On the Eigenfunctions of the Finite Hankel Transform

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Abstract

In this paper, we study the properties of the eigenfunctions of the finite Hankel transform. We deduce a sampling series in terms of these functions for Hankel-band-limited signals and derive bounds for the truncation error of the sampling series.

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1 Introduction

The classical Whittaker–Kotel’nikov–Shannon (WKS) sampling theorem says that any function f from the Paley–Wiener space PW_Ω , i.e. any $f \in \mathcal{L}^2(-\infty, \infty)$ representable through its Fourier transform $F[f]$ as

$$f(x) = \frac{1}{2\pi} \int_{-\Omega}^{\Omega} e^{iyx} F[f](y) dy,$$

can be reconstructed from its equally spaced samples,

$$f(x) = \sum_{k=-\infty}^{\infty} f\left(\frac{\pi k}{\Omega}\right) \frac{\sin \Omega(x - \pi k/\Omega)}{\Omega(x - \pi k/\Omega)}. \quad (1)$$

The series in the right-hand side converges both in the $\mathcal{L}_2$ -norm and uniformly on $(-\infty, \infty)$. This follows from the fact that $F[f] \in \mathcal{L}_2(\Omega, \Omega)$ and hence

$$F[f](y) = \frac{\pi}{\Omega} \sum_{k=-\infty}^{\infty} f\left(\frac{\pi k}{\Omega}\right) \exp(-i\pi ky/\Omega).$$

Another sampling formula for $f(t)$, which was invented and studied in [30, 31], is based on the eigenfunctions $\psi_l(\Omega, x) = \psi_l(x)$ of the Finite (Truncated)

Fourier Transform (TFT)¹ $F_{\Omega}[f](x) = \int_{-\Omega}^{\Omega} \exp(ixy) f(y) dy$. The eigenfunctions $\psi_l(x)$ are first defined inside the interval $I_{\Omega} = [-\Omega, \Omega]$ by

$$F_{\Omega}[\psi_l](x) = \int_{-\Omega}^{\Omega} \exp(ixy) \psi_l(y) dy = \mu_l(\Omega) \psi_l(x), \quad (2)$$

and are then continued analytically to the entire real axis according to (2). The associated TFT eigenvalues $\mu_l = \mu_l(\Omega)$ are ordered by magnitude, $\sqrt{2\pi} > |\mu_0| > |\mu_1| > \dots > 0$.

Due to their unusual and interesting features, the TFT eigenfunctions, also known as 'prolates', have long attracted attention; we refer the reader to the classical works by Slepian, Pollak, and Landau [27, 16, 17, 18], as well as the papers [30, 31, 34] and the literature cited therein. This list of references is far from complete. The properties of prolates ψ_l are widely discussed in the literature. Nevertheless we shall proceed with a short survey of those properties which are related to the fact that prolates form an orthonormal basis in a reproducing kernel space [5]. In such spaces Kramer's generalization of the WKS sampling theorem is valid. Kramer's theorem expands the variety of functions that can be recovered from their sampled values by means of orthogonal sampling formulae, and also establishes general conditions allowing such a recovery (see e.g. [14, 12, 35, 8, 7, 23]).

The essence of the general theory is summarized in a recent paper [36], where an algorithm to generate systems of functions possessing the above-mentioned properties of prolates is provided. We shall illustrate the general description of [36] with the example of the Finite (Truncated) Hankel Transform (THT) and its eigenfunctions. These functions are extensively used now in various areas that range from applied physics to computer science. The number of publications in which they are employed is so large that any comprehensive survey of modern applications would be a very complicated task going far beyond the present study. Here we only note that historically the THT eigenfunctions were introduced almost simultaneously in three pioneering works [26, 9, 33], each being a source of many subsequent publications on numerical optical analysis, such as both forward and inverse analysis of high numerical aperture focusing systems, calculation of mode patterns and losses in optical resonators with spherical mirrors. Among the most recent papers see, e. g. [25, 6, 3].

In the present work we pay special attention to the bounds of the error induced by the sampling series truncation. In [30, 31, 21, 22] the truncation error estimates were obtained for particular classes of band-limited functions and

¹In order to avoid ambiguity, the abbreviation TFT (THT) stands here for the Finite Fourier (Hankel) Transform, as the acronym FFT is usually used for the Fast Fourier Transform.

some illustrative examples were given. In what follows we shall be concerned with another class of band-limited functions, namely with the range of the operator $G_\Omega := (1/2\pi) F_\Omega^* \circ F_\Omega$, i.e.

$$G_\Omega[f](x) = \frac{1}{2\pi} F_\Omega^* \circ F_\Omega[f](x) = \int_{-\Omega}^{\Omega} \frac{\sin \Omega(x-y)}{\pi(x-y)} f(y) dy.$$

One can also interpret the operator G_Ω as a successive application of the truncated direct and truncated inverse Fourier transforms.

Similar to the examples studied in [30, 31, 21, 22], functions from the range of G_Ω , $\text{Rg}(G_\Omega)$, are well approximated by the elements of the span of the first few prolates ψ_l , $l = 0, 1, \dots, L$. The number L that provides a sufficient approximation accuracy is often called the number of degrees of freedom and is explicitly expressed in terms of Ω : $L \sim 2\Omega^2/\pi$.

A generalization of the results obtained for prolates to other reproducing kernel spaces is thought to be straightforward [16]. However, even in the case of Hankel-band-limited functions, the number of degrees of freedom and the truncation error bounds have not thus far been computed. The relevant analysis will be presented below. The discussion will primarily focus on the Hankel transform analogue of the operator G_Ω and its range.

2 TFT eigenfunctions

Let us return to prolates and their definition via Eq. (2). The eigenvalues μ_l can be shown to be simple. However except for the first $L \sim 2\Omega^2/\pi$ of them, these eigenvalues are numerically indistinguishable from zero, while for the first L eigenvalues $|\mu_l| \approx \sqrt{2\pi}$ [27, 16].

For prolates ψ_l , the Fourier inversion formula takes the form

$$\frac{\mu_l}{2\pi} \int_{-\infty}^{\infty} \exp(-ixy) \psi_l(y) dy = \psi_l(x) \chi_\Omega(x), \quad \chi_\Omega(x) = \begin{cases} 1, & x \in I_\Omega, \\ 0, & x \in \mathbb{R} \setminus I_\Omega \end{cases}. \quad (3)$$

If the functions ψ_l are normalized by

$$\int_{-\Omega}^{\Omega} |\psi_l(x)|^2 dx = 1,$$

they form an orthonormal basis in $\mathcal{L}_2(I_\Omega)$,

$$\int_{-\Omega}^{\Omega} \psi_l(x) \psi_p(x) dx = \delta_{lp}. \quad (4)$$

Their analytical continuations then form an orthogonal basis in PW_Ω as well:

$$\int_{-\infty}^{\infty} \psi_l(x) \psi_p(x) dx = \frac{2\pi}{\mu_l \bar{\mu}_p} \int_{-\Omega}^{\Omega} \psi_l(x) \psi_p(x) dx = \frac{\delta_{lp}}{\gamma_l}, \quad (5)$$

where $\gamma_l = \frac{|\mu_l|^2}{2\pi}$, $l = 0, 1, \dots$, are the eigenvalues of the operator G_Ω , so that

$$\gamma_l \approx \begin{cases} 1 & \text{for } l \ll 2\Omega^2/\pi, \\ 0 & \text{for } l \gg 2\Omega^2/\pi. \end{cases} \quad (6)$$

One can choose the functions ψ_l to be real, then

$$\int_{-\infty}^{\infty} \psi_l^2(x) dx - \int_{-\Omega}^{\Omega} \psi_l^2(x) dx = \int_{|x|>\Omega} \psi_l^2(x) dx = \frac{1 - \gamma_l}{\gamma_l}, \quad (7)$$

and therefore

$$\gamma_l = \frac{\int_{-\Omega}^{\Omega} \psi_l^2(x) dx}{\int_{-\infty}^{\infty} \psi_l^2(x) dx} = \frac{\|\psi_l\|_\Omega^2}{\|\psi_l\|_\mathbb{R}^2}.$$

