Filtering from PNS2 Sampling

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Abstract

Periodic Nonuniform Sampling of order 2 (PNS2) is defined by two sequences with same period and some delay between them. PNS2 is known to suppress aliasing of multiband signals. Even if PNS2 reconstructs the signal by a linear combination of samples, the problem of retrieving a filtered version of the signal is more complicated. The simplest solution begins by a signal reconstruction through a sampling formula, followed by an analog filter. However new sampling formulas can solve this problem in a single stage. We give equivalent digital circuits which provide these formulas. Examples are provided, particularly when looking to retrieve subbands in communications.

Key words and phrases : stationary process, irregular sampling, sub-bands, linear filtering.

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1 Introduction

We consider a stationary process $\mathbf{Z} = \{Z(t), t \in \mathbb{R}\}$ with power spectral density $s_Z(\omega)$ defined by [1], [11]

$$\mathbb{E}[Z(t)Z^*(t-\tau)] = \int_{-\infty}^{\infty} e^{i\omega\tau} s_Z(\omega) d\omega$$

where $\mathbb{E}[\cdot]$ stands for the mathematical expectation and the superscript $*$ for the complex conjugate. The power spectrum $s_Z(\omega)$ is assumed regular enough to allow the inversions of limits or summations or integrals which intervene in computations. PNS2 (Periodic Nonuniform Sampling of order 2) deals with two periodic samplings $\mathbf{t}_1$ and $\mathbf{t}_2$

$$\mathbf{t}_1 = \{n + a, n \in \mathbb{Z}\}, \mathbf{t}_2 = \{n + b, n \in \mathbb{Z}\}$$

where the sampling rate (equal to 2) is fitted to a bandwidth 4π [10]. For $l \in \mathbb{N}^*$, and provided that

$$2l(b-a) \notin \mathbb{Z}$$

this sampling plan is able to reconstruct processes such that

$$\begin{aligned} s_Z(\omega) &= 0, \omega \notin \Delta \\ \Delta &= ((-2l-1)\pi, (-2l+1)\pi) \cup ((2l-1)\pi, (2l+1)\pi). \end{aligned} \quad (1)$$

This spectral shape is currently found in the case of telecommunications. More generally it is possible to reconstruct processes with no more than two foldings and for well-chosen $b-a$ [8], [9]. For example, it is not possible for a power spectrum filling

$$\left(-\frac{5\pi}{2}, -\frac{3\pi}{2}\right) \cup \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \cup \left(\frac{3\pi}{2}, \frac{5\pi}{2}\right)$$

which implies three foldings (with respect to $(-\pi, \pi)$) on $(-\frac{\pi}{2}, \frac{\pi}{2})$ even though the bandwidth is smaller than 4π . Whatever the bandwidth, it is possible to find the (linear) mean-square (m.s) estimate for the PNS2 sampling in the shape of two digital linear invariant filters (dLIF) [8], [9]. We find complex gains $\psi_t(\omega), \eta_t(\omega)$ defined on $(-\pi, \pi)$ (t takes any real value) such that we have the correspondences

$$\begin{cases} \eta_t(\omega) = \sum_n \alpha_n(t) e^{-in\omega}, \psi_t(\omega) = \sum_n \beta_n(t) e^{-in\omega} \\ \tilde{Z}(t+n) = \sum_k \alpha_k(t) Z(n+a-k) + \sum_k \beta_k(t) Z(n+b-k) \end{cases} \quad (2)$$

$\tilde{Z}(t)$ is the best linear estimation of $Z(t)$ i.e it minimizes σ_t^2 defined by

$$\sigma_t^2 = E \left[\left| Z(t) - \tilde{Z}(t) \right|^2 \right].$$

As explained in the appendix, the same process can be applied to a neighbouring problem. We look for the best estimation $\tilde{\mathbf{U}}$ of $\mathbf{U} = \mathcal{F}[\mathbf{Z}]$ where $\mathcal{F}$ is an analog linear invariant filter (aLIF) with complex gain $F(\omega)$ [7]. This means that (for regular enough functions)

$$\begin{cases} F(\omega) = \int_{-\infty}^{\infty} f(u) e^{-i\omega u} du \\ U(t) = \int_{-\infty}^{\infty} f(u) Z(t-u) du. \end{cases}$$

$f(t)$ is the impulse response of the filter $\mathcal{F}$.

