4.1 follows from Lemma 4.4 and the proof of Lemma 4.4 follows from the Lemmas 4.2-4.3 (which are independent of any kind of operator) and Lemmas 2.2-2.3. Since the Lemmas 2.2-2.3 are identical with Lemmas 3.2-3.3 (making abstraction of the kernels), in the case of the truncated max-product operator based on Whittaker-type kernel we easily obtain corresponding results for Theorems 4.1 and 5.1 and for Corollaries 5.2 and 5.3, which are the main results of the previous two sections.

In conclusion, for $W_n^{(M)}$ we can state the following local inverse saturation result, localization result and local direct saturation result, respectively.

Theorem 6.1. *Let $f : [0, \pi] \rightarrow \mathbb{R}_+$ and $0 < \alpha < \beta < \pi$ be such that f is continuous on $[\alpha, \beta]$. If there exists a constant $M > 0$ (independent of n but depending on f, α and β) such that*

$$\|W_n^{(M)}(f) - f\|_{[\alpha, \beta]} \leq M/n, \text{ for all } n \in \mathbb{N},$$

then $f|_{[\alpha, \beta]} \in \text{Lip}[\alpha, \beta]$, that is f is a Lipschitz function on $[\alpha, \beta]$.

Theorem 6.2. *Let $f, g : [0, \pi] \rightarrow \mathbb{R}_+$ be both bounded on $[0, \pi]$ with strictly positive lower bounds and suppose that there exists $a, b \in [0, \pi]$, $0 < a < b < \pi$ such that $f(x) = g(x)$ for all $x \in [a, b]$. Then for all $c, d \in [a, b]$ satisfying $a < c < d < b$ there exists $\tilde{n} \in \mathbb{N}$ which depends only on f, g, a, b, c, d such that $W_n^{(M)}(f)(x) = W_n^{(M)}(g)(x)$ for all $x \in [c, d]$ and $n \in \mathbb{N}$, $n \geq \tilde{n}$.*

Corollary 6.3. *Let $f : [0, \pi] \rightarrow \mathbb{R}_+$ be bounded on $[0, \pi]$ with strictly positive lower bound and suppose that there exists $a, b \in [0, \pi]$, $0 < a < b < \pi$ and the constant C_0 which depends only on a and b , such that*

$$|(f)(x) - f(y)| \leq C_0 |x - y| \text{ for all } x \in [a, b],$$

that is $f|_{[a, b]} \in \text{Lip}([a, b])$. Then, for any $c, d \in [0, 1]$ satisfying $a < c < d < b$, we have

$$\left| W_n^{(M)}(f)(x) - f(x) \right| \leq \frac{C}{n} \text{ for all } n \in \mathbb{N} \text{ and } x \in [c, d],$$

where the constant C depends only on f and the values a, b, c, d .

Corollary 6.4. *Let $f : [0, \pi] \rightarrow \mathbb{R}_+$ be bounded on $[0, \pi]$ with strictly positive lower bound and suppose that there exists $a, b \in [0, \pi]$, $0 < a < b < \pi$ such that f is constant on $[a, b]$ with the constant value $\alpha > 0$. Then for any $c, d \in [a, b]$ with $a < c < d < b$, there exists $\tilde{n} \in \mathbb{N}$ which depends only on a, b, c, d and f such that $W_n^{(M)}(f)(x) = \alpha$ for all $x \in [c, d]$ and $n \in \mathbb{N}$, $n \geq \tilde{n}$.*

Remarks. 1) Similar to the case of the operator $T_n^{(M)}$, the results in the Theorem 6.2 and Corollary 6.4 show the nice properties of the truncated max-product operator $W_n^{(M)}$ to reproduce locally with great accuracy the graph of a non-smooth strictly positive, bounded and locally continuous function f , with important potential applications in the signals' representation. Note that the truncated linear sampling operator W_n does not have these properties.

2) Although the max-product operators $W_n^{(M)}$ and $T_n^{(M)}$ studied by this paper are well-defined only for bounded functions with values in $\mathbb{R}_+$, they can easily be used to generate max-product type operators for the approximation of functions of arbitrary sign $f : [0, \pi] \rightarrow \mathbb{R}$, as follows : if $a < \min\{f(x); x \in [0, \pi]\}$, then the new operators $P_n^{(M)}(f)(x) = W_n^{(M)}(f - a)(x) + a$ and $Q_n^{(M)}(f)(x) = T_n^{(M)}(f - a)(x) + a$, approximate f with the same Jackson-type order $\omega_1(f; 1/n)_{[0, \pi]}$ and keep the interpolation properties $P_n^{(M)}(f)(j\pi/n) = Q_n^{(M)}(f)(j\pi/n) = f(j\pi/n)$, $j = 0, 1, \dots, n$.

ACKNOWLEDGEMENT

The work of both authors was supported by a grant of the Romanian National Authority for Scientific Research, CNCS-UEFISCDI, project number PN-II-ID-PCE-2011-3-0861.

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