As shown in [27], γ_0 yields the largest possible value for the ratio $\frac{\|f\|_\Omega^2}{\|f\|_\mathbb{R}^2}$ among the elements of PW_Ω . In general, we denote by PW_Ω^l the orthogonal complement to $\text{Span}\{\psi_0, \psi_1, \dots, \psi_{l-1}\}$ in PW_Ω , then

$$\gamma_l = \max_{f \in PW_\Omega^l} \frac{\|f\|_\Omega^2}{\|f\|_\mathbb{R}^2}.$$

For the convolution of ψ_l with a *sinc*-kernel we have

$$G[\psi_l](x) := \psi_l(x) * \frac{\sin \Omega(x)}{\pi x} = \int_{-\infty}^{\infty} \frac{\sin \Omega(x-y)}{\pi(x-y)} \psi_l(y) dy = \psi_l(x),$$

and hence

$$\frac{\sin \Omega(x-y)}{\pi(x-y)} = \sum_{l=0}^{\infty} \gamma_l \psi_l(x) \psi_l(y). \quad (8)$$

This, in addition to (4)–(5), induces one more orthogonality relation [30, 31]:

$$\sum_{l=0}^{\infty} \gamma_l \psi_l\left(\frac{\pi k}{\Omega}\right) \psi_l\left(\frac{\pi m}{\Omega}\right) = \frac{\Omega}{\pi} \delta_{km},$$

along with the equality

$$\sum_{l=0}^{\infty} \gamma_l \psi_l^2(x) = \frac{\Omega}{\pi}. \quad (9)$$

Integrating the latter equality over I_Ω yields for the trace of G_Ω

$$\text{tr } G_\Omega = \sum_{l=0}^{\infty} \gamma_l = \frac{2\Omega^2}{\pi}, \quad (10)$$

the same value is given by integration of the squared *sinc*-function, namely,

$$\int_{-\Omega}^{\Omega} \int_{-\infty}^{\infty} \frac{\sin^2 \Omega(x-y)}{[\pi(x-y)]^2} dy dx = \frac{2\Omega^2}{\pi} = \text{tr } G_\Omega.$$

On the other hand, Eq. (8) also gives

$$\text{tr } G_\Omega^2 = \sum_{l=0}^{\infty} \gamma_l^2 = \int_{-\Omega}^{\Omega} \int_{-\Omega}^{\Omega} \frac{\sin^2 \Omega(x-y)}{[\pi(x-y)]^2} dy dx$$

and hence the following estimate holds

$$\sum_{l=0}^{\infty} (\gamma_l - \gamma_l^2) = \sum_{l=0}^{\infty} \gamma_l (1 - \gamma_l) = \int_{-\Omega}^{\Omega} \int_{|y|>\Omega} \frac{\sin^2 \Omega(x-y)}{[\pi(x-y)]^2} dy dx \leq B_1 \ln \left(\frac{2\Omega^2}{\pi} \right) + B_2 \quad (11)$$

with some positive constants B_1 and B_2 .

Combined with (10), the estimate (11) explains (6): each summand $\gamma_l (1 - \gamma_l)$ is small, i.e. γ_l is close either to zero or to one, besides in view of Eq. (10) the number of eigenvalues γ_l close to one is restricted to $L = 2\Omega^2/\pi$, for details see [16].

The same L defines the number of the essential samples in the reconstruction formula written in terms of TFT eigenfunctions [30, 31]. On substituting expansion (8) in (1) one obtains for $f(x) \in PW_\Omega$

$$\begin{aligned} f(x) &= \frac{\pi}{\Omega} \sum_{k=-\infty}^{\infty} f\left(\frac{\pi k}{\Omega}\right) \sum_{l=0}^{\infty} \gamma_l \psi_l\left(\frac{\pi k}{\Omega}\right) \psi_l(x) \\ &= \frac{\pi}{\Omega} \sum_{l=0}^{\infty} \gamma_l \left\{ \sum_{k=-\infty}^{\infty} f\left(\frac{\pi k}{\Omega}\right) \psi_l\left(\frac{\pi k}{\Omega}\right) \right\} \psi_l(x). \end{aligned} \quad (12)$$

Here, the order of double summation is interchangeable, both double series converging pointwise and in the $\mathcal{L}_2$ -norm. Because of the double summation, series

in the sampling formula (12) look more cumbersome than (1), yet in practical calculations the latter series may be more suitable than the classical one, provided that the function $f(x)$ decays rapidly outside the interval I_Ω . If this is the case, then to a very high accuracy the infinite sums in (12) can be truncated to $l \leq \frac{2\Omega^2}{\pi}$ and $|k| \leq \frac{\Omega^2}{\pi}$. As was mentioned above, the relevant proofs for particular classes of band-limited functions can be found in [30, 31, 21, 22].

The new sampling formula (12) applied to ψ_p provides one more orthogonality relation [30, 31], namely

$$\psi_p(x) = \frac{\pi}{\Omega} \sum_{l=0}^{\infty} \gamma_l \left\{ \sum_{k=-\infty}^{\infty} \psi_p\left(\frac{\pi k}{\Omega}\right) \psi_l\left(\frac{\pi k}{\Omega}\right) \right\} \psi_l(x),$$

and as a result

$$\sum_{k=-\infty}^{\infty} \psi_l\left(\frac{\pi k}{\Omega}\right) \psi_p\left(\frac{\pi k}{\Omega}\right) = \frac{\Omega}{\gamma_l \pi} \delta_{lp}.$$

3 Hankel-band-limited Functions

By a Hankel-band-limited function we understand a function expressed through the Finite (Truncated) Hankel Transform:

$$h(r) = \int_0^a \sqrt{r\rho} J_m(r\rho) g(\rho) d\rho, \quad (13)$$

where J_m is the m -th order Bessel function of the 1st kind, $m = 0, 1, 2, \dots$, and $g(\rho) = H_m[h](\rho)$ stands for the Hankel transform of $h(r)$, i.e. $g(\rho) = \int_0^\infty \sqrt{\rho\zeta} J_m(\rho\zeta) h(\zeta) d\zeta$; evidently $H_m[h](\rho) = 0$ for $\rho > a$. Hankel-band-limited functions serve as radial parts of rotationally symmetric 2D-functions with Fourier transform supported on a domain confined to a circle of radius a ,

$$f(\mathbf{r}) = \frac{1}{(2\pi)^2} \int_{v=|\mathbf{v}| \leq a} \exp(i \langle \mathbf{r}, \mathbf{v} \rangle) F[f](\mathbf{v}) d\mathbf{v}. \quad (14)$$

Eq. (13) is obtained by substituting $f(\mathbf{r})$ in the form

$$f(\mathbf{r}) = f(r, \varphi) = r^{-1/2} h(r) e^{im\varphi}$$

into Eq. (14). One also takes into account the integral representation of the Bessel function:

$$J_m(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ix \sin \xi + im\xi} d\xi,$$

as well as the identity

$$h(r) = \int_0^a \int_0^\infty \zeta \sqrt{r\rho} J_m(r\zeta) J_m(\rho\zeta) h(\rho) d\rho d\zeta, \quad (15)$$

that holds for any Hankel-band-limited function [32, 28]. Changing the order of integration in the right-hand side of (15) yields

$$h(r) = \int_0^\infty \sqrt{r\rho} h(\rho) \int_0^a \zeta J_m(r\zeta) J_m(\rho\zeta) d\zeta d\rho = \int_0^\infty \mathcal{K}_{am}(r, \rho) h(\rho) d\rho, \quad (16)$$

where

$$\begin{aligned} \mathcal{K}_{am}(r, \rho) &= \sqrt{r\rho} \int_0^a \zeta J_m(r\zeta) J_m(\rho\zeta) d\zeta \\ &= \frac{a\sqrt{r\rho}}{r^2 - \rho^2} \left[r J'_m(ar) J_m(a\rho) - \rho J_m(ar) J'_m(a\rho) \right] \end{aligned} \quad (17)$$

$$= \frac{a\sqrt{r\rho}}{r^2 - \rho^2} \left[r J_{m+1}(ar) J_m(a\rho) - \rho J_m(ar) J_{m+1}(a\rho) \right], \quad (18)$$

see e.g. [24].