In section 2, we give dLIF complex gains ψ_t, η_t [7] matched to sequences

$$\tilde{\mathbf{U}}_t = \left\{ \tilde{U}(t+n), n \in \mathbb{Z} \right\}.$$

η_t, ψ_t are complex gains of digital filters with inputs

$$\mathbf{Z}_a = \{Z(n+a), n \in \mathbb{Z}\}, \mathbf{Z}_b = \{Z(n+b), n \in \mathbb{Z}\}.$$

The sum of filters output is $\tilde{\mathbf{U}}_t$ (see Fig. 2). We give general formulas allowing to determine $\tilde{\mathbf{U}}_t$. We study the special case of spectra in two symmetric intervals particularly adapted to the communications domain.

Figure 1: orthogonal scheme

Figure 2: “symmetric” scheme

2 The sampling formulas

Figure 2 is a symmetric device with two dLIF with complex gains η_t, ψ_t . Inputs are the sequences $\mathbf{Z}_a, \mathbf{Z}_b$. The outputs are the sequences

$$\tilde{\mathbf{U}}_{1t} = \left\{ \tilde{U}_{1t}(n), n \in \mathbb{Z} \right\}, \tilde{\mathbf{U}}_{2t} = \left\{ \tilde{U}_{2t}(n), n \in \mathbb{Z} \right\}$$

and

$$\tilde{U}_t(n) = \tilde{U}(t+n) = \tilde{U}_{1t}(n) + \tilde{U}_{2t}(n)$$

is the (linear) m.s estimation of $\mathbf{U} = \mathcal{F}[\mathbf{Z}]$ for the PNS2 $\mathbf{t}_1, \mathbf{t}_2$. Complex gains are given in function of folded functions S_x, S_x^F defined on $(-\pi, \pi)$ by

$$\begin{cases} S_x(\omega) = \sum_n e^{2i\pi n x} s_Z(\omega + 2n\pi) \\ S_x^F(\omega) = \sum_n e^{2i\pi n x} [s_Z F](\omega + 2n\pi). \end{cases} \quad (3)$$

We obtain the symmetric formulas (see appendix) which are true whatever the power spectra supports

$$\begin{cases} \eta_t(\omega) = e^{i\omega(t-a)} \frac{S_{t-a}^F S_0 - S_{t-b}^F S_{b-a}}{S_0^2 - |S_{a-b}|^2} \\ \psi_t(\omega) = e^{i\omega(t-b)} \frac{S_{t-b}^F S_0 - S_{t-a}^F S_{a-b}}{S_0^2 - |S_{a-b}|^2} \end{cases} \quad (4)$$

Like in (2) this means that we have the equivalences

$$\begin{cases} \eta_t(\omega) = \sum_n \alpha_n(t) e^{-in\omega}, \psi_t(\omega) = \sum_n \beta_n(t) e^{-in\omega} \\ \tilde{U}(t+n) = \sum_k \alpha_k(t) Z(n+a-k) + \sum_k \beta_k(t) Z(n+b-k). \end{cases}$$

Formulae (4) are simplified for the symmetric spectral support Δ defined in (1):

$$\begin{aligned} \eta_t(\omega) &= i e^{i\omega(t-a)} \frac{F(\omega + 2l\pi) e^{2i\pi l(t-b)} - F(\omega - 2l\pi) e^{-2i\pi l(t-b)}}{2 \sin 2\pi l(b-a)} \\ \psi_t(\omega) &= i e^{i\omega(t-b)} \frac{F(\omega + 2l\pi) e^{2i\pi l(t-a)} - F(\omega - 2l\pi) e^{-2i\pi l(t-a)}}{2 \sin 2\pi l(a-b)} \end{aligned} \quad (5)$$

In this situation we know that the initial process $\mathbf{Z}$ can be reconstructed without error from the sequences $\mathbf{Z}_a, \mathbf{Z}_b$ of samples when $2l(b-a) \notin \mathbb{Z}$. The property is obviously true for the filtered process $\mathbf{U} = \mathcal{F}[\mathbf{Z}]$ and the filters do not depend on the true value of $s_Z(\omega)$ but on the support Δ . Hereafter, examples are treated highlighting filters used in communications domain in particular. Power spectra are contained in Δ (defined in (1)) for examples 1 to 5 but examples 6, 7 and 8 study more general spectra.