Thus Hankel-band-limited functions form a reproducing kernel Hilbert space with a reproducing kernel $\mathcal{K}_{am}(r, \rho)$, for details see, e.g. [36, 29, 10, 11]. As follows from the representation (16), $\sqrt{r} h(r) \in C^\infty(\mathbb{R}_+)$.

4 Bessel functions

Before turning to the sampling theorem in the space $\mathcal{B}_{am}$ of Hankel-band-limited functions, we shall address a few more properties of the Bessel functions of the 1st kind necessary for future discussion. Most of them are well-known and have been proved elsewhere (see e.g. [32, 28, 24]), and we shall only mention them briefly; the others, like a modification of the Prüfer angle associated with Bessel's equation, will be considered in more detail.

4.1 Modified Prüfer angle associated with bounded solutions of Bessel's equation

Hereafter $m = 0, 1, 2, \dots$, is fixed. Let us introduce functions $\theta(r)$ and $\mathcal{A}(r)$, $r > 0$, as solutions to the ODEs

$$\theta'(r) = -\frac{m^2 - 1/4}{r^2} \cos^2(r + \theta), \quad (19)$$

and

$$\mathcal{A}'(r) = -\mathcal{A}(r) \frac{m^2 - 1/4}{r^2} \sin(r + \theta) \cos(r + \theta), \quad (20)$$

respectively. By direct substitution, one verifies that the expression

$$\mathcal{G}(r) := -\frac{\mathcal{A}(r)}{\sqrt{r}} \cos \Phi(r) \quad (21)$$

satisfies Bessel's equation

$$\frac{1}{r} \left(r \mathcal{G}'(r) \right)' + \left(1 - \frac{m^2}{r^2} \right) \mathcal{G}(r) = 0, \quad (22)$$

and therewith

$$\mathcal{G}'(r) = \frac{\mathcal{A}(r)}{\sqrt{r}} \left[\sin \Phi(r) + \cos \Phi(r) / 2r \right]. \quad (23)$$

Here $\Phi(r) := r + \theta(r)$ is the modified Prüfer angle of $\mathcal{G}(r)$.

All solutions of Bessel's equation which are finite at $r = 0$ possess the same Prüfer angle, $\Phi(r)$, that in the neighborhood of $r = 0$ satisfies the condition

$$2r \tan \left(\Phi(r) \right) = - \sum_{k=0}^{\infty} a_k r^{2k}, \quad (24)$$

with $a_0 = m + \frac{1}{2}$, $a_1 = -\frac{1}{2(m+1)}$, and $a_k = -\frac{\sum_{j=1}^{k-1} a_j a_{k-j}}{2(m+k)}$, $k = \overline{2, \infty}$. The series in the right-hand side of (24) converges absolutely and uniformly on $[0, r_0]$, provided that $|a_1| r_0^2 / 2 < 1$, since for any k we have $\sqrt[k]{|a_k|} < |a_1| / 2$. Note that the Prüfer angle is defined up to a summand $k\pi$, $k \in \mathbb{Z}$. By integrating Eq. (19) with the initial condition

$$\Phi(r_0) = - \arctan \left(\sum_{k=0}^{\infty} a_k r_0^{2k-1} / 2 \right), \quad (25)$$

the Prüfer angle associated with bounded solutions of (22) is uniquely defined on $[r_0, \infty)$.

Unlike the Prüfer angle, the function $\mathcal{A}(r)$ specifies a particular bounded solution of Bessel's equation. Thus the conventional Bessel function of the 1st kind $J_m(r)$ is specified according to its asymptotic behaviour at infinity

$$J_m(r) = \sqrt{\frac{2}{\pi r}} \cos \left(r - \frac{\pi}{2} m - \frac{\pi}{4} \right) + O \left(\frac{1}{r^{3/2}} \right), \quad (26)$$

which means that for the corresponding function $A(r)$

$$|\mathcal{A}(r)| = \sqrt{\frac{\pi}{2}} \exp \left(\int_r^{\infty} \frac{m^2 - 1/4}{2 \xi^2} \sin 2 \Phi(\xi) d\xi \right), \quad (27)$$

cf. Eq. (20).

In what follows the functions $\theta(r)$ and $\mathcal{A}(r)$ stand for the solutions of Eqs. (19), (20) subject to the conditions (25), (27); as well as the Prüfer angle $\Phi(r)$, they correspond to the function $J_m(r)$. Evidently $\mathcal{A}(r)$ is bounded on the interval $[1, \infty)$: $\underline{A} \leq |\mathcal{A}(r)| \leq \bar{A}$ for some $\bar{A}$ and $\underline{A}$, say $\bar{A} = \pi / (2 \underline{A}) = \sqrt{\pi} / 2 e^{(4m^2+1)/8}$. We shall use the representation of $J_m(r)$ through its Prüfer angle and $\mathcal{A}(r)$ in order to obtain some useful estimates of integrals containing Bessel functions.

4.2 Lommel integrals

The integrals arising most frequently in the subsequent discussion are of the Lommel type [24]:

$$\begin{aligned}\mathcal{I}_L &:= \int_0^a r \rho J_m^2(r \rho) d\rho = \frac{a^2 r}{2} \left\{ J_m^2(ar) + \left(1 - \frac{m^2}{a^2 r^2}\right) J_m^2(ar) \right\} \quad (28) \\ &= \frac{a^2 r}{2} (J_m^2(ar) + J_{m+1}^2(ar)) - a m J_m(ar) J_{m+1}(ar).\end{aligned}$$

Obviously, if $r < 1/a$, $\mathcal{I}_L$ is less than $1/a$, since $|J_k| \leq 1$, $k = 0, 1, \dots$. In order to obtain an upper bound on $\mathcal{I}_L$ for $r > 1/a$, one substitutes the functions Φ and $\mathcal{A}$ into (28):

$$\begin{aligned}\int_0^a r \rho J_m^2(r \rho) d\rho &= a \mathcal{A}^2(ar) \left(1 + \frac{\sin 2\Phi(ar)}{2ar} - \frac{m^2 - 1/4}{a^2 r^2}\right) \quad (29) \\ &\leq a \bar{A}^2 \left(1 + \frac{1}{2ar} + \frac{1}{(2ar)^2}\right) \leq B = 2a \bar{A}^2.\end{aligned}$$

4.3 Zeros of the Bessel functions

Let us denote the zeros of the Bessel function $J_m(r)$ by r_{mn} , $n = 0, 1, 2, \dots$. As is well-known, $r_{mn} = \pi(n + m + 1/4) + O(1)$, $r \rightarrow \infty$. Using Eq. (19) one deduces a sharper asymptotic behaviour of the zeros r_{mn} at infinity. Each time when $J_m(r) = 0$, the Prüfer angle crosses the line $\Phi = \pi k$, $k \in \mathbb{Z}$, so that

$$\pi = \Phi(r_{m,n+1}) - \Phi(r_{m,n}) = r_{m,n+1} - r_{m,n} + \int_{r_{m,n}}^{r_{m,n+1}} \theta'(\xi) d\xi.$$

If $m = 1, 2, \dots$, we have $\theta'(\xi) \leq 0$, and hence the distance between two neighboring zeros $r_{m,n}$ and $r_{m,n+1}$ is never less than π . In the case $m = 0$, we still can write $\int_{r_{0,n}}^{r_{0,n+1}} \theta'(\xi) d\xi \leq (r_{0,n+1} - r_{0,n})/4r_{0,n}^2$. As a result we obtain

$$r_{m,n+1} - r_{m,n} \geq \begin{cases} \pi (1 - 1/(4r_{0,n}^2)), & m = 0, \\ \pi, & m = 1, 2, \dots \end{cases} \quad (30)$$