3 Examples

3.1 Example 1: rectangular filter

We assume that $\mathcal{F}_{\alpha\beta}$ is the band-pass filter with complex gain

$$F_{\alpha\beta}(\omega) = \begin{cases} 1, \omega \in (-\beta, -\alpha) \cup (\alpha, \beta) \\ 0 \text{ elsewhere.} \\ 2l\pi \leq \alpha < \beta \leq (2l+1)\pi \end{cases}$$

and that $s_Z(\omega)$ verifies (1). Using (5), we have for example

$$\psi_t(\omega) = \begin{cases} e^{i\omega(t-b)} \frac{ie^{2i\pi l(t-a)}}{2 \sin 2\pi l(a-b)}, \omega \in (\alpha - 2l\pi, \beta - 2l\pi) \\ e^{i\omega(t-b)} \frac{-ie^{-2i\pi l(t-a)}}{2 \sin 2\pi l(a-b)}, \omega \in (-\beta + 2l\pi, -\alpha + 2l\pi) \end{cases}$$

The Fourier development of $\psi_t(\omega)$ leads to $\tilde{U}_{2t}(n)$. The same for $\eta_t(\omega) = [\mu_t - \phi\psi_t](\omega)$ with respect to $\tilde{U}_{1t}(n)$. Finally we obtain

$$U(t) = \sum_k \alpha_k(t) Z(a-k) + \sum_k \beta_k(t) Z(b-k) \quad (6)$$

$$\alpha_k(t) = \frac{\alpha - \beta}{2\pi \sin 2\pi l (b - a)} \sin \left[\frac{\beta + \alpha}{2} (t - a + k) + 2\pi l (a - b) \right] \operatorname{sinc} \frac{\beta - \alpha}{2} (t - a + k)$$

$$\beta_k(t) = \frac{\alpha - \beta}{2\pi \sin 2\pi l (a - b)} \sin \left[\frac{\beta + \alpha}{2} (t - b + k) + 2\pi l (b - a) \right] \operatorname{sinc} \frac{\beta - \alpha}{2} (t - b + k)$$

with $\operatorname{sinc} x = (\sin x)/x$. Whatever the place of α, β with respect to $2l\pi$, we find the same result. Because $F(\omega)$ has discontinuities, the series have a poor convergence rate (in k^{-1}).

3.2 Example 2: a raised cosine filter

Raised cosine filters have a continuous and derivable (one time) complex gain. This leads to reconstruction formulas with a good convergence celerity. In the context of a symmetric filtering we have for instance (in the limit case of a "roll-off" equal to 1)

$$F(\omega) = \begin{cases} \cos^2 \frac{\pi}{2\beta} (\omega - \alpha), & |\omega - \alpha| \leq \beta \\ \cos^2 \frac{\pi}{2\beta} (\omega + \alpha), & |\omega + \alpha| \leq \beta \\ 0 & \text{elsewhere} \end{cases}$$

with $(2l - 1)\pi \leq \alpha - \beta \leq \alpha + \beta \leq (2l + 1)\pi$. When (1) is true we find with respect to (6)

$$\beta_k(t) = \frac{\pi \sin [\alpha (t + k - b) + 2\pi l (b - a)] \sin [\beta (t + k - b)]}{2\beta^2 (t + k - b) \left[(t + k - b)^2 - \left(\frac{\pi}{\beta} \right)^2 \right] \sin 2\pi l (a - b)}$$

$\alpha_k(t)$ is obtained by symmetry, changing a in b and reciprocally. As predicted by the Fourier series theory, the coefficients $\alpha_k(t), \beta_k(t)$ decrease like k^{-3} . The same property is true for a roll-off different from 0 or 1.

3.3 Example 3: Hilbert filter

We know that the Hilbert filter allows the transformation of a real process in the "analytic signal" which has lost the negative part of the spectrum. The "analytic signal" $\mathbf{U} = \mathcal{F}[\mathbf{Z}]$ is defined by

$$F(\omega) = \begin{cases} 2, & \omega > 0 \\ 0, & \omega < 0. \end{cases}$$

Easy calculations from (5) lead to

$$\alpha_k(t) = \frac{ie^{2i\pi l(t-b)}}{\sin 2\pi l (b - a)} \operatorname{sinc} \pi (t - a + k).$$

Coefficients decrease in k^{-1} , because $e^{i\omega t}$, as a periodic function in ω of period 2π , is not continuous (except when $t \in \mathbb{Z}$).

3.4 Example 4: amplitude demodulation

We assume that

$$Z(t) = V(t) \cos \omega_0(t + \Phi)$$

where $\mathbf{V} = \{V(t), t \in \mathbb{R}\}$ is stationary with spectrum in baseband and bandwidth 2δ , with

$$(\omega_0 - \delta, \omega_0 + \delta) \subset ((2l - 1)\pi, (2l + 1)\pi).$$

ω_0 is an unknown parameter. Φ is uniformly distributed on $(0, 2\pi/\omega_0)$ (for instance), independent of $\mathbf{V}$ and is introduced to stationarize the process $\mathbf{Z}$ (suppressing phase references):

$$\mathbb{E}[V(t)V^*(t - \tau)] = \int_{-\delta}^{\delta} e^{i\omega\tau} s_V(\omega) d\omega.$$

The $\mathbf{V}$ -spectrum is sent inside Δ by the amplitude modulation. $\mathbf{V}$ cannot be linearly retrieved from the observed sequences $\mathbf{Z}_a, \mathbf{Z}_b$ because $V(t)$ is orthogonal to samples $Z(n + a), Z(n + b)$. However, we have the decomposition

$$\begin{cases} Z(t) = Z_+(t) + Z_-(t) \\ s_{Z_+}(\omega) = 0, \omega \notin ((2l - 1)\pi, (2l + 1)\pi) \\ s_{Z_-}(\omega) = 0, \omega \notin ((-2l - 1)\pi, (-2l + 1)\pi). \end{cases}$$

The processes $\mathbf{Z}_+$ and $\mathbf{Z}_-$ are filterings of $\mathbf{Z}$ on bandpass filters, and can be retrieved from $\mathbf{Z}_a, \mathbf{Z}_b$. As we have

$$2Z_+(t) = V(t)e^{i\omega_0(t+\Phi)}, 2Z_-(t) = V(t)e^{-i\omega_0(t+\Phi)}$$

then

$$W(t) = Z_+(t)e^{-2il\pi t} + Z_-(t)e^{2il\pi t} = V(t) \cos[(\omega_0 - 2\pi l)t + \omega_0\Phi].$$

which is in baseband. An amplitude demodulation in low frequency allows an extraction of $\mathbf{V}$. $\mathbf{Z}_+$ and $\mathbf{Z}_-$ can be retrieved from $\mathbf{Z}_a, \mathbf{Z}_b$ ($2\mathbf{Z}_+$ is the "analytic signal"). With respect to Fig. 2 with inputs $\mathbf{Z}_a, \mathbf{Z}_b$ and output $\mathbf{W}$, we have, from (5) (with $2l(a - b) \notin \mathbb{Z}$)

$$\psi_t(\omega) = 2e^{i\omega(t-b)} \frac{\sin 2\pi l a}{\sin 2\pi l(a - b)}.$$

The filter in the first branch is obtained by symmetry. Since $\mathbf{Z}_+, \mathbf{Z}_-$ are errorless reconstructions, the filters do not depend on the $\mathbf{Z}$ -spectrum and particularly on ω_0 . Taking $a = 0$ for instance implies that the second branch has no effect ($\psi_t(\omega) = 0$). Consequently, $\mathbf{W}$ can be obtained with only one periodic sampling ($\mathbf{Z}_b$ for instance with $b \neq 0$).