Later we will also use the inequality

$$r_{m,n} J_{m+1}^2(r_{m,n}) = r_{m,n} (J'_m(r_{m,n}))^2 = \mathcal{A}^2(r_{m,n}) > \underline{A}^2. \quad (31)$$

5 The Finite Hankel Transform and its eigenfunctions

Let H_{am} be the THT operator

$$H_{am}[g](r) = \int_0^a \sqrt{r\rho} J_m(r\rho) g(\rho) d\rho,$$

defined on functions $g(r) \in \mathcal{L}^2(I_a)$, with $I_a = (0, a)$. Let also $T_{aml}(r) = T_l(r)$ be its eigenfunctions,

$$\Upsilon_l T_l(r) = \int_0^a \sqrt{r\rho} J_m(r\rho) T_l(\rho) d\rho, \quad l = 0, 1, 2, \dots, \quad (32)$$

associated with the eigenvalues $\Upsilon_{aml} = \Upsilon_l$ ordered in accordance with their absolute magnitude:

$$|\Upsilon_0| \geq |\Upsilon_1| \geq \dots \geq |\Upsilon_l| \dots$$

Evidently $H_{am} = H_{am}^*$ and the functions T_l can be chosen real. They are at the same time eigenfunctions of the integral operator $K_{am} := H_{am} \circ H_{am}^* = H_{am}^2$:

$$K_{am}[T_l](r) = \int_0^a \mathcal{K}_{ma}(r, \rho) T_l(\rho) d\rho = \Upsilon_l^2 T_l(r).$$

The similarity between the properties of TFT and THT eigenfunction was discussed in detail in [26], see also [15, 13].

Being the eigenfunctions of a self-adjoint compact operator, T_l form an orthonormal basis in $\mathcal{L}^2(I_a)$, provided that they are normalized by

$$\int_0^a T_l^2(r) dr = 1. \quad (33)$$

Further, Eq. (32) not only defines T_l inside the interval I_a , but also its analytic continuation onto the positive semi-axis $\mathbb{R}_+$. For the Hankel transform of the function T_l continued to $\mathbb{R}_+$

$$H_m[T_l](\rho) = \int_0^\infty \sqrt{r\rho} J_m(r\rho) T_l(r) dr = \frac{1}{\Upsilon_l} T_l(\rho) \chi_a(\rho), \quad (34)$$

holds, where $\chi_a(\rho) = \begin{cases} 1, & \rho \in I_a, \\ 0, & \rho > a. \end{cases}$

In addition to the orthogonality on I_a , T_l are pairwise orthogonal on $\mathbb{R}_+$:

$$\int_0^\infty T_l(r) T_p(r) dr = \frac{1}{\Upsilon_l \Upsilon_p} \int_0^a T_l(r) T_p(r) dr = \frac{1}{\Upsilon_l \Upsilon_p} \delta_{lp}. \quad (35)$$

In particular, this means that

$$\Upsilon_l^2 = \frac{\|T_l\|_a^2}{\|T_l\|_{\mathbb{R}_+}^2} \leq 1,$$

where $\|\cdot\|_a$ and $\|\cdot\|_{\mathbb{R}_+}$ are the norms in $\mathcal{L}^2(I_a)$ and in $\mathcal{B}_{am}$, respectively.

Similarly to ψ_l , the eigenfunction T_l yields a maximum of the ratio $\|h\|_a/\|h\|_{\mathbb{R}_+}$ among all $h \in \mathcal{B}_{am}$ orthogonal to $\text{Span}\{T_0, T_1, \dots, T_{p-1}\}$ [26].

The completeness of the functions T_l in $\mathcal{L}^2(I_a)$ implies that they also form a basis in $\mathcal{B}_{am}$. Indeed, let $h \in \mathcal{B}_{am}$ and let h_l be the Fourier coefficients of its Hankel transform in terms of T_l in $\mathcal{L}^2(I_a)$, i.e. $h_l = \langle H_m[h], T_l \rangle_a$, $l = 0, 1, \dots$. Then in view of (15)–(32), and (34), for any $L \in \mathbb{N}$

$$\left\| h - \sum_{l=0}^L \Upsilon_l h_l T_l \right\|_{\mathbb{R}_+}^2 = \left\| H_{am}[h] - \sum_{l=0}^L h_l T_l \right\|_a^2.$$

As an immediate consequence of the completeness, the following evident but important expansions hold:

$$\sqrt{r\rho} J_m(r\rho) = \sum_{l=0}^{\infty} \Upsilon_l T_l(r) T_l(\rho), \quad (36)$$

$$\mathcal{K}_{am}(r, \rho) = \sum_{l=0}^{\infty} \Upsilon_l^2 T_l(r) T_l(\rho). \quad (37)$$

Fixing one variable, say $r \in \mathbb{R}_+$, the expansions (36) and (37) are valid in $\mathcal{L}^2(I_a)$ and $\mathcal{L}^2(\mathbb{R}_+)$, respectively in terms of the remaining variable ρ . Moreover, the series in the right-hand side of (37) converges uniformly on $\rho \in \mathbb{R}_+$. This is readily seen from the obvious relations (below B is the same as in (29)):

$$\begin{aligned} & \left| \mathcal{K}_{am}(r, \rho) - \sum_{l=0}^L \Upsilon_l^2 T_l(r) T_l(\rho) \right| \\ &= \left| \int_0^a \sqrt{\rho\zeta} J_m(\rho\zeta) \left[\sqrt{r\zeta} J_m(r\zeta) - \sum_{l=0}^L \Upsilon_l T_l(r) T_l(\zeta) \right] d\zeta \right| \\ &\leq \left\{ \int_0^a \rho \zeta J_m^2(\rho\zeta) d\zeta \int_0^a \left[\sqrt{r\zeta} J_m(r\zeta) - \sum_{l=0}^L \Upsilon_l T_l(r) T_l(\zeta) \right]^2 d\zeta \right\}^{1/2} \\ &\leq B \left\| \sqrt{r\zeta} J_m(r\zeta) - \sum_{l=0}^L \Upsilon_l T_l(r) T_l(\zeta) \right\|_a. \end{aligned} \quad (38)$$

Expansion (36) allows us to compute the trace of H_{am} , namely for $\rho = r$

$$r J_m(r^2) = \sum_{l=0}^{\infty} \Upsilon_l T_l^2(r),$$

and hence

$$\text{tr } H_{am} = \sum_{l=0}^{\infty} \Upsilon_l = \int_0^a r J_m(r^2) dr = \frac{1}{2} \int_0^{a^2} J_m(r) dr.$$

Similarly from (37) we obtain that

$$\begin{aligned} \text{tr } K_{am} &= \sum_{l=0}^{\infty} \Upsilon_l^2 = \int_0^a \mathcal{K}_{am}(r, r) dr = \int_0^a \int_0^a r \rho J_m^2(r \rho) d\rho dr \\ \sum_{l=0}^{\infty} \Upsilon_l^3 &= \int_0^a \int_0^a \int_0^a r \rho \zeta J_m(r \rho) J_m(\rho \zeta) J_m(\zeta r) d\zeta d\rho dr, \\ \sum_{l=0}^{\infty} \Upsilon_l^4 &= \int_0^a \int_0^a \mathcal{K}_{am}^2(r, \rho) d\rho dr \\ &= \int_0^a \int_0^a \int_0^a \int_0^a r \rho \zeta \xi J_m(r \rho) J_m(\rho \zeta) J_m(\zeta \xi) J_m(\xi r) d\xi d\zeta d\rho dr. \end{aligned} \tag{39}$$

In [15] the asymptotic behaviour of the eigenvalues $\Upsilon_{a\nu l}$ was studied both for large l and for large a , and it was found that

$$\Upsilon_{a\nu l} \rightarrow 0, \text{ as } l \rightarrow \infty \quad \text{and} \quad \Upsilon_{a\nu l} \rightarrow (-1)^l \text{ as } a \rightarrow \infty ;$$

here ν is not necessarily an integer, $\nu = 1/2$ corresponds to the TFT case. The arguments of [15] did not involve any estimates generalizing formulae (10), (11), however such a generalization seems reasonable: similarly to the TFT case, the relevant estimates would allow us to determine the number of eigenvalues Υ_l essentially different from zero, i.e. to determine the number of degrees of freedom in $\mathcal{B}_{am}$.

6 Number of degrees of freedom

Hereafter we assume that $a \gg 1$. The internal integral in (39) is the same Lommel integral as in (28):

$$\text{tr } K_{am} = \frac{a^2}{2} \int_0^a r (J_m^2(ar) + J_{m+1}^2(ar)) dr - a m \int_0^a J_m(ar) J_{m+1}(ar) dr$$

$$\begin{aligned}
&= \frac{a^4}{4} \left(J_m'^2(a^2) + J_m^2(a^2) + J_{m+1}'^2(a^2) + J_{m+1}^2(a^2) \right) \\
&- \frac{m^2}{4} J_m^2(a^2) - \frac{(m+1)^2}{4} J_{m+1}^2(a^2) - \frac{m}{2} J_{m+1}^2(a^2) \\
&- m(m+1) \int_0^a \frac{J_{m+1}^2(ar)}{r} dr.
\end{aligned} \tag{40}$$

On substituting the asymptotics (26) and the normalization condition [32]

$$\int_0^\infty \frac{J_k^2(ar)}{r} dr = \frac{1}{2k} \quad k = 1, 2, \dots, \tag{41}$$

into (40), we conclude that

$$\text{tr} K_{am} = \sum_{l=0}^\infty \Upsilon_l^2 = \frac{a^2}{\pi} - \frac{m}{2} + O\left(\frac{1}{a^2}\right). \tag{42}$$

As the next step we formulate and prove an analogue of the inequality (11). To this end, we first show that

$$\sum_{l=0}^\infty (\Upsilon_l^2 - \Upsilon_l^4) = \int_0^a \int_a^\infty \mathcal{K}_{am}^2(r, \rho) d\rho dr. \tag{43}$$

This equality follows from Eqs. (39), along with Parseval's theorem which in the case of the Hankel transform claims that $\|h\|_\infty = \|H_m[h]\|_a, \forall h \in \mathcal{B}_{am}$ [28]. In particular,

$$\begin{aligned}
\int_0^a r \int_0^a \xi J_m^2(r\xi) d\xi dr &= \int_0^a r \int_0^\infty \rho \left(\int_0^a \xi J_m(r\xi) J_m(\xi\rho) d\xi \right)^2 d\rho dr \\
&= \int_0^a \int_0^\infty \mathcal{K}_{am}^2(r, \rho) d\rho dr,
\end{aligned}$$

which implies (43). One can rewrite the right-hand side of Eq. (43) differently:

$$\mathcal{K}_{am}(r, \rho) = a^2 \sqrt{r\rho} \int_0^1 \zeta J_m(ar\zeta) J_m(a\rho\zeta) d\zeta = a^2 \mathcal{K}_{1m}(ar, a\rho),$$

and hence

$$\sum_{l=0}^\infty (\Upsilon_l^2 - \Upsilon_l^4) = \int_0^a \int_a^\infty \mathcal{K}_{am}^2(r, \rho) d\rho dr = \int_0^{a^2} \int_{a^2}^\infty \mathcal{K}_{1m}^2(r, \rho) d\rho dr. \tag{44}$$

We use Eq. (44) in the proof of the following proposition which generalizes the inequality (11).

Proposition 1. *There exist constants $\tilde{B}_1$ and $\tilde{B}_2$ such that*

$$\sum_{l=0}^{\infty} (\Upsilon_l^2 - \Upsilon_l^4) = \sum_{l=0}^{\infty} \Upsilon_l^2 (1 - \Upsilon_l^2) \leq \tilde{B}_1 \ln a + \tilde{B}_2. \quad (45)$$

Proof. We first consider the subdomain $[0, 1] \times [a^2, \infty)$ partitioned off from the whole integration area in the right hand side of (44). Since for any k , $|J_k(r)| \leq 1$, and due to (26) there exists a constant C_1 such that

$$\begin{aligned} \mathcal{I}_1 &:= \int_0^1 \int_{a^2}^{\infty} \mathcal{K}_{1m}^2(r, \rho) d\rho dr \\ &= \int_0^1 \int_{a^2}^{\infty} \frac{\rho r}{(\rho^2 - r^2)^2} \left[\rho J_{m+1}(\rho) J_m(r) - r J_m(\rho) J_{m+1}(r) \right]^2 d\rho dr \\ &\leq 2 \int_{a^2}^{\infty} \frac{\rho}{(\rho^2 - 1)^2} \left[\rho^2 J_{m+1}^2(\rho) + J_m^2(\rho) \right] d\rho \leq \frac{C_1}{a^2}. \end{aligned}$$

Substituting (21) and (23) into the remaining part of the integral in (44) yields:

$$\begin{aligned} &\int_1^{a^2} \int_{a^2}^{\infty} \mathcal{K}_{1m}^2(r, \rho) d\rho dr \\ &= \int_1^{a^2} \int_{a^2}^{\infty} \frac{\rho r}{(\rho^2 - r^2)^2} \left[\rho J_m'(\rho) J_m(r) - r J_m(\rho) J_m'(r) \right]^2 d\rho dr \\ &= \int_1^{a^2} \int_{a^2}^{\infty} \frac{\mathcal{A}^2(\rho) \mathcal{A}^2(r) [\rho \sin \Phi(\rho) \cos \Phi(r) - r \sin \Phi(r) \cos \Phi(\rho)]^2}{(\rho^2 - r^2)^2} d\rho dr \\ &\leq \int_1^{a^2} \int_{a^2}^{\infty} \frac{\mathcal{A}^2(\rho) \mathcal{A}^2(r)}{(\rho - r)^2} \sin^2 \left(\Phi(\rho) - \Phi(r) \right) d\rho dr \\ &+ \int_1^{a^2} \int_{a^2}^{\infty} \frac{\mathcal{A}^2(\rho) \mathcal{A}^2(r)}{(\rho + r)^2} \sin^2 \left(\Phi(\rho) + \Phi(r) \right) d\rho dr. \end{aligned}$$

To obtain an estimate of the integral

$$\mathcal{I}_2 := \int_1^{a^2} \int_{a^2}^{\infty} \frac{\mathcal{A}^2(\rho) \mathcal{A}^2(r)}{(\rho - r)^2} \sin^2 \left(\Phi(\rho) - \Phi(r) \right) d\rho dr,$$

we use the definition of the Prüfer angle, namely that $\Phi(\rho) - \Phi(r) = (\rho - r) + (\theta(\rho) - \theta(r))$ and the boundedness of the function $\mathcal{A}$ on the interval $[1, \infty)$:

$$\mathcal{I}_2 \leq 2\bar{A}^4 \int_1^{a^2} \int_{a^2}^{\infty} \left\{ \frac{\cos^2(\theta(\rho) - \theta(r))}{(\rho - r)^2} \sin^2(\rho - r) + \frac{\sin^2(\theta(\rho) - \theta(r))}{(\rho - r)^2} \cos^2(\rho - r) \right\} d\rho dr.$$

For the first term of the latter integral we apply the same estimate as in (11):

$$\int_1^{a^2} \int_{a^2}^{\infty} \frac{\sin^2(\rho - r)}{(\rho - r)^2} \cos^2(\theta(\rho) - \theta(r)) d\rho dr \leq \int_0^{a^2} \int_{a^2}^{\infty} \frac{\sin^2(\rho - r)}{(\rho - r)^2} d\rho dr \leq C_2 \ln(a^2) + C_3.$$