3.5 Example 5: realistic sampling

Let $\mathbf{U}$ be a process with power spectrum on the set Δ defined in (1) and let $\mathbf{Z}$ be such that

$$Z(t) = \frac{1}{2\varepsilon} \int_{t-\varepsilon}^{t+\varepsilon} U(u) du.$$

The sampling sequences $\mathbf{Z}_a, \mathbf{Z}_b$ are temporal means of $\mathbf{U}$ around the points $n+a, n+b$. The process $\mathbf{Z}$ is the output of an aLIF $\mathcal{G}$ with input $\mathbf{U}$, impulse response $g(t)$ and complex gain $G(\omega)$

$$g(t) = \begin{cases} 1/2\varepsilon & \text{for } |t| < \varepsilon \\ 0 & \text{elsewhere} \end{cases}$$

$$G(\omega) = \text{sinc} \varepsilon \omega$$

Let assume that $\text{sinc} \varepsilon \omega$ does not cancel when $\omega \in \Delta$. Conversely we can consider that $\mathbf{U}$ is the output of an aLIF $\mathcal{F}$ with input $\mathbf{Z}$ and complex gain $F(\omega)$ defined by

$$F(\omega) = \frac{\varepsilon \omega}{\sin \varepsilon \omega}.$$

$\mathbf{U}$ will be reconstructed without error from the sampling sequences $\mathbf{Z}_a, \mathbf{Z}_b$ by the device of Fig. 2 with

$$\psi_t(\omega) = \frac{ie^{i\omega(t-b)}}{2 \sin 2\pi l(a-b)} \left[\frac{e^{2i\pi l(t-a)}}{\text{sinc}[\varepsilon(\omega + 2l\pi)]} - \frac{e^{-2i\pi l(t-a)}}{\text{sinc}[\varepsilon(\omega - 2l\pi)]} \right].$$

We have $\eta_t(\omega)$ by symmetry. Of course, non-rectangular shapes of samplings can be treated with the same method.

3.6 Example 6: a three bands case

Actually, it is possible to extract $\mathbf{U} = \mathcal{F}[\mathbf{Z}]$ when the foldings number of $s_Z(\omega)$ on $(-\pi, \pi)$ is not larger than 2. This means that the infinite sum defining $S_0(\omega)$ (see formulas 3) is reduced to two elements at most whatever ω . This can happen for $s_Z(\omega)$ occupying pieces of more than two bands. As an example

$$s_Z(\omega) = 0, \omega \notin (-3\pi, -2\pi) \cup (-\pi, \pi) \cup (2\pi, 3\pi).$$

where we have three (two partially) occupied bands and we sample at the Landau rate (the bandwidth is 4π). The piece $(2\pi, 3\pi)$ is folded on $(0, \pi)$ and the piece $(-3\pi, -2\pi)$ on $(-\pi, 0)$. Then $S_0(\omega)$ has only two terms which implies an errorless interpolation when $\pi(a-b) \notin \mathbb{Z}$. Using (3), (4) leads to (we take an even $F(\omega)$)

$$\beta_k(t) = \frac{1}{2\pi \sin \pi(a-b)} \int_0^\pi \{F(\omega) \sin(\omega(t-b+k) + \pi(a-b))\}$$

$$-F(\omega + 2\pi) \sin(\omega(t - b + k) - \pi(a + b) + 2\pi t)\} d\omega$$

As usual, coefficients do not depend on the true values of $s_Z(\omega)$ but on the spectral support shape.

3.7 Example 7: unlimited spectrum

Let assume that

$$s_Z(\omega) = \frac{c}{2\pi} \text{sinc}^2\left(\frac{c\omega}{2}\right)$$

where c is an integer. It is a common spectrum in physics for Gaussian process or for NRZ (the basic signal in telecommunications). Because it has more than two foldings on $(-\pi, \pi)$ (actually an infinity), an errorless interpolation of $\mathbf{Z}$ is not possible. The best one is [9]

$$\tilde{Z}(t) = \begin{cases} \frac{t-b+1}{1-b+a} Z(\bar{t}+a) + \frac{a-t}{1-b+a} Z(\bar{t}-1+b), & 0 < \underline{t} < a \\ \frac{b-t}{b-a} Z(\bar{t}+a) + \frac{t-a}{b-a} Z(\bar{t}+b), & a < \underline{t} < b \\ \frac{t-b}{1-b+a} Z(\bar{t}+1+a) + \frac{1-t+a}{1-b+a} Z(\bar{t}+b), & b < \underline{t} < 1 \end{cases}$$

where $0 < a < b < 1$, $t = \bar{t} + \underline{t}$, $\bar{t} \in \mathbb{Z}$, $0 \leq \underline{t} < 1$ ($\bar{t}$ is the entire part and $\underline{t}$ is the fractional part of t). $\tilde{Z}(t)$ only depends on the nearest samples. The interpolation error is given by