Further, due to Eq. (19) for $\rho > r$

$$\left| \sin(\theta(\rho) - \theta(r)) \right| \leq |\theta(\rho) - \theta(r)| \leq \int_r^{\rho} |\theta'(x)| dx \leq (m^2 + 1/4) \frac{\rho - r}{\rho r},$$

holds, and hence a constant C_4 can be found such that

$$\int_1^{a^2} \int_{a^2}^{\infty} \frac{\sin^2(\theta(\rho) - \theta(r))}{(\rho - r)^2} \cos^2(\rho - r) d\rho dr \leq (m^2 + 1/4)^2 \int_1^{a^2} \int_{a^2}^{\infty} \frac{d\rho dr}{\rho^2 r^2} \leq \frac{C_4}{a^2},$$

Finally,

$$\mathcal{I}_3 := \int_1^{a^2} \int_{a^2}^{\infty} \frac{\mathcal{A}^2(\rho) \mathcal{A}^2(r)}{(\rho + r)^2} \sin^2(\Phi(\rho) + \Phi(r)) d\rho dr \leq \int_1^{a^2} \int_{a^2}^{\infty} \frac{\bar{A}^4 d\rho dr}{(\rho + r)^2} \leq C_5,$$

which completes the proof of the inequality (45), since the integrals $\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3$ compose the right hand side of Eq. (44), and as a result, the sum $\sum_{l=0}^{\infty} (\Upsilon_l^2 - \Upsilon_l^4)$. $\square$

Thus we can conclude that the number of the eigenvalues Υ_l essentially different from zero does not exceed $\frac{a^2}{\pi} - \frac{m}{2}$ and

$$\Upsilon_l \approx \begin{cases} 1, & l \ll \frac{a^2}{\pi} - \frac{m}{2}, \\ 0, & l \gg \frac{a^2}{\pi} - \frac{m}{2}. \end{cases} \quad (46)$$

The proof repeats that for TFT in [16].

7 Kramer's Sampling Theorem in $\mathcal{B}_{ma}$

For an arbitrary function $h \in \mathcal{B}_{ma}$, Kramer's sampling formula can be written as

$$h(r) = \sum_{n=0}^{\infty} h\left(\frac{r_{mn}}{a}\right) \frac{K_{am}\left(\frac{r_{mn}}{a}, r\right)}{K_{am}\left(\frac{r_{mn}}{a}, \frac{r_{mn}}{a}\right)} = \sum_{n=0}^{\infty} h\left(\frac{r_{mn}}{a}\right) \frac{2\sqrt{a r_{mn} r} J_m(ar)}{(r_{mn}^2 - (ar)^2) J_{m+1}(r_{mn})}. \quad (47)$$

see [29, 10, 11].

The representation (47) is easy to obtain by expanding the function $g = H_m[h]$ into the Fourier series in the functions $\sqrt{\frac{r_{mn}r}{a}} J_m\left(\frac{r_{mn}r}{a}\right) / K_{am}\left(\frac{r_{mn}}{a}, \frac{r_{mn}}{a}\right)$, $n = 0, 1, 2, \dots$, (which form an orthonormal basis in $\mathcal{L}_2(I_a)$), and then integrating the series versus the Hankel transform kernel. According to Eq. (13), the Fourier coefficients of the function g are the samples $h\left(\frac{r_{mn}}{a}\right)$, and

$$g(\rho) = \sum_{n=0}^{\infty} h\left(\frac{r_{mn}}{a}\right) \sqrt{\frac{r_{mn}\rho}{a}} \frac{J_m\left(\frac{r_{mn}\rho}{a}\right)}{K_{am}\left(\frac{r_{mn}}{a}, \frac{r_{mn}}{a}\right)}. \quad (48)$$

The above series converges in $\mathcal{L}_2(I_a)$ and by Parseval's theorem, this implies the convergence of the sampling series (47) in $\mathcal{L}_2(\mathbb{R}_+)$

$$\begin{aligned} \varsigma_N^2 &:= \int_0^{\infty} \left\{ h(r) - \sum_{n=0}^N h\left(\frac{r_{mn}}{a}\right) \frac{K_{am}\left(\frac{r_{mn}}{a}, r\right)}{K_{am}\left(\frac{r_{mn}}{a}, \frac{r_{mn}}{a}\right)} \right\}^2 dr \\ &= \int_0^a \left\{ g(\rho) - \sum_{n=0}^N h\left(\frac{r_{mn}}{a}\right) \sqrt{\frac{r_{mn}\rho}{a}} \frac{J_m\left(\frac{r_{mn}\rho}{a}\right)}{K_{am}\left(\frac{r_{mn}}{a}, \frac{r_{mn}}{a}\right)} \right\}^2 d\rho, \end{aligned}$$

as well as the uniform convergence on $\mathbb{R}_+$ (cf. Section 3),

$$\varepsilon_N := \left| h(r) - \sum_{n=0}^N h\left(\frac{r_{mn}}{a}\right) \frac{K_{am}\left(\frac{r_{mn}}{a}, r\right)}{K_{am}\left(\frac{r_{mn}}{a}, \frac{r_{mn}}{a}\right)} \right| \leq B \varsigma_N,$$

with the constant B being the same as in (28).

On substituting $\rho = r_{mn}/a$ in (37) one rewrites the sampling formula (47) in terms of $T_l(r)$:

$$h(r) = \frac{2}{a} \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} \Upsilon_l^2 \frac{h\left(\frac{r_{mn}}{a}\right) T_l\left(\frac{r_{mn}}{a}\right)}{r_{mn} J_{m+1}^2(r_{mn})} T_l(r). \quad (49)$$

Similar to the case of prolates, the sampling formula with respect to the THT eigenfunctions provides new discrete orthogonality relations. Let us multiply both sides of the equality

$$T_p(r) = \frac{2}{a} \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} \Upsilon_l^2 \frac{T_p\left(\frac{r_{mn}}{a}\right) T_l\left(\frac{r_{mn}}{a}\right)}{r_{mn} J_{m+1}^2(r_{mn})} T_l(r)$$

by $T_s(r)$, and integrate them over $[0, a]$. This yields

$$\frac{2 \Upsilon_s^2}{a} \sum_{n=0}^{\infty} \frac{T_p\left(\frac{r_{mn}}{a}\right) T_s\left(\frac{r_{mn}}{a}\right)}{r_{mn} J_{m+1}^2(r_{mn})} = \delta_{ps}. \quad (50)$$

Another discrete orthogonality relation is easily derived from the expansion of the reproducing kernel $\mathcal{K}_{am}$:

$$\sum_{l=0}^{\infty} \Upsilon_l^2 T_l\left(\frac{r_{mn}}{a}\right) T_l\left(\frac{r_{mk}}{a}\right) = \mathcal{K}_{am}\left(\frac{r_{mn}}{a}, \frac{r_{mk}}{a}\right) = \frac{a r_{mn}}{2} J_{m+1}^2(r_{mn}) \delta_{nk}. \quad (51)$$

8 Truncation numbers

As in the modified WKS formula, the summations over n and l in (49) are interchangeable, i.e.

$$h(r) = \frac{2}{a} \sum_{l=0}^{\infty} \Upsilon_l^2 \left\{ \sum_{n=0}^{\infty} \frac{h\left(\frac{r_{mn}}{a}\right) T_l\left(\frac{r_{mn}}{a}\right)}{r_{mn} J_{m+1}^2(r_{mn})} \right\} T_l(r). \quad (52)$$

One derives the latter sampling formula from the Fourier expansion of the function $g = H_m[h]$ in terms of T_l , the Fourier coefficients then being

$$g_l := \int_0^a g(\rho) T_l(\rho) d\rho = \sum_{n=0}^{\infty} \Upsilon_l T_l\left(\frac{r_{mn}}{a}\right) \int_0^a \sqrt{\frac{r_{mn} \rho}{a}} \frac{g(\rho) J_m\left(\frac{r_{mn} \rho}{a}\right)}{K_{am}\left(\frac{r_{mn}}{a}, \frac{r_{mn}}{a}\right)} d\rho,$$

where inside the interval I_a the functions T_l are replaced by their expansions (48).

Similarly to (47), the sampling series (49) and (52) converge both in $\mathcal{L}_2(\mathbb{R}_+)$ and uniformly on $\mathbb{R}_+$. However because of the double summation, they look much more complicated than (47), and without some additional assumptions on the sampled function seem not so worthy. Yet, for the functions from the range of K_{am} , the representation (49) becomes more useful than (47). The proposition below shows that the contribution from T_l at indices $l > L$ in the Fourier expansion of $h \in \text{Rg}(K_{am})$ is at most Υ_{L+1} as large as that in the associated function $G = K_{am}^{-1}(h)$.

Proposition 2. *Let $h \in \text{Rg}(K_{am})$, i.e. $h(r) = \int_0^a \mathcal{K}_{am}(r \rho) G(\rho) d\rho$ for some $G \in \mathcal{L}_2(I_a)$. Denote by h_l , G_l the Fourier coefficients (in terms of T_l) of the function h and G , respectively (we remind that for h its Fourier expansion is valid on the $\mathbb{R}_+$, while for G only on I_a). Then for the truncation error caused by neglect of the contribution from T_l at $l > L$*

$$\varepsilon_L := \left\| h - \sum_{l=0}^L h_l T_l \right\|_{\mathbb{R}_+} \leq \Upsilon_{L+1} \left\| G - \sum_{l=0}^L G_l T_l \right\|_a \leq \Upsilon_{L+1} \|G\|_a.$$

Proof. For the proof, we notice that inside the interval I_a the Hankel transform $H_m[h](\eta)$ coincides with the band-limited function $g(\eta) := \int_0^a \sqrt{\eta\rho} J_m(\eta\rho) G(\rho) d\rho$ and hence is itself representable as

$$H_m[h](\eta) = \frac{2}{a} \sum_{l=0}^{\infty} \Upsilon_l^2 \left\{ \sum_{n=0}^{\infty} \frac{g\left(\frac{r_{mn}}{a}\right) T_l\left(\frac{r_{mn}}{a}\right)}{r_{mn} J_{m+1}^2(r_{mn})} \right\} T_l(\eta), \quad \eta \in I_a, \quad (53)$$

so that $h_l = \Upsilon_l^2 G_l$. Therefore on account of Parseval's theorem and the orthogonality relation (33),

$$\begin{aligned} \varepsilon_L^2 &= \int_0^a \left(g(\eta) - \sum_{l=0}^L \Upsilon_l G_l T_l(\eta) \right)^2 d\eta = \sum_{l>L} \Upsilon_l^2 G_l^2 \\ &\leq \Upsilon_{L+1}^2 \left\| G - \sum_{l=0}^L G_l T_l \right\|_a^2. \end{aligned}$$

□

Looking back at the relations (46), we see that the reasonable truncation number is $L \sim \frac{a^2}{\pi} - \frac{m}{2}$.

Now let us compute the error caused by the truncation of the inner sum in (52),

$$\varepsilon_{LN} := \left\| \frac{2}{a} \sum_{l=0}^L \sum_{n>N} \Upsilon_l^2 \frac{h\left(\frac{r_{mn}}{a}\right) T_l\left(\frac{r_{mn}}{a}\right)}{r_{mn} J_{m+1}^2(r_{mn})} T_l(r) \right\|_{\mathbb{R}_+}.$$

Here again the functions from $\text{Rg}(K_{am})$ are advantageous over others in $\mathcal{B}_{am}$ because of the convergence rate of their samples.

Proposition 3. *Let $h \in \text{Rg}(K_{am})$, then samples $h\left(\frac{r_{mn}}{a}\right)$ decay at infinity as $1/r_{mn}$.*

Proof. To show that, we write for $a^2 < r_{mN} < r_{mn}$:

$$\begin{aligned} h^2\left(\frac{r_{mn}}{a}\right) &= \left\{ \int_0^a \mathcal{K}_{am}\left(\frac{r_{mn}}{a}, \rho\right) G(\rho) d\rho \right\}^2 \\ &\leq \|G\|_a^2 \left(\frac{r_{mn}}{a}\right)^2 r_{mn} J_{m+1}^2(r_{mn}) \int_0^a \frac{a \rho J_m^2(a\rho)}{\left(\left(\frac{r_{mn}}{a}\right)^2 - \rho^2\right)^2} d\rho \\ &\leq \frac{\|G\|_a^2 B r_{mn} J_{m+1}^2(r_{mn})}{\left(1 - \frac{a^4}{r_{mN}^2}\right)^2} \left(\frac{a}{r_{mn}}\right)^2. \end{aligned} \quad (54)$$

□

Note that T_l are also in $\text{Rg}(K_{am})$, and hence the above estimate holds for them with Υ_l^{-2} replacing $\|G\|_a$.

As follows from (30), $r_{mn} = r_{mN} + \pi(n - N) + \mathcal{O}(1/n^2)$, and

$$\mathcal{R}_N = \sum_{n=N}^{\infty} \frac{1}{r_{mn}^2} \leq \frac{1}{r_{mN}}. \quad (55)$$

For the truncation error ε_{LN} one writes

$$\begin{aligned} \varepsilon_{LN}^2 &= \int_0^\infty \left(\frac{2}{a} \sum_{l=0}^L \sum_{n>N} \Upsilon_l^2 \frac{h\left(\frac{r_{mn}}{a}\right) T_l\left(\frac{r_{mn}}{a}\right)}{r_{mn} J_{m+1}^2(r_{mn})} T_l(r) \right)^2 dr \\ &= \sum_{l=0}^L \frac{4 \Upsilon_l^2}{a^2} \left(\sum_{n>N} \frac{h\left(\frac{r_{mn}}{a}\right) T_l\left(\frac{r_{mn}}{a}\right)}{r_{mn} J_{m+1}^2(r_{mn})} \right)^2 \\ &\leq \frac{4}{a^2} \sum_{n>N} \frac{h^2\left(\frac{r_{mn}}{a}\right)}{r_{mn} J_{m+1}^2(r_{mn})} \sum_{l=0}^L \sum_{n>N} \frac{\Upsilon_l^2 T_l^2\left(\frac{r_{mn}}{a}\right)}{r_{mn} J_{m+1}^2(r_{mn})} \end{aligned} \quad (56)$$

Summing up the relations (54), (55) and (56), we obtain an upper bound on the error ε_{LN} ,

$$\varepsilon_{LN} \leq \frac{2 B \|G\|_a}{a \left(1 - \frac{a^4}{r_{mN}^2}\right)^2} \left[\sum_{l=0}^L \frac{1}{\Upsilon_l^2} \right]^{1/2} \left(\frac{a^2}{r_{mN}} \right). \quad (57)$$

Here it is worth noticing that, if a is large and $r_{mN} > a^2$, the contribution from

$$\Sigma_l := \sum_{n>N} \frac{\Upsilon_l^2 T_l^2\left(\frac{r_{mn}}{a}\right)}{r_{mn} J_{m+1}^2(r_{mn})}$$

into the truncation error ε_{LN} is essential only for $l \sim \frac{a^2}{\pi} - \frac{m}{2}$ and the following proposition holds:

Proposition 4. *Let $r_{mN} > a^2$. There exists a constant D such that for any $l = 0, 1, \dots$,*

$$\Sigma_l \leq D \Upsilon_l \sqrt{1 - \Upsilon_l^2}.$$

Proof. To begin with, any THT eigenfunction satisfies the inequality

$$\left| \int_{r>a} T_l^2(r) dr - \sum_{n>N} \frac{r_{mn+1} - r_{mn}}{a} T_l^2\left(\frac{r_{mn}}{a}\right) \right| \leq 2 \left\{ \int_{r>a} T_l^2(r) dr \int_{r>a} (T_l'(r))^2 dr \right\}^{1/2}. \quad (58)$$

The proof of (58) is almost identical to that of Lemma 4 in [30], and is therefore omitted here.