$$\mathbb{E} \left[|Z(t) - \tilde{Z}(t)|^2 \right] = \begin{cases} \frac{2(a-\underline{t})(1-b+\underline{t})}{c(1-b+a)}, & 0 < \underline{t} < a \\ \frac{2(\underline{t}-a)(b-\underline{t})}{c(1-b+a)}, & a < \underline{t} < b \\ \frac{2(1-\underline{t}+a)(\underline{t}-b)}{c(1-b+a)}, & b < \underline{t} < 1. \end{cases}$$

Whatever the filter $\mathcal{F}$, $\mathbf{U} = \mathcal{F}[\mathbf{Z}]$ cannot be reconstructed without error from $\mathbf{Z}_a, \mathbf{Z}_b$.

3.8 Example 8

However it may be possible to reconstruct perfectly some $\mathbf{U} = \mathcal{F}[\mathbf{Z}]$ but not $\mathbf{Z}$. For example let consider

$$s_Z(\omega) = \begin{cases} 1, & \omega \in (-\pi, \pi) \\ c, & \omega \in (-2\pi, -\pi) \cup (2\pi, 3\pi) \\ 0 & \text{elsewhere.} \end{cases}$$

Because it has three foldings on $(0, \pi)$, the exact reconstruction of $\mathbf{Z}$ is impossible. However it is possible to recover $\mathbf{U} = \mathcal{F}[\mathbf{Z}]$ for $\mathcal{F}$ such that

$$F(\omega) = 0, \omega \in (-2\pi, -\pi) \cup (0, \pi) \cup (2\pi, 3\pi).$$

We can take for instance $\psi_t(\omega) = 0$ whatever ω and

$$\mu_t(\omega) = \begin{cases} e^{i\omega(t-a)} F(\omega), & \omega \in (-\pi, 0) \\ 0, & \omega \in (0, \pi) \end{cases}$$

which leads to

$$U(t) = \sum_n a_n(t) Z(n+a)$$

$$a_n(t) = \frac{1}{2\pi} \int_{-\pi}^0 e^{i\omega(t-a-n)} F(\omega) d\omega.$$

4 Conclusion

Mixing periodic samplings is not a recent sampling procedure. J. L. Yen studied this case in baseband [13] (generalized when using different periods in [5]). More general irregular sampling schemes exist for an errorless reconstruction in baseband [4], [7]. Formulas for processes which are not in baseband were first proposed by J. R. Higgins [2], [3], [12], [14], in the case of PNS n (periodic nonuniform sampling of order n) which is a mixing of n periodic sequences of same period. This paper addresses a neighbouring problem, the direct reconstruction of a filtered version of the sampled process, without a previous reconstruction of the latter. We give solutions and examples in the PNS2 frame. The method can be easily generalized for PNS of higher order, at the price of a heavier load of computation.

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Appendix:

We note $\mathbf{H}_a$ and $\mathbf{H}_b$ the Hilbert spaces of linear combinations of elements taken in the sampling sequences $\mathbf{Z}_a$ and $\mathbf{Z}_b$. We look for

$$\tilde{U}(t) = \text{pr}_{\mathbf{H}_a + \mathbf{H}_b} U(t).$$

This means that $\tilde{U}(t)$ is the orthogonal projection of $U(t)$ on $\mathbf{H}_a + \mathbf{H}_b$, i.e $\tilde{U}(t)$ is the linear combination of samples which is the closest of $U(t)$ in the mean-square sense. The difficulty is that both Hilbert spaces $\mathbf{H}_a$ and $\mathbf{H}_b$ are not orthogonal most of the time. Then the projection on the sum of spaces is not the sum of projections. To bypass the difficulty, we look for the Hilbert space $\mathbf{K}$ such that

$$\mathbf{H}_a + \mathbf{H}_b = \mathbf{H}_a + \mathbf{K}, \mathbf{H}_a \perp \mathbf{K}.$$

Let $\mathbf{K}$ be the linear space generated by the sequence $\mathbf{D} = \{D_n, n \in \mathbb{Z}\}$ where [8], [9],

$$D_n = Z(n+b) - \phi[\mathbf{Z}_a](n) \tag{A-1}$$

$$\phi(\omega) = e^{i\omega(b-a)} \frac{S_{b-a}}{S_0}(\omega)$$

$S_x(\omega)$, which is defined in (3) together with $S_x^F(\omega)$, is a generalization of the familiar folded spectrum, and $S_0(\omega)$ is the spectral density of the sequences $\mathbf{Z}_a$ and $\mathbf{Z}_b$. $\phi(\omega)$ is the complex gain of a dLIF. $\phi[\mathbf{Z}_a]$ is the stationary sequence defined by the correspondence [7]

$$\phi(\omega) = \sum_k \alpha_k e^{-ik\omega} \Leftrightarrow C_n = \phi[\mathbf{Z}_a](n) = \sum_k \alpha_k Z(n+a-k).$$

$\mathbf{K}$ is a closed subspace of $\mathbf{H}_a + \mathbf{H}_b$ and then is a Hilbert space. Now, the problem is to find $\tilde{V}_{1t}(n) = \text{pr}_{\mathbf{H}_a} U(t+n)$ and $\tilde{V}_{2t}(n) = \text{pr}_{\mathbf{K}} U(t+n)$ such that

$$\tilde{U}(t+n) = \text{pr}_{\mathbf{H}_a + \mathbf{H}_b} U(t+n) = \tilde{V}_{1t}(n) + \tilde{V}_{2t}(n)$$

which gives $\tilde{U}(t) = \text{pr}_{\mathbf{H}_a + \mathbf{H}_b} U(t)$ as particular case ($n = 0$). The main result is that the sequences $\tilde{\mathbf{V}}_{1t} = \{\tilde{V}_{1t}(n), n \in \mathbb{Z}\}$ and $\tilde{\mathbf{V}}_{2t} = \{\tilde{V}_{2t}(n), n \in \mathbb{Z}\}$ are outputs of dLIF of complex gains

$$\mu_t(\omega) = e^{i\omega(t-a)} \frac{S_{t-a}^F}{S_0}(\omega) \quad (\text{A-2})$$

$$\psi_t(\omega) = e^{i\omega(t-b)} \frac{S_{t-b}^F S_0 - S_{t-a}^F S_{a-b}}{S_0^2 - |S_{a-b}|^2} \quad (\text{A-3})$$

with inputs $\mathbf{Z}_a$ and $\mathbf{D}$. For sets of ω such that $S_0(\omega) = |S_t(\omega)|$, the value of $\psi_t(\omega)$ is indifferent. The proof is a straightforward derivation of proofs in [8] or [9]. For instance, we have to write that $U(t+n) - \tilde{V}_{1t}(n)$ is orthogonal to the $Z(m+a), m \in \mathbb{Z}$, i.e to $\mathbf{H}_a$. This operation allows to determine the complex gain $\mu_t(\omega)$ using the unicity property of Fourier series. Figure 1 depicts the equivalent circuit which gives $\tilde{\mathbf{U}}_t$ from the sampling sequences $\mathbf{Z}_a, \mathbf{Z}_b$ in two orthogonal branches. Figure 2 summarizes the operations in two symmetric branches with outputs the sequences $\tilde{\mathbf{U}}_{1t}, \tilde{\mathbf{U}}_{2t}$ and

$$\eta_t(\omega) = [\mu_t - \phi\psi_t](\omega) = e^{i\omega(t-a)} \frac{S_{t-a}^F S_0 - S_{t-b}^F S_{b-a}}{S_0^2 - |S_{a-b}|^2}.$$

The filters complex gains $\eta_t(\omega), \psi_t(\omega)$ have symmetric writings with respect to a, b . Moreover the error verifies

$$\mathbb{E} \left[|U(t) - \tilde{U}(t)|^2 \right] = \mathbb{E} \left[|U(t)|^2 \right] - \mathbb{E} \left[|\tilde{V}_{1t}(0)|^2 \right] - \mathbb{E} \left[|\tilde{V}_{2t}(0)|^2 \right]$$

by application of the Pythagorean theorem. In the case where the spectral support is Δ (defined in (1)), it is well-known that the interpolation is errorless for $Z(t)$ and then also for any $U(t)$ [3].