If $m \geq 1$, we have in the right-hand side of (58)

$$\begin{aligned}
\int_a^\infty (T_l'(r))^2 dr &\leq - \int_0^\infty T_l(r) T_l''(r) dr \\
&= -\frac{1}{\Upsilon_l^2} \int_0^a \int_0^a T_l(\rho) T_l(\xi) \int_0^\infty \sqrt{r\rho} J_m(r\rho) \frac{\partial^2}{\partial r^2} \left(\sqrt{r\xi} J_m(r\xi) \right) dr d\rho d\xi \\
&= \frac{1}{\Upsilon_l^2} \int_0^a \int_0^a \sqrt{\xi\rho} T_l(\rho) T_l(\xi) \int_0^\infty \left(\xi^2 - \frac{m^2 - 1/4}{r^2} \right) r J_m(r\rho) J_m(r\xi) dr d\rho d\xi \\
&\leq \frac{1}{\Upsilon_l^2} \int_0^a \int_0^a \sqrt{\xi\rho} T_l(\rho) T_l(\xi) \xi \delta(\rho - \xi) d\rho d\xi = \frac{1}{\Upsilon_l^2} \int_0^a (\rho T_l(\rho))^2 d\rho \leq \frac{a^2}{\Upsilon_l^2}.
\end{aligned}$$

Here we have used Eq. (41), Bessel's equation written in the form

$$\frac{d^2}{dr^2} (\sqrt{r} J_m(r)) + \left(1 - \frac{m^2 - 1/4}{r^2} \right) \sqrt{r} J_m(r) = 0$$

and the relation $\int_0^\infty r J_m(r\rho) J_m(r\xi) dr = \delta(\rho - \xi)/\xi$.

Even if $m = 0$, one can still obtain a similar estimate, e.g. by proving that

$$\int_a^\infty (T_l'(r))^2 dr \leq T_l(a) T_l'(a) + (\Upsilon_l^{-2} - 1) \int_0^a (\rho T_l(\rho))^2 d\rho + \frac{1}{4} \int_a^\infty \left(\frac{T_l(r)}{r} \right)^2 dr \leq \frac{D_1}{\Upsilon_l^2},$$

with a constant D_1 independent of l .

Let Δ_m be the smallest distance between two zeros of $J_m(r)$, then it follows from (58)

$$\begin{aligned}
\frac{\Delta_m}{a} \sum_{n>N} T_l^2 \left(\frac{r_{mn}}{a} \right) &\leq \sum_{n>N} \frac{r_{mn+1} - r_{mn}}{a} T_l^2 \left(\frac{r_{mn}}{a} \right) \\
&\leq \int_{r>a} T_l^2(r) dr + 2 \left\{ \int_{r>a} T_l^2(r) dr \int_{r>a} (T_l'(r))^2 dr \right\}^{1/2}.
\end{aligned}$$

In view of (31) we write (cf. [30]) for $m \geq 1$

$$\Sigma_l \leq \frac{1}{\underline{A}^2} \sum_{n>N} \Upsilon_l^2 T_l^2 \left(\frac{r_{mn}}{a} \right) \leq \frac{a}{\underline{A}^2 \Delta_m} \left\{ \Upsilon_l^2 (1 - \Upsilon_l^2) + 2a \Upsilon_l \sqrt{1 - \Upsilon_l^2} \right\}. \quad (59)$$

The first term in the right-hand side of (59) is dominated by the second one, whence the result. For $m = 0$ the parameter a is to be replaced with $\sqrt{D_1}$. $\square$

In any case $\Upsilon_l \sqrt{1 - \Upsilon_l^2}$ is very small for large a and l essentially different from $\frac{a^2}{\pi} - \frac{m}{2}$. Thus the optimal truncation number N (i.e. the number of employed samples of the reconstructed function $h \in \text{Rg}(K_{am})$) is the smallest one for which $r_{mN} > a^2$, that is $N \sim \frac{a^2}{\pi} - m$.

9 Important remarks

Our first important remark concerns Hankel-band-limited functions that are not in the range of K_{am} . If such a function can still be well approximated by a linear combination of the first $L \sim \frac{a^2}{\pi} - \frac{m}{2}$ THT eigenfunctions T_l , then again the main contribution to the truncation error ε_{LN} comes from T_l at $l \sim L$, and although the samples $h\left(\frac{r_{mn}}{a}\right)$ may decay not as fast as previously, one can still write

$$\begin{aligned} \varepsilon_{LN}^2 &\leq \frac{4}{a^2} \sum_{n>N} \frac{h^2\left(\frac{r_{mn}}{a}\right)}{r_{mn} J_{m+1}^2(r_{mn})} \sum_{l=0}^L \sum_{n>N} \frac{\Upsilon_l^2 T_l^2\left(\frac{r_{mn}}{a}\right)}{r_{mn} J_{m+1}^2(r_{mn})} \\ &\sim \frac{4B \|h\|_{\mathbb{R}_+}^2}{A^2 \left(1 - \frac{a^4}{r_{mN}^2}\right)^2} \left(\frac{1}{\Upsilon_L^2 r_{mN}}\right). \end{aligned}$$

Not every Hankel-band-limited function is well approximated by a truncated sampling sum (49). As an obvious counter example one can consider here the kernel $\mathcal{K}_{am}(\tilde{r}, r)$. For any fixed $\tilde{r} > 0$ this function is evidently Hankel-band limited, however, as is readily seen from (37),

$$\varepsilon_L^2 = \sum_{l>L} \Upsilon_l^2 T_l^2(\tilde{r}) = \left\| \sqrt{\tilde{r}} J_m(\tilde{r} r) - \sum_{l=0}^L \left(\int_0^a T_l(\rho) \sqrt{\tilde{r} \rho} J_m(\tilde{r} \rho) d\rho \right) T_l(r) \right\|_a^2.$$

Here the error ε_L converges to zero as L increases, but not as fast as Υ_L .

Additionally we remark that by (46) we have obtained an explicit bound $M \sim 2a^2/\pi$ on the number of effective angular momenta m required in sampling of two-dimensional band-limited signals.

Finally we would like to emphasize that the range of operator K_{am} plays in $\mathcal{B}_{am}$ the role similar to that of $\text{Rg}(G_\Omega)$ in the Paley–Wiener space. With minor changes the results obtained for $\text{Rg}(K_{am})$ can be transferred to $\text{Rg}(G_\Omega)$ and the functions from $\text{Rg}(G_\Omega)$ to a high accuracy can be reconstructed from their $2\Omega^2/\pi$ samples through the first $2\Omega^2/\pi$ THT eigenfunctions.

10 Evaluation of the THT eigenfunctions

As proved in [26], $T_l(r)$ are the bounded solutions of the equation

$$\frac{d}{dr}(a^2 - r^2) \frac{d}{dr} T_l(r) + a^2 \left[\lambda_l - r^2 - \frac{m^2 - 1/4}{r^2} \right] T_l(r) = 0, \quad 0 < r < a, \quad (60)$$

i.e., the eigenfunctions of a singular self-adjoint Sturm–Liouville problem. The author of [26] called these solutions generalized prolate spheroidal functions, and investigated various asymptotic cases, e.g. for both small and large radii a (see also [15, 13]). Based on the discussion in [26], as well as on the approach developed in [1, 2, 20] for numerical solution of singular self-adjoint Sturm–Liouville problems, a new efficient, robust and accurate numerical technique was recently invented for evaluation of THT eigenfunctions and eigenvalues. In [19] its tentative version was briefly discussed and first preliminary calculations illustrating the facilities of the method were presented. In [4] various numerical methods for solving the problem (60) are discussed in detail and compared. The advantages are that the eigenvalues of problem (60) are well-resolved and that the original integral equation (32) is not used for calculations but rather for their verification. The numerical approaches presented in [4] are accurate, robust and efficient and allow calculation of the THT eigenfunctions for a wide range of parameters.